INVARIANT MEASURES AND QUANTUM UNIQUE ERGODICITY
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Abstract. Notes for my lecture series where I discussed the measure classification result due to E. Lindenstrauss: *Invariant measures and arithmetic quantum unique ergodicity*, Annals of Mathematics, **163** (2006), 165-219. The presentation concentrates to the instance of this measure classification which is used to prove AQUE for congruence lattices over $\mathbb{Q}$. All hand-drawn pictures are contained in an appendix. Their original locations are indicated in the text.

Contents

1. Goal and preliminaries

The goal of this lecture series is to prove the measure classification theorem, which has been a key step in Lindenstrauss' proof of arithmetic quantum unique ergodicity for congruence lattices over $\mathbb{Q}$. The relation between this measure classification and quantum limits will be explained in Pablo's lecture series. Throughout we use the following notation. We set

$$H := \mathrm{PGL}_2(\mathbb{R}).$$

(We could also work in a PSL_2-setting as Lindenstrauss does. Things wouldn't change, but for consistency with Pablo's lectures, we decided to use PGL_2.) For any prime number p we set

$$L_p := \mathrm{PGL}_2(\mathbb{Q}_p), \quad \text{and} \quad K := \mathrm{PGL}_2(\mathbb{Z}_p).$$

For any congruence lattice Λ over $\mathbb{Q}$ and almost all primes p there is a subgroup $\Gamma \in H \cap L_p$ such that

$$\Lambda\backslash \mathrm{PSL}_2(\mathbb{R}) \cong \Gamma\backslash(H \times L_p)/K_p, \qquad \Lambda g \mapsto \Gamma(g, \mathrm{id})K_p. \tag{1}$$

Here, the action of Γ on $H \times L_p$ is diagonal (from the left), and Γ is a lattice in $H \times L_p$, and K_p only acts on L_p (from the right). For example, if $\Lambda = \mathrm{PSL}_2(\mathbb{Z})$, then $\Gamma = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$. If $\Lambda = \mathrm{P}\Gamma(N)$ and $(p, N) = 1$, then

$$\Gamma = \{g \in \mathrm{PGL}_2(\mathbb{Z}[\tfrac{1}{p}]) \mid g \equiv \mathrm{id} \mod N\}.$$

For later use, we remark that for any congruence lattice Λ over $\mathbb{Q}$ we have $\Gamma K_p = L_p$ and $\Gamma \cap K_p = \Lambda$. For the provided examples, this can be checked easily. An indication of the proof of the general statement will appear in Lior's lecture series. Let

$$G := H \times L_p.$$

We let

$$A := \{a_t \mid t \in \mathbb{R}\} \quad \text{with} \quad a_t := \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$$

be the one-parameter diagonal subgroup of G in the $\mathrm{PGL}_2(\mathbb{R})$-factor (or the $\mathrm{SL}_2(\mathbb{R})$-factor). The group A is often identified with $A \times \{\mathrm{id}_L\}$. Then A acts from the right on $\Lambda\backslash \mathrm{SL}_2(\mathbb{R})$ resp. from the right in the H-factor of $\Gamma\backslash(H \times L_p)/K_p$:

$$\Lambda g.a := \Lambda ga \quad \text{resp.} \quad \Gamma(g_\infty, g_p)K_p.a := \Gamma(g_\infty a, g_p)K_p.$$

We drop the subscript p from L_p and K_p.

Theorem 1.1. *Suppose Γ arises from a congruence lattice over $\mathbb{Q}$. Let*

$$S := L/K \quad \text{and} \quad X := \Gamma\backslash H \times S.$$

Let μ be a probability measure on X (on the Borel σ-algebra $\mathcal{B}_X$ of X) such that

(i) *μ is A-invariant,*
(ii) *μ is S-recurrent,*
(iii) *μ-almost all A-ergodic components of μ have positive entropy under the A-flow.*

Then μ is the Haar measure m_X on X.

Conjecture 1.2. Theorem 1.1 should be true without the requirement (iii).

p-adic numbers. Let p be a prime number. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$ on $\mathbb{Q}$ given by

$$\left|p^k \frac{n}{m}\right|_p = p^{-k}, \quad |0|_p = 0,$$

where $k \in \mathbb{Z}$, $(p, nm) = 1$. This means that if x is "divided" by high powers of p, then its p-norm becomes very small. This norm extends to $\mathbb{Q}_p$. We may identify $\mathbb{Q}_p$ with the Laurent series

$$\left\{ \sum_{j=k}^{\infty} a_j p^j \;\middle|\; k \in \mathbb{Z},\ a_j \in \{0, 1, \ldots, p-1\},\ a_k \neq 0 \right\}.$$

Calculations with these Laurent series are canonical, and

$$\left| \sum_{j=k}^{\infty} a_j p^j \right|_p = p^{-k}.$$

The space $\mathbb{Q}_p$ is locally compact, second countable, complete metric. Moreover,

$$\mathbb{Z}_p = \{|x|_p \leq 1\} \cong \left\{ \sum_{j=0}^{\infty} a_j p^j \right\}.$$

Recurrence. The measure μ is called *S-recurrent* if for all $B \in \mathcal{B}_X$ and μ-almost all $x \in B$ the following is satisfied:

picture: recurrent S-leaf

Let $(g_\infty, g_p) \in H \times L$ be a representative of x, i.e., $x = \Gamma(g_\infty, g_p)K$. Then there exists a sequence (h_n) in L such that $h_n \to \infty$ (meaning, the sequence (h_n) leaves any compact subset of L) and

$$\Gamma(g_\infty, g_p h_n)K \in B$$

for all $n \in \mathbb{N}$.

One easily sees that this definition does not depend on the choice of representative for x. In other words, S-recurrence means that for almost any $x = \Gamma(g_\infty, g_p)K \in B$, its S-leaf $\Gamma(g_\infty, S)$ visits B again and again.

A-invariance and A-ergodicity. The measure μ is called A-*invariant* if for all $a \in A$ we have $a_*\mu = \mu$. This means that for all $a \in A$ and all $B \in \mathcal{B}_X$, we have

$$(a_*\mu)(B) = \mu(\{x \in X \mid xa \in B\}) = \mu(Ba^{-1}) = \mu(B).$$

The measure μ is called A-*ergodic* if μ is A-invariant, and whenever $B = Ba$ for all $a \in A$ and some $B \in \mathcal{B}_X$, then $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Ergodic decomposition. Invoking one-point-compactification we can embed X into a compact second countable space $\overline{X}$ with a single additional point. Moreover, we can extend any given probability measure μ on X trivially to a probability measure $\overline{\mu}$ on $\overline{X}$. Formally we should apply ergodic decomposition to $\overline{\mu}$ and observe that almost all ergodic components will give zero mass to the additional point. This in turn allows us to restrict the ergodic decomposition to μ. For simplicity, we ignore this additional step here.

So let μ be an A-invariant probability measure on X. Then there exists a probability space (Ξ, ν) and a measurable map

$$\Xi \to M^1(X) = \{\text{probability measures on } X\}, \xi \mapsto \mu_\xi,$$

such that each μ_ξ is an A-invariant and ergodic probability measure and

$$\mu = \int_\Xi \mu_\xi d\nu(\xi).$$

Here, $M^1(X)$ is endowed with the weak* topology and the Borel σ-algebra induced by this topology. The *ergodic decomposition* of μ is essentially unique.

Ergodic decompositions can be constructed via conditional measures, which we explain further below.

Entropy. Suppose first that μ is an a_t-invariant probability measure on X for some fixed $t \in \mathbb{R}$. Roughly speaking, the entropy of μ with respect to a_t can be understood as a measurement how much information the iterated action of a_t reveals on average about the constellation of X.

Let us imagine that we are asked to find a certain point x in X. We are allowed to perform experiments on X. This – mathematically – translates to putting a countable partition $\mathcal{P}$ on X, consisting of measurable subsets. For any partition we would be told the partition element which contains x.

picture: partition

What amount of information do we gain on (measure theoretic) average on X and how do we measure it? Imposing a number of reasonable conditions on a measuring function (such as, if some partition element has full measure, the information gained is zero; gained information is maximal if all partition element have the same measure; some kind of continuity; etc.), one finds that there is up to scaling a unique function, namely,

$$H_\mu(\mathcal{P}) := -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P),$$

the *static entropy of* $\mathcal{P}$, which may be finite or infinite. This is the average of the *information function*

$$I_\mu(\mathcal{P})(x) := -\log\mu(P) \quad \text{for } x \in P \in \mathcal{P}.$$

Now we take into account the evolution of X under a_t (here, t is fixed), and we can perform our experiment after each step with a_t.

picture: iteration

Performing this procedure k times is equivalent to considering the partition

$$\mathcal{P}_0^{k-1} := \bigvee_{j=0}^{k-1} a_t^{-j}.\mathcal{P} := \{P_{i_0} \cap a_t^{-1}.P_{i_1} \cap \ldots \cap a_t^{-(k-1)}.P_{i_{k-1}} \mid P_{i_j} \in \mathcal{P}\}.$$

Then

$$\lim_{k\to\infty} \frac{1}{k} H_\mu(\mathcal{P}_0^{k-1}) =: h_\mu(a_t,\mathcal{P})$$

exists, it is even an infimum, and its limit is called the *entropy of* a_t *with respect to* $\mathcal{P}$. Then, changing the partition, we get

$$h_\mu(a_t) := \sup\{h_\mu(a_t,\mathcal{P}) \mid \mathcal{P} \text{ partition of } X,\, H_\mu(\mathcal{P}) < \infty\},$$

the *entropy of* a_t. If μ is an A-invariant probability measure, then for $t \neq 0$, the value

$$\frac{1}{|t|} h_\mu(a_t)$$

is independent of t. We call this common value the *entropy of* μ *with respect to* $A = \{a_t\}$, denoted $h_\mu(a_\bullet)$. We remark that $h_\mu(a_\bullet)$ may depend on the parametrization of A (and it does unless it vanishes).

2. Very brief outline of the proof of Theorem 1.1

We set
$$N := \left\{ n_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$
and consider it as a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ for the left hand side of (1), and as a subgroup in the H-factor for the right hand side of (1), or as identified with $N \times \{\mathrm{id}_L\}$ for the right hand side of (1) (adapted to personal taste). Then N acts from the right on either side of (1), and these actions are equivariant.

Step 1: If μ is N-invariant, then $\mu = m_X$.

Proof of Step 1. We take advantage of the $\mathrm{PSL}_2(\mathbb{R})$-equivariance of the isomorphism in (1) and actually show on the left hand side that $\mu = m_X$. By hypothesis, μ is N- and A-invariant. By symmetry, μ is also invariant under
$$U = N^- = \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$$
(either one conjugates the whole set-up such that U becomes N and uses the hypothesis for the conjugate measure μ', or one recalls that N is the Lie group to one of the two simple roots of $\mathrm{PSL}_2(\mathbb{R})$ and U is the Lie group to the other simple root, and since here the whole group A is acting, there is no difference between these two situation. All arguments to prove that μ is actually N-invariant are also valid for a proof that μ is U-invariant). Thus, μ is invariant under N and U. Thus, by generation (look at the Lie algebra), μ is $\mathrm{PSL}_2(\mathbb{R})$-invariant, and hence equals m_X. □

Step 2: The action on N on X (from the right) provides a foliation of X into N-orbits. We define a family of Radon measures (defined on Borel σ-algebra, locally finite, inner regular) $\{\mu_x^N\}_{x\in X}$ such that $x_*\mu_x^N$ is supported on xN almost surely. If μ_x^N is N-invariant for μ-almost all $x \in X$, then μ is N-invariant.

Step 3: For μ-almost all $x \in X$, the leafwise measure μ_x^N is N-invariant.

Remark 2.1. The key point of Theorem 1.1 is that invariance under the geodesic flow and the assumptions on recurrence and entropy imply invariance under the horocycle flow. This shows that the geodesic flow and the horocycle flow, though *a priori* flows of very different behavior, are linked to each other.

Remark 2.2. Lindenstrauss' measure classification theorem is actually more general. For L, he allows any S-algebraic group, for K any compact subgroup of L, for Γ any discrete subgroup of $H \times L$ (not necessarily a lattice) such that $\Gamma \cap \{\mathrm{id}_H\} \times L = \{id\}$. In this generality one cannot expect to conclude that μ is Haar. The proof of this more general theorem is parallel to the specification we consider here (which is the version one needs for AQUE). In the following we will see all ingredients of the proof. Hence also in the general setting, one concludes that μ is N-invariant, and then uses the Ratner Theorems to deduce that μ is a linear combination of homogeneous measures.

3. Leafwise measures and proof of step 2

As a tool for the construction of N-leafwise measures we need conditional measures.

picture: N-leaves

Conditional measures. Let μ be a finite (or probability) measure on X.

Example 3.1. If $\mathcal{A}$ is the σ-algebra on X generated by the finite partition $\{A_1, \dots, A_n\}$ with $\mu(A_j) > 0$,

picture: finite partition

then the *conditional (probability) measures* are given by the family $\{\mu_x^{\mathcal{A}}\}_{x\in X}$,

$$\mu_x^{\mathcal{A}}(B) := \frac{\mu(B\cap A_j)}{\mu(A_j)}, \quad \text{for } B\in\mathcal{B}_X, \text{ if } x\in A_j,$$

and

$$\int_A f d\mu = \sum_{A_j\subseteq A} \mu(A_j)\cdot\frac{1}{\mu(A_j)}\int_{A_j} 1_{A_j} f(y)d\mu(y) = \int_A\int_X f(y)d\mu_x^{\mathcal{A}}(y)d\mu(x)$$

for all $f\in\mathcal{L}^1(X,\mathcal{B}_X,\mu)$ and all $A\in\mathcal{A}$. The conditional measure $\mu_x^{\mathcal{A}}$ takes the set A_j with $x\in A_j$ and restricts and normalizes the original measure μ to A_j. Iterated integration then first averages f over the partition elements and then sums over all partition elements belonging to A.

