Geometric Fourier analysis on Riemannian symmetric spaces

Summer School "Eigenfct estimates and related topics" in Matruz, July 2015

Goal: overview/survey of Riem globally symm spaces and their Fourier analysis as needed for Simon's lecture series

Def: $S_p : X \ni p \mapsto p \cdot x$.

Geodesic symmetry at $p$:

$S_p(g(t)) = g(-t)$ (defined in at least some nbh of $p$)

where $g$ is any geodesic on $X$ with $g(0) = p$. 
X \text{ Riem symm space } \Rightarrow \text{ for each } p \in X,
the geodesic symmetry \text{ } s_p \text{ is an isometry of } X.

Remark:

Study of Riemann symmetric spaces was initiated by E. Cartan in 1926 and vigorously developed in late 1920's.
In particular, there exists a complete classification. Spaces have a lot of structure:

\begin{align*}
\text{Space} \quad \Rightarrow \quad \text{Isometry groups} \\
\text{Riem symm spaces} \quad \Rightarrow \quad \text{Semisimple Lie groups} \\
\text{study problems on} \quad \Rightarrow \quad \text{Semisimple Lie algebras} \\
\text{spaces via its isometries} \quad (\text{Klein's Erlanger program})
\end{align*}
de Rham decomposition for Riemann symmetric space \( X \):

\[
X = X^\text{eul} \times X_{p,1} \times \cdots \times X_{p,m} \times X_{n,1} \times \cdots \times X_{n,m}
\]

as Riemannian product \( \mathbb{R}^k \) of irreducible, non-empty sectional curvature.

i.e., doesn't decompose nontrivially into a Riemann product.

From now on assume that \( X \) is irreducible, nonpositive sectional curvature ( = 0 of noncompact type)

A specific example:

the hyperbolic plane: upper half plane model

\[
H = \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \}
\]

\[
ds^2 = \frac{dx^2 + dy^2}{y^2}
\]

\[\text{geodesics}\]
group of Riemann isometries:

covn comp (= orientation-preserving): 

\[ \text{PSL}_2(\mathbb{R}) = \frac{\text{SL}_2(\mathbb{R})}{\{ \pm I \}} \]

may also use \( \text{SL}_2(\mathbb{R}) \)

action by Möbius transformations/fractional linear transformations:

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \]

\( \text{PSL}_2(\mathbb{R}) \) \( \text{H} \)

note:

* action extends to

\[ \text{SH} = \{ v \in T\text{H} \mid \|v\| = 1 \} \]

(unit tangent bundle)

by taking derivatives

* \( \text{PSL}_2(\mathbb{R}) \) is a Lie group.
different elements of $\text{PSL}_2(\mathbb{R})$ can act in qualitatively different ways.

We will use this observation to introduce coordinate systems on $\text{PSL}_2(\mathbb{R})$.

We will also see: $H$ is a homogeneous space.

To that end consider $i$ as origin of $H$.

$\delta y/\delta i$ as origin of $SH$.
\(M = \{ \pm 1 \}\) acts trivially on \(N\) and \(H\).

\[a_i \cdot (2r_f) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}\]

\(k_{\gamma r} : \gamma r \mapsto \gamma r\) is the rotation with angle \(2\pi\).

\(a \cdot (\gamma l)^i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}\)

A orbit of \(i\) produces unpaired quark pairs through:

\((\gamma l)^i + (\gamma l)^i\)
3) \( N := \{ n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{R} \} \)

\[ n_x \cdot i = i + x \]

\[ n_x \cdot (\partial y | i) = \partial y | i + x \]

- Orbit of \( i \) produces horocycle through \( i \) with center \( \infty \).

Moving of generic tangent vectors:

Method 1:

\[ \text{goal} \]

- Step 1: produce correct angle with \( K \)
- Step 2: move with \( A \) to correct height
- Step 3: slide sideways with \( N \)

\( \Rightarrow \quad \text{PSL}_2(\mathbb{R}) = \text{NAK} \)

Iwasawa decomposition

(by-product): \( H \cong G/K \) (Gauss algorithm)
Method 2 (easier to draw in model)

Step 1: Use $K$ to produce correct angle

Step 2: Use $A$ to move along standard geodesic into Pos A or Pos B to avoid choices restrict to

$$\overline{A^t} = \{ a_t \mid t \geq 0 \} \quad (\rightarrow \text{Pos A})$$

Step 3: Use $K$ to rotate

$$\Rightarrow PSL_2(\mathbb{R}) = K \overline{A^t} K$$

Castan decomp
(polar decomp)
For general Riemannian spaces $X$ of non-empty\textsuperscript{3} type

Fix $o \in X$ "origin"

