ON LINEARIZED VERSIONS OF MATRIX INEQUALITIES

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Abstract. In this note, we prove linear versions of the Aleksandrov-Fenchel inequality and the Brunn-Minkowski inequality for positive semidefinite matrices. With this aim, given a positive semidefinite matrix A and a linear subspace L, we consider a family of matrices having the same projection onto L, obtaining a linear version of the Aleksandrov-Fenchel inequality. In the case of the Brunn-Minkowski inequality, the milder assumption of having equal determinant of the projection of A onto L will be enough to obtain a linearized version of this inequality.

1. Introduction and background

Positive semidefinite and definite matrices are considered, in many fields of science, as a generalization of non-negative real numbers. In the vector space of real symmetric matrices $\mathcal{M}^n$, the set of symmetric positive semidefinite ones $\mathcal{S}^n_+$ is a closed, convex cone. In this paper, we will focus mainly on mixed discriminant for matrices in $\mathcal{S}^n_+$, but for completeness, we state the definition of the mixed discriminant first for arbitrary $n \times n$ matrices.

Following [3], let $n \in \mathbb{N}$, with $n \geq 1$, and $A_1, \cdots, A_n$ be arbitrary $n \times n$ matrices. If $A_j^{(i)}$ denotes the i-th column of the matrix A_j, then

$$\mathrm{D}(A_1, \ldots, A_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \det\left(A^{(1)}_{\sigma(1)}, \ldots, A^{(n)}_{\sigma(n)}\right), \tag{1.1}$$

where S_n denotes the symmetric group of permutations of $\{1, \ldots, n\}$. In the literature, further different approaches to mixed discriminants can be found, see e.g. [1, 9, 14, 15]. In [3], we have also the following relations to the determinant for arbitrary matrices $A_1 \cdots, A_n$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$:

$$\mathrm{D}(A_1, \ldots, A_n) = \frac{1}{n!} \frac{\partial^n}{\partial\lambda_1 \ldots \partial\lambda_n} \det(\lambda_1 A_1 + \ldots + \lambda_n A_n), \tag{1.2}$$

and

$$\mathrm{D}(A_1, \ldots, A_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \det(A_{i_1} + \ldots + A_{i_k}). \tag{1.3}$$

Since we are going to deal mainly with symmetric and positive semidefinite matrices, we estate the following result of the determinant for this class of matrices.

Theorem A. [11] *Let $n \in \mathbb{N}$, with $n \geq 1$. There exists a unique symmetric real-valued function*

$$\mathrm{D}\colon (\mathcal{S}^n_+)^n \longrightarrow \mathbb{R},$$

satisfying

$$\det(\lambda_1 A_1 + \ldots + \lambda_m A_m) = \sum_{i_1, \ldots, i_n = 1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \mathrm{D}(A_{i_1}, \ldots, A_{i_n}), \tag{1.4}$$

for every $m \geq 1$, $A_1, \ldots, A_m \in \mathcal{S}^n_+$ and $\lambda_1, \ldots, \lambda_m \geq 0$.

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Theorem A states that the determinant of a linear combination of m real $n \times n$ matrices $A_1, \dots, A_m$, with non-negative reals $\lambda_1, \dots, \lambda_m \geq 0$, is a homogeneous polynomial of degree n, in $\lambda_1, \dots, \lambda_m$, whose coefficients are the mixed discriminants of $A_1, \dots, A_m$. Observe that m may not coincide with n.

For connections of the mixed discriminant to other notions we refer to [4, 5], and the references therein. The following results provide us with fundamental inequalities for positive semidefinite matrices.

Theorem B. [13, Theorem 7.8.21] *Let $A, B \in \mathcal{M}^n$ be positive definite matrices. Then*

$$\det((1-\lambda)A) + \lambda B)^{1/n} \geq (1-\lambda)\det(A)^{1/n} + \lambda \det(B)^{1/n} \tag{1.5}$$

for any $\lambda \in [0,1]$.

The latter (see also [6]) is usually referred to as the (Brunn-)Minkowski inequality for the determinant, because of its analogy with the far-reaching and powerful Brunn-Minkowski inequality for the volume; see e.g. [12].

The following inequality, also known as the (first) Minkowski inequality, can be proven directly as a consequence of Theorem B, and establishes an inequality between the mixed discriminant $D(\underbrace{A, \dots, A}_{(n-1)-\text{times}}, B)$, and $\det(A)$ and $\det(B)$.

Theorem C. [16] *Let $A, B \in \mathcal{M}^n$ be positive definite matrices. Then*

$$\mathrm{D}(\underbrace{A, \dots, A}_{(n-1)-times}, B)^n \geq \det(A)^{n-1}\det(B). \tag{1.6}$$

There are deep existing analogies between mixed discriminants and mixed volumes within the framework of Convex Geometry, for which we refer, e.g., to [14, 15, 19], and the references therein. There are also remarkable differences between them too, as [2] displays.

The next inequality also receives in the literature the same name as a fundamental geometric inequality within the realm of Convex Geometry due to their analogy, the Aleksandrov-Fenchel inequality.

Theorem D. [19, Theorem 5.5.4] *Let $A, B, C, A_2, \dots, A_n \in \mathcal{M}^n$ be real $n \times n$ symmetric matrices, where $A, A_2, \dots, A_n$ are positive definite and C is positive semidefinite. Then*

$$\mathrm{D}(C, A_2, \dots, A_n) \geq 0. \tag{1.7}$$

Equality holds if and only if $C = 0$. Further,

$$\mathrm{D}(A, B, A_3, \dots, A_n)^2 \geq \mathrm{D}(A, A, A_3, \dots, A_n)\mathrm{D}(B, B, A_3, \dots, A_n). \tag{1.8}$$

Equality holds if and only if $B = \lambda A$ for $\lambda \in \mathbb{R}$.

The main aim of this note is to investigate conditions under which the latter two inequalities, i.e., the Brunn-Minkowski, and the Aleksandrov-Fenchel inequalities for positive semidefinite matrices, have linearized versions following the spirit of [19, Section 7.7]. By a linear version of the inequalities (1.5) and (1.8) we mean inequalities of the form:

$$\det((1-\lambda)A + \lambda B) \geq (1-\lambda)\det(A) + \lambda \det(B), \text{ and} \tag{1.9}$$

$$2\mathrm{D}(A, B, A_3, \dots, A_n) \geq \mathrm{D}(A, A, A_3 \dots, A_n) + \mathrm{D}(B, B, A_3, \dots, A_n), \tag{1.10}$$

for suitable matrices $A, B, A_3, \dots, A_n$.

It is immediate to verify that inequalities of the type (1.9) and (1.10) cannot hold for all positive semidefinite matrices. For instance, the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

violate both inequalities, (1.9) for all $\lambda \in (0,1)$, and (1.10).

Within the theory of Convex Bodies there exist several results pursuing linearized versions of geometric inequalities, in particular, for the Brunn-Minkowski inequality and the Aleksandrov-Fenchel inequality. We refer to [19, Section 7] for a wealth of information on this.