Example 3.2. We can also define families of conditional probability measures which may be supported on null sets related to a sub σ-algebra $\mathcal{A}$ of $\mathcal{B}_X$.

Let $Z=[0,1]^2$, endowed with the usual Lebesgue (Haar) measure m. Consider the sub σ-algebra $\mathcal{A} := \mathcal{B}_{[0,1]}\times\{\emptyset,[0,1]\}$. Then the atoms (minimal sets of $\mathcal{A}$ are given by $\{x\}\times[0,1]$, $x\in[0,1]$. These are m-null sets.

picture: null sets as atoms

Then $\{\mu_{(x,y)}^{\mathcal{A}}\}_{(x,y)\in Z}$, $\mu_{(x,y)}^{\mathcal{A}} = \delta_x\times m_{[0,1]}$, provides a family of conditional measures and again

$$\int_A f dm = \int_A\int_Z f(u,v)d\mu_{(x,y)}^{\mathcal{A}}(u,v)d\mu(x,y)$$

for all $f\in\mathcal{L}^1(Z,\mathcal{B}_Z,m)$, all $A\in\mathcal{A}$.

Proposition 3.3. *Let $\mathcal{A}$ a sub σ-algebra of $\mathcal{B}_X$ and μ a finite measure on X. Then there exists a system of probability measures $\{\mu_x^{\mathcal{A}}\}_{x\in X}$ (called* conditional measures*) and a set $X'\in\mathcal{A}$, $\mu(X\setminus X')=0$, such that*

(i) *the map $x\mapsto\mu_x^{\mathcal{A}}$ is $\mathcal{A}$-measurable on X'. That means, for any $f\in\mathcal{L}^1(X,\mathcal{B}_X,\mu)$, the map $x\mapsto\int fd\mu_x^{\mathcal{A}}$ is $\mathcal{A}$-measurable on X'.*
(ii) *For all $f\in\mathcal{L}^1(X,\mathcal{B}_X,\mu)$ and all $A\in\mathcal{A}$, we have*

$$\int_A f d\mu = \int_A\int f d\mu_x^{\mathcal{A}} d\mu(x).$$

The conditional measures are uniquely characterized up to changes on null sets.

Note that we have to work with real (onest) functions (not equivalence classes of functions) in Proposition 3.3 because $\mu_x^{\mathcal{A}}$ might be singular with respect to μ.

In the following we will (almost exclusively) work with countably generated σ-algebras. A σ-algebra $\mathcal{A}$ on X is called *countably generated* if there is a countable set $\mathcal{A}_0$ of subsets of X such that $\sigma(\mathcal{A}_0) = \mathcal{A}$, where $\sigma(\mathcal{A}_0)$ denotes the smallest σ-algebra which contains $\mathcal{A}_0$. If $\mathcal{A}_0$ is a countable algebra (which we may assume without loss of generality), then the $\mathcal{A}$-*atom* at $x \in X$ is

$$[x]_{\mathcal{A}} := \bigcap_{x \in A \in \mathcal{A}} A = \bigcap_{x \in B \in \mathcal{A}_0} B.$$

In particular, $[x]_{\mathcal{A}} \in \mathcal{A}$. This need not be the case for uncountably generated σ-algebras.

Proposition 3.4. *Hypotheses as in Proposition 3.3. Then, by possibly shrinking X' slightly,*

(i) *If $\mathcal{A} = \sigma(\{A_1, A_2, \ldots\})$ is countably generated and $\mathcal{A}_n := \sigma(\{A_1, \ldots, A_n\})$ is the finite σ-algebra generated by the first n generators of $\mathcal{A}$, then $\mu_x^{\mathcal{A}_n} \to \mu_x^{\mathcal{A}}$ in the weak* topology for μ-almost all $x \in X$.*

(ii) *If $\mathcal{A}$ is countably generated, then $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$ for every $x \in X'$. Moreover, if $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$ for some $x, y \in X'$, then $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$.*

Ergodic decompositions revisited. Let

$$\mathcal{E} := \{B \in \mathcal{B}_X \mid B \text{ is } A\text{-invariant}\}$$

be the σ-algebra of all A-invariant subsets. Then any family $\{\mu_x^{\mathcal{E}}\}$ of conditional measures on $\mathcal{E}$ provides an ergodic decomposition of μ.

Leafwise measures for N. We would like to find a countably generated sub σ-algebra of $\mathcal{B}_X$ whose atoms are μ-almost surely the N-leaves. In the situations we consider, this is not possible. However, it is possible to locally model N-leaves by atoms of appropriately chosen sub σ-algebras. Thus, locally we can define measures on N-leaves by conditional measures. Then we patch together these local descriptions to Radon measures defined on entire N-leaves. Since we want to take advantage of the additional structure on the N-leaves given by the groups, we will actually define these leafwise measures on N.

picture: local N-leaves

As preparation for this construction, we need a few definitions.

Metrics. We endow H with a left H-invariant Riemannian metric d_H, and S with an L-invariant metric d_S. Then

$$d_{H\times S}\big((h_1,s_1),(h_2,s_2)\big) := \max\{d_H(h_1,h_2), d_S(s_1,s_2)\}$$

is a metric on $H\times S$, and

$$d_X\big(\Gamma(h_1,s_1),\Gamma(h_2,s_2)\big) := \inf_{\gamma\in\Gamma} d_{H\times S}\big(\gamma.(h_1,s_1),(h_2,s_2)\big)$$

is a metric on X. With this metric, X becomes a locally compact, second countable metric space, and the projection map

$$H\times S \to X$$

is locally an isometry.

Definition 3.5. Let $x\in X$ and $A\subseteq xN$. We say that A is an *open N-plaque* if and only if the set

$$\{n\in N \mid xn\in A\}$$

is open and bounded.

Theorem 3.6 (Definition and existence of leafwise measures for $T=N$)**.** *There is a family $\{\mu_x^N\}_{x\in X}$ of Radon measures on N and a set $X'\in\mathcal{B}_X$, $\mu(X\setminus X')=0$, satisfying the following properties:*

(i) *Let $Z\in\mathcal{B}_X$ and $\mathcal{A}$ be a countably generated σ-algebra on Z such that for any $x\in Z$, the atom $[x]_{\mathcal{A}}$ is an open N-plaque, say $[x]_{\mathcal{A}} = xU_{x,\mathcal{A}}$ with*

$$U_{x,\mathcal{A}} = \{n\in N \mid xn\in[x]_{\mathcal{A}}\}.$$

Then, for μ-almost every $x\in Z$, we have

$$\big(\mu|_Z\big)_x^{\mathcal{A}} \propto x_*\big(\mu_x^N|_{U_{x,\mathcal{A}}}\big).$$

Here, $x_*\big(\mu_x^N|_{U_{x,\mathcal{A}}}\big)$ is the push-forward of $\mu_x^N|_{U_{x,\mathcal{A}}}$ under the map $N\to X$, $n\mapsto xn$.

(ii) *For all $x\in X'$ we have $\mu_x^N(B_1^N)=1$.*

(iii) *For every $x\in X'$ and $n\in N$ such that $xn\in X'$, we have*

$$\mu_x^N \propto n_*\big(\mu_{xn}^N\big),$$

where $n_*\big(\mu_{xn}^N\big)$ is the push-forward of μ_{xn}^N under the map $N\to N, m\mapsto nm$ (note $nm=mn$).

This family of measures is unique (already by (i) and (ii)) up to redefining on null sets in $\mathcal{B}_X$. It is called the family of leafwise measures.

picture: compatibility

The remainder of this subsection serves to prove Theorem 3.6. Its proof is split into several subresults.

Definition 3.7. Let $\delta > 0$, $R \geq 1$ and $x_0 \in X$. We say that $C \in \mathcal{B}_X$ is an (R, δ)-*cross section for* N *at* x_0 if and only if

(i) $B_\delta(x_0) \subseteq CB_1^N$,
(ii) the map $C \times B_R^N \to CB_R^N$, $(x, n) \mapsto xn$, is injective and bi-measurable.

Definition 3.8. Let C be a (R, δ)-cross section for N at $x_0 \in X$. Let $\mathcal{A}$ be a countably generated σ-algebra $\mathcal{A}$ on $CB_R^N =: Z$ all of whose atoms are of the form xB_R^N for some $x \in C$. Then we call $(\mathcal{A}, Z)$ an (N, R)-*flower with base* $B_\delta(x_0)$.

Proposition 3.9. *Let $x_0 \in X$ and $R \geq 1$ and suppose that the map*

$$\overline{B_R^N} \to X, \ n \mapsto x_0 n,$$

is injective. Then there exists an (R, δ)-cross section C for N at x_0 for some $\delta > 0$. Moreover, there exists a countably generated σ-algebra $\mathcal{A}$ on CB_R^N for which

$$[xn]_{\mathcal{A}} = xB_R^N$$

for all $x \in C$ and $n \in B_R^N$. Hence $(\mathcal{A}, CB_R^N)$ is an (N, R)-flower with base $B_\delta(x_0)$.

picture: flowers

For the proof of Proposition 3.9 (and also later on) we want to have a kind of application of elements in $H \times S = G/K$ on X.

***S*-structure.** Let

$$\widetilde{X} := \Gamma \backslash G$$

and let $\pi_K \colon \widetilde{X} \to X = \widetilde{X}/K$ denote the canonical quotient map. Clearly, L (and G) acts on $\widetilde{X}$ from the right, but $S = L/K$ (and G/K) cannot act on $X = \widetilde{X}/K$, even though S-leaf are well-defined subsets of X.

We claim that for any $x \in X$ there is an open neighborhood U of x in X and a continuous local section

$$\tau_U \colon U \to \widetilde{X}$$

of π_K. Given such a local section τ_U, we can define an action-like continuous map $t_U \colon U \times (G/K) \to X$ by

$$t_U(y, gK) := \pi_K(\tau_U(y)g).$$

For the construction of these local continuous sections we use that $\Gamma K = L$ and $\Gamma \cap K = \Lambda$. More precisely we claim that there is an open cover $\mathcal{S}$ of X such that

(i) For all $U \in \mathcal{S}$, $x \in U$, we have $t_U(x, (id, K)) = x$.
(ii) For any $U \in \mathcal{S}$, $x \in U$, $y \in t_U(x, G/K)$ and $V \in \mathcal{S}$ with $y \in V$, there exists $g \in G$ such that
$$t_V(y, g\cdot) = t_U(x, \cdot).$$
Moreover, the action of g stays in the T-component.
(iii) If $U \in \mathcal{S}$ is relatively compact, then there exists $r = r_U > 0$ such that for all $x \in U$, $t_U(x, \cdot)$ is injective on $\overline{B}_r^{G/K}$.

Let $x \in X$ and pick a presentation of the form

$$x = \Gamma(h, \mathrm{id})K.$$

Let

$$\pi_\Lambda \colon H \to \Lambda \backslash H$$

be the canonical quotient map. Since Λ is a lattice in H, we find an injectivity radius $r > 0$ at $\pi_\Lambda(h)$. Hence the map

$$B_r^H \to B_r^{\Lambda\backslash H}(\Lambda h), \quad q \mapsto \Lambda h q$$

is an isometry. Set

$$U := \{\Gamma(hq, \mathrm{id})K \mid q \in B_r^H\}.$$

We claim that

$$\tau_U \colon U \to \Gamma\backslash G = \widetilde{X}, \quad y = \Gamma(g, \mathrm{id})K \mapsto \Gamma(g, \mathrm{id})$$

is a local continuous section of π_K. To see that τ_U is well-defined, assume that

$$U \ni y = \Gamma(g, \mathrm{id})K = \Gamma(g', \mathrm{id})K.$$

Hence there exist $\gamma \in \Gamma$, $k \in K$ such that

$$(g', \mathrm{id}) = \gamma.(g, \mathrm{id})k = (\gamma g, \gamma k).$$

Thus,

$$k = \gamma^{-1} \in \Gamma \cap K = \Lambda.$$

By the definition of U, we have $g = hq$, $g' = hq'$ for some $q, q' \in B_r^H$. Thus,

$$\gamma hq = \gamma hq'$$

with $\gamma \in \Lambda$. Since r is an injectivity radius, it follows $q = q'$. In turn, $\gamma = \mathrm{id}$. This shows that τ_U is well-defined.

We prove that U is open. Let

$$\pi_\Gamma \colon G \to \Gamma\backslash G$$

be the canonical quotient map. Then

$$\pi_K^{-1}(U) = \{\Gamma(hg, k) \mid q \in B_r^H,\ k \in K\} = \pi_\Gamma(hB_r^H \times K),$$

which is open, since K is open in L.

One immediately checks that τ_U is continuous and a π_K-section. Moreover, one easily proves the other requirements.

For simplicity, we set

$$xB_r^T := t_U(x, B_r^T).$$

This definition does not depend on the choice of U with $x \in U$.

