Resonances of Schottky surfaces

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(joint work with O. Bandtlow, T. Schick, A. Weiße)

The investigation of L^2-Laplace eigenvalues and eigenfunctions for hyperbolic surfaces of *finite area* is a classical and exciting topic at the intersection of number theory, harmonic analysis and mathematical physics. In stark contrast, for geometrically finite hyperbolic surfaces of *infinite area*, the discrete L^2-spectrum is finite. A natural replacement are the resonances of the considered hyperbolic surface, which are the poles of the meromorphically continued resolvent

$$R(s) = (\Delta - s(1-s))^{-1}$$

of the hyperbolic Laplacian Δ. These spectral entities also play an important role in number theory and various other fields, and many fascinating results about them have already been found; the generalization of Selberg's 3/16-theorem by Bourgain, Gamburd and Sarnak [4] is a well-known example. However, an enormous amount of the properties of such resonances, also some very elementary ones, is still undiscovered. Prominent open questions include the existence of a Weyl law, the fractal Weyl law conjecture by Lu, Sridhar and Zworski [6], and the essential spectral gap conjecture by Jakobson and Naud [5].

A few years ago, by means of numerical experiments, Borthwick [2] noticed for some classes of Schottky surfaces (hyperbolic surfaces of infinite area without cusps and conical singularities) that their sets of resonances exhibit unexcepted and nice patterns, which are not yet fully understood. He used the method of *periodic orbit expansion*, which is well-suited for investigations of resonances with positive real part and of Schottky surfaces with large funnel widths and Euler characteristic near -1.

We discussed an alternative method, termed *domain-refined Lagrange–Chebychev approximation*, which has some advantages over the method of period orbit expansion. Figure 1 displays a part of the resonance set of a so-called funneled torus Schottky surface, calculated with this method.

Observation ([1]). *The method of domain-refined Lagrange–Chebychev approximation allows us to calculate resonances also for Schottky surfaces with smaller Euler characteristic or small funnel widths as well as resonances with negative real part. This method is efficient and does not require any specific properties (e.g., additional symmetries) of the Schottky surfaces.*

The methods of periodic orbit expansion and of domain-refined Lagrange–Chebychev approximation have the same starting point as both take advantage of the interpretation of resonances as zeros of the Selberg zeta function and of a transfer-operator-based representation of this zeta function. For any Schottky surface X, the Selberg zeta function Z_X is given by the Euler product

$$Z_X(s) = \prod_{\ell \in L_X} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right) \qquad \text{for } \operatorname{Re} s \gg 1, \tag{1}$$

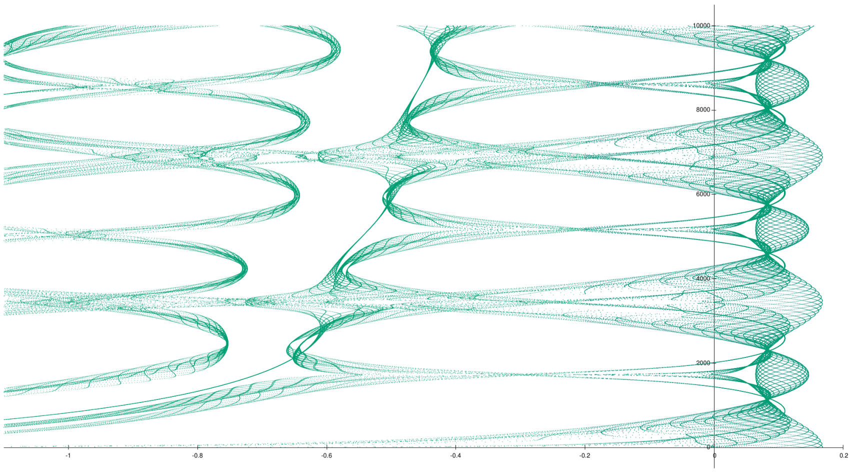

FIGURE 1. Resonances for a Schotty surface

and its holomorphic continuation to all of $\mathbb{C}$. Here, the multiset L_X in the first product of (1) refers to the primitive geodesic length spectrum of X. There exists a family of transfer operators $(\mathcal{L}_{X,s})_{s\in\mathbb{C}}$ for X, which derives from a discretization of the geodesic flow on X and whose Fredholm determinant equals the Selberg zeta function of X:

$$Z_X(s) = \det(1 - \mathcal{L}_{X,s})\,. \tag{2}$$

For the method of periodic orbit expansion one infers from (2) a series expansion

$$Z_X(s) = \sum_{n=0}^{\infty} d_n(s)\,,$$

whose coefficients $(d_n(s))_{n\in\mathbb{N}_0}$ are defined and calculated recursively in terms of the traces $(\operatorname{Tr}\mathcal{L}_{X,s}^m)_{m\in\mathbb{N}}$ of the transfer operator $\mathcal{L}_{X,s}$. The zeros of the truncated series approximate the zeros of Z_X, and hence the resonances of X.

For the method of domain-refined Lagrange–Chebychev approximation we note that the transfer operator $\mathcal{L}_{X,s}$ has an integral kernel. Thus,

$$(\mathcal{L}_{X,s}f)w = \int_\Omega K_s(z,w)f(z)\,dz\,,$$

where Ω is a finite union of certain open subsets of $\mathbb{C}$, the map f belongs to a well-chosen function space, and the integral kernel K_s has a rather simple structure. We use the Gauss–Chebychev quadrature rule to approximate K_s or, equivalently, Lagrange–Chebychev interpolation for the functions f. Then the transfer operator $\mathcal{L}_s$ gets approximated by a finite matrix, say M_s, and hence the Selberg zeta function $Z_X(s) = \det(1-\mathcal{L}_{X,s})$ is approximated by $D(s) := \det(1-M_s)$. The zeros of D serve as an approximation of the zeros of Z_X, and in turn of the resonances of X.

The method of periodic orbit expansion allowed us to discover that the resonance set exhibits astonishing structures in the positive half-plane, as shown by Borthwick's seminal work [2] and subsequent investigations (we refer to [3, 1] for extensive references). With the method of domain-refined Lagrange–Chebychev approximation we see that these structures not just extend to the negative half-plane but show new patterns there.

References

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