In order to motivate the research on such linearized versions of the inequalities we observe that from Theorem B, we directly obtain

$$\det(A+B)=\Big(\det(A+B)^{1/n}\Big)^n\geq\Big(\det(A)^{1/n}+\det(B)^{1/n}\Big)^n\geq\det(A)+\det(B),$$

for any $A,B\in\mathcal{M}^n$ positive definite matrices. However, when $\lambda\in[0,1]$ comes into play, we *only* have

$$\det((1-\lambda)A+\lambda B)\geq(1-\lambda)^n\det(A)+\lambda^n\det(B),$$

which is clearly not as sharp as inequality (1.9).

The present note is organized as follows. In Section 2, we recall some known results, which are used in the rest of the note. Then, in Section 3, we consider projections of matrices onto linear subspaces, by restricting the corresponding quadratic form of a matrix onto a subspace. In Section 4, we introduce a *canal class* for positive semidefinite matrices inspired by the analogue notion in Convex Geometry, and we establish some linearized versions of the Aleksandrov-Fenchel inequality for suitable matrices, under the assumption of positivity of a suitable mixed discriminant of $(n-1)\times(n-1)$ matrices closely connected to the given ones. Finally, in Section 5, we prove linearized versions of the Brunn-Minkowski inequality for positive semidefinite matrices having the same determinant of their projection onto a hyperplane, and related results.

2. Known results

In this section, we establish several inequalities for mixed discriminants, whose proofs do follow essentially the exact steps of their analogs within the Theory of Convex Bodies. Some of the proofs are provided for the sake of completeness. Further, at the end of the section, we recall some inequalities relating to the determinant of positive semidefinite matrices and their submatrices.

The next lemma gathers some fundamental properties of mixed discriminants. Most of the proofs follow from the definition given by (1.1). We denote by I_n the identity matrix.

Lemma 2.1. [3] *Let $A_1,\dots,A_n,B,R,S$ be $n\times n$ arbitrary matrices. Then the following assertions hold:*

i) $\mathrm{D}(I_n,\dots,I_n)=1$.
ii) $\mathrm{D}(A_1,\dots,A_n)=\mathrm{D}(A_{\sigma(1)},\dots,A_{\sigma(n)})$, *for all* $\sigma\in S_n$.
iii) $\mathrm{D}(A_1,\dots,A_n)=D(A_1^T,\dots,A_n^T)$.
iv) $\mathrm{D}(aA_1+bB,A_2,\dots,A_n)=a\mathrm{D}(A_1,A_2,\dots,A_n)+b\mathrm{D}(B,A_2,\dots,A_n)$, *for all* $a,b\in\mathbb{R}$.
v) $\mathrm{D}(SA_1R,\dots,SA_nR)=\det(R)\det(S)\mathrm{D}(A_1,\dots,A_n)$.

As pointed out in Theorem D, it is sometimes fundamental to know whethera mixed discriminant of positive semidefinite matrices $A_1,\dots,A_n$ vanishes. The following result provides us with equivalent conditions for the positivity of the mixed discriminants given by $A_1,\dots,A_n\in\mathcal{S}^n_+$. Let $P(A_i)$ denote the linear subspace of $\mathbb{R}^n$ spanned by the eigenvectors of A_i associated with strictly positive eigenvalues of A_i, $i=1,\dots,n$.

Proposition 2.2. [3, 11] *Let $A_1,\dots,A_n\in\mathcal{S}^n_+$ be positive semidefinite matrices. Then, the following assertions are equivalent:*

i) $\mathrm{D}(A_1,\dots,A_n)>0$,
ii) *There are linearly independent vectors $v_i\in P(A_i)$, $i=1,\dots,n$.*
iii) $\dim(P(A_{i_1})+\ldots+P(A_{i_k}))\geq k$ *for each choice of indices* $1\leq i_1<\ldots<i_k\leq n$ *and for all* $k\in\{1,\dots,n\}$.

We observe that (1.7) is consistent with the latter for $C\neq 0$ (cf. Corollary 2.3).

The following corollary of Proposition 2.2 provides us with some useful tools for the computation of the mixed discriminant.

Corollary 2.3. *Let $n\in\mathbb{N}$, and let $A,A_1,\dots,A_n\in\mathcal{S}^n_+$.*

i) *If* $A_1, \dots, A_n$ *are positive definite, then* $\mathrm{D}(A_1, \dots, A_n) > 0$.
ii) *If the positive semidefinite matrix* A *has rank one, then*

$$\mathrm{D}(A, A, A_3, \dots, A_n) = 0.$$

Proof. The positivity of the mixed discriminant in i) follows immediately from Proposition 2.2. Indeed, as $P(A_i) = \mathbb{R}^n$, for $i = 1, \dots, n$, condition iii) in Proposition 2.2 is directly fullfilled.

In order to prove ii) we use Proposition 2.2 iii). It is enough to observe, that condition $\dim(P(A_{i_1}) + \dots + P(A_{i_k})) \geq k$ for each choice of indices $1 \leq i_1 < \dots < i_k \leq n$ and for $1 \leq k \leq n$, is violated for $k = 2$ and $A_1 = A_2 = A$, as $\dim(P(A) + P(A)) = \dim(P(A)) = 1 < 2$. □

Next, we establish two results, also parallel to existing results within the framework of convex bodies, which provide us with a general version of Theorem B, for mixed discriminants, and a characterization of the linearity of the latter. For completeness, and since the authors have not been able to find the results stated explicitly within the context of positive semidefinite matrices, we will provide the proofs of both of them, expressly noticing that they are a one-to-one translation into matrices, determinants, and mixed discriminants of the referred to results in [19]. For the sake of brevity, we introduce the notation $\mathrm{D}(A[k], A_{k+1}, \dots, A_n) := \mathrm{D}(\underbrace{A, \dots, A}_{k-\text{times}}, A_{k+1} \dots, A_n)$ for the mixed discriminant of the matrices $A, A_{k+1}, \dots, A_n$, where the matrix A appears k times.

Theorem 2.4. *Let* $n, m \in \mathbb{N}$ *be such that* $n \geq 1$ *and* $1 \leq m \leq n$. *Let* $A_0, A_1, A_{m+1}, \dots, A_n \in \mathcal{S}^n_+$, *and* $A_\lambda = (1-\lambda)A_0 + \lambda A_1 \in \mathcal{S}^n_+$. *Then the function*

$$f(\lambda) := \mathrm{D}(A_\lambda[m], A_{m+1}, \dots, A_n)^{1/m}$$

is concave on $[0, 1]$.

Notice that the case $m = n$ coincides with Theorem B.

Proof. (Follows the same lines of the Proof of Theorem 7.4.5 in [19]). First, we prove that the second derivative of f at 0 is non-positive. Then, we will argue that this implies the concavity of f on $[0, 1]$. For the sake of brevity we write $\mathcal{A} = (A_{m+1}, \dots, A_n)$.