Proof of Proposition 3.9. Thickening of piece of N-leaf: Let $U \in \mathcal{S}(x_0)$. We claim that there exists $\eta > 0$ such that

$$B_\eta^{G/K}\overline{B_R^N} \to X,\ u \mapsto t_U(x_0, u) \tag{2}$$

is injective. To seek a contraction, suppose that for any $n \in \mathbb{N}$ we find $u_1^{(n)} = \varepsilon_1^{(n)} n_1^{(n)}, u_2^{(n)} = \varepsilon_2^{(n)} n_2^{(n)} \in B_{\frac{1}{n}}^{G/K}\overline{B_R^N}$ with $u_1^{(n)} \neq u_2^{(n)}$ and $t_U(x_0, u_1^{(n)}) = t_U(x_0, u_2^{(n)})$. Passing to a subsequence, we may assume that

$$n_1^{(n)} \to n_1,\ n_2^{(n)} \to n_2 \quad \text{with } n_1, n_2 \in \overline{B_R^N}.$$

Then

$$t_U(x_0, u_1^{(n)}) = t_U(x_0, u_2^{(n)}) \to x_0 n_1 = x_0 n_2$$

and, by injectivity, $n_1 = n_2$. Hence $u_1^{(n)} \to (n_1, K)$ and $u_2^{(n)} \to (n_1, K)$. This contradicts to the existence of an injectivity radius at $x_0 n_1$: We find $V \in \mathcal{S}(x_0 n_1)$ relatively compact and a local section τ_V such that

$$t_V(x_0 n_1, n_1^{-1} u_j^{(n)}) = t_U(x_0, u_j^{(n)}).$$

Now $n_1^{-1} u_1^{(n)}, n_1^{-1} u_2^{(n)} \to (\mathrm{id}, K)$, but $n_1^{-1} u_1^{(n)} \neq n_1^{-1} u_2^{(n)}$.

Choice of cross section: Let $r > 0$ be an injectivity radius at x_0. Hence, the canonical projection

$$B_r^{G/K} \to B_r^X(x_0)$$

is an isometry. Fix an Iwasawa decomposition $H = K_\infty AN$. Pick sufficiently small $\delta \in (0, \eta)$ and $\varepsilon > 0$ such that, with

$$C_{G/K} := (K_\infty A \times S) \cap B_\delta^{G/K},$$

we have

$$C_{G/K} B_\varepsilon^N \subseteq B_r^{G/K} \cap B_\eta^{G/K}\overline{B_R^N}.$$

Then

$$C_{G/K} \times B_\varepsilon^N \to C_{G/K} B_\varepsilon^N$$

is an isomorphism (a diffeomorphism). Set

$$C := t_U(x_0, C_{G/K}).$$

Then

$$C \times B^N_\varepsilon \to CB^N_\varepsilon, \quad (y,n) \mapsto yn \tag{3}$$

is injective, and we find some $\xi > 0$ such that $B_\xi(x_0) \subseteq CB^N_1$. We claim that (3) stays injective for $\varepsilon = R$. To that end suppose that we find $(x_1,n_1),(x_2,n_2) \in C \times B^N_R$ such that $x_1n_1 = x_2n_2$. Pick $h_1,h_2 \in C_{G/K}$ such that $t_U(x_0,h_j) = x_j$. Then

$$t_U(x_0, h_1n_1) = x_1n_1 = x_2n_2 = t_U(x_0,h_2n_2).$$

By injectivity of (2), we find $h_1n_1 = h_2n_2$. Comparing entries of these elements in G/K and envoking that the Iwasawa decomposition provides an diffeomorphism $H \to K_\infty \times A \times N$, it follows immediately that $n_1 = n_2$. The map in (2) is obviously measurable. Bi-measurability follows from a general principle (and can also be proven by hand).

Definition of σ-algebra: We consider the σ-algebra

$$\mathcal{B}_C \times \{\emptyset, B^N_R\}$$

on $C \times B^N_R$. This is clearly countably generated. Let $\mathcal{A}$ be the image of this σ-algebra unter the map in (2) (for $\varepsilon = R$). Then $\mathcal{A}$ is a countably generated σ-algebra on CB^N_R and

$$[xt]_{\mathcal{A}} = xB^N_R$$

for any $x \in C$ and $t \in B^N_R$. □

For the definition of the leafwise measures we need the following property: Whenever $B \in \mathcal{B}_N$ is bounded, then for μ-almost all $x \in X$, we find an (N,R)-flower $\mathcal{A}$ with an atom "covering" B, that is $xB \subseteq [x]_{\mathcal{A}}$. For that we need to show that in Proposition 3.9 we can have abitrarily large R.

Lemma 3.10. *For μ-almost all $x \in X$, the N-leaf xN is* embedded. *This means that the map $N \to X$, $n \mapsto xn$, is injective.*

Proof. Let $D \in \mathcal{B}_X$ be compact. We will show the statement on D. Since X is σ-compact, the statement then follows on all of X. Since μ is A-invariant, Poincaré recurrence yields that for μ-almost all $x \in D$ there exists a sequence $(t_j)_{j\in\mathbb{N}} \nearrow \infty$ in $\mathbb{R}$ such that $xa_{t_j} \in D$ for all $j \in \mathbb{N}$. Let $x \in D$ be such that this conclusion holds and let (t_j) be such a sequence. By passing to a subsequence, we may assume that (xa_{t_j}) converges to, say, $x_0 \in D$. Suppose that the N-leaf xN is not embedded. Then there exists $n \in N$, $n \neq \mathrm{id}$, such that $x = xn$. Hence also

$$xa_{t_j} = xna_{t_j} = xa_{t_j}(a_{-t_j}na_{t_j}). \tag{4}$$

Now $a_{-t_j}na_{t_j} \to \mathrm{id}$, and hence both sides of (4) converge to x_0. Let $r > 0$ be an injectivity radius at x_0. For sufficiently large j, the point $xa_{t_j} = xa_{t_j}(a_{-t_j}na_{t_j})$ is in $B^X_r(x_0)$, but $a_{-t_j}na_{t_j} \neq \mathrm{id}$, contradicting to r being an injectivity radius. In turn, the N-leaf xN is embedded. □

If $B \in \mathcal{B}_N$ is a bounded and $(\mathcal{A}, Z)$ an (N,R)-flower such that $xB \in [x]_{\mathcal{A}}$ for some $x \in X$, then we would like to define

$$\mu_x^N(B) := \frac{\mu_x^{\mathcal{A}}(xB)}{\mu_x^{\mathcal{A}}(xB_1^N)}.$$

Therefore we need $\mu_x^{\mathcal{A}}(xB_1^N) > 0$ for μ-almost all x.

Lemma 3.11. *Let $U \subseteq N$ be an open neighborhood of* id *and $(\mathcal{A}, Z)$ be an (N,R)-flower. For μ-almost all $x \in Z$, we have $\mu_x^{\mathcal{A}}(xU) > 0$.*

Proof. Let Z' be a full-measure subset of Z such that for the conditional measures $\mu_x^{\mathcal{A}}$, $x \in Z'$, the properties of Theorem 3.3 and Proposition 3.4 hold. Define

$$B := \{x \in Z' \mid \mu_x^{\mathcal{A}}(xU) = 0\}.$$

We want to show that $\mu(B) = 0$. By the definition of conditional measures (see Theorem 3.3), we have

$$\mu(B) = \int 1_B d\mu = \int \int 1_B(y) d\mu_x^{\mathcal{A}}(y) d\mu(y).$$

Hence, it suffices to show that

$$\mu_x^{\mathcal{A}}(B) = \mu_x^{\mathcal{A}}([x]_{\mathcal{A}} \cap B) = 0$$

for each $x \in Z'$. The goal is to cover $[x]_{\mathcal{A}} \cap B$ with countably many $\mu_x^{\mathcal{A}}$-null sets.

Let $x \in Z'$. Since $\mathcal{A}$-atoms are open N-plaques, we have

$$[x]_{\mathcal{A}} = xU_x$$

for some open bounded subset $U_x \subseteq N$. Let

$$V_x := \{n \in U_x \mid xn \in [x]_{\mathcal{A}} \cap B\}.$$

The familiy $\{nU\}_{n \in V_x}$ is an open cover of V_x. Since V_x is second countable, there is a countable subfamily $\{n_j U\}_{j \in \mathbb{N}}$, which covers V_x. Now

$$\mu_x^{\mathcal{A}}([x]_{\mathcal{A}} \cap B) \leq \mu_x^{\mathcal{A}}\left(\bigcup_{j \in \mathbb{N}} xn_j U\right) \leq \sum_{j \in \mathbb{N}} \mu_x^{\mathcal{A}}(xn_j U).$$

Since $xn_j \in B$, we have $0 = \mu_{xn_j}^{\mathcal{A}}(xn_j U) = \mu_x^{\mathcal{A}}(xn_j U)$, and hence $\mu_x^{\mathcal{A}}(B) = 0$. □

Proof of Theorem 3.6. By the combination of Lemma 3.10 and Proposition 3.9, for each $R \in \mathbb{N}$ we pick a countable family of (N,R)-flowers such that their bases cover X. The union of these families provides us with a countable family $\mathcal{F}$ of flowers. Let $X'' \in \mathcal{B}_X$ be a full measure set such that the N-leaf through any point of X'' is embedded and $\mu_x^{\mathcal{A}}(xB_1^N) > 0$ for all $x \in X''$ and $\mathcal{A} \in \mathcal{F}$ and such that whenever $\mathcal{A} \in \mathcal{F}$ and $x \in X''$, we have $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$, and whenever $x, y \in X''$ with $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$, then $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$.

Let $x \in X''$. Let $B \in \mathcal{B}_N$ be bounded. Then there exists $\mathcal{A} \in \mathcal{F}$ such that $xB \subseteq [x]_{\mathcal{A}}$. We define

$$\mu_x^N(B) := \frac{\mu_x^{\mathcal{A}}(xB)}{\mu_x^{\mathcal{A}}(xB_1^N)}. \tag{5}$$

To show that this definition is well-defined in all aspects (choice of $\mathcal{F}$, choice of $\mathcal{A}$) and to show (i) one uses that countably generated σ-algebra all of whose atoms are open N-plaques are countably equivalent on their common domain. Now for countably equivalent σ-algebra the conditional measures on the atoms of the common refinement are proportional. These proofs are not difficult, we omit them here.

To prove (iii), let $n \in N$ and $B \in \mathcal{B}_X$. Pick $\mathcal{A} \in \mathcal{F}$ such that $[x]_{\mathcal{A}} = [xn]_{\mathcal{A}}$ (there are many such $\mathcal{A}$). Then, by Proposition 3.4, we have $\mu_x^{\mathcal{A}} = \mu_{xn}^{\mathcal{A}}$. Hence

$$\begin{aligned} n_*\mu_{xn}^N(B) = \mu_{xn}^N(n^{-1}B) &= \frac{\mu_{xn}^{\mathcal{A}}(xB)}{\mu_{xn}^{\mathcal{A}}(xnB_1^N)} \\ &= \frac{\mu_x^{\mathcal{A}}(xB_1^N)}{\mu_x^{\mathcal{A}}(xnB_1^N)} \cdot \frac{\mu_x^{\mathcal{A}}(xB)}{\mu_x^{\mathcal{A}}(xB_1^N)} \\ &= c\mu_x^N(B). \end{aligned}$$

This completes the proof. □

Proposition 3.12 (Step 2). *If μ_x^N is N-invariant for μ-almost all $x \in X$, then μ is N-invariant.*

Proof. Let $n \in N$. We have to show that $n_*\mu = \mu$. Let $B \in \mathcal{B}_X$. Suppose $n \in B_R^N$. In the proof of Theorem 3.6 we have seen that we can cover X with countably many sets $Z_k \in \mathcal{B}_X$ such that on each Z_k there is based a $(N, 2R)$-flower. By appropriate choices of the Z_k we may that we can cover B with countably many sets $B_k \in \mathcal{B}_X$ such that $B_k \subseteq Z_k$ and also $B_k n^{-1} \subseteq Z_k$. This allows us to take advantage of Theorem 3.6(i). To simplify notation, we omit all restriction to subsets. Let $\mathcal{A}$ be an appropriate (N, R)-flower.

Let $x \in X$ and suppose that μ_x^N is N-invariant. We claim that $\mu_x^{\mathcal{A}}$ is N-invariant. Recall $\mu_x^{\mathcal{A}} = cx_*\mu_x^N$ for some $c > 0$. We have

$$\begin{aligned} n_*\mu_x^{\mathcal{A}}(B) &= \mu_x^{\mathcal{A}}(Bn^{-1}) \\ &= c\mu_x^N(\{m \in N \mid xm \in Bn^{-1}\}) \\ &= c\mu_x^N(\{m \in N \mid xmn \in B\}) \\ &= c\mu_x^N(\{m \in N \mid xnm \in B\}) \\ &= c\mu_x^N(n^{-1}\{m \in N \mid xm \in B\}) \\ &= cx_*\big(n_*\mu_x^N\big)(B) \\ &= cx_*\mu_x^N(B) \\ &= \mu_x^{\mathcal{A}}(B). \end{aligned}$$

Hence, the conditional measures $\mu_x^{\mathcal{A}}$ are N-invariant for μ-almost all $x \in X$. Then

$$\begin{aligned} n_*\mu(B) = \mu(Bn^{-1}) = \int_X 1_{Bn^{-1}} d\mu = \int_X \mu_x^{\mathcal{A}}(Bn^{-1}) d\mu(x) = \int_X \mu_x^{\mathcal{A}}(B) d\mu(x) \\ = \mu(B). \end{aligned}$$

Thus, μ is N-invariant. □

4. Preparations for step 3

The remaining section are devoted to the proof of the following proposition, the final step in the proof of Theorem 1.1.

Proposition 4.1. *For μ-almost all $x \in X$, the leafwise measure μ_x^N is N-invariant.*

Idea of the proof: Before we start with the details of the proof of Proposition 4.1, we illustrate its global idea. The proof makes essential use of the Ratner H-property, the observation that leafwise measures coincide almost surely if the anchor points are in the same S-leaf and the fact that the map $x \mapsto \mu_x^N$ is "sufficiently" continuous. More precisely, the idea of the proof is as follows. Qualitatively, the Ratner H-property is the following phenomenon. When we take to close-by point x, y in X and consider their trajectories under the horocycle flow (here: the N-flow), then there are two possibilities for their behavior:

(1) They stay close to each other all the time. This can only happen if they are on the same N-orbit.
(2) At some point xn_s, yn_s differ significantly. Then most of their distance is in the flow direction. In particular, there is $n_{s'}$ (significant) such that yn_s and $xn_{s+s'}$ are close.

