Geometric Fourier analysis on Riemannian symmetric spaces

Summer School "Eigenfct estimates and related topics" in Marburg, July 2015

Goal: overview / survey of Riem globally symm spaces and their Fourier analysis as needed for Simon's lecture series

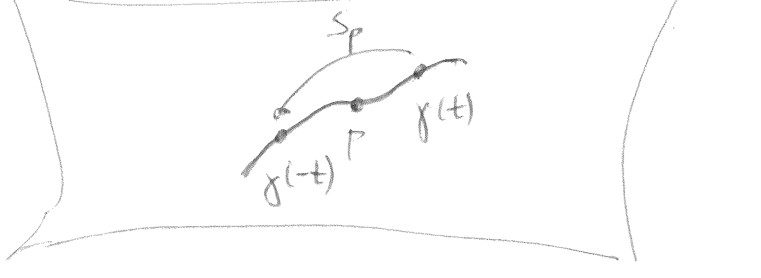
Def: Sps X Riem mfd, take $p \in X$.

geodesic symmetry at p

$$S_p(\gamma(t)) := \gamma(-t)$$

(defined in at least some nbh of p)

where γ is any geodesic on X with $\gamma(0) = p$.



X <u>Riem symm space</u> $:\Leftrightarrow$ for each $p \in X$, the geodesic symmetry S_p is an isometry of X.

<u>Rem:</u>

Study of Riem symm spaces was initiated by É. Cartan in 1926 and vigorously developed in late 1920's.

In particular, there exists a complete classification. Spaces have a lot of structure:

$$\begin{tikzcd}
\text{Space} & \text{Isometry groups} \\
\boxed{\text{Riem symm spaces}} \arrow[r, leftrightarrow] & \boxed{\begin{array}{c}\text{semisimple}\\ \text{Lie groups}\end{array}} \arrow[d, leftrightarrow] \\
& \boxed{\begin{array}{c}\text{semisimple}\\ \text{Lie algebras}\end{array}}
\end{tikzcd}$$

study problems on spaces via its isometries

(Klein's Erlangen program)

de Rham decomp for Riem symm space X:

$$X \underset{\substack{\uparrow \\ \text{as Riem} \\ \text{product}}}{\cong} \underset{\substack{\mathbin{\rotatebox{90}{$\cong$}} \\ \mathbb{R}^k}}{X_{eucl}} \times \underbrace{X_{p,1} \times \dots \times X_{p,m_p}}_{\substack{\text{irred,} \\ \text{nonneg sectional} \\ \text{curvature}}} \times \underbrace{X_{n,1} \times \dots \times X_{n,m_n}}_{\substack{\text{irred,} \\ \text{nonpos sec} \\ \text{curv}}}$$

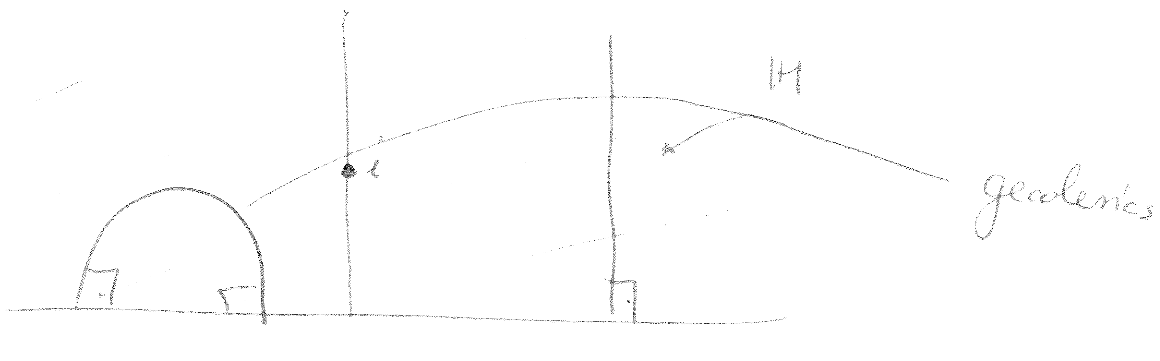
i.e., doesn't decomp nontrivially into a Riem product.