Let us denote by $D_{(i)} = \mathrm{D}(A_0[m-i], A_1[i], \mathcal{A})$, and define $g(\lambda) := f(\lambda)^m$. Then, using multilinearity of the mixed discriminant, we have

$$g(\lambda) = f(\lambda)^m = \sum_{j=0}^{m} (1-\lambda)^{m-j} \lambda^j \binom{m}{j} D_{(j)}.$$

Using standard computations, the second derivative of f at zero satisfies

$$f''(0) = (m-1) D_{(0)}^{\frac{1}{m}-2} \left[-D_{(1)}^2 + D_{(2)} D_{(0)} \right].$$

Indeed,

$$\begin{aligned} f'(\lambda) &= \frac{1}{m} g(\lambda)^{\frac{1}{m}-1} g'(\lambda), \quad \text{and} \\ f''(\lambda) &= \frac{1}{m}\left(\frac{1}{m} - 1\right) g(\lambda)^{\frac{1}{m}-2} (g'(\lambda))^2 + \frac{1}{m} g(\lambda)^{\frac{1}{m}-1} g''(\lambda). \end{aligned}$$

Direct calculation on $g(\lambda) = \sum_{j=0}^{m} (1-\lambda)^{m-j} \lambda^j \binom{m}{j} D_{(j)}$ provides us with

$$g(0) = D_{(0)}, \quad g'(0) = m(D_{(1)} - D_{(0)}), \quad \text{and} \quad g''(0) = m(m-1)(D_{(0)} - 2D_{(1)} + D_{(2)}).$$

Then,

$$\begin{aligned} f''(0) &= \frac{1}{m}\left(\frac{1}{m} - 1\right) D_{(0)}^{\frac{1}{m}-2} (-mD_{(0)} + mD_{(1)})^2 + (m-1) D_{(0)}^{\frac{1}{m}-1} (D_{(0)} - 2D_{(1)} + D_{(2)}) \\ &= (m-1) D_{(0)}^{\frac{1}{m}-2} \left[-D_{(1)}^2 + D_{(2)} D_{(0)} \right]. \end{aligned}$$

We observe that $-D_{(1)}^2 + D_{(2)}D_{(0)} < 0$ by the Aleksandrov-Fenchel inequality (1.8), which yields $f''(0) \leq 0$.

Next we complete the proof showing that the latter is enough to show $f''(\lambda) \leq 0$ for all $\lambda \in [0,1]$. Let $\lambda' \in (0,1)$ be fixed, we define $\bar{A}_\tau = (1-\tau)A_{\lambda'} + \tau A_1$ and $h\colon [0,1] \to \mathbb{R}$, as $h(\tau) = \mathrm{D}(\bar{A}_\tau[m], \mathcal{A})^{1/m}$, for $\tau \in [0,1]$.

Defining $\mu := \tau(1-\lambda')$, for $0 \leq \tau \leq 1$, we have $0 \leq \mu \leq 1-\lambda' < 1$ and

$$h(\tau) = h(\frac{\mu}{1-\lambda'}) = \mathrm{D}(\bar{A}_{\frac{\mu}{1-\lambda'}}[m], \mathcal{A})^{1/m}. \tag{2.1}$$

Observing that

$$\begin{aligned}\bar{A}_{\frac{\mu}{1-\lambda'}} &= (1-\frac{\mu}{1-\lambda'})\Big((1-\lambda')A_0 + \lambda' A_1\Big) + \frac{\mu}{1-\lambda'}A_1 \\ &= (1-(\lambda'+\mu))A_0 + (\lambda'+\mu)A_1,\end{aligned}$$

we get $h(\tau) = h(\frac{\mu}{1-\lambda'}) = f(\lambda'+\mu)$. We remark also that $f(\lambda'+\mu)$ is well-defined since $0 \leq \mu \leq 1-\lambda' < 1$. Moreover $h(0) = f(\lambda')$. Computing the first and second derivatives of f with respect to λ at 0, we get

$$\frac{d}{d\lambda}|_{\lambda=\lambda'}\ f(\lambda) = \frac{1}{1-\lambda'}\frac{d}{d\tau}|_{\tau=0}\ h(\tau), \quad \text{and} \quad \frac{d^2}{d\lambda^2}|_{\lambda=\lambda'}\ f(\lambda) = \frac{1}{(1-\lambda')^2}\frac{d^2}{d\tau^2}|_{\tau=0}\ h(\tau).$$

We have $\frac{d^2}{d\tau^2}|_{\tau=0}\ h(\tau) \leq 0$ from the same argument of $f''(0) = \frac{d^2}{d\lambda^2}|_{\lambda=0}\ f(\lambda) \leq 0$, where instead of A_0 we consider $A_{\lambda'}$. Thus, we have $\frac{d^2}{d\lambda^2}|_{\lambda=\lambda'}\ f(\lambda) \leq 0$.
It only remains $\frac{d^2}{d\lambda^2}|_{\lambda=1}\ f(\lambda)$ (left derivative), for which we consider $g(\lambda) = f(1-\lambda)$. This yields $\frac{d^2}{d\lambda^2}|_{\lambda=1}\ f(\lambda) = \frac{d^2}{d\lambda^2}|_{\lambda=0}\ g(\lambda) \leq 0$.

□

As briefly mentioned before, the next theorem contains a characterization of the linearity case in Theorem 2.4.

Theorem 2.5. *Let $n, m \in \mathbb{N}$ be such that $n \geq 1$ and $1 \leq m \leq n$, $A_0, A_1, A_{m+1}, \ldots, A_n \in \mathcal{S}_+^n$, $A_\lambda = (1-\lambda)A_0 + \lambda A_1 \in \mathcal{S}_+^n$, and let $\mathcal{A} = (A_{m+1}, \ldots, A_n)$. Let further*

$$f(\lambda) := \mathrm{D}(A_\lambda[m], A_{m+1}, \ldots, A_n)^{1/m} = \mathrm{D}(A_\lambda[m], \mathcal{A})^{1/m}$$

for $0 < \lambda < 1$. Under the assumption $\mathrm{D}(A_0[m], \mathcal{A}) > 0$, and $\mathrm{D}(A_1[m], \mathcal{A}) > 0$, the following conditions are equivalent:

i) *The function f is linear.*
ii) $\mathrm{D}(A_0[m-i], A_1[i], \mathcal{A})^2 = \mathrm{D}(A_0[m-i+1], A_1[i-1], \mathcal{A})\mathrm{D}(A_0[m-i-1], A_1[i+1], \mathcal{A})$ *for* $i = 1, \ldots, m-1$.
iii) $\mathrm{D}(A_0[m-1], A_1[1], \mathcal{A})^m = \mathrm{D}(A_0[m], \mathcal{A})^{m-1}\mathrm{D}(A_1[m], \mathcal{A})$.

Proof. (The proof follows the same lines of the proof of [19, Theorem 7.4.6]).