For the proof of Proposition 4.1 we will note that whenever x, y are on the same S-leaf, then $\mu_x^N = \mu_y^N$ (μ-almost surely). Moreover, by S-recurrence, we find any pairs (x, y) of close-by points which are on the same S-leaf but not on the same N-leaf. Let us fix one such pair (x, y) and suppose that xn_s, yn_s differ significantly. Since xn_s and yn_s are still on the same S-leaf (the N-action preserves the S-leaves), we have

$$\mu_{xn_s}^N = \mu_{yn_s}^N.$$

Now $xn_{s+s'}$ is close to yn_s and hence the leafwise measures $\mu_{xn_{s+s'}}^N$, $\mu_{yn_s}^N$ should be similar (by Lusin). We will see that we actually may assume

$$\mu_{xn_{s+s'}}^N = \mu_{yn_s}^N.$$

Then

$$\mu_{xn_s}^N = \mu_{yn_s}^N = \mu_{xn_{s+s'}}^N \propto (n_{-s'})_* \mu_{xn_s}^N$$

We will see that this is as good as being N-invariant.

We will now first provide a conditional proof of Proposition 4.1, where we proceed under the assumption of a doubling condition (to be explained below) to illustrate the ideas. Finally, we will provide an unconditional proof where the doubling condition is substituted a local version. We start with a few preparations.

The first step is to reduce the proof of Proposition 4.1 to the proof that on a non-null set, the leafwise measures μ_x^N are proportional to their push-forward $n_*\mu_x^N$ for some $n \in N$, $n \neq \mathrm{id}$.

Lemma 4.2. *For μ-almost all $x \in X$ and all $a \in A$ we have $\mu_{xa}^N \propto \theta_* \mu_x^N$, where $\theta\colon N \to N$, $n \mapsto a^{-1}na$.*

Proof. We restrict X to a full-measure subset X' on which leafwise measures are defined via (5) and on which the statement of Lemma 3.11 holds. Let $B \in \mathcal{B}_N$ be bounded. Pick $R \geq 1$ such that $B, \theta(B) \subseteq B_R^N$, and pick an (N, R)-flower $(\mathcal{A}, CB_R^N)$ of the form as in the definition (5). Then also the σ-algebra $\mathcal{A}a$ is such an (N, R)-flower (possibly, one has to shrink the cross section C a bit to avoid problems with injectivity radii). Using the definition of conditional measures on $\mathcal{A}$ and $\mathcal{A}a$, one easily finds

$$a_*\mu_x^{\mathcal{A}} = \mu_{xa}^{\mathcal{A}a},$$

where $a_*\mu_x^{\mathcal{A}}$ is the push-forward of $\mu_x^{\mathcal{A}}$ under the map $X \to X$, $x \mapsto xa$. Then

$$\begin{aligned}
\mu_{xa}^N(B) &= \frac{\mu_{xa}^{\mathcal{A}a}(xaB)}{\mu_{xa}^{\mathcal{A}a}(xaB_1^N)} = \frac{\big(a_*\mu_x^{\mathcal{A}}\big)(xaB)}{\big(a_*\mu_x^{\mathcal{A}}\big)(xaB_1^N)} \\
&= \frac{\mu_x^{\mathcal{A}}(xaBa^{-1})}{\mu_x^{\mathcal{A}}(xaB_1^Na^{-1})} = \frac{\mu_x^{\mathcal{A}}(xaBa^{-1})}{\mu_x^{\mathcal{A}}(xB_1^N)} \cdot \frac{\mu_x^{\mathcal{A}}(xB_1^N)}{\mu_x^{\mathcal{A}}(xaB_1^Na^{-1})} \\
&= c\big(\theta_*\mu_x^{\mathcal{A}}\big)(B)
\end{aligned}$$

for an obvious constant $c > 0$. □

Lemma 4.3. *Let*

$$\begin{aligned}
Z &:= \{x \in X \mid \forall\, n \in N \colon \mu_x^N = n_*\mu_x^N\}, \\
Y &:= \{x \in X \mid \exists\, n \in N \setminus \{\mathrm{id}\} \colon \mu_x^N \propto n_*\mu_x^N\}.
\end{aligned}$$

Then $\mu(Y \setminus Z) = 0$.

Proof. For $y \in Y$ we set

$$R_y := \{s > 0 \mid \mu_y^N \propto (n_s)_*\mu_y^N\} \quad \text{and} \quad r(y) := \inf R_y.$$

We aim to show that $R_y = \mathbb{R}^+$ for μ-almost all $y \in Y$.

For $s \in R_y$ it follows with Lemma 4.2 that

$$\mu_{ya_t}^N \propto \theta_*\mu_y^N \propto \theta_*(n_s)_*\mu_y^N = \big(n_{e^{-2t}s}\big)_*\theta_*\mu_y^N \propto \big(n_{e^{-2t}s}\big)_*\mu_{ya_t}^N. \tag{6}$$

Hence $r(ya_t) = e^{-2t}r(y)$. Since μ is A-invariant, Poincaré recurrence applied to Y yields $r(y) = 0$ for μ-almost all $y \in Y$.

We fix a test function $\varphi \in C_c(N)$ which is non-negative everywhere and positive on some neighborhood of id_N. For $y \in Y$ define

$$L_y := \left\{ s \geq 0 \;\middle|\; \int \varphi d(n_s)_*\mu_y^N > 0 \right\}.$$

For μ-almost all $y \in Y$, the set L_y is nonempty and contains R_y. Note that L_y depends on φ. The map

$$s \mapsto \int \varphi d(n_s)_*\mu_y^N = \int \varphi(nn_s)d\mu_y^N(n)$$

is clearly continuous. This yields that whenever $R_y \subseteq L_y$ and $r(y) = 0$ (which is true μ-almost surely), this map is positive on an interval $[0, b]$ for some $b = b(y, \varphi) > 0$. Define the map $k_y \colon L_y \to \mathbb{R}$ by

$$\exp(k_y(s)) := \frac{\int \varphi d(n_s)_*\mu_y^N}{\int \varphi d\mu_y^N} = \frac{\int \varphi(nn_s)d\mu_y^N(n)}{\int \varphi(n)d\mu_y^N(n)}.$$

We claim that $L_y = [0, \infty)$ and $k_y(rs) = rk_y(s)$ for any $r, s > 0$.

If $s_1 \in R_y$ and $s_2 \in L_y$, then

$$\begin{aligned}\frac{\int \varphi(nn_{s_1+s_2})d\mu_y^N(n)}{\int \varphi(n)d\mu_y^N(n)} &= \frac{\int \varphi(nn_{s_1+s_2})d\mu_y^N(n)}{\int \varphi(nn_{s_1})d\mu_y^N(n)} \cdot \frac{\int \varphi(nn_{s_1})d\mu_y^N(n)}{\int \varphi(n)d\mu_y^N(n)} \\ &= \frac{\int \varphi(nn_{s_2})d(n_{s_1})_*\mu_y^N(n)}{\int \varphi(n)d(n_{s_1})_*\mu_y^N(n)} \cdot \exp(k_y(s_1)) \\ &= \frac{\int \varphi(nn_{s_2})d\mu_y^N(n)\cdot c}{\int \varphi(n)d\mu_y^N(n)\cdot c} \cdot \exp(k_y(s_1)) \\ &= \exp(k_y(s_1+s_2)).\end{aligned}$$

Thus, $s_1 + s_2 \in L_y$ and $k_y(s_1+s_2) = k_y(s_1) + k_y(s_2)$. If $r(y) = 0$, then since R_y contains arbitrarily small $s > 0$, this special kind of additivity yields that $L_y = [0,\infty)$ (hence we can move $[0,b]$ by small amounts, and then repeat) and hence k_y is continuous on $[0,\infty)$. Moreover, R_y is dense in L_y (note that $s_1 + s_2 \in R_y$ for $s_1, s_2 \in R_y$. Adding up arbitrarily small elements sufficiently often produces elements arbitrarily close to any given $s_0 \in L_y$). (Note that we cannot conclude that $R_y = L_y$ because L_y depends on φ.) This now implies that

(1) $\forall\, s \in R_y,\ \forall\, m \in \mathbb{N}\colon k_y(ms) = mk_y(s)$,
(2) $\forall\, s \in L_y,\ \forall\, m \in \mathbb{N}\colon k_y(ms) = mk_y(s)$, (by continuity)
(3) $\forall\, s \in L_y,\ \forall\, q \in \mathbb{Q}\colon k_y(qs) = qk_y(s)$,
(4) $\forall\, s \in L_y,\ \forall\, r > 0\colon k_y(rs) = rk_y(s)$.

Let $s \in R_y$. As seen in (6), then $e^{-2t}s \in R_{ya_t}$. Then

$$\begin{aligned}\exp(k_y(s)) &= \frac{\int \varphi d(n_s)_*\mu_y^N}{\int \varphi d\mu_y^N} = \frac{\int \varphi d\theta_*(n_s)_*\mu_y^N}{\int \varphi d\theta_*\mu_y^N} \\ &= \frac{\int \varphi d\big(n_{e^{-2t}s}\big)_*\theta_*\mu_y^N}{\int \varphi d\theta_*\mu_y^N} = \frac{\int \varphi d\big(n_{e^{-2t}s}\big)_*\mu_{ya_t}^N}{\int \varphi d\mu_{ya_t}^N} \\ &= \exp(k_{ya_t}(e^{-2t}s)).\end{aligned}$$

Invoking continuity, we find

$$k_y(s) = k_{ya_t}(e^{-2t}s)$$

for all $s \in L_y$, all $t \in \mathbb{R}$.

The map $f\colon Y' \to \mathbb{R}$, $y \mapsto k_y(1)$, is measurable on some full measure subset Y' of Y. By Lusin's Theorem and regularity of μ, for any $\varepsilon > 0$ we find a compact subset K_ε of Y' with $\mu(K) \geq \mu(Y') - \varepsilon$ such that $f|_{K_\varepsilon}$ is continuous. In particular, it is bounded. Applying Poincaré recurrence to K_ε yields that

$$k_y(1) = e^{-2t}k_{ya_t}(1) = 0$$

for μ-almost all $y \in K_\varepsilon$. Letting $\varepsilon \to 0$, shows $k_y \equiv 0$ for μ-almost all $y \in Y$. Thus, we have

$$\int \varphi d(n_s)_*\mu_y^N = \int \varphi d\mu_y^N$$

for μ-almost all $y \in Y$ and all $s > 0$ and all test functions φ. Therefore $Y \setminus Z$ is a null set. $\square$

Remark 4.4. The proof of Lemma 4.3 shows that Z and Y are A-invariant μ-almost surely.

First reduction of proof of Proposition 4.1. Let Z, Y be as in Lemma 4.3. The statement of Proposition 4.1 is equivalent to $\mu(X \setminus Z) = 0$. Assume that $\mu(X \setminus Z) > 0$ and set

$$\mu' := \mu|_{X\setminus Z}.$$

Then μ' is S-recurrent, and by Remark 4.4 also A-invariant. Clearly, (since Z is A-invariant) the entropy condition is satisfied. In the construction of (N, R)-flowers we can restrict to A-invariant sets because A-invariant sets essentially consist of complete N-leaves (easily seen using standard Iwasawa decomposition). Then we can further restrict to the flowers (conditional measures of conditional measures). Since $X \setminus Z$ is A-invariant, we have

$$(\mu')^{\mathcal{A}}_x = \mu^{\mathcal{A}}_x$$

μ-almost surely. Therefore, for the proof of Proposition 4.1 it suffices to show that the assumption $\mu(Z) = 0$ or $\mu(Y) = 0$ leads to a contradiction. $\square$

Proposition 4.5 (Identical leafwise measures)**.** *There exists $X' \in \mathcal{B}_X$, $\mu(X \setminus X') = 0$, such that for all $x, y \in X'$ with $x \overset{S}{\sim} y$ we have $\mu^N_x = \mu^N_y$.*

For $x \in X$ let $\mathcal{S}(x) := \{U \in \mathcal{S} \mid x \in U\}$, where $\mathcal{S}$ is the family of open sets from the local continuous sections. For all pratical purposes we stay here with the family of local continuous sections we constructed explicitly.

Sketch of proof of Proposition 4.5.

- We start by showing that we can define leafwise measures for S-leaves in analogy to Theorem 3.6. For that we need the thickening property, cross sections, (S, R)-flowers and long embedded S-leaves.
- We show the existence of thickening and cross sections. Let $x_0 \in X$, $U \in \mathcal{S}(x_0)$, $R \geq 1$ and suppose that

  $$t_U(x_0, \cdot)\colon \overline{B}^S_R \to X, \quad s \mapsto t_U(x_0, s)$$

  is injective. We claim that there exists $\varepsilon > 0$ such that $x_0 B^H_\varepsilon \subseteq U$ and

  $$x_0 B^H_\varepsilon \times B^S_R \to X, \quad (x, s) \mapsto t_U(x, s)$$

  is injective. Note that by our choices of local sections, we have

  $$t_U(x_0 h, s) = t_U(x_0, s)h$$

  for sufficiently small h. To seek a contradiction we suppose that for all $n \in \mathbb{N}$ we find $h^{(n)}_1, h^{(n)}_2 \in B^H_{\frac{1}{n}}$, $s^{(n)}_1, s^{(n)}_2 \in B^S_R$ such that

  $$t_U(x_0 h^{(n)}_1, s^{(n)}_1) = t_U(x_0 h^{(n)}_2, s^{(n)}_2)$$

  and $h^{(n)}_1 s^{(n)}_1 \neq h^{(n)}_2 s^{(n)}_2$. Since $\overline{B}^S_R$ is compact, we may assume that

  $$s^{(n)}_1 \to s_1,\ s^{(n)}_2 \to s_2 \quad \text{for some } s_1, s_2 \in \overline{B}^S_R.$$

  Since t_U is continuous, it follows

  $$t_U(x_0 h^{(n)}_1, s^{(n)}_1) = t_U(x_0 h^{(n)}_2, s^{(n)}_2) \to t_U(x_0, s_1) = t_U(x_0, s_2).$$

Hence, by injectivity, $s_1 = s_2$. Using an appropriate local section at $x_0 s_1$, we see find a contraction to the existence of an injectivity radius at $x_0 s_1$. This shows that the existence of $\varepsilon > 0$ such that

$$t_U(x_0, \cdot)\colon B_\varepsilon^H B_R^S \to X$$

is injective. Now, if $x_1 = x_0 h_1, x_2 = x_0 h_2 \in x_0 B_\varepsilon^H$ and $s_1, s_2 \in B_R^S$ with $t_U(x_1, s_1) = t_U(x_2, s_2)$, we have

$$t_U(x_1, s_1) = t_U(x_0, s_1)h_1 = t_U(x_0, h_1 s_1) = t_U(x_0, h_2 s_2).$$

Hence, $h_1 s_1 = h_2 s_2$. By comparing the components, $h_1 = h_2$, $s_1 = s_2$.

