Generalized transfer matrices and automorphic functions

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As known for a long time, transfer matrix techniques prove to be powerful in the study of lattice spin systems, for example for deriving exact solutions of one- and two-dimensional systems such as the Onsager solution. Reflection positivity in lattice spin systems is intimately related to the existence of self-adjoint positive definite transfer matrices.

Also known for a long time, the correspondence principle of quantum mechanics suggests close relations between geometric and spectral entities of Riemannian manifolds (and, more generally, of Riemannian orbifolds). In particular, one expects strong interdependencies between geodesics (classical mechanical objects) on the one side and L^2-eigenfunctions and L^2-eigenvalues of the Laplacian, and more generally, resonances and resonant states (quantum mechanical objects) on the other side.

Over the last century much effort was spend on establishing instances of such interdependencies in a mathematically rigorous way. An ever increasing number of results were found and were seen to be of great importance for various areas of mathematics, including dynamical systems, spectral theory, harmonic analysis, representation theory, number theory, and mathematical physics. However, the full scope and depth of the relation between geometric and spectral objects of Riemannian orbifolds is still mysterious.

In the talk we restricted to the case of non-elementary hyperbolic surfaces $X = \Gamma\backslash\mathbb{H}$ with at most finitely many ends (of finite and infinite area). Here, $\mathbb{H}$ denotes the hyperbolic plane, and Γ is a discrete non-cyclic geometrically finite subgroup of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$. For these spaces, a relation between periodic geodesics and resonances is shown by the Selberg zeta function which is the dynamical zeta function given by

$$Z_X(\beta) := \prod_{\ell\in L(X)} \prod_{k=0}^{\infty} \left(1 - e^{-(\beta+k)\ell}\right) \qquad (\beta\in\mathbb{C},\ \mathrm{Re}(\beta)\gg 1) \tag{1}$$

where $L(X)$ denotes the primitive geodesic length spectrum of X, counted with multiplicities. The infinite product in (1) converges if $\mathrm{Re}\,\beta$ is sufficiently large, and it has a meromorphic continuation to all of $\mathbb{C}$. The zeros of the Selberg zeta function consist of the resonances of X and some well-understood 'trivial' zeros (of rather topological nature). Thus, the Selberg zeta function Z_X establishes a relation between the geodesic length spectrum and the Laplace spectrum of X, or, in other words, a relation between geodesics and resonant states on the spectral level.

We discussed a construction of generalized transfer matrices and showed that these allow us to establish a relation between periodic geodesics and L^2-eigenfunctions beyond the spectral level, thereby improving on the connection provided by means of the Selberg zeta function. The construction of the generalized transfer matrices rely on a good choice of a discretization for the geodesic flow on X. We took advantage of the discretizations provided in [16], which are particularly well-suited for our purposes. Each such discretization provides a discrete dynamical system $F\colon D \to D$ on a union of certain intervals in $\mathbb{R}$ that is semi-conjugate to the geodesic flow on X and that branches into finitely many 'submaps' given by the Möbius action of some element in Γ. The associated generalized transfer matrix with parameter $\beta \in \mathbb{C}$ (transfer operator in the sense of Ruelle and Mayer) is

$$\mathcal{L}_\beta f(x) := \sum_{y \in F^{-1}(x)} e^{-\beta \ln |F'(y)|} f(y),$$

acting on functions $f\colon D \to \mathbb{C}$.

Some of the major results regarding the role of these transfer operators in the study of the interdependencies of geodesics and eigenfunctions and resonant states of X are roughly as follows:

- If X has finite area and at least one cusp, that is, an end of finite area, and if $\operatorname{Re}\beta \in (0,1)$ then the space of rapidly decaying L^2-eigenfunctions on X (Maass cusp forms) is isomorphic to the space of sufficiently regular eigenfunctions with eigenvalue 1 of $\mathcal{L}_\beta$ [12, 15, 14, 18, 13]. The isomorphism is given by an explicit integral transform. Up to date, transfer operator techniques are the only tool known to provide such a deep relation between geometric and spectral entities of hyperbolic surfaces.
- If X has finite or infinite area and at least one cusp then an induction procedure of the discretization of the geodesic flow used for the construction of $\mathcal{L}_\beta$ provides a uniformly expanding, infinitely branched discrete dynamical system. The associated transfer operator $\widetilde{\mathcal{L}}_\beta$ acts on a certain Banach space of holomorphic functions. As such it is nuclear of order zero and hence admissible for the thermodynamic formalism. Its Fredholm determinant equals the Selberg zeta function

  $$Z_X(\beta) = \det\big(1 - \widetilde{\mathcal{L}}_\beta\big).$$

  The possibility to represent Z_X as a Fredholm determinant of a transfer operator family suggests that many results obtained with the help of the Selberg zeta function and the Selberg trace formula should follow as a 'shadow' from results obtained via transfer operators. Moreover, transfer operator techniques provide an alternative proof of meromorphic extendability of the Selberg zeta function. See [12, 18, 17, 19] for all of these results.
- Eigenfunctions with eigenvalue 1 of $\mathcal{L}_\beta$ and $\widetilde{\mathcal{L}}_\beta$ are isomorphic, see [1] for Hecke triangle groups and forthcoming manuscripts for general Γ. This result together with the previously mentioned allows us to recover already a part of the spectral interpretations of the zeros of the Selberg zeta function without relying on the Selberg trace formula.
- Twists by finite-dimensional unitary representations can easily be accommodated by the transfer operators as additional weights. The results on

the connection between Selberg zeta functions and $\widetilde{\mathcal{L}}_\beta$ as well as on the relation between $\mathcal{L}_\beta$ and $\widetilde{\mathcal{L}}_\beta$ extend to the twisted objects [19, 1].

- Also twists by finite-dimensional representations χ with non-expanding cusp monodromies (representations which are not necessarily unitary but have controlled behavior in cusps) can be accommodated by transfer operators. Transfer operator techniques are currently the only known method to prove meromorphic extendability of the χ-twisted Selberg zeta functions [8].
- These results recover, illuminate and refine the seminal transfer operator techniques for the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ by Mayer [10, 11], Chang–Mayer [3], Efrat [7], Lewis–Zagier [9], Bruggeman [2], and its extension to certain finite-index subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ [4, 5, 6].

It is expected that the mentioned results, in particular the isomorphism between eigenfunctions of transfer operators and Maass cusp forms, can be generalized to eigenfunctions of other regularity, to (Γ,χ)-twisted and vector-valued eigenfunctions, and to general resonant states. Moreover, generalizations to more general locally symmetric spaces are expected. The relation between these generalized transfer matrices and reflection positivity remains to be understood.

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