

Fractal Weyl bounds via transfer operators

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Resonances of Riemannian orbifolds play an important role in many areas of mathematics, e.g., analysis, dynamical systems, mathematical physics, and number theory. In this extended abstract we focus on geometrically finite hyperbolic orbisurfaces, i.e., on two-dimensional Riemannian orbifolds of the form $\Gamma \backslash \mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic plane and Γ is a finitely generated Fuchsian group, acting on $\mathbb{H}$ by Riemannian isometries. We set $X := \Gamma \backslash \mathbb{H}$ and let Δ_X denote the hyperbolic Laplacian on X . The resolvent

$$R_X(s) := (\Delta_X - s(1-s))^{-1} : L^2(X) \rightarrow H^2(X)$$

of Δ_X is defined for $s \in \mathbb{C}$ with $\Re s > 1/2$ and $s(1-s)$ not being an L^2 -eigenvalue of Δ_X . It extends meromorphically to all of $\mathbb{C}$ as operators

$$R_X(s) : L^2_{\text{comp}}(X) \rightarrow H^2_{\text{loc}}(X).$$

The resonances of X are the poles of this meromorphic family. We denote by $\mathcal{R}(X)$ the multiset of resonances of X , where each resonance is repeated according to its multiplicity (i.e., the rank of the residue operator at this resonance). We let

$$N_X(r) := \# \left\{ s \in \mathcal{R}(X) : \left| s - \frac{1}{2} \right| \leq r \right\}, \quad r > 0,$$

denote the counting function of resonances in balls (centered at $1/2$).

For *compact* hyperbolic orbisurfaces X , all resonances originate from L^2 -eigenvalues. Up to finitely many exceptions, they are located at the critical axis $\Re s = 1/2$, and the Weyl law for their asymptotic counting is well-known:

$$(1) \quad N_X(r) \sim \frac{\text{vol}(X)}{2\pi} r^2 \quad \text{as } r \rightarrow \infty.$$

For non-compact hyperbolic orbisurfaces X of *finite area*, not all resonances originate from L^2 -eigenvalues. In this situation, also scattering resonances make an appearance, which has the effect that the resonance set spreads out more. However, it is confined to the strip $\Re s \in [0, 1]$. This difference to compact spaces complicates the counting of resonances. Nevertheless, by work of Selberg [13] and W. Müller [7], the same asymptotics for the resonance set as for compact hyperbolic orbisurfaces was established. Thus, also for these orbisurfaces, the Weyl law (1) is known to be valid. We emphasize that it is a Weyl law for the resonance set, not necessarily for the L^2 -eigenvalue set.

For hyperbolic surfaces X of *infinite area*, in stark contrast, it is not yet known if such a Weyl law for the resonance set should be expected. For geometrically finite hyperbolic orbisurfaces X of infinite area with at least one periodic geodesic, Guilloupé and Zworski [4, 5] showed

$$N_X(r) \asymp r^2 \quad \text{as } r \rightarrow \infty.$$

Thus, the order of growth of the resonance counting function is as for hyperbolic orbisurfaces of finite area, but (non-)equality of the implied constants could not

yet be decided. A few results regarding the finer structure of these constants are known, e.g., as in [1, 11].

A further significant difference to the situation of finite-area orbisurfaces is the location of the resonance set. For infinite-area orbisurfaces it is not confined to a strip of finite width, but may distribute all over a certain right half-plane in $\mathbb{C}$ (with the Hausdorff dimension δ of the limit set of X being the right-most resonance). This makes it interesting to consider a resonance counting function with a “vertical counting direction.” More precisely, for $\sigma \in \mathbb{R}$ and $T > 0$ we define

$$N_X(\sigma, T) := \# \{s \in \mathcal{R}(X) : \Re s \geq \sigma, |\Im s| \leq T\}$$

to be the function counting the resonances in the box $[\sigma, \infty) + i[-T, T]$, with an interest of understanding its asymptotics for $T \rightarrow \infty$. Motivated by Sjöstrand’s work [14] and numerical experiments, Lu–Sridhar–Zworski [6] conjectured a *fractal Weyl law* of the form

$$(2) \quad N_X(\sigma, T) \sim c_\sigma T^{1+\delta} \quad \text{as } T \rightarrow \infty,$$

with δ being the right-most resonance of X (and c_σ a suitable implied constant, potentially depending on everything other than T). An important result towards this conjecture was achieved by Zworski [15] and Guillopé–Lin–Zworski [3], with two different proofs. They showed that for Schottky surfaces X (i.e., geometrically finite hyperbolic orbisurfaces of infinite area without cusps and elliptic points), for all $\sigma \in \mathbb{R}$ we have

$$N_X(\sigma, T) \ll_\sigma T^{1+\delta} \quad \text{as } T \rightarrow \infty.$$

We now turned to the case of geometrically finite hyperbolic orbisurfaces of infinite area with cusps, at least one periodic geodesic and potentially elliptic points, in which we could establish the following result.