- We claim that all S-leaves are embedded. This means that for any $x \in X$ and $U \in \mathcal{S}(x)$, the map $t_U(x, \cdot)\colon S \to X$ is injective. Let $s_1, s_2 \in S$ with $t_U(x, s_1) = t_U(x, s_2)$. Suppose $x = \Gamma(h, \mathrm{id})K$. Then

$$\Gamma(h, s_1) = t_U(x, s_1) = t_U(x, s_2) = \Gamma(h, s_2).$$

Since Γ acts diagonally, it follows $s_1 = s_2$.

- Now we can define families of S-leafwise measures as in Theorem 3.6 with the difference that we have to pay attention to different local sections. This means, that for $x \in U \in \mathcal{S}$, the leafwise measure at x depends on the local section on U, resulting in families $\{\mu_{x,U}^S\}$. Moreover, instead of Theorem 3.6(iii), for $x \overset{S}{\sim} y$ with $x \in U \in \mathcal{S}(x)$, $y \in V \in \mathcal{S}(y)$ and any isometry λ on S with

$$t_U(x, \cdot) = t_V(y, \cdot) \circ \lambda, \tag{7}$$

we have

$$\mu_{x,U}^S \propto \lambda_* \mu_{y,V}^S.$$

- We also can achieve all these statements and constructions for $N \times S$-leaves by using a hyprid version of the construction of N-leafwise and S-leafwise measures. This hybrid structure shows (for more general situations, it is proven by Einsiedler-Katok) that for μ-almost all $x \in X$ and $U \in \mathcal{S}(x)$ we have

$$\mu_{x,U}^{N \times S} \propto \mu_x^N \times \mu_{x,U}^S.$$

- If now $x \overset{S}{\sim} y$ with $x \in U \in \mathcal{S}(x)$, $y \in V \in \mathcal{S}(y)$ and λ an isometry on S as in (7), then

$$\mu_x^N \times \mu_{x,U}^S \propto \mu_{x,U}^{N\times S} \propto \lambda_* \mu_{y,V}^{N\times S} \propto \mu_y^N \times \lambda_* \mu_{y,V}^S \propto \mu_y^N \times \mu_{x,U}^S.$$

Hence, $\mu_x^N = \mu_y^N$ (equality because of normalization).

□

Lemma 4.6. *For every $\varepsilon > 0$, every $B \in \mathcal{B}_X$ and μ-almost every $x \in B$, there is some*

$$y \in B \cap B_\varepsilon(x) \setminus x B_1^{N\times S}$$

such that $y \overset{S}{\sim} x$. Here, $x B_1^{N\times S} := t_U(x, B_1^{N\times S})$ does not depend on the choice of $U \in \mathcal{S}(x)$.

Proof. A picture proof:

picture: long S-leaves

From the arguments in the proof of Proposition 4.5 we know that almost every $N \times S$-leaf is embedded, i.e., for μ-almost all $x \in X$. Pick $U \in \mathcal{S}(x)$. Let $\varepsilon > 0$ and cover B with a countable family $\{B_j\}_{j\in\mathbb{N}}$ of $\varepsilon/2$-balls in X. Let $j \in \mathbb{N}$. By S-recurrence, we know that for μ-almost every $x \in B \cap B_j$ there exists $s \in S \setminus B_1^S$ such that $y := t_U(x, s) \in B \cap B_j$. Since the $N \times S$-leaf through x is embedded μ-almost surely, we have $y \notin x B_1^{N\times S}$ μ-almost surely. Noting $B_j \subseteq B_\varepsilon(x)$ finishes the proof. □

5. A conditional proof of Proposition 4.1

Lemma 5.1 (Ratner H-property, quantitatively)**.** *Let $X' \subseteq X$ be compact and $\varrho \in (0, 1)$. Then we find $C, \eta_0 > 0$ such that for all $\delta \in (0, \eta_0)$ and $x, x' \in X'$ with*

$$x' \in B_\delta(x) \setminus x B_1^{N\times S}$$

there exists $\eta > 0$ such that for all $r \in \mathbb{R}$ with

$$\varrho\eta < |r| < \eta$$

there exists $r' \in \mathbb{R}$ with

$$\frac{1}{C} < |r - r'| < C$$

such that we have

$$x' n_r \in B_{C\delta^{\frac{1}{2}}}(x n_{r'})$$

Conditional proof of Proposition 4.1. Let $\varepsilon > 0$ sufficiently small ($\varepsilon < 1/16$ is sufficient for the conditional proof, the unconditional one may require a smaller ε). We fix a compact set $X_1 \in \mathcal{B}_X$ with $\mu(X_1) > 1 - \varepsilon$ such that

(a) $X_1 \cap Y = \emptyset$ (possible since Y is a null set),
(b) the map $x \mapsto \mu_x^N$ is continuous on X_1 (possible by Lusin's Theorem),
(c) for all $x \in X_1$, the N-leafwise measure μ_x^N satisfies Theorem 3.6(i)-(iii) (possible by Theorem 3.6), and
(d) $\forall\, x, x' \in X_1 \colon (x \overset{S}{\sim} x' \Rightarrow \mu_x^N = \mu_{x'}^N)$ (possible by Proposition 4.5).

We will deduce a contradiction to (a) under the following **doubling assumption**: There is a constant $\varrho \in (0,1)$ such that for μ-almost every $x \in X$ and all $r > 1$ we have

$$\mu_x^N(B_r^N) > 2\mu_x^N(B_{\varrho r}^N).$$

We **claim** that there is a (fixed) compact interval $I \subseteq \mathbb{R}_{>0}$ (in particular, bounded away from 0) such that, if ε is sufficiently small, for any sufficiently small $\delta > 0$ we find points $y, y' \in X_1$ with

$$y' \in B_\delta(yn_t) \tag{8}$$

for some $|t| \in I$, and

$$\mu_y^N = \mu_{y'}^N.$$

If we suppose this claim to be true for the moment, then we apply it to a sequence $(\delta_j) \searrow 0$ to get sequences $(y_j), (y_j')$ in X_1. Since X_1 is compact, we may assume

$$y_j \to y,\ y_j' \to y' \quad \text{for some } y, y' \in X_1.$$

Since

$$d(y_j', y_j n_{t_j}) < \delta_j \to 0$$

and the n_{t_j} are contained in the compact set $\{n_t \mid |t| \in I\}$, there exists $t \in I \cup -I$ with

$$y' = yn_t.$$

By (b), the map $x \mapsto \mu_x^N$ is continuous on X_1. Since

$$\mu_{y_j}^N = \mu_{y_j'}^N,$$

we find

$$\mu_y^N = \mu_{y'}^N = \mu_{yn_t}^N.$$

Using (c), we have

$$\mu_{yn_t}^N \propto \left(n_{-t}\right)_* \mu_y^N.$$

Hence either y or y' (depending on whether $t > 0$ or not) is in Y, which is a contradiction to (a).

It remains to prove the claim. By a special maximal ergodic theorem by Lindenstrauss-Rudolph (note that μ_x^N is not yet known to be N-invariant), we find a compact set $X_2 \in \mathcal{B}_X$, $X_2 \subseteq X_1$, $\mu(X_2) \geq 1 - c_1 \varepsilon^{\frac{1}{2}}$, where c_1 is some universal constant, such that

$$\forall\, x \in X_2\ \forall\, r > 0\colon \int_{B_r^N} 1_{X_1}(xn) d\mu_x^N(n) \geq (1 - \varepsilon^{\frac{1}{2}}) \mu_x^N(B_r^N). \tag{9}$$

Moreover, we may assume that the doubling assumption holds on X_2 without exceptions. Let C, η_0 be as in Lemma 5.1 applied to X_2 and ϱ (pick a ϱ in the doubling assumption). For $\delta \in (0, \eta_0)$ pick $\eta > 0$ as in Lemma 5.1. Pick δ sufficiently small, so that $\eta > 1$. By Lemma 4.6, for μ-almost every $x \in X_2$ we find

$$x' \in X_2 \cap B_\delta(x) \setminus x B_1^{N \times S}$$

with $x' \overset{S}{\sim} x$. We fix such a pair (x, x') and consider

$$G_1 := \{s \in \mathbb{R} \mid xn_s \in X_1\}, \quad G_2 := \{s \in \mathbb{R} \mid x'n_s \in X_1\}.$$

We want to show that

$$\left(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}\right) \cap G_1 \cap G_2 \neq \emptyset.$$

We estimate its measure. For that we use the identification $\mathbb{R} \to N$, $s \mapsto n_s$. Since $x, x' \in X_1$ and $x \overset{S}{\sim} x'$, (d) shows

$$\mu_x^N = \mu_{x'}^N.$$

Since $x, x' \in X_2$, we have

$$\mu_x^N\big((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \setminus G_j\big) = \mu_x^N(B_\eta^{\mathbb{R}}) - \Big[\mu_x^N(\overline{B}_{\varrho\eta}^{\mathbb{R}}) + \mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \cap G_j)\Big].$$

With (9) and $\mu_x^N = \mu_{x'}^N$ it follows that

$$\begin{aligned}\mu_x^N(\overline{B}_{\varrho\eta}^{\mathbb{R}}) + \mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \cap G_j) &\geq \mu_x^N(B_\eta^{\mathbb{R}} \cap G_j)\\ &\geq (1-\varepsilon^{\frac{1}{2}})\mu_x^N(B_\eta^{\mathbb{R}}).\end{aligned}$$

Hence

$$\mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \setminus G_j) \leq \varepsilon^{\frac{1}{2}}\mu_x^N(B_\eta^{\mathbb{R}}).$$

By the doubling assumption, we have

$$\mu_x^N(B_\eta^{\mathbb{R}}) = \mu_x^N(\overline{B}_{\varrho\eta}^{\mathbb{R}}) + \mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \leq \frac{1}{2}\mu_x^N(B_\eta^{\mathbb{R}}) + \mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}).$$

Thus

$$\frac{1}{2}\mu_x^N(B_\eta^{\mathbb{R}}) \leq \mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}).$$

Therefore,

$$\mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \setminus G_j) \leq 2\varepsilon^{\frac{1}{2}}\mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}).$$

By (c),

$$\mu_x^N(B_\eta^{\mathbb{R}}) > 0.$$

Now

$$\begin{aligned}\mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) &\leq \mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \setminus G_1) + \mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \setminus G_2) + \mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \cap G_1 \cap G_2)\\ &\leq 4\varepsilon^{\frac{1}{2}}\mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) + \mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \cap G_1 \cap G_2).\end{aligned}$$

Thus,

$$\mu_x^N((B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \cap G_1 \cap G_2) \geq (1-4\varepsilon^{\frac{1}{2}})\mu_x^N(B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}),$$

which is positive for sufficiently small ε.

Now pick $s_0 \in (B_\eta^{\mathbb{R}} \setminus \overline{B}_{\varrho\eta}^{\mathbb{R}}) \cap G_1 \cap G_2$ and consider

$$y := xn_{s_0},\ y' := x'n_{s_0}\ \in X_1.$$

By Lemma 5.1,

$$y' \in B_{C\delta^{\frac{1}{2}}}(yn_{s_1})$$

for some s_1 such that $|s_1|$ is in some fixed compact interval $I \subseteq \mathbb{R}_{>0}$ (which does not depend on x, x' or δ). Since $\mu_x^N = \mu_{x'}^N$ and $x, x', y, y' \in X_1$, we have

$$\mu_y^N = \mu_{xn_{s_0}}^N \propto \big(n_{-s_0}\big)_*\mu_x^N = \big(n_{-s_0}\big)_*\mu_{x'}^N \propto \mu_{x'n_{s_0}}^N = \mu_{y'}^N.$$

From $\mu_y^N(B_1^N) = 1 = \mu_{y'}^N(B_1^N)$ it follows that $\mu_y^N = \mu_{y'}^N$. This completes the proof. □

Remark 5.2.
- As soon as δ, η are fixed, we need Lemma 5.1 just for the pair (x, x'), and the doubling assumption just for x and η. At these points one can twist the conditional proof to make an unconditional one.
- We have not used the entropy condition in Theorem 1.1 yet.

6. An unconditional proof of Proposition 4.1

For $\varrho \in (0,1)$, $C, \gamma > 0$ and $x, x' \in X$ we define

$$R_\varrho(x) := \{r > 0 \mid \mu_x^N(B_r^N) > 2\mu_x^N(B_{\varrho r}^N)\},$$
$$D_{\varrho,C,\gamma}(x,x') := \{r \in \mathbb{R} \mid \forall\, s \in \mathbb{R}, \varrho r < |s| < r \;\exists\, s' \in \mathbb{R}, C^{-1} < |s - s'| < C \colon x' n_s \in B_\gamma(x n_{s'})\}.$$

Thus, $R_\varrho(x)$ is the set on which the doubling condition holds for μ_x^N, and $D_{\varrho,D,\gamma}(x,x')$ the set on which the "near displacement condition" holds for x, x'.