From now on assume that X is irred, nonpos sec curv ($\Rightarrow$ of noncpt type)

A specific example:

the hyperbolic plane: upper half plane model

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}, \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}$$



<u>group of Riem isometries:</u>

conn comp (= orientation-preserving):

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm \mathbb{1}\}$$ [may also use $SL_2(\mathbb{R})$]

action by Möbius transformations / fractional linear transformations:

$$\underset{PSL_2(\mathbb{R})}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \cdot \underset{\mathbb{H}}{z} = \frac{az+b}{cz+d}$$

<u>note:</u>

* action extends to

$$S\mathbb{H} = \{ v \in T\mathbb{H} \mid \|v\| = 1 \}$$

(unit tangent bundle)

by taking derivatives

* $PSL_2(\mathbb{R})$ is a Lie group.

- different elems of $PSL_2(\mathbb{R})$ can act in qualitatively different ways

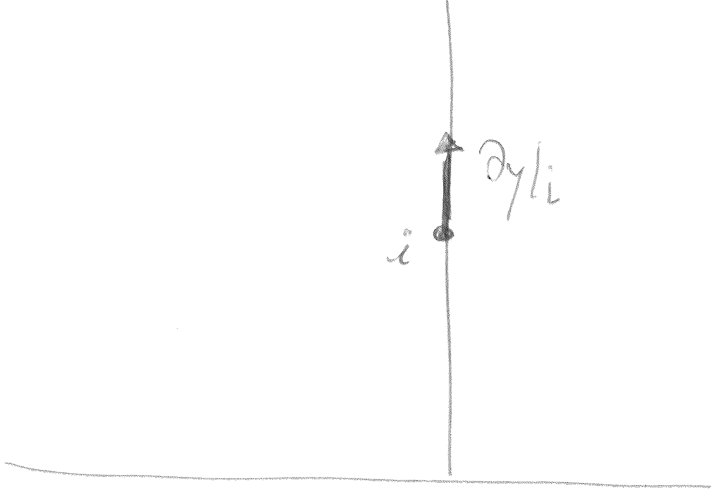
- will use this observation to introduce coordinate systems on $PSL_2(\mathbb{R})$

- will also see: $\mathbb{H}$ is a homogeneous space

To that end consider

$$i \quad \text{as origin of} \quad \mathbb{H}$$

$$\partial_y|_i \quad \text{as origin of} \quad S\mathbb{H}$$



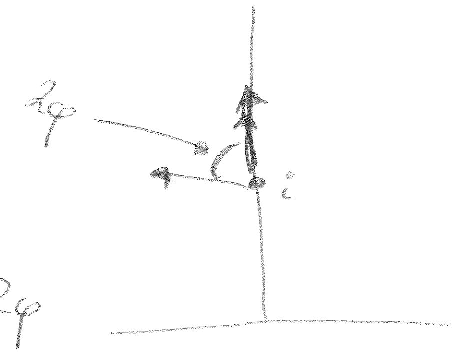
F6

0) if working with $SL_2(\mathbb{R})$:

$M := \{\pm \mathbb{1}\}$ acts trivially on $\mathbb{H}$ and $S\mathbb{H}$.

1) $\operatorname{Stab}_{PSL_2(\mathbb{R})}(i) = \left\{ k_\varphi := \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \;\middle|\; \varphi \in \mathbb{R} \right\}$