As in the proof of Theorem 2.4, we will denote by $D_{(i)} = \mathrm{D}(A_0[m-i], A_1[i], \mathcal{A})$. Firstly, assume that f is linear. Therefore,

$$\mathrm{D}(A_\lambda[m], \mathcal{A})^{1/m} = (1-\lambda)\mathrm{D}(A_0[m], \mathcal{A})^{1/m} + \lambda\mathrm{D}(A_1[m], \mathcal{A})^{1/m}.$$

By taking the m-th power and using multilinearity of the mixed discriminant, we obtain

$$\begin{aligned}\sum_{i=0}^{m}\binom{m}{i}(1-\lambda)^{m-i}\lambda^i D_{(i)} &= \mathrm{D}(A_\lambda[m], \mathcal{A}) \\ &= \Big((1-\lambda)\mathrm{D}(A_0[m], \mathcal{A})^{1/m} + \lambda\mathrm{D}(A_1[m], \mathcal{A})^{1/m}\Big)^m \\ &= \sum_{i=0}^{m}\binom{m}{i}(1-\lambda)^{m-i}\lambda^i D_{(0)}^{(m-i)/m} D_{(m)}^{i/m}.\end{aligned}$$

Hence, we get $D_{(i)}^m = D_{(0)}^{m-i} D_{(m)}^i$ for every $i = 0, \dots, m$.
Assume the latter equality holds for every $i = 0, \dots, m$. Considering the equality for $i-1, i, i+1$ and squaring we obtain:

$$\begin{aligned} D_{(i)}^{2m} &= D_{(0)}^{2(m-i)} D_{(m)}^{2i}, \\ D_{(i-1)}^{m} &= D_{(0)}^{m-i+1} D_{(m)}^{i-1}, \text{ and} \\ D_{(i+1)}^{m} &= D_{(0)}^{m-i-1} D_{(m)}^{i+1}. \end{aligned}$$

Thus, taking the product of the mixed discriminants corresponding to $i-1$ and $i+1$ yields

$$D_{(i-1)}^m D_{(i+1)}^m = D_0^{2(m-i)} D_m^{2i} = D_{(i)}^{2m}.$$

Next, we show ii) implies iii). We have that

$$\frac{D_{(i)}}{D_{(i-1)}} = \frac{D_{(i+1)}}{D_{(i)}} \quad \text{for all } i \in \{1, \dots, m-1\}.$$

Therefore, we get immediately the desired equality

$$\left(\frac{D_{(1)}}{D_{(0)}}\right)^{m-1} = \frac{D_{(2)}}{D_{(1)}} \cdot \frac{D_{(3)}}{D_{(2)}} \cdots \frac{D_{(m)}}{D_{(m-1)}} = \frac{D_{(m)}}{D_{(1)}}.$$

It remains to prove that i) follows from iii). As we showed in Theorem 2.4, f is concave. Hence, we get $f'(0) \geq f(1) - f(0)$ with equality if and only if f is linear. On the other hand, from Minkowski's first inequality, Theorem C, we receive that

$$D_{(1)}^m \geq D_{(0)}^{(m-1)} D_{(m)},$$

with equality if and only if f is linear.

□

Next, we state two well-known inequalities, the Bergstrom and the Ky Fan inequalities for the determinant. They are, a priori, not parallel to inequalities within the realm of Convex Geometry and will be key to prove linearized versions of the Brunn-Minkowski and the Aleksandrov-Fenchel inequalities for positive semidefinite matrices.

Theorem 2.6. [6, 7, 8] *Let A and B be two $n \times n$ positive definite matrices, we denote by A_i and B_i the two $(n-1) \times (n-1)$ matrices given by A and B deleting the i-th row and the i-th column. Then we have*

$$\frac{\det(A+B)}{\det(A_i + B_i)} \geq \frac{\det(A)}{\det(A_i)} + \frac{\det(B)}{\det(B_i)}, \tag{2.2}$$

for every $i \in \{1, \dots, n\}$.

Theorem 2.7. [6, 10] *Let A and B be two $n \times n$ positive definite matrices, we denote by $A_{(k)}$ and $B_{(k)}$ the principal $k \times k$ matrices of A and B obtained by taking the first k rows and the k columns. Then we have*

$$\left(\frac{\det(A+B)}{\det(A_{(k)} + B_{(k)})}\right)^{\frac{1}{n-k}} \geq \frac{\det(A)^{\frac{1}{n-k}}}{\det(A_{(k)})^{\frac{1}{n-k}}} + \frac{\det(B)^{\frac{1}{n-k}}}{\det(B_{(k)})^{\frac{1}{n-k}}}, \tag{2.3}$$

for every $k \in \{1, \dots, n-1\}$.

3. Projection of a matrix onto a linear subspace

As briefly mentioned before, linearized versions of the Brunn-Minkowski and Aleksandrov-Fenchel inequalities within the context of Convex Geometry can be found in the literature (see e.g. [19, Section 7.7]). For those results, assumptions on equal projections of the involved convex sets have shown to yield fruitful conclusions. Inspired by that fact, we consider next projections of positive semidefinite matrices, which will be used in the next section with the purpose of linearizing the matrix versions of the Brunn-Minkowski and Aleksandrov-Fenchel inequalities.

We consider $\mathbb{R}^n$ endowed with the Euclidean structure. We write $u^T v$ for the scalar product of $u, v \in \mathbb{R}^n$ and $||\cdot||$ for the Euclidean norm. We denote by e_i the i-th vector of the standard basis in $\mathbb{R}^n$. Further, we denote by $\mathbb{S}^{n-1}$ the Euclidean unit sphere in $\mathbb{R}^n$, and for a vector $u \in \mathbb{R}^n$, we denote by $u^\perp$ the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to u, which will refer to just as a hyperplane.

The following notion of projection of a matrix onto a subspace has been considered in [4, 5], and it is inherited from the definition of restriction of a quadratic form to a linear subspace.

Let L be a linear subspace of $\mathbb{R}^n$, $Q \in \mathcal{M}^n$ a real square symmetric matrix, and let q_Q be the quadratic form on $\mathbb{R}^n$ associated to Q, i.e., $q_Q(x) = \langle x, Qx\rangle$. The projection of the matrix Q onto L will be defined as the matrix associated to the restriction of q to the subspace $L \subset \mathbb{R}^n$, and denoted by $Q|L \in \mathcal{M}^{\dim L}$. The matrix $Q|L$ is well defined, and if Q is positive semidefinite, then so is $Q|L$.

For the matrix terminology, as usual, we need to consider orthonormal bases of L and $L^\perp$, B_L and $B_{L^\perp}$, and the basis of $\mathbb{R}^n$ given by union of those, $B_{L,L^\perp} = B_L \dot\cup B_{L^\perp}$. Whenever we will be dealing with projection of matrices, every matrix $Q \in \mathcal{M}^n$ will be considered as the matrix of the associated linear map, with respect to the bases $B_L, B_{L^\perp}$, and $B_{L,L^\perp}$, fixed in advance. With some abuse of notation, once we have fixed bases of $L, L^\perp$ and $\mathbb{R}^n$ as just mentioned, given a matrix T, we will denote by T also the linear map $T : U \longrightarrow V$ where $U, V \in \{L, L^\perp, \mathbb{R}^n\}$ are appropriately chosen. The other way around, given a map $T : U \longrightarrow V$ for $U, V \in \{L, L^\perp, \mathbb{R}^n\}$ we also denote by T the matrix of T, with respect to the bases $B_L, B_{L^\perp}$ or $\mathbb{R}^n$, accordingly. With that, the projection of a matrix onto the subspace L -implicitly w.r.t. the orthonormal bases B_L of L, $B_{L^\perp}$ of $L^\perp$, and $B_{L,L^\perp}$ of $\mathbb{R}^n$ amounts to the following.