Let $\varepsilon > 0$ sufficiently small (will become clear during the proofs) and define the sets X_1, X_2 as in the conditional proof of Proposition 4.1.

Lemma 6.1. *Recall the set X_2. For any $\delta > 0$, any $C, \gamma > 0$ and any $x, x' \in X_2$ with*

(i) $d(x,x') < \delta$,
(ii) $x \overset{S}{\sim} x'$,
(iii) $D_{\varrho,C,\gamma}(x,x') \cap R_\varrho(x) \neq \emptyset$,

we find $s, s' \in \mathbb{R}$ with $C^{-1} < |s'| < C$ such that

(a) $y := x n_s, y' := x' n_s \in X_1$,
(b) $y \in B_\gamma(y' n_{s'})$,
(c) $\mu_y^N = \mu_{y'}^N$.

Proof. Note that $y \overset{S}{\sim} y'$, since the action of N preserves S-leaves and by (ii) x, x' are on the same S-leaf. Then $y, y' \in X_1$ immediately implies (c). To prove (a) and (b), we set (as before)

$$G_1 := \{s \in \mathbb{R} \mid x n_s \in X_1\}, \qquad G_2 := \{s \in \mathbb{R} \mid x' n_s \in X_1\}.$$

Pick $r \in D_{\varrho,C,\gamma}(x,x') \cap R_\varrho(x)$. Since $x, x' \in X_1$ and $x \overset{S}{\sim} x'$, we have $\mu_x^N = \mu_{x'}^N$. Then, since $x, x' \in X_2$, it follows as before that

$$\mu_x^N\big((B_r^{\mathbb{R}} \setminus \overline{B}_{\varrho r}^{\mathbb{R}}) \setminus G_j\big) \leq \varepsilon^{\frac{1}{2}} \mu_x^N(B_r^{\mathbb{R}}).$$

Invoking $r \in R_\varrho(x)$, it follows further

$$\leq 2\varepsilon^{\frac{1}{2}} \mu_x^N\big((B_r^{\mathbb{R}} \setminus \overline{B}_{\varrho r}^{\mathbb{R}})\big).$$

to x, which contradicts to the fact that μ-almost all N-leaves are embedded. Therefore, as before, we find

$$s_0 \in \big(B_r^{\mathbb{R}} \setminus \overline{B}_{\varrho r}^{\mathbb{R}}\big) \cap G_1 \cap G_2.$$

Set

$$y := x n_{s_0}, \quad y' := x' n_{s_0}.$$

Then $y, y' \in X_1$ and, since $r \in D_{\varrho,D,\gamma}$, we find $s' \in \mathbb{R}$ with $C^{-1} < |s'| < C$ such that $y \in B_\gamma(y' n_{s'})$. $\square$

The following proposition is the only place where one uses the entropy condition in Theorem 1.1.

Proposition 6.2. *For all $\varepsilon > 0$ there exists $\varrho \in (0,1)$ such that the set*

$$X_\varrho := \{x \in X \mid \mu_x^N(B_1^N) > 2\mu_x^N(B_\varrho^N)\}$$

has measure $\mu(X_\varrho) > 1 - \varepsilon$.

The relation between $R_\varrho(x)$ and X_ϱ is given by

$$e^{2t} \in R_\varrho(x) \Leftrightarrow xa_t \in X_\varrho$$

because of

$$\mu_{xa_t}^N(B_s^N) = c\theta_*\mu_x^N(B_s^N) = c\mu_x^N(a_tB_s^Na_{-t}) = c\mu_x^N(B_{e^{2t}s}^N).$$

Sketch of proof of Proposition 6.2.

- We claim that if μ-almost every a_1-ergodic component of μ has positive entropy, then μ-almost every N-leafwise measure μ_x^N is infinite. We note that it does not make a big difference, whether we work with a-ergodic or A-ergodic components.

  Let

  $$E_1 := \{x \in X \mid \mu_x^N \text{ is finite}\}$$

  and

  $$E_2 := \{x \in X \mid \mu_x^N = \delta_{\mathrm{id}_N}\}.$$

  Then $E_2 \subseteq E_1$ and we claim that $\mu(E_1 \setminus E_2) = 0$. To that end let

  $$r(x) := \begin{cases} \inf\{r > 0 \mid \mu_x^N(B_r^N) > \frac{1}{2}\mu_x^N(B_\infty^N)\} & \text{if } x \in E_1, \\ 0 & \text{otherwise.} \end{cases}$$

  As in Lemma 4.3 we see $r(x) = e^{-1}r(xa_1)$. Using Poincaré recurrence as in Lemma 4.3 it follows that $r(x) = 0$ μ-almost surely. This shows $\mu(E_1 \setminus E_2) = 0$.

  Now let $\nu = \mu_x^{\mathcal{E}}$ be an ergodic component of μ. We find a countably generated Borel σ-algebra $\mathcal{A}$ on X such that
  (1) for μ-almost every $x \in X$ we have

  $$xB_\varepsilon^N \subseteq [x]_{\mathcal{A}} \subseteq xB_r^N,$$

  (2) $a_1\mathcal{A} \subseteq \mathcal{A}$,
  (3)

  $$h_\nu(a_1) = H_\nu(\mathcal{A} \mid a_1\mathcal{A}) = -\int \log \nu_x^{a_1\mathcal{A}}([x]_{\mathcal{A}})d\nu(x).$$

  We omit the proof of its existence here. Then

  $$h_\nu(a_1) = -\int \log \frac{x_*\nu_x^N([x]_{\mathcal{A}})}{x_*\nu_x^N([x]_{a_1\mathcal{A}})}d\nu(x). \tag{10}$$

  Since E_2 is a_1-invariant (up to a null set) and ν is ergodic, we have $\nu(E_2) = 0$ or $\nu(E_2) = 1$. Note that $\nu_x^N = \mu_x^N$ μ-almost surely. If $\nu(E_2) = 1$, then (10), we have $h_\nu(a_1) = 0$, which is a contradiction. Hence μ-almost each leafwise measure is infinite.

- We claim that if μ-almost each leafwise measure is infinite, then μ is N-recurrent.

picture proof: N-recurrent

- We now prove the statement of the proposition. To seek a contradiction, we assume that we find $\varepsilon_0 > 0$ such that for all $\varrho \in (0,1)$ such that
$$\mu(X_\varrho) < 1 - \varepsilon_0.$$
We set
$$\begin{aligned} Y_\varrho := X' \setminus X_\varrho &= \{x \in X' \mid \mu_x^N(B_1^N) \leq 2\mu_x^N(B_\varrho^N)\} \\ &= \{x \in X' \mid \frac{1}{2} \leq \mu_x^N(B_\varrho^N)\}. \end{aligned}$$
Note that for $\varrho_1 < \varrho_2$ we have
$$Y_{\varrho_2} \supseteq Y_{\varrho_1}.$$
Hence
$$\mu\Big(\bigcap_{n\in\mathbb{N}} Y_{\frac{1}{n}}\Big) = \lim_{n\to\infty} \mu(Y_{\frac{1}{n}}) \geq \varepsilon_0.$$
Hence, there exists $B \in \mathcal{B}_X$, $\mu(B) > 0$ such that for μ-almost all $x \in B$ we find a sequence $(n_k) \to \infty$ in $\mathbb{N}$ such that
$$\frac{1}{2} \leq \mu_x^N(B^N_{\frac{1}{n_k}})$$
for all $k \in \mathbb{N}$. Thus, on a set of positive measure, we have
$$\mu_x^N(\{\mathrm{id}\}) \geq \frac{1}{2}.$$
We pick an (N, R)-flower whose basis intersects this set in a positive measure set. Then
$$\mu(\{x\}) > 0$$
for all x in a positive measure set. For all these x, by N-recurrence, the N-orbit through x has to return to x, which is a contradiction to the fact that μ-almost all N-leaves are embedded.

□

From now on fix some $\varrho \in (0,1)$ as in Proposition 6.2 adapted to our previous choice of ε.

Lemma 6.3. *There exists a constant $C_0 > 0$ such that for any sufficiently small $\delta \in (0,1)$ and any $x, x' \in X_1$ such that $d(x,x') < \delta$ and $x \overset{S}{\sim} x'$, we find $\xi_1 > C_0^{-1}\delta^{-\frac{1}{2}}$ such that at least one of the following properties is satisfied:*

(i) *For all $t \in (0, \kappa \log \xi_1)$ we have*
$$\xi_1 \in D_{\varrho, C_0, \delta^{\frac{1}{4}}}(xa_t, x'a_t).$$
Here, κ is an absolute positive constant.

(ii) *For all $t \in (\kappa' \log \xi_1, 2\kappa' \log \xi_1)$ we have*
$$e^{-t}\xi_1 \in D_{\varrho, C_0, \delta^{\frac{1}{4}}}(xa_t, x'a_t).$$
Here, κ' is again an absolute positive constant.

Proof. We start by considering the difference between x and x' in the H-direction. Suppose that
$$x = \Gamma(g,s), \quad x' = \Gamma(g', s').$$
Then
$$\delta > d_X(x,x') = \inf_{\gamma \in \Gamma} d_{H\times S}(\gamma.(g,s), (g', s')).$$
Without loss of generality, we may assume that
$$\delta > d_{H\times S}((g,s),(g',s')) = \max\{d_H(g,g'), d_S(s,s')\}.$$
Hence
$$\delta > d_H(g,g') = d_H(\mathrm{id}_H, g^{-1}g').$$
Let
$$\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in H$$
and
$$U := \sigma N \sigma = \left\{ u_s = \begin{pmatrix} 1 & \\ s & 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}.$$
Then the identity component H_0 of H has the Bruhat decomposition
$$H_0 = UNA \cup NA\sigma,$$
where UNA is an open neighborhood of id_H. We suppose that δ is so small that $g^{-1}g' \in UNA$. Define s_-, s_+, s_a by
$$g^{-1}g' = u_{s_-} n_{s_+} a_{s_a}.$$
Then $|s_-|, |s_+|, |s_a| < C\delta$ for some absolute constant $C > 0$ (which only depends on the parametrization of the metric). Note that x, x' are not in the same N-leaf. If they were, they could not be both in X_1 since $X_1 \cap Y = \emptyset$. Hence at least one of s_-, s_a does not vanish. The relation between s_- and s_a will result in the consideration of the two cases.

Now
$$x' \overset{H\times S}{\sim} xu_{s_-} n_{s_+} a_{s_a} = \Gamma(g', s)$$
and
$$d(x', xu_{s_-} n_{s_+} a_{s_a}) < \delta.$$
Then, for any $t, \xi \in \mathbb{R}$ we have
$$x' a_t n_\xi \overset{H\times S}{\sim} xu_{s_-} n_{s_+} a_{s_a} a_t n_\xi$$

and

$$d(x'a_tn_\xi, xu_{s_-}n_{s_+}a_{s_a}a_tn_\xi) < \delta.$$

We **claim** that for $t > 0, \xi > 1, |\xi^2 e^{2t} s_-| \le 1$ we have

$$xu_{s_-}n_{s_+}a_{s_a}a_tn_\xi \in xa_tn(\xi - 2s_a\xi - e^{2t}s_-\xi^2)B_\sigma^H$$

with

$$\sigma = C_2 \max\{\xi e^{2t}|s_-|, \xi^{-1}, \xi|s_a|\delta, \delta\}$$

for some global constant C_2. If this is the case, then

$$x'a_tn_\xi \in B_\sigma(xa_tn_{\xi'})$$

with $\xi' := \xi - 2s_a\xi - e^{2t}s_-\xi^2$. We have

$$u_{s_-}n_{s_+}a_{s_a}a_tn_\xi = a_tu_{e^{2t}s_-}n_{e^{-2t}s_+ + e^{s_a}\xi}a_{s_a}$$

and claim that

$$u(e^{2t}s_-)n(e^{-2t}s_+ + e^{-2s_a}\xi)a_{s_a} \in n(\xi - 2s_a\xi - e^{2t}s_-\xi^2)B_\sigma^H$$

Note that any displacement caused by a_{s_a} is contained in $B_{C_3\sigma}^H$ for some C_3. Hence a_{s_a} can be neglected in the following calculation. We have

$$\begin{aligned} &n(\xi - 2s_a\xi - e^{2t}s_-\xi^2)^{-1}u(e^{2t}s_-)n(e^{-2t}s_+ + e^{-2s_a}\xi) \\ &= \begin{pmatrix} 1 + e^{2t}s_-\xi(-1 + 2s_a) + (e^{2t}s_-\xi)^2 & b \\ e^{2t}s_- & 1 + s_-s_+ + e^{-2s_a}e^{2t}s_-\xi \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} b := e^{-2t}s_- &+ (e^{2t}s_-\xi^2)(-e^{-2s_a} + 1 + 2e^{-s_a}s_-s_a) + (e^{2t}s_-\xi)^2e^{-2t}s_+ + (e^{2t}s_-\xi^2)e^{-2s_a}\xi^{-1} \\ &+ (e^{-2s_a}\xi - (1 + s_-s_+)\xi + 2s_a\xi(1 + s_-s_+)). \end{aligned}$$

Now we use that $\xi > 1$, $t > 0$, $|e^{2t}s_-\xi^2| \le 1$, and $|s_-|, |s_+|, |s_a| \le C\delta$, and

$$|e^{-2s_a}\xi - (1 + s_-s_+)\xi + 2s_a\xi(1 + s_-s_+)| \le C'\xi|s_a|\delta$$

for some global constant C' to see that the claim holds true.