Theorem (Naud–P.–Soares). *For certain geometrically finite hyperbolic orbisurfaces X of infinite area with cusps, for all $\sigma \in \mathbb{R}$, we have*

$$(3) \quad N_X(\sigma, T) \ll_\sigma T^{1+\delta} (\log T)^{2-\delta} \quad \text{as } T \rightarrow \infty.$$

The proof is based on transfer operator techniques, following the strategy of the proof in [3]. However, it is more involved due to the presence of a cusp. Every cusp has the effect that thickenings of the limit set do not have uniform contraction properties. Further, the required one-parameter transfer operator families $(\mathcal{L}_s)_s$ for orbisurfaces with cusps are *a priori* valid only for $\Re s \gg 1$, and hence we need to work within the domain of the meromorphic continuation of these families in s . The realm of this theorem heavily depends on the existence of representations of the Selberg zeta function of X as the Fredholm determinant of a well-structured transfer operator family for X . Such families are, e.g., provided in [10, 2] and, in particular by combination of [9] and [12] for a descent class of Fuchsian groups. An announcement of the theorem above with a sketch of the proof for non-cofinite Hecke triangle group appeared in [8]. The result in full detail will be available soon, including also a discussion of the extended setting involving finite-dimensional unitary representations of the fundamental group of X (i.e., a vector-valued situation).

Comparing (3) to the conjectured asymptotics (2) we notice the additional factor of $(\log T)^{2-\delta}$. It is not yet understood if this factor is immanent to the setting and, if so, if the exponent is best possible.

REFERENCES

- [1] D. Borthwick, *Sharp geometric upper bounds on resonances for surfaces with hyperbolic ends*, Anal. PDE **5** (2012), no. 3, 513–552.
- [2] R. Bruggeman and A. Pohl, *Eigenfunctions of transfer operators and automorphic forms for Hecke triangle groups of infinite covolume*, arXiv:1909.11432, to appear in AMS Memoirs.
- [3] L. Guillopé, K. Lin, and M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups*, Commun. Math. Phys. **245** (2004), no. 1, 149–176.
- [4] L. Guillopé and M. Zworski, *Upper bounds on the number of resonances for non-compact Riemann surfaces*, J. Funct. Anal. **129** (1995), no. 2, 364–389.
- [5] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. Math. (2) **145** (1997), no. 3, 597–660.
- [6] W. Lu, S. Sridhar and M. Zworski, *Fractal Weyl laws for chaotic open systems*, Phys. Rev. Lett. **91** (2003), 154101.
- [7] W. Müller, *Spectral geometry and scattering theory for certain complete surfaces of finite volume*, Invent. Math. **109** (1992), no. 2, 265–305.
- [8] F. Naud, A. Pohl and L. Soares, *Fractal Weyl bounds and Hecke triangle groups*, Electron. Res. Announc. Math. Sci. **26** (2019), 24–35.
- [9] A. Pohl, *Symbolic dynamics for the geodesic flow on two-dimensional hyperbolic good orbifolds*, Discrete Contin. Dyn. Syst., Ser. A **34** (2014), no. 5, 2173–2241.
- [10] A. Pohl, *A thermodynamic formalism approach to the Selberg zeta function for Hecke triangle surfaces of infinite area*, Commun. Math. Phys. **337** (2015), no. 1, 103–126.
- [11] A. Pohl and L. Soares, *Density of Resonances for covers of Schottky surfaces*, J. Spectr. Theory **10** (2020), no. 3, 1053–1101.
- [12] A. Pohl and P. Wabnitz, *Selberg zeta functions, cuspidal accelerations, and existence of strict transfer operator approaches*, arXiv:2209.05927.
- [13] A. Selberg, *Göttingen lectures*, unpublished.
- [14] J. Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J. **60** (1990), no. 1, 1–57.
- [15] M. Zworski, *Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces*, Invent. Math. **136** (1999), no. 2, 353–409.