$G := \text{Isom}_c(X)$ Lie gp, ss, finite center

$K := \text{Stab}_G(o) \leq G$ max.opt

$\flat := \text{totally geodesic flat submanifold of } X$ of max. dim

$r := \text{rank } G = \text{rank } X := \text{dim } \flat$

ask for all flats containing $o$

\text{typical picture ($r = 2$)}

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) -- (2,2);
\draw[thick] (0,0) -- (-2,2);
\draw[thick] (0,0) -- (2,-2);
\draw[thick] (0,0) -- (-2,-2);
\draw[thick] (0,0) -- (0,2);
\draw[thick] (0,0) -- (0,-2);
\draw[thick] (0,0) -- (2,0);
\draw[thick] (0,0) -- (-2,0);
\end{tikzpicture}
\end{center}

\text{finitely many lines of intersection}
for some $F$-parameter abelian (scheme)

A subgroup of $\mathcal{R}$-diagonalizable elements

Weyl group

$W = \frac{N_K(A)}{Z_K(A)}$

W acts transitively on Weyl chambers $\mathcal{C}$

Fix one $A$, fix one Weyl chamber $\mathcal{C}$

$A^+ := \{ x \in A | x \cdot \emptyset \in \mathcal{C} \}$
$G = \bar{K} A^+ K$

(some geometric interpretation as before)

Moreover, there exists a concept of horocycles which are submanifolds in a certain sense transversal to the flats. These are given by orbits. Those containing are orbits of unipotent subgroups $N$ of $G$.

Iwasawa decomposition

$G = NA \bar{K}$
Differential operators

\[ D(X) := \text{algebra of diff ops } D \text{ on } X \text{ which are invariant under the action of } G \]

Let \( \lambda : D(X) \to \mathbb{C} \) be a homomorphism.

associated joint eigenspace:

\[ E_\lambda := \{ f \in C_c^\infty(X) \mid \forall D \in D(X) : Df = \lambda(D)f \} \]

- Thm (Harish-Chandra):

1) The homomorphs \( \lambda \) from above are parametrized by \( \mathfrak{a}^* \backslash \mathbb{C} / \mathbb{W} \) \((\mathfrak{a} = \text{Lie } A)\) via the Harish-Chandra homomorphisms

\[ \begin{array}{c}
\mathfrak{a}^* \quad \text{via the Harish-Chandra homomorphisms} \\
\mathbb{C} \quad \text{homom } \\
\lambda \quad \gamma(\cdot)(i\lambda) : D(X) \to \mathbb{C}
\end{array} \]
2) For $\lambda \in \sigma^*_D$ let

$$E_{\lambda} := \left\{ f \in E(X) \mid \forall D \in D(X): \quad Df = \gamma(D)(i\lambda)f \right\}$$

The spaces $E_{\lambda}$ are all joint eigenspaces of $D(X)$.

Fourier transform on $\mathbb{R}^n$ in polar coordinates:

$$\tilde{F}(\lambda w) = \int_{\mathbb{R}^n} F(x) e^{-i\lambda \langle x, w \rangle} \, dx, \quad |w| = 1, \; \lambda \in \mathbb{R}$$

Geometrically:
here (non-euclidean version):

\[ A(x, b) \in \alpha \]

\[ X \times \mathcal{B} = \frac{G}{K} \times \frac{K}{M} \]

\[ G \rightarrow \text{space} \rightarrow \text{directions} \]

\[ g \in \mathcal{N} \exp \left( A(q) \right) \mathcal{K} \]

Fact:

\[ A(q^k, kM) = A(k^{-1}q) \]

elementary waves:

\[ e_{x,b} : X \rightarrow C, \quad x \mapsto e^{(i\lambda + \xi^j)A(x,b)} \]

\[ \lambda = \frac{1}{2} \sum (\text{dim } \alpha_j) \lambda_i : \alpha_i \rightarrow \mathbb{R} \]

positive roots (restricted roots; roots of \( (q, \alpha) \))
Fact: The functions \( e_{w \cdot x, b} \), \( w, x, b \), are constant on each horocycle with center \( b \) and satisfy

\[ (** ) \quad \mathcal{D} e_{w \cdot x, b} = f(D) (iD) e_{w \cdot x, b} \quad \forall D \in \mathbb{R}^X \]

If \( \lambda \) is regular, the \( e_{w \cdot x, b} \) form a basis of the solutions to (**).

**Fourier transform:** \( \mathcal{F}: X \rightarrow \mathbb{C} \) for

\[ \tilde{f}(\lambda, b) := \int_{\mathcal{O}^* \times B} f(x) e^{-\lambda \cdot x, b(x)} \, dx \]

if makes sense.

If makes sense \( \mathcal{F} \) at least for \( f \in \mathcal{D}(X) = \mathcal{S}(X) \).