Case 1: Let $|s_a| > |s_-|^{\frac{10}{21}}$. We set $\xi_1 := |s_a|^{-1}$. Then $\xi_1 > C^{-1}\delta^{-1}$, and in particular > 1 for sufficiently small δ. Take

$$t \in \left(0, \frac{\log \xi_1}{100}\right), \quad \xi \in (\varrho\xi_1, \xi_1),$$

and set $\xi' := \xi - 2s_a\xi - e^{2t}s_-\xi^2$. Then

$$|\xi' - \xi| = |2s_a\xi + e^{2t}s_-\xi^2|$$

Now

$$|\xi^2e^{2t}s_-| \le |\xi_1^{\frac{202}{100}}s_-| = |s_a|^{-\frac{202}{100}}|s_-| \le |s_a|^{-\frac{202}{100}+\frac{210}{100}} = |s_a|^{\frac{8}{100}},$$

which becomes arbitrarily small for sufficiently small δ. Moreover,

$$2\varrho \le |2s_a\xi| \le 2.$$

Therefore

$$2\varrho - (\text{small}) \le |\xi' - \xi| = |2s_a\xi + e^{2t}s_-\xi^2| \le 2 + (\text{small}).$$

For an appropriate choice of C_0 (only depending on ε and ϱ and absolute constants), we have

$$\xi_1 \in D_{\varrho,C_0,\sigma}(xa_t, x'a_t).$$

Finally,

$$\sigma = C_2 \max\{\xi e^{2t}|s_-|, \xi^{-1}, \xi|s_a|\delta, \delta\} \leq C_3\delta$$

for an absolute constant C_3. This is qualitatively even better than $\leq \delta^{\frac{1}{4}}$.

Case 2: Let $|s_a| \leq |s_-|^{\frac{10}{21}}$ and set $\xi_1 := |s_-|^{-\frac{1}{2}}$. Then

$$\xi_1 > C^{-\frac{1}{2}}\delta^{-\frac{1}{2}} > 1$$

for sufficiently small δ. Let

$$t \in \left(\frac{\log \xi_1}{20}, \frac{\log \xi_1}{10}\right), \quad \xi \in \left(\varrho e^{-t}\xi_1, e^{-t}\xi_1\right).$$

As before set $\xi' := \xi - 2s_a\xi - e^{2t}s_-\xi^2$ and consider

$$|\xi' - \xi| = |2s_a\xi + e^{2t}s_-\xi^2|.$$

Now

$$\begin{aligned}|2s_a\xi| &\leq 2|s_a|e^{-t}\xi_1 \leq 2|s_a|\xi_1^{-\frac{1}{20}+1}\\ &\leq 2|s_-|^{\frac{10}{21}-\frac{19}{40}} < C'\delta^{\frac{10}{21}-\frac{19}{40}}.\end{aligned}$$

Note that $\frac{10}{21} - \frac{19}{40} > 0$. Moreover,

$$|e^{2t}s_-\xi^2| \leq e^{2t}|s_-|e^{-2t}\xi_1^2 = 1$$

and

$$|e^{2t}s_-\xi^2| \geq \varrho^2.$$

Hence,

$$\varrho - (\text{small}) \leq |\xi' - \xi| \leq 1 + (\text{small}).$$

For appropriate constant C_0 (only depending on ε, ϱ and absolute constants) we have

$$e^{-t}\xi \in D_{\varrho,C_0,\sigma}(xa_t, x'a_t).$$

Finally,

$$\sigma \leq C_3\delta^{\frac{41}{42}},$$

which again is qualitatively better than $\delta^{\frac{1}{4}}$. This completes the proof. □

Relative to our choice of $\varepsilon > 0$ and $\varrho \in (0,1)$ we fix a compact subset X_3 of X_2 with $\mu(X_3) > 1 - C_2\varepsilon^{\frac{1}{4}}$ (for some absolute constant C_2) such that for any $x \in X_3$ and any $t > 0$ we have

$$\frac{1}{t}\int_0^t 1_{X_2}(xa_s)ds \geq 1 - \varepsilon^{\frac{1}{4}}, \tag{11}$$

$$\frac{1}{t}\int_{-t}^0 1_{X_2}(xa_s)ds \geq 1 - \varepsilon^{\frac{1}{4}}, \tag{12}$$

$$\frac{1}{t}\int_0^t 1_{X_\varrho}(xa_s)ds \geq 1 - \varepsilon^{\frac{1}{4}}, \tag{13}$$

$$\frac{1}{t}\int_{-t}^0 1_{X_\varrho}(xa_s)ds \geq 1 - \varepsilon^{\frac{1}{4}}. \tag{14}$$

Note that for almost all $x \in X_3$ and all $\delta > 0$, we find $x' \in X_3$ with $d(x, x') < \delta$ and $x \overset{S}{\sim} x'$ by Lemma 4.6.

Lemma 6.4. *Let $\delta > 0$ be sufficiently small and suppose that $x, x' \in X_3$ with $d(x, x') < \delta$ and $x \overset{S}{\sim} x'$. Then there is some $t \geq 0$ such that*

$$D_{\varrho, C_0, \delta^{\frac{1}{4}}}(xa_t, x'a_t) \cap R_\varrho(xa_t) \neq \emptyset$$

and

$$xa_t, x'a_t \in X_2.$$

Moreover, $xa_t \overset{S}{\sim} x'a_t$ and $d(xa_t, x'a_t) < \delta^{\frac{1}{4}}$.

Proof. (i) Suppose that $\xi_1 > C_0^{-1}\delta^{-1/2}$ is as in Lemma 6.3(i). Then for all $t \in (0, \kappa \log \xi_1)$, we have

$$\xi_1 \in D_{\varrho, C_0, \delta^{\frac{1}{4}}}(xa_t, x'a_t).$$

From (11) it follows that measurewise at least

$$\int_0^{k \log \xi_1} 1_{X_2}(xa_s) 1_{X_2}(x'a_s) ds \geq \left(1 - 2\varepsilon^{\frac{1}{4}}\right) \kappa \log \xi_1$$

common displacements (with $t \in (0, \kappa \log \xi_1)$) of x and x' are simultaneously in X_2. In more detail: Let λ denote the Lebesgue measure on $\mathbb{R}$ and set

$$L_1 := \{s \in (0, \kappa \log \xi_1) \mid xa_s \in X_2\}$$
$$L_2 := \{s \in (0, \kappa \log \xi_1) \mid x'a_s \in X_2\}.$$

Then

$$\begin{aligned} \lambda(L_1 \cap L_2) &= \lambda(L_1) + \lambda(L_2) - \lambda(L_1 \cup L_2) \\ &\geq \left[(1 - \varepsilon^{\frac{1}{4}}) + (1 - \varepsilon^{\frac{1}{4}}) - 1\right] \kappa \log \xi_1. \end{aligned}$$

To estimate for how many (measurewise) displacements (with $t \in (0, \kappa \log \xi_1)$) of x, the point ξ_1 is not in $R_\varrho(xa_t)$, we recall that

$$\xi_1 \in R_\varrho(xa_t) \quad \Leftrightarrow \quad xa_t a_{\frac{1}{2}\log \xi_1} = xa_{t + \frac{1}{2}\log \xi_1} \in X_\varrho.$$

From (13) it now follows

$$\begin{aligned} \int_0^{\kappa \log \xi_1} 1_{X \setminus X_\varrho}(xa_{s + \frac{1}{2}\log \xi_1}) ds &= \int_{\frac{1}{2}\log \xi_1}^{(\kappa + \frac{1}{2})\log \xi_1} 1_{X \setminus X_\varrho}(xa_t) dt \\ &= \left(\int_0^{(\kappa + \frac{1}{2})\log \xi_1} - \int_0^{\frac{1}{2}\log \xi_1} \right) 1_{X \setminus X_\varrho}(xa_t) dt \\ &\leq \int_0^{(\kappa + \frac{1}{2})\log \xi_1} 1_{X \setminus X_\varrho}(xa_t) dt \leq \varepsilon^{\frac{1}{4}}(\kappa + \frac{1}{2}) \log \xi_1. \end{aligned}$$

Now

$$\left(1 - 2\varepsilon^{\frac{1}{4}}\right)\kappa \log \xi_1 - \varepsilon^{\frac{1}{4}}\left(\kappa + \frac{1}{2}\right) \log \xi_1 > 0$$

if and only if

$$\varepsilon^{\frac{1}{4}} < \frac{\kappa}{3\kappa + \frac{1}{2}}.$$

Thus, if ε is chosen below this absolute constant (and also below the absolute constants from before), then we find some t as in the statement of the lemma.

(ii) Suppose now that $\xi_1 > C_0^{-1}\delta^{-\frac{1}{2}}$ is as in Lemma 6.3(ii). Hence, for all $t \in (\kappa' \log \xi_1, 2\kappa' \log \xi_1)$ we have

$$e^{-t}\xi_1 \in D_{\varrho, C_0, \delta^{\frac{1}{4}}}(xa_t, x'a_t).$$

From (11) it follows

$$\begin{aligned}\int_{\kappa' \log \xi_1}^{2\kappa' \log \xi_1} 1_{X_2}(xa_s)1_{X_2}(xa_s)ds &= \left(\int_0^{2\kappa' \log \xi_1} - \int_0^{\kappa' \log \xi_1}\right) 1_{X_2}(xa_s)1_{X_2}(x'a_s)ds \\ &\geq \left(1 - 2\varepsilon^{\frac{1}{4}}\right)2\kappa' \log \xi_1 - \kappa' \log \xi_1 \\ &= \left(1 - 4\varepsilon^{\frac{1}{4}}\right)\kappa' \log \xi_1.\end{aligned}$$

Recall that

$$e^{-t}\xi_1 \in R_\varrho(xa_t) \quad \Leftrightarrow \quad xa_t a_{-\frac{1}{2}t+\frac{1}{2}\xi_1} = xa_{\frac{1}{2}t+\frac{1}{2}\xi_1} \in X_\varrho.$$

With (13) it follows that

$$\begin{aligned}\int_{\kappa' \log \xi_1}^{2\kappa' \log \xi_1} 1_{X\setminus X_\varrho}(xa_{\frac{1}{2}s+\frac{1}{2}\xi_1})ds &= 2\int_{\frac{1}{2}(\kappa'+1)\log \xi_1}^{(\kappa'+\frac{1}{2})\log \xi_1} 1_{X\setminus X_\varrho}(xa_t)dt \\ &\geq 2\int_0^{(\kappa'+\frac{1}{2})\log \xi_1} 1_{X\setminus X_\varrho}(xa_t)dt \\ &\geq 2\varepsilon^{\frac{1}{4}}\left(\kappa' + \frac{1}{2}\right)\log \xi_1.\end{aligned}$$

Now

$$\left(1 - 4\varepsilon^{\frac{1}{4}}\right)\kappa' \log \xi_1 - 2\varepsilon^{\frac{1}{4}}\left(\kappa' + \frac{1}{2}\right)\log \xi_1 > 0$$

if and only if

$$\frac{\kappa'}{6\kappa' + 1} > \varepsilon^{\frac{1}{4}},$$

which can be satisfied for sufficiently small ε. Note that all bounds on ε are given by absolute constants. The remaining statements follow immediately from the calculations in Lemma 6.3. Hence, the proof of this lemma is complete.

□

Proof of Proposition 4.1. As in the condition proof of Proposition 4.1 we claim to find a fixed compact interval $I \subseteq \mathbb{R}_{>0}$ such that for sufficiently small $\delta > 0$ we find points $y, y' \in X_3$ with

$$y' \in B_\delta(yn_t)$$

for some $|t| \in I$ and $\mu_y^N = \mu_{y'}^N$. Then we conclude as in the conditional proof that $X_3 \cap Y \neq \emptyset$ which is a contradiction.