Proposition and Definition 3.1. [4, 5] *Let $Q \in \mathcal{M}^n$ be a positive semidefinite matrix, and let $L \subseteq \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$ of dimension $1 \le k \le n$. The following definitions of the projection of a matrix are equivalent.*

i) *Let $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the quadratic form $x \to x^T Qx$. Then, the projection of the matrix Q onto the subspace L is the positive semidefinite $k \times k$ matrix of the restriction of q to the subspace L.*
ii) *Let B_L and $B_{L^\perp}$ be bases of L and $L^\perp$ respectively, and let $B_{L,L^\perp} = B_L \dot\cup B_{L^\perp}$ be an orthonormal basis of $\mathbb{R}^n$. Then, the projection of the matrix Q onto L is defined as the self-adjoint operator*
$$Q|L = P^* QP,$$
where $P\colon L \to \mathbb{R}^n$ denotes the inclusion of L into $\mathbb{R}^n$, and $P^*\colon \mathbb{R}^n \to L$ the orthogonal projection of $\mathbb{R}^n$ onto L.

Remark 3.2. *We observe that for the equivalence of the two definitions of the projection of a matrix onto a subspace we have used the conventions made before, namely for a subspace $L \subset \mathbb{R}^n$, with a fixed orthonormal basis B_L, we consider the inclusion and projection maps $\mathrm{P} : L \longrightarrow \mathbb{R}^n$, and $P^* : \mathbb{R}^n \longrightarrow L$, and the representation matrices P, and P^*, respectively, with respect to the orthonormal basis of L, B_L, and the standarbasis of $\mathbb{R}^n$.*

Remark 3.3. *Let $C = (c_{ij})_{1\le i,j\le n} \in \mathcal{M}^n$ be a positive semidefinite matrix, which is also the matrix of the linear map $C : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with respect to the standardbasis in $\mathbb{R}^n$. Let L be a linear subspace of $\mathbb{R}^n$, and let $B_L, B_{L^\perp}, B_{L,L^\perp} = B_L \dot\cup B_{L^\perp}$ be orthonormal bases of $L, L^\perp$, and $\mathbb{R}^n$, respectively. Then, if the matrix A gives the linear map C in the basis $B_{L,L^\perp}$, then the projection of C onto L is*

$$P^*CP = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \cdot A \cdot \begin{pmatrix} 1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

that is, the $\dim L \times \dim L$ *principal submatrix of A is given by the first* $\dim L$ *columns and rows.*

We observe that the projection of a matrix onto a subspace L, as a linear map does not depend on the choice of the bases for $L, L^{\perp}$ or $\mathbb{R}^n$, while it does when we *only* consider the matrix.

The next lemma provides us with a connection between the projection of a matrix, and the principal matrices of A and B, involved in Theorems 2.6 and 2.7, which are indeed appropriate projections of A and B.

Lemma 3.4. *Let $A \in \mathcal{S}^n_+$ be a positive definite matrix and $1 \leq i \leq n$. Let A_i denote the $(n-1)\times(n-1)$ matrix obtained from A by removing the i-th row and the i-the column, and let $A_{(i)}$ denote the $i \times i$ matrix obtained from A by taking the first i columns and i rows. Then,*

i) *$A_i = A|L$ for $L = e_i^{\perp}$,*

ii) *$A_{(i)} = A|L$ for $L = \operatorname{lin}\{e_1, \dots, e_i\}$.*

Proof. Let $A \in \mathcal{S}^n_+$. Both statements follow the same idea, namely, an appropriate permutation of the canonical basis and Remark 3.3.

In order to prove i) it is enough to consider the basis $B = (e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n, e_i)$, which is just a reordering of the orthonormal canonical basis. Let C be the matrix representation of A with respect to the basis B. The projection matrix of C onto $e_i^{\perp}$, taking Remark 3.3 into account, is performed by deleting the last column and the last row from C, which is exactly removing the i-th column and i-th row of A.

Analogously, for the proof of ii) we consider the standard basis $B' = (e_1, \dots, e_n)$ and project A onto L.

□

We list now some properties of the projection of matrices, in connection with mixed discriminants. Let $H \subseteq \mathbb{R}^n$ be a hyperplane. Let $Q_1, \dots, Q_{n-1} \in \mathcal{S}^n_+$ be positive semidefinite matrices. By using Proposition and Definition 3.1 i) the projection matrices $Q_1|H, \dots, Q_{n-1}|H \in \mathcal{S}^{n-1}_+$, onto H are $(n-1)\times(n-1)$ positive semidefinite matrices. We will denote by D^{n-1} *the mixed discriminant* on $(\mathcal{S}^{n-1}_+)^{n-1}$, whose existence is provided by Theorem A. Thus, $\mathrm{D}^{n-1}(Q_1|H, \dots, Q_{n-1}|H)$ is the mixed discriminant of the $(n-1)\times(n-1)$ positive semidefinite matrices $Q_1|H, \dots, Q_{n-1}|H$. Lemma 3.6 states a connection between mixed discriminants in $(\mathcal{S}^n_+)^n$ and in $(\mathcal{S}^n_+)^{n-1}$. Before we can state the mentioned result, we need the following standard fact about quadratic forms, described by positive semidefinite matrices of rank 1.

Remark 3.5. *Let $A \in \mathcal{S}^n_+$ be a positive semidefinite matrix with* $\operatorname{rank}(A) = 1$. *Then, there exists $u \in \mathbb{S}^{n-1}$ and $\lambda > 0$, such that $A = \lambda u u^T$. Thus, the quadratic form defined by A is given by $x^T A x = x^T \lambda u u^T x = \lambda (x^T u)^2$.*

Now, we can state the connection of mixed discriminant of appropriate $n \times n$ matrices, to the mixed discriminant D^{n-1} of the $(n-1)\times(n-1)$ projection matrices onto the kernel of the one having rank one.

Lemma 3.6. [5, Lemma 2.4] *Let $u \in \mathbb{S}^{n-1}$, $Q_1, \dots, Q_n \in \mathcal{S}^n_+$ positive semidefinite matrices, so that $Q_n = \lambda u u^T$ for $\lambda > 0$. Let $H = u^{\perp}$ be the hyperplane orthogonal to u, and let $Q_1|H, \dots, Q_{n-1}|H$ denote the projection matrices of $Q_1, \dots, Q_{n-1}$ onto H. Then,*

$$\mathrm{D}(Q_1, \dots, Q_n) = \frac{\lambda}{n} \mathrm{D}^{n-1}(Q_1|H, \dots, Q_{n-1}|H). \tag{3.1}$$

Remark 3.7. *We observe that the choice of an orthonormal basis for the hyperplane H in Lemma 3.6 would not alter the validity of the result. Indeed, if B_1 and B_2 are orthonormal bases of H, and $Q|H_{B_1}$, and $Q|H_{B_2}$ denote the projection matrices of Q onto H with respect to the basis B_1, and B_2, there is an orthogonal matrix, such that $Q|H_{B_1} = OQ|H_{B_2}O^T$. Thus, Lemma 2.1 v) yields*

$$\mathrm{D}^{n-1}(Q_1|H_{B_1}, \dots, Q_{n-1}|H_{B_2}) = \mathrm{D}^{n-1}(OQ_1|H_{B_1}O^T, \dots, OQ_{n-1}|H_{B_2}O^T).$$

We end this section with the following remark on projection of a matrix onto two hyperplanes, which will be useful in the coming sections.