**Inversion formula (Harish-Chandra):**

\[ f(x) = \frac{1}{|W|} \int_{\mathcal{O}^* \times B} e^{\gamma(x)} \tilde{f}(\lambda, b) \, d\lambda \, db, \]
\[ dA = \frac{dx}{|c(x)|^2} \]

\[ C: \sigma_*^0 \rightarrow C \]

Harish-Chandra c-fct, picks out the roots and assigns them weights explicitly known (Gindikin-Kapustin product formula)

Paley-Wiener Theorem (describes range of $\hat{f}(x)$):

Def: For $R > 0$ let $\mathbf{H}^R(\sigma^* \times B)$ denote the space of smooth ($C^\infty$) fct $\psi: \sigma^* \times B \rightarrow \mathbb{C}$ with

1) $\psi$ is holomorphic in the $\sigma^*$ variable

2) For $N \in \mathbb{N}_0$:

\[ \psi(\lambda, b) \ll_N (1 + |\lambda|)^{-N} R^{-1} |\text{Im} \lambda| \]

where for $\lambda = \xi + i\eta \in \sigma^* \subset \mathbb{C}$, $|\text{Im} \lambda| = \eta$ and

\[ |\lambda| = (\xi^2 + \eta^2)^{1/2} \]
\( H^R_W(\mathbb{R}^n \times B) := \text{space of fcts } \psi \in X^R(\mathbb{R}^n \times B) \) satisfying

\[
\int e^{x \cdot b(x)} \psi(x, \mathbf{b}) d\mathbf{b} = \int e^{\omega \cdot b(x)} \psi(\omega, \mathbf{b}) d\mathbf{b} \quad \forall \omega \in \mathbb{R}^n
\]

\( H^R_W(\mathbb{R}^n \times B) = \bigcup_{R>0} H^R_W(\mathbb{R}^n \times B) \)

Paley-Wiener Theorem:

The Fourier transform \( f \rightarrow \hat{f} \) is a bijection

\( \mathcal{D}(X) \rightarrow H^R_W(\mathbb{R}^n \times B) \).

For each \( R>0 \), \( \hat{f} \) restricts to a bijection of the space of fcts in \( \mathcal{D}(X) \) with support in \( B_R(0) \) onto \( H^R_W(\mathbb{R}^n \times B) \).
Plancherel formula:

The Fourier transform \( f \rightarrow \hat{f} \) extends to an isometry

\[
L^2(\mathbb{X}) \rightarrow L^2(\mathbb{A}^* \times \mathbb{B}, \mathcal{H} \, d\alpha d\beta) = L^2(\mathbb{A}^* \times \mathbb{B})^W
\]

Moreover,

\[
\int_{\mathbb{X}} \overline{f_1(x)} f_2(x) \, dx = \frac{1}{|W|} \int_{\mathbb{A}^* \times \mathbb{B}} \overline{\hat{f}_1(x,b)} \hat{f}_2(x,b) \, d\alpha d\beta
\]

Con.: \( \mathbb{X} \cong \mathbb{A}^* \times \mathbb{B} \) is self-dual under the Fourier transform.

Spherical functions:

\( \varphi: \mathbb{X} \rightarrow \mathbb{C} \) smooth, \( \varphi(0) = 1 \)

Spherical: \( \varphi \) satisfies

* \( \forall k \in \mathbb{K} \forall x \in \mathbb{X}: \varphi(kx) = \varphi(x) \) [K-inv]

* \( \forall \mathcal{D} \in \mathcal{D}(\mathbb{X}) : \mathcal{D} \varphi = \lambda \varphi \) [joint eigenf.]

[or \( \varphi: \mathbb{G} \rightarrow \mathbb{C} \) bi-inv.]
Harish-Chandra: Spherical functions are parametrized by \( a \in \mathbb{C}/W \):

\[
\varphi_a(g) = \int_{G/K} e^{(ix \cdot g)A(k)} \, dk.
\]

Spherical transform (= Fourier transform for \( f \) \( \text{bi-K-inv} \))

Let \( f: X \rightarrow Y \) bi-K-inv

\[
\tilde{f}(\lambda) = \int_{G} f(g) \varphi_{-\lambda}(g) \, dg.
\]

Inversion formula:

\[
f(g) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \tilde{f}(\lambda) \varphi_{\lambda}(g) \, d\lambda.
\]

Blower-Pollak: \( B \leq A \) bdd, \( C \) convex hull of \( W \mathfrak{a} \)

\[
\begin{array}{rcl}
A \subseteq B & \Rightarrow & A \subseteq \lambda + \mathfrak{a}^* + i C \mathcal{E} \\
\varphi_{\mathfrak{a}}(k, ak_2) = \varphi_{\mathfrak{a}}(a) & \ll & B \left( 1 + \| \lambda \|_1 \cdot \| \alpha \| \right)^{-\frac{1}{2}}
\end{array}
\]
trivial bound: \leq 1

Note: bound uniform when approaching id in the argument and when approaching walls of Weyl chambers in the spectral parameter.

Rem: non-uniform bounds (sharper, but non-uniform) are proven by S. Marshall and Duistermaat–Kolk–Varadarajan.