For any $\delta > 0$ we can find a pair $x, x' \in X_3$ with $d(x, x') < \delta$ and $x \overset{S}{\sim} x'$ by Lemma 4.6. Then Lemma 6.4 shows that there exists $t \geq 0$ such that

$$D_{\varrho, C_0, \delta^{\frac{1}{4}}}(xa_t, x'a_t) \cap R_\varrho(xa_t) \neq \emptyset$$

and

$$xa_t, x'a_t \in X_2.$$

Moreover, $xa_t \overset{S}{\sim} x'a_t$ and $d(xa_t, x'a_t) < \delta^{\frac{1}{4}}$. By Lemma 6.1 we find $s, s' \in \mathbb{R}$ with $C_0^{-1} < |s'| < C_0$ such that

$$\begin{aligned}
&y := xa_t n_s,\ y' := x'a_t n_s \in X_1,\\
&y \in B_{\delta^{\frac{1}{4}}}(yn_{s'}),\\
&\mu_y^N = \mu_{y'}^N.
\end{aligned}$$

This completes the proof. □

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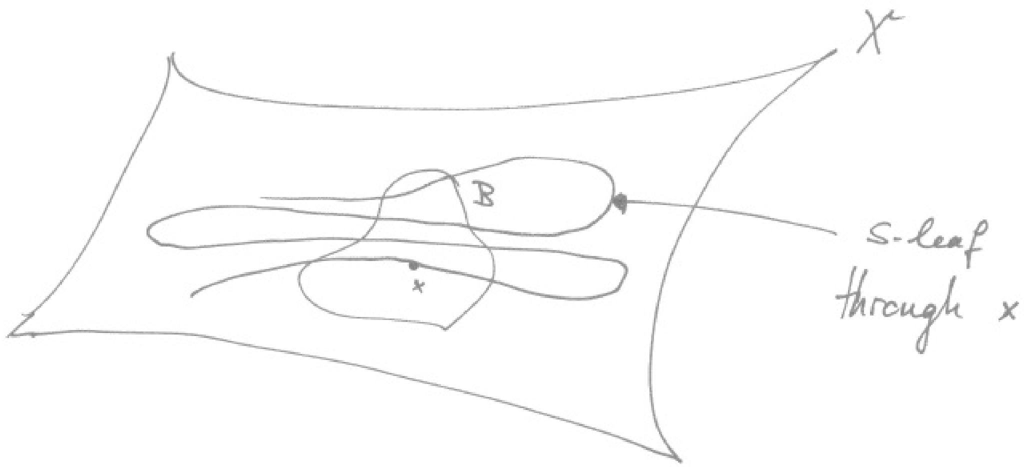
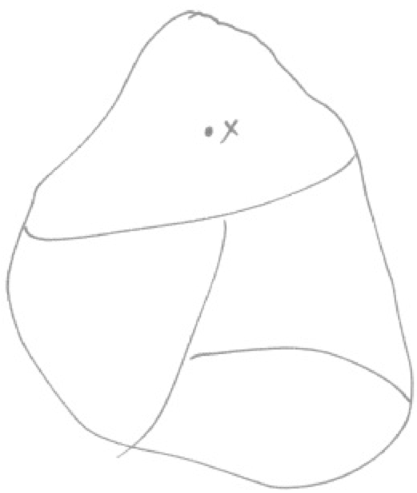
Appendix: Pictures

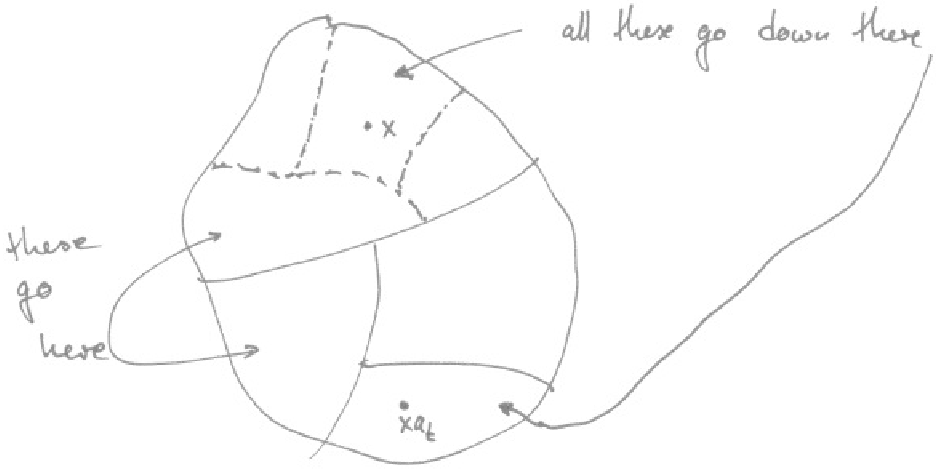
recurrent S-leaf:



partition:

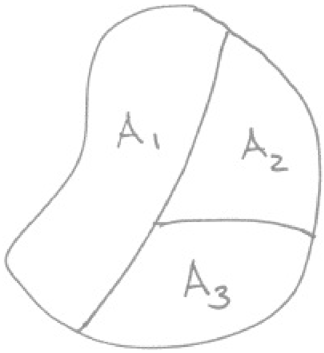


iteration:

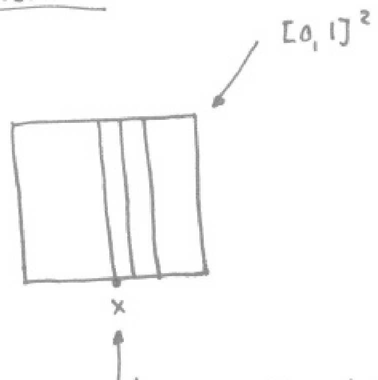


N-leaves:

finite partition:

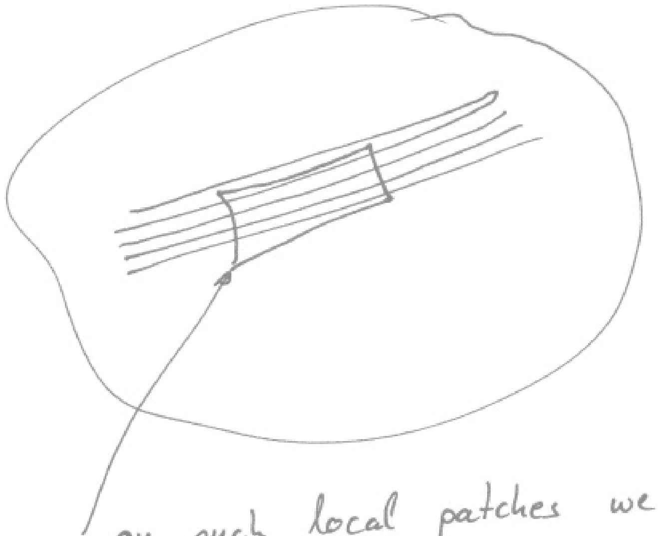


these partition elements coincide with the atoms null sets as atoms:



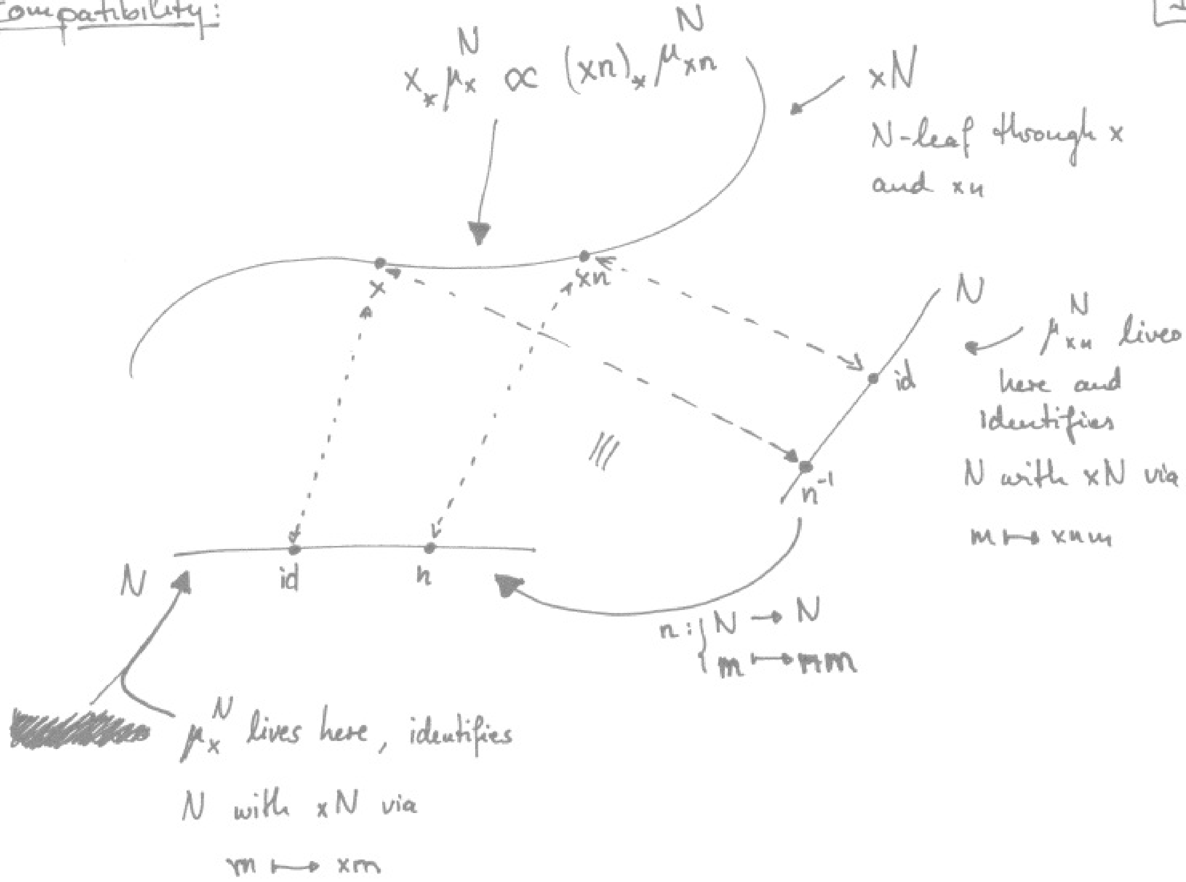
atom: $[x] = \{x\} \times [0,1]$

local N-leaves:

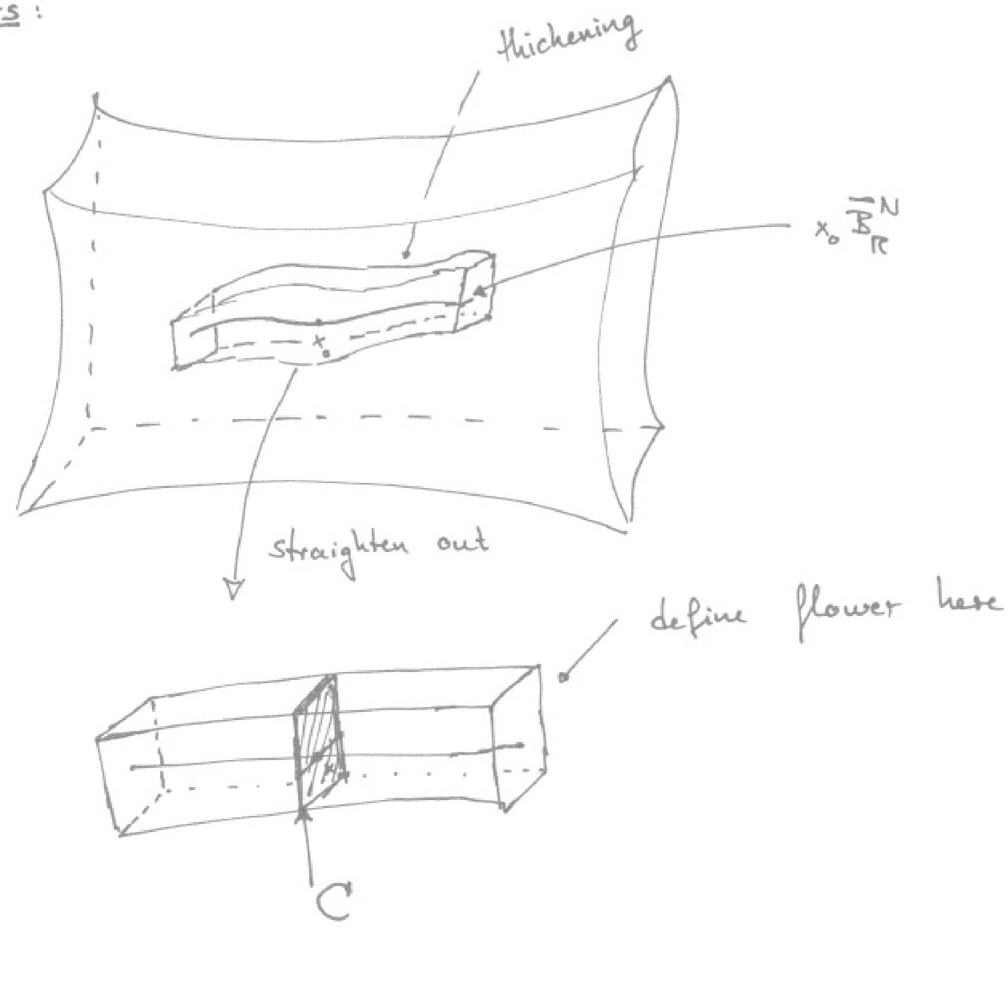


on such local patches we can put conditional measures

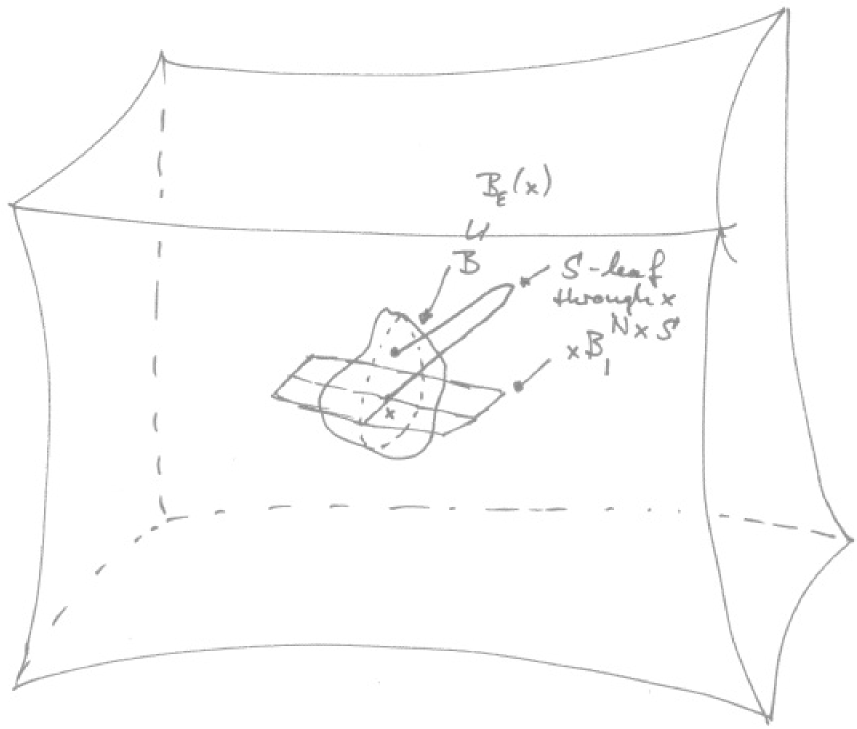
compatibility:



flowers:



long S^1-leaves:



N-recurrent:

think in patches and conditional measures