DYNAMICAL CHARACTERIZATION OF MAASS FORMS

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The interdependence of the geometric and the spectral data of Riemannian manifolds is of great interest in various areas, including dynamical systems, spectral theory, harmonic analysis, representation theory, number theory, and mathematical physics, in particular, quantum chaos. Over the last few years, this relation has been studied using an ever increasing number of methods which focus on the dynamics of the manifolds rather than on their (static) geometry. Among these dynamical methods are transfer operator techniques.

We discussed the development of transfer operator techniques for Riemannian surfaces (rather orbifolds) $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic plane and Γ is a geometrically finite, non-elementary Fuchsian group with at least one cusp.

The discretization for the geodesic flow on $\Gamma\backslash\mathbb{H}$ provided in [11, 8] gives rise to a discrete dynamical system (D_Γ, F_Γ), where D_Γ is a *finite* disjoint union of (the cuspidal-free and funnel-free part of) intervals in $\mathbb{R}$, and F_Γ is piecewise given by fractional linear transformations by certain elements in Γ.

The associated transfer operator $\mathcal{L}_{\Gamma,s}$ with parameter $s \in \mathbb{C}$ is given by

$$\mathcal{L}_{\Gamma,s} f(x) := \sum_{y \in F_\Gamma^{-1}(x)} \left|F_\Gamma'(y)\right|^{-s} f\big(F_\Gamma(y)\big),$$

a priori acting on functions $f \in \operatorname{Fct}(D_\Gamma; \mathbb{C})$. The structure of F_Γ yields that $\mathcal{L}_{\Gamma,s}$ is a finite sum of slash-actions $|_s g$ (multiplied with characteristic functions), where g runs through a finite subset of Γ. From this it follows immediately that $\mathcal{L}_{\Gamma,s}$ also acts on functions defined on certain domains larger than D_Γ. Of major interest to us are eigenfunctions with eigenvalue 1 of $\mathcal{L}_{\Gamma,s}$.

For the modular group $\mathrm{PSL}_2(\mathbb{Z})$ we have $D := D_{\mathrm{PSL}_2(\mathbb{Z})} = (0,\infty) \setminus \mathbb{Q}$. The self-map $F_{\mathrm{PSL}_2(\mathbb{Z})} \colon D \to D$ decomposes into the two branches

$$(0,1) \setminus \mathbb{Q} \to (0,\infty) \setminus \mathbb{Q}, \quad x \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} .x$$

and

$$(1,\infty) \setminus \mathbb{Q} \to (0,\infty) \setminus \mathbb{Q}, \quad x \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} .x.$$

The associated transfer operator $\mathcal{L}_s := \mathcal{L}_{\mathrm{PSL}_2(\mathbb{Z}),s}$ reads

$$\mathcal{L}_s f(x) = f(x+1) + (x+1)^{-2s} f\left(\frac{x}{x+1}\right), \quad x > 0,$$

or, equivalently,

$$\mathcal{L}_s = |_s \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + |_s \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

acting on $\mathrm{Fct}(\mathbb{R}_{>0};\mathbb{C})$. Eigenfunctions f with eigenvalue 1 of $\mathcal{L}_s$ satisfy the *Lewis equation*

$$f(x) = f(x+1) + (x+1)^{-2s} f\left(\frac{x}{x+1}\right), \quad x > 0.$$

As shown in [6, 2], the space of real-analytic functions f to the Lewis equation for which

$$x \mapsto \begin{cases} f(x) & \text{if } x > 0 \\ -|x|^{-2s} f\left(-\frac{1}{x}\right) & \text{if } x < 0 \end{cases} \tag{1}$$

extends C^∞ to 0 ('period functions') is isomorphic to the space of Maass cusp forms for $\mathrm{PSL}_2(\mathbb{Z})$ with spectral parameter s (see also [4]).

This kind of relation generalizes to other Fuchsian groups.

Theorem 1 ([7, 10, 9, 8])**.** *Suppose that Γ is cofinite and $s \in \mathbb{C}$, $\mathrm{Re}\, s \in (0,1)$. Then the space of Maass cusp forms for Γ with spectral parameter s is isomorphic to the space of sufficiently regular eigenfunctions with eigenvalue 1 of the transfer operator $\mathcal{L}_{\Gamma,s}$.*

The regularity required in Theorem 1 is similar to the one for the case $\mathrm{PSL}_2(\mathbb{Z})$. The isomorphism from Maass cusp forms to $\mathcal{L}_s$-eigenfunctions is given by an integral transform, making it reasonable to consider the $\mathcal{L}_s$-eigenfunctions as *period functions*. The proof of Theorem 1 takes advantage of the characterization of Maass cusp forms in parabolic 1-cohomology by Bruggeman–Lewis–Zagier [3].

Theorem 1 naturally leads to several conjectures. It is reasonable to expect that also other Laplace eigenfunctions can be characterized as $\mathcal{L}_{\Gamma,s}$-eigenfunctions. Moreover, the construction of the transfer operators applies to non-cofinite Γ. In view of the results on representing Selberg zeta functions as Fredholm determinants of transfer operators [12, 1], we should expect that residues at (scattering) resonances are determined by $\mathcal{L}_{\Gamma,s}$-eigenfunctions.

Furthermore, finite-dimensional representations $\chi\colon \Gamma \to \mathrm{GL}(V)$ can be accommodated by a transfer operator as a weight. In regard of the transfer operator approaches to χ-twisted Selberg zeta functions for χ having non-expanding cusp monodromy (e. g., if χ is unitary) an analogue of Theorem 1 for (Γ,χ)-automorphic functions or cusp forms should be expected [13, 1, 5].

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