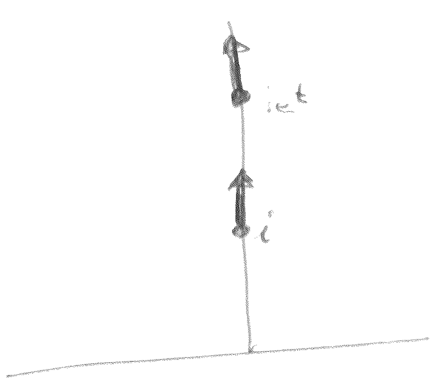
$=: K$



$k_\varphi . i = i$

$k_\varphi . (\partial y|_i) =$ rotation with angle 2φ

2) $A := \left\{ a_t := \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$



$a_t . i = ie^t$

$a_t . (\partial y|_i) = t^2 \, \partial y|_{ie^t}$

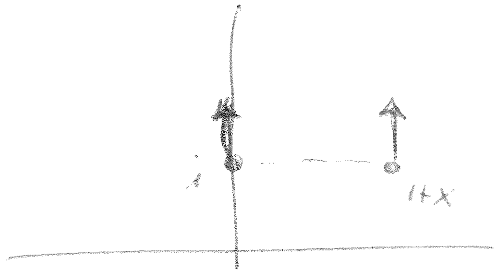
A-orbit of i produces standard geodesic through i (in the direction of $\partial y|_i$)

[F7]

3) $N := \left\{ n_x := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$

$n_x . i = i + x$

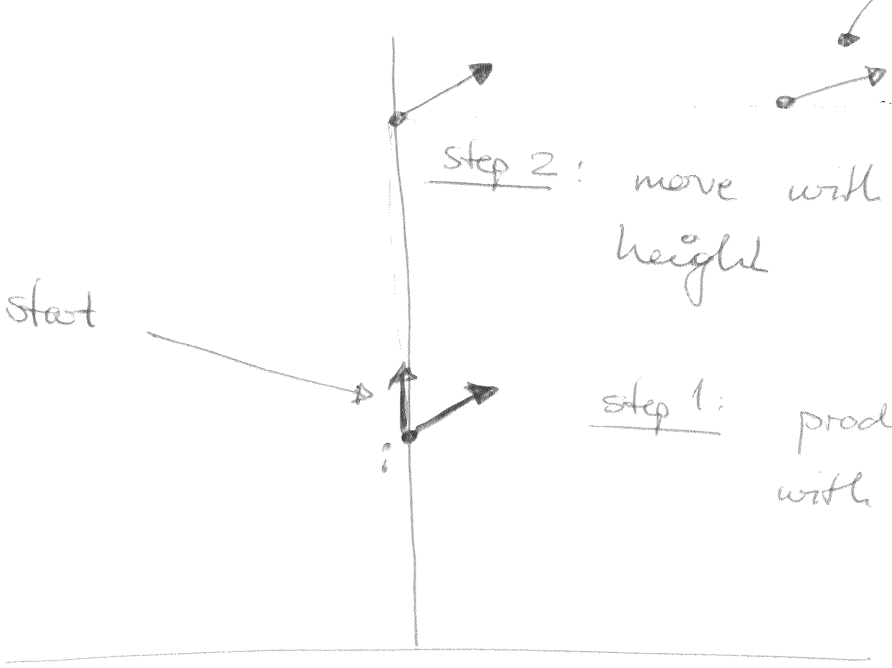
$n_x . \left(\partial_y|_i \right) = \partial_y|_{i+x}$



N-orbit of i produces horocycle through i with center ∞.