Remark 3.8. [4, Proof of Lemma 2.3.1] *Let $A \in \mathcal{M}^n$ and let $u \in \mathbb{S}^{n-1}$. Let O be an orthogonal matrix mapping u to e_i, and thus, mapping $u^{\perp}$ to $e_i^{\perp}$. Then* $\det\left(A|u^{\perp}\right) = \det\left((OAO^T)|e_i^{\perp}\right)$.

4. Linearized version of the Aleksandrov-Fenchel inequality

In this section, we aim to obtain a linearized version of the Aleksandrov-Fenchel inequality for mixed discriminants. The following theorem and its proof, which provides us with the mentioned goal, is a one-to-one adaptation of [19, Theorem 7.4.3] to the matrix context. We provide detailed proof in this framework for completeness.

Theorem 4.1. *Let $A_0, A_1, A_2, A_3 \dots, A_n$ be positive semidefinite matrices, and $\mathcal{A} := (A_3, \dots, A_n)$. If $\mathrm{D}(A_1, A_0, \mathcal{A})$, $\mathrm{D}(A_2, A_0, \mathcal{A}) > 0$, then*

$$\frac{\mathrm{D}(A_1, A_1, \mathcal{A})}{\mathrm{D}(A_1, A_0, \mathcal{A})^2} - \frac{2\mathrm{D}(A_1, A_2, \mathcal{A})}{\mathrm{D}(A_1, A_0, \mathcal{A})\mathrm{D}(A_2, A_0, \mathcal{A})} + \frac{\mathrm{D}(A_2, A_2, \mathcal{A})}{\mathrm{D}(A_2, A_0, \mathcal{A})^2} \leq 0. \tag{4.1}$$

Proof. (The proof follows the same lines of the proof of [19, Theorem 7.4.3]) We consider first the case in which all matrices involved are positive definite matrices. We observe that the assumption $\mathrm{D}(A_1, A_0, \mathcal{A}) = \mathrm{D}(A_2, A_0, \mathcal{A}) > 0$ is in this case redundant.

Following the notation of the proof of Theorem 7.4.3 in [19], we denote

$$U_{ij} := \mathrm{D}(A_i, A_j, \mathcal{A}),$$

with $i, j = 0, 1, 2$. The following statement is a direct adaptation to mixed discriminant of [19, Lemma 7.4.1].

Claim. The following inequality

$$\left(U_{00}U_{12} - U_{01}U_{02}\right)^2 \leq \left(U_{01}^2 - U_{00}U_{11}\right)\left(U_{02}^2 - U_{00}U_{22}\right)$$

holds.

Indeed, by the Aleksandrov-Fenchel inequality for mixed discriminants, i.e., Theorem D, direct computations for fixed $\lambda_1, \lambda_2 \geq 0$ show

$$\begin{aligned} 0 &\leq \mathrm{D}(\lambda_1 A_1 + \lambda_2 A_2, A_0, \mathcal{A})^2 - \mathrm{D}(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 A_1 + \lambda_2 A_2, \mathcal{A})\mathrm{D}(A_0, A_0, \mathcal{A}) \\ &= \lambda_1^2(\underbrace{U_{10}^2 - U_{11}U_{00}}_{A}) - 2\lambda_1\lambda_2(\underbrace{U_{12}U_{00} - U_{10}U_{20}}_{B}) + \lambda_2^2(\underbrace{U_{20}^2 - U_{22}U_{00}}_{C}). \end{aligned}$$

Denoting by $A = U_{10}^2 - U_{11}U_{00}$, $B = U_{12}U_{00} - U_{10}U_{20}$, and $C = U_{20}^2 - U_{22}U_{00}$ we obtain the validity of the inequality

$$Ax^2 + 2Bx + C \geq 0,$$

for every $x \in \mathbb{R}$. Therefore, $4B^2 - 4AC \leq 0$, i.e.,

$$\begin{aligned} \left(U_{00}U_{12} - U_{01}U_{02}\right)^2 &\leq \left(U_{01}^2 - U_{00}U_{11}\right)\left(U_{02}^2 - U_{00}U_{22}\right) \\ &= U_{01}^2U_{02}^2\left(1 - \frac{U_{00}U_{11}}{U_{01}^2}\right)\left(1 - \frac{U_{00}U_{22}}{U_{02}^2}\right). \end{aligned}$$

By Theorem D, we have $1 - \frac{U_{00}U_{22}}{U_{02}^2} > 0$ and $1 - \frac{U_{00}U_{11}}{U_{01}^2} > 0$, and since all matrices involved are positive definite, we also have $U_{00}, U_{01}, U_{02} > 0$. If we take now the negative squared root on the left-hand side, and the positive square root on the right-hand side, then applying the inequality $4ab \leq (a+b)^2$, i.e, considering $-2ab \leq a + b$ with $a = 1 - \frac{U_{00}U_{22}}{U_{02}^2} > 0$, and $b = \frac{U_{00}U_{11}}{U_{01}^2} > 0$, we get

$$U_{01}U_{02}\left(\frac{U_{11}}{U_{01}^2} - 2\frac{U_{12}}{U_{01}^2} + \frac{U_{22}}{U_{02}}\right) \leq 0,$$

which yields (4.1).

Next, we perform a standard approximation argument, which allows us to remove the definiteness assumption and consider positive semidefinite matrices. We include the main steps for completeness. In the previous argument for positive definite matrices, we needed $\mathrm{D}(A_0, A_0, \mathcal{A}) = U_{00} > 0$, $\mathrm{D}(A_0, A_1, \mathcal{A}) = U_{01} > 0$, $\mathrm{D}(A_0, A_2, \mathcal{A}) = U_{02} > 0$. From the assumptions follow directly that $U_{01}, U_{02} > 0$. Now we prove that if $U_{00} = 0$, we can also argue as above after performing a suitable approximation.

We observe first, that it is sufficient to consider A_0 diagonal because of Lemma 2.1. On the one hand, there exists k_0 and $A_{i_1}, \dots, A_{i_{k_0}}$ such that $\dim(P(A_{i_1}) + \ldots + P(A_{i_{k_0}})) < k_0$. On the other hand, if S an orthogonal matrix, such that $S^T A_0 S = A_0'$ is diagonal, then $U_{00} = 0$ and vice versa. Thus, $U_{00} = 0$ is equivalent to $\mathrm{D}(A_0', A_0', \mathcal{A}) = 0$.