<u>moving of generic tangent vectors:</u>

<u>method 1:</u>



<u>step 1:</u> produce correct angle with K

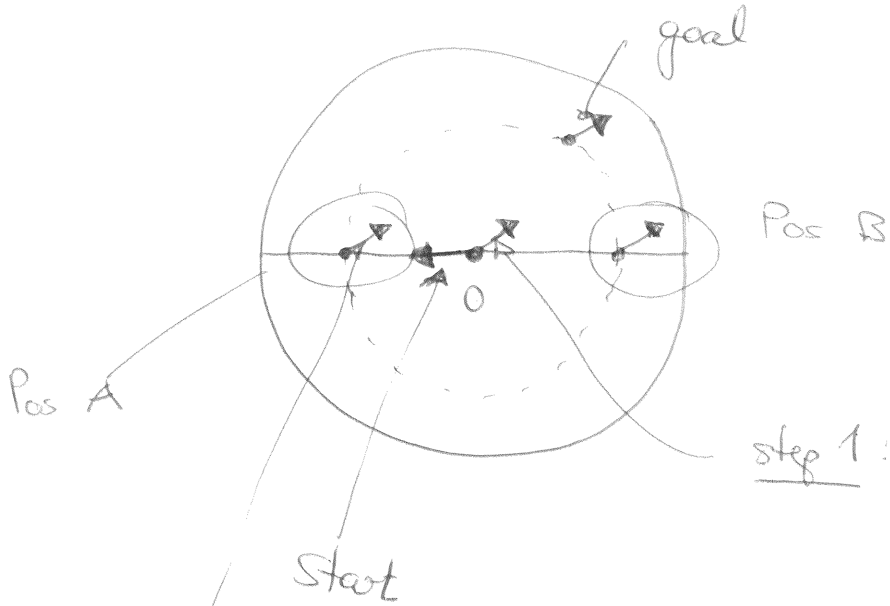
<u>step 2:</u> move with A to correct height

<u>step 3:</u> slide sidewards with N

$\Rightarrow \quad PSL_2(\mathbb{R}) = NAK$ <u>Iwasawa decomp</u> (Gauss algorithm)

by-product: $\mathbb{H} \cong G/K$

<u>method 2</u> (easier to draw in ~~[illegible]~~ disc model)



<u>step 1:</u> use K to produce correct angle

<u>step 2:</u> use A to move along standard geodesic into Pos A or Pos B

to avoid choices restrict to

$$\overline{A^+} := \{ a_t \mid t \geq 0 \} \qquad (\rightsquigarrow \text{Pos A})$$

<u>step 3:</u> use K to rotate

$$\Rightarrow \quad PSL_2(\mathbb{R}) = K \, \overline{A^+} \, K$$

<u>Cartan decomp</u>

(polar decomp)

<u>For general Riem symm spaces X of noncpt</u>

<u>type</u>

Fix $o \in X$ "origin"

$G := \mathrm{Isom}_o(X)$ Lie grp, ss, finite center

$K := \mathrm{Stab}_G(o) \le G$ max cpt

<u>flat</u> ~~[illegible]~~ $:=$ totally geodesic flat submfd of X of max dim

$r := \mathrm{rank}_{\mathbb{R}} G = \mathrm{rank}\, X := \dim$ flat

ask for all flats containing o:

typical picture $(r = 2)$:

finitely many lines of intersection

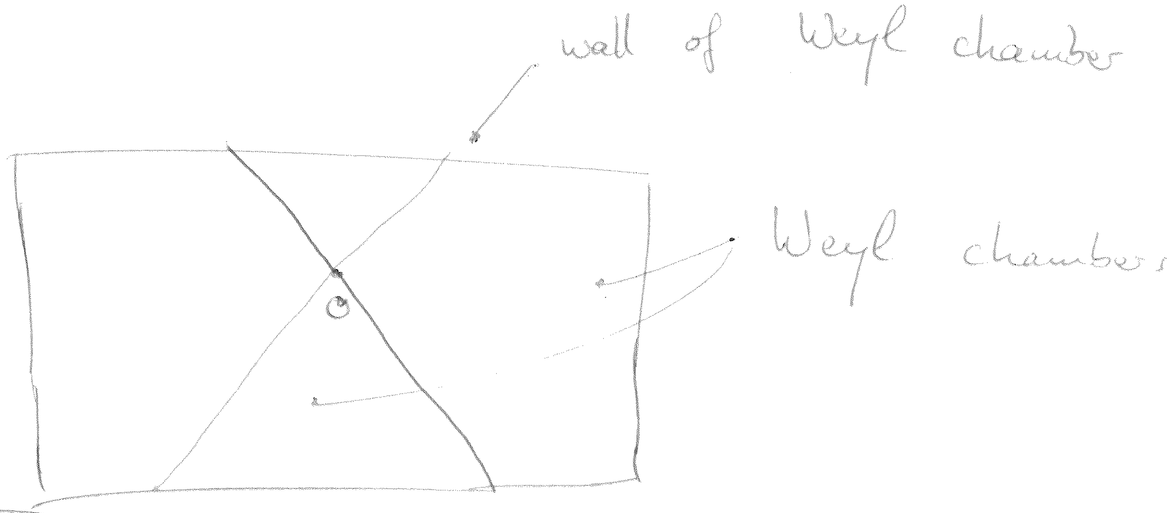
F 10

each such flat is an orbit $A.o$

for some r-parameter abelian (noncpt)

subgrp A of $\mathbb{R}$-diagonalizable elems

<u>view from top:</u>



~~Fix one A~~

Weyl group $W :=$ generated by reflections at walls

$$= N_K(A) \big/ Z_K(A)$$

W acts transitively on Weyl chambers

Fix one A, fix one Weyl chamber c_0:

$$A^+ := \{ a \in A \mid a.o \in c_0 \}$$

$\rightsquigarrow$ Cartan decomp

$$G = K\overline{A^+}K$$

(same geom interpretation as before)

~~For each~~

Moreover, there exists a concept of horocycles which are submanifolds in a certain sense transversal to the flats. ~~These are given by orbits~~ Those containing o are orbits of unipotent subgrps N of G.

$\rightsquigarrow$ Iwasawa decomp

$$G = NAK$$

Fourier analysis

Differential operatoren

$\mathbb{D}(X) :=$ algebra of diff ops D on X which are invariant under the action of G

Let $\chi : \mathbb{D}(X) \to \mathbb{C}$ be a homomorphism.

associated joint eigenspace:

$$E_\chi := \{ f \in \underset{\overset{\shortparallel}{C^\infty(X)}}{\mathcal{E}(X)} \mid \forall D \in \mathbb{D}(X) : Df = \chi(D) f \}$$

Thm (Harish-Chandra):

1) The homoms χ from above are parametrized by $\mathfrak{a}^*_{\mathbb{C}}/W$ ($\mathfrak{a} = \mathrm{Lie}\, A$)

[via the Harish-Chandra homomorphismus

$$\mathfrak{a}^*_{\mathbb{C}} \to \{\text{homom}\}$$
$$\lambda \mapsto \gamma(\cdot)(i\lambda) : \mathbb{D}(X) \to \mathbb{C}$$
]

2) For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ let

$$\mathcal{E}_{\lambda} := \left\{ f \in \mathcal{E}(X) \;\middle|\; \forall D \in \mathbb{D}(X) : \; Df = \gamma(D)(i\lambda) f \right\}$$

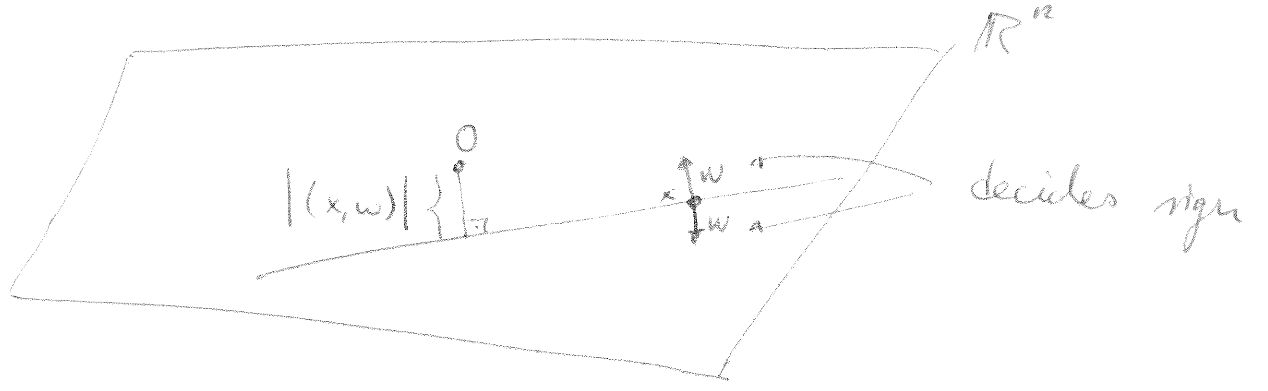
spectral parameter

<u>note:</u>
The spaces $\mathcal{E}_\lambda$ are all joint eigenspaces of $\mathbb{D}(X)$.

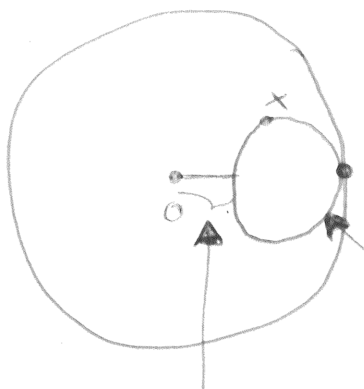
Fourier transform on $\mathbb{R}^n$ in polar coordinates:

$$\tilde{F}(\lambda w) = \int_{\mathbb{R}^n} F(x) e^{-i\lambda(x,w)} dx, \qquad |w| = 1, \quad \lambda \in \mathbb{R}$$

geometrically:



here (noneuclidean version):



boundary point ($\hat{=} w$)

$b \in B = K/M \qquad M = Z_K(A)$

horocycle

$A(x,b) \in \mathfrak{a}$ (distance expressed as elem of $\mathfrak{a}$)

$$X \times B = \underset{\substack{\uparrow \\ \text{space}}}{G/K} \times \underset{\substack{\uparrow \\ \text{directions}}}{K/M}$$

$$G \ni g \in N \exp(A(g)) K$$

Fact:

$$A(gK, kM) = A(k^{-1}g)$$

elementary waves:

$$e_{\lambda,b} : X \longrightarrow \mathbb{C}, \quad x \longmapsto e^{(i\lambda+\rho)A(x,b)}$$

$$\rho = \frac{1}{2} \sum_{\lambda \in \Lambda^+} (\dim \mathfrak{g}_\lambda)\, \lambda : \mathfrak{a} \longrightarrow \mathbb{R}$$

positive roots (restricted roots; roots of $(\mathfrak{g}, \mathfrak{a})$)

<u>Fact:</u> The fns $e_{w.\lambda,b}$, $w \in W$, are constant on each horocycle with center b and satisfy

$$(**) \quad D e_{w.\lambda,b} = \gamma(D)(i\lambda) e_{w.\lambda,b} \qquad \forall D \in \mathbb{D}(X)$$

If λ is regular, the $e_{w.\lambda,b}$ form a basis of the solutions to $(**)$.

<u>Fourier transform:</u> $f : X \to \mathbb{C}$ fct.

$$\underset{\mathfrak{a}_{\mathbb{C}}^* \times B}{\underset{\cap}{\tilde{f}(\lambda, b)}} := \int_X f(x)\, e_{-\lambda,b}(x)\, dx$$

if makes sense.

Makes sense ~~for~~ at least for $f \in \mathcal{D}(X) = C_c^\infty(X)$.

<u>Inversion formula</u> (Harish-Chandra):

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} e_{\lambda,b}(x)\, \tilde{f}(\lambda,b)\, \frac{d\lambda}{|c(\lambda)|^2}\, db,$$

$$\bar{d}\lambda = \frac{d\lambda}{|c(\lambda)|^2},$$

$c: \mathfrak{a}_{\mathbb{C}}^* \longrightarrow \mathbb{C}$ Harish-Chandra c-fct, picks out the roots and assigns them weights; explicitly known (Gindikin-Karpelevich product formula)

Paley-Wiener Thm (describes range of $\tilde{\mathcal{D}}(X)$):

Def: For $R>0$ let $\mathcal{H}^R(\mathfrak{a}_{\mathbb{C}}^* \times B)$ denote the space of smooth (C^∞) fct $\psi: \mathfrak{a}_{\mathbb{C}}^* \times B \longrightarrow \mathbb{C}$ s.th.

1) ψ is holomorphic in the $\mathfrak{a}_{\mathbb{C}}^*$-variable

2) $\forall N \in \mathbb{N}_0$:

$$\psi(\lambda, b) \ll_N (1+|\lambda|)^{-N} e^{R \cdot |\mathrm{Im}\,\lambda|},$$

where for $\lambda = \xi + i\eta \in \mathfrak{a}_{\mathbb{C}}^*$ $\mathrm{Im}\,\lambda = \eta$ and $|\lambda| = (|\xi|^2 + |\eta|^2)^{1/2}$

$\mathcal{H}^R_W(\mathfrak{a}^*_{\mathbb{C}} \times B) :=$ space of fcts $\psi \in \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)$ satisfying

$$\int_B e_{\lambda,b}(x)\,\psi(\lambda,b)\,db = \int_B e_{w\lambda,b}(x)\,\psi(w\lambda,b)\,db \quad \forall w \in W$$

$$\mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}} \times B) := \bigcup_{R>0} \mathcal{H}^R_W(\mathfrak{a}^*_{\mathbb{C}} \times B).$$

<u>Paley-Wiener Thm:</u>

The Fourier transform $f \longrightarrow \tilde{f}$ is a bijection

$$\mathcal{D}(X) \longrightarrow \mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}} \times B).$$

For each $R>0$, it restricts to a bijection of the space of fcts in $\mathcal{D}(X)$ with support in $\overline{B_R(0)}$ onto $\mathcal{H}^R_W(\mathfrak{a}^*_{\mathbb{C}} \times B)$.

Plancherel formula:

The Fourier transform $f \longrightarrow \tilde{f}$ extends to an isometry

$$L^2(X) \longrightarrow L^2(\mathfrak{a}^*_+ \times B, \, d\!\!\!{}^-\lambda \, db) = L^2(\mathfrak{a}^* \times B)^W$$

Moreover,

$$\int_X f_1(x)\overline{f_2(x)}\,dx = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} \tilde{f}_1(\lambda,b)\overline{\tilde{f}_2(\lambda,b)}\, d\!\!\!{}^-\lambda\, db.$$

Cor: $X \cong A^+ \times B$ is self-dual under the Fourier transform.

Spherical functions:

$\varphi : X \longrightarrow \mathbb{C}$ smooth, $\varphi(o) = 1$

[or $\varphi : G \longrightarrow \mathbb{C}$ bi-K-inv]

spherical $:\Leftrightarrow$

* $\forall\, k \in K \ \forall\, x \in X : \ \varphi(kx) = \varphi(x)$ [K-inv]

* $\forall\, D \in \mathbb{D}(X) : \ D\varphi = \lambda_D \varphi$ [joint eigenfct]

Harish-Chandra: ~~All~~ Spherical fcts are parametrized by $\mathfrak{a}^*_{\mathbb{C}}/W$:

$$\varphi_{\lambda}(g) := \int_K e^{(i\lambda+\rho)A(kg)}\,dk$$

$\lambda \in \mathfrak{a}^*_{\mathbb{C}}$

Spherical transform (= Fourier trafo for f K-inv):
(Harish-Chandra transform)

Let ~~f K-inv~~ $f: G \to \mathbb{C}$ bi-K-inv

$$\tilde{f}(\lambda) = \int_G f(g)\,\varphi_{-\lambda}(g)\,dg.$$

Inversion formula:

$$f(g) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \tilde{f}(\lambda)\,\varphi_\lambda(g)\,d\lambda.$$

Blomer-Pohl: $B \subseteq A$ bdd, C_ρ convex hull of $W.\rho$

$\Rightarrow \forall\, a \in B\ \forall\, \xi = \lambda + i\eta \in \mathfrak{a}^* + iC_\rho$:

$$\varphi_\xi(k_1 a k_2) = \varphi_\xi(a) \ll_B (1 + \|\lambda\| \cdot \|a\|)^{-1/2}$$

trivial bound : ≤ 1

note: bound uniform when approaching id in the argument and when approaching walls of Weyl chambers in the ~~sp~~ spectral parameter.

rem: non-uniform bounds (sharper, but non-uniform) are proven by

S. Marshall

and

Duistermaat - Kolk - Varadarajan.