Therefore, we can assume that $A_0 = diag(a_{11}, \dots, a_{kk}, 0, \dots, 0)$, for some $k \in \{1, \dots, n\}$, and $a_{ii} > 0$, $i \in \{1, \dots, k\}$. Defining $A_{0,j} = diag(a_{11}, \dots, a_{kk}, 1/j, \dots, 1/j)$, with $0 < j \in \mathbb{N}$, we obtain that $A_{0,j}$ is positive definite for every j and it converges to A_0 in the Euclidean metric. Hence, $\mathrm{D}(A_{0,j}, A_{0,j}, \mathcal{A}) > 0$ and by the assumption, we also have $\mathrm{D}(A_1, A_{0,j}, \mathcal{A}) > 0$, $\mathrm{D}(A_2, A_{0,j}, \mathcal{A}) > 0$, for every $0 < j \in \mathbb{N}$. Applying the previous argument to $\mathrm{D}(A_{0,j}, A_{0,j}, \mathcal{A}), \mathrm{D}(A_1, A_{0,j}, \mathcal{A})$ and $\mathrm{D}(A_2, A_{0,j}, \mathcal{A})$, we have for every $0 < j \in \mathbb{N}$

$$\frac{\mathrm{D}(A_1, A_1, \mathcal{A})}{\mathrm{D}(A_1, A_{0,j}, \mathcal{A})^2} - \frac{2\mathrm{D}(A_1, A_2, \mathcal{A})}{\mathrm{D}(A_1, A_{0,j}, \mathcal{A})\mathrm{D}(A_2, A_{0,j}, \mathcal{A})} + \frac{\mathrm{D}(A_2, A_2, \mathcal{A})}{\mathrm{D}(A_2, A_{0,j}, \mathcal{A})^2} \leq 0.$$

Using (1.3) and the continuity of the determinant we get

$$\lim_{j\to+\infty} \mathrm{D}(A_{0,j}, A_{0,j}, \mathcal{A}) = \mathrm{D}(A_0, A_0, \mathcal{A}), \qquad \lim_{j\to+\infty} \mathrm{D}(A_1, A_{0,j}, \mathcal{A}) = \mathrm{D}(A_1, A_1, \mathcal{A}),$$
$$\lim_{j\to+\infty} \mathrm{D}(A_2, A_{0,j}, \mathcal{A}) = \mathrm{D}(A_2, A_2, \mathcal{A}),$$

which finishes the proof. □

From this result, we obtain immediately the first *linearization* version of the Aleksandrov-Fenchel inequality. We need the assumption of $\mathrm{D}(A_1, A_0, \mathcal{A}) = \mathrm{D}(A_2, A_0, \mathcal{A}) > 0$, which is only relevant in the case that the matrices are not positive definite.

Corollary 4.2. *Let $A_0, A_1, A_2, A_3 \dots, A_n \in \mathcal{S}_+^n$ be positive semidefinite matrices. We will write $\mathcal{A} := (A_3, \dots, A_n)$. If $\mathrm{D}(A_1, A_0, \mathcal{A}) = \mathrm{D}(A_2, A_0, \mathcal{A}) > 0$ holds, then*

$$2\mathrm{D}(A_1, A_2, \mathcal{A}) \geq \mathrm{D}(A_1, A_1, \mathcal{A}) + \mathrm{D}(A_2, A_2, \mathcal{A}). \tag{4.2}$$

The previous results enable us to establish the linearized Aleksandrov-Fenchel inequality for matrices sharing the same projection.

Theorem 4.3. *Let $C_3 \dots, C_n, C \in \mathcal{S}_+^n$ be positive semidefinite matrices, and let $L = u^\perp$ be the $(n-1)$-dimensional subspace orthogonal to u. If $\mathrm{D}^{n-1}(C|L, C_3|L, \dots, C_n|L) > 0$, then,*

$$2\mathrm{D}(A, C, C_3, \dots, C_n) \geq \mathrm{D}(A, A, C_3, \dots, C_n) + \mathrm{D}(C, C, C_3, \dots, C_n), \tag{4.3}$$

for every $A \in \mathcal{S}_+^n$ with $A|L = C|L$, the Aleksandrov-Fenchel inequality (1.8) *can be linearized.*

Proof. We consider the rank one positive semidefinite matrix $M = uu^T$. Since $\mathrm{rank}(M) = 1$, then by Lemma 3.6 we have

$$\mathrm{D}(C, M, C_3, \dots, C_n) = \frac{1}{n}\mathrm{D}^{n-1}(C|L, C_3|L, \dots, C_n|L) > 0.$$

Moreover, for every $A \in \mathcal{S}_+^n$ with $A|L = C|L$ we have also

$$\mathrm{D}(A, M, C_3, \dots, C_n) = \mathrm{D}(C, M, C_3, \dots, C_n) = \frac{1}{n}\mathrm{D}^{n-1}(C|L, C_3|L, \dots, C_n|L) > 0.$$

By Corollary 4.2, with $A_0 = M$, we obtain the result. □

Next, as seems to be natural after the last result, we introduce the following family of matrices, depending on a linear subspace and a given matrix. Not only this family allows us to have a linearized Aleksandrov-Fenchel inequality, but (a generalization of) it will allow us to linearize the Brunn-Minkowski inequality in the next section. We will call this family, canal class, following the terminology used for the analog notion within the context of convex bodies. For the latter, we refer the reader to [19, Section 7.7], and the references therein.

Definition 4.4. *Let $C \in \mathcal{S}_+^n$ be a positive semidefinite matrix, and let L be a linear subspace in $\mathbb{R}^n$. Then, the canal class of C, relative to the subspace L, is the following family of positive semidefinite matrices:*

$$M_C^L = \left\{ A \in \mathcal{S}_+^n \, : \, C|L = A|L \right\}.$$

The following corollary is now an immediate consequence.

Corollary 4.5. *Let $C_3 \dots, C_n, C, M \in \mathcal{S}^n_+$ be positive semidefinite matrices, $u \in \mathbb{S}^{n-1}$, and let $L = u^\perp$. We assume further that $\mathrm{D}^{n-1}(C|L, C_3|L, \dots, C_n|L) > 0$. Then,*

$$2\mathrm{D}(A, B, C_3, \dots, C_n) \geq \mathrm{D}(A, A, C_3, \dots, C_n) + \mathrm{D}(B, B, C_3, \dots, C_n),$$

for every $A, B \in M^L_C$.

We finish the section by stating one more inequality for mixed discriminants, which can be derived from the latter results, and which will be needed in the next. We follow [19, p. 399-400]. We recall, that a finite sequence of real numbers $(a_0, a_1, \dots, a_m)$ is called concave if

$$a_{i-1} - 2a_i + a_{i+1} \leq 0 \quad \text{ for } i = 1, \dots, m-1,$$

which is equivalent to

$$a_0 - a_1 \leq a_1 - a_2 \leq \cdots \leq a_{m-1} - a_m.$$

By taking arithmetic means the following inequalities are also satisfied by a concave sequence

$$\frac{a_i - a_j}{j - i} \leq \frac{a_j - a_k}{k - j} \quad \text{ and } \quad (k-j)a_i + (i-k)a_j + (j-i)a_k \leq 0.$$

In the latter, equality holds if and only if

$$a_{r-1} - 2a_r + a_{r+1} = 0 \quad \text{for } r = i+1, \dots, k-1.$$

Lemma 4.6. *Let $1 \leq m \leq n$, let $A_0, A_1, A_2, A_{m+1} \dots, A_n$ be positive semidefinite matrices, and write $\mathcal{A} := (A_{m+1}, \dots, A_n)$. If $\mathrm{D}(A_0[m-i], A_1[i], \mathcal{A}) > 0$ for $0 \leq i \leq m$, then the finite sequence $(\mathrm{D}_{(0)}, \dots, \mathrm{D}_{(m)})$, where $D_{(i)} = \mathrm{D}(A_0[m-i], A_1[i], \mathcal{A})$, $1 \leq i \leq m$, is a concave sequence. Moreover,*

$$(k-j)\mathrm{D}_{(i)} + (i-k)\mathrm{D}_{(j)} + (j-i)\mathrm{D}_{(k)} \leq 0$$

for every $0 \leq i < j < k \leq m$, with equality holds if and only if

$$\mathrm{D}_{(k-1)} - 2\mathrm{D}_{(k)} + \mathrm{D}_{(k+1)} = 0$$

holds for every $k = 1, \dots, m-1$.

Proof. The discussion preceding the lemma states the inequality and the equality case as long as we prove that the sequence $(\mathrm{D}_{(0)}, \dots, \mathrm{D}_{(m)})$ is concave. The concavity follows from Corollary 4.2. □

5. Linearized Brunn-Minkowski inequality

In this section, we focus on linearizations of the Brunn-Minkowski inequality and related inequalities. We begin with a linearized version of the general Brunn-Minkowski inequality, which relies on the linearized Aleksandrov-Fenchel inequality for matrices sharing a common projection on a hyperplane.

Theorem 5.1. *Let $u \in \mathbb{S}^{n-1}$ and $L = u^\perp$ be the $(n-1)$-dimensional subspace orthogonal to u. Let $C \in \mathcal{S}^n_+$ be a positive semidefinite matrix, fix $m \in \{2, \dots, n\}$, and $C_{m+1}, \dots, C_n \in \mathcal{S}^n_+$ be such that $\mathrm{D}^{n-1}(C|L[m-1], C_{m+1}|L, \dots, C_n|L) > 0$. Let $A_0, A_1 \in M^L_C$, and denote by $A_\lambda = (1-\lambda)A_0 + \lambda A_1$ the convex combination of A_0 and A_1, with $\lambda \in [0, 1]$. Then, the function*

$$f(\lambda) = \mathrm{D}(A_\lambda[m], C_{m+1}, \dots, C_n)$$

is concave on $[0, 1]$.

Proof. (The proof of this result follows the same lines of the proof of [19, Theorem 7.7.2]).

We denote by $\mathcal{C} := (C_{m+1}, \dots, C_n)$, then $f(\lambda) = \mathrm{D}(A_\lambda[m], \mathcal{C})$. We observe first, that $A_\lambda \in M^L_C$. Using now the linearity of the mixed discriminant in each entry in Proposition 2.1, we obtain (see the function g in the proof of Theorem 2.4):

$$f''(0) = m(m-1)\big[\mathrm{D}(A_0[m], \mathcal{C}) - 2\mathrm{D}(A_1, A_0[m-1], \mathcal{C}) + \mathrm{D}(A_1, A_1, A_0[m-2], \mathcal{C})\big].$$

Applying now Theorem 4.3, we obtain that $f''(0) \leq 0$, which establishes the concavity of the function f in $[0, 1]$. □

In the case $m = n$ we obtain the following immediate corollary.

Corollary 5.2. *Let $u \in \mathbb{S}^{n-1}$, and let $u^\perp$ be the $(n-1)$-dimensional subspace orthogonal to u. Let $A, B \in \mathcal{S}^n_+$ be such that $A|u^\perp = B|u^\perp$. Then,*

$$\det((1-\lambda)A + \lambda B) \geq (1-\lambda)\det(A) + \lambda \det(B),$$

for every $\lambda \in [0,1]$. Moreover, we have equality if and only if there exists a matrix $R \in \mathcal{S}^n_+$ of rank at most 1, such that $B = A + R$.

The equality case follows from the following result and remark.

Theorem 5.3. [17, Theorem 5.1] *Let $A, B \in \mathcal{S}^n_+$ be two positive semidefinite matrices. Then*

$$\det((1-\lambda)A + \lambda B) = (1-\lambda)\det(A) + \lambda \det(B)$$

if and only if there exists a matrix $R \in \mathcal{S}^n_+$ of rank at most 1, such that $B = A + R$.

Remark 5.4. *We observe that if there exists a matrix $R \in \mathcal{S}^n_+$ of rank at most 1, such that $B = A + R$, then there exists $u \in \mathbb{S}^{n-1}$ with $R = \lambda u u^T$, for a suitable $\lambda \in \mathbb{R}$. This implies that $R|u^\perp$ is the $(n-1)\times(n-1)$ zero matrix. Hence, $A|u^\perp = B|u^\perp$.*

Next, we deal with the equality case in Theorem 5.1, i.e., the equality case in the linearized Brunn-Minkowski inequality, under the assumption of equal matrix projections onto a hyperplane.

Theorem 5.5. *Under the same notations and hypotheses of Theorem 5.1, let $f(\lambda) := \mathrm{D}(A_\lambda[m], \mathcal{C})$ for $1 \leq m \leq n$, and $C_{m+1}, \ldots, C_n \in \mathcal{S}^n_+$. The following assertions are equivalent:*

i) *The function f is linear;*
ii) *We have the following chain of equalities:*

$$\begin{aligned}&\mathrm{D}(A_0[m], \mathcal{C}) - \mathrm{D}(A_0[m-1], A_1, \mathcal{C})\\ &\quad = \mathrm{D}(A_0[m-1], A_1, \mathcal{C}) - \mathrm{D}(A_0[m-2], A_1[2], \mathcal{C}) = \ldots = \mathrm{D}(A_0, A_1[m-1], \mathcal{C}) - \mathrm{D}(A_1[m], \mathcal{C});\end{aligned}$$

iii) $(m-1)\mathrm{D}(A_0[m], \mathcal{C}) - m\mathrm{D}(A_0[m-1], A_1, \mathcal{C}) + \mathrm{D}(A_1[m], \mathcal{C}) = 0.$

Proof. (The proof of this result follow the same lines of the proof of [19, Theorem 7.7.2]).

Let $f(\lambda) := \mathrm{D}(A_\lambda[m], \mathcal{C})$, and for $1 \leq i \leq m \leq n$, let $D_{(i)} = \mathrm{D}(A_0[m-i], A_1[i], \mathcal{C})$. From Theorem 5.1 we have that the function f is concave, which yields $f'(0) \geq f(1) - f(0)$, with equality if and only if the function is linear. Writing the latter explicitly, having into account that $f'(0) = -mD_{(0)} + mD_{(1)}$, we get

$$\mathrm{D}_{(m)} - m\mathrm{D}_{(1)} + \mathrm{D}_{(0)}(m-1) \leq 0$$

with equality if and only if f is linear, which is exactly the equivalence between i) and iii).

The equivalence of ii) and iii) follows from the equality in Lemma 4.6. ☐

5.1. **Equal determinant of projections.** In this subsection, we aim to *improve* the linearized version of the Brunn-Minkowski inequality for matrices under the assumption of equal projections, by considering a milder assumption, namely, that the matrices have equal determinant of the projections onto a hyperplane.

We prove first the result, when the projections are assumed to be onto hyperplanes orthogonal to the standard basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$.

Proposition 5.6. *Let $1 \leq i \leq n$, and let $A, B \in \mathcal{S}^n_+$ be two positive definite matrices. If $\det\left(A|e_i^\perp\right) = \det\left(B|e_i^\perp\right)$, then*

$$\det((1-\lambda)A + \lambda B) \geq (1-\lambda)\det(A) + \lambda \det(B),$$

for every $\lambda \in [0,1]$.

Proof. Let $\lambda \in [0,1]$. We use Lemma 3.4 and apply Bergstrom's inequality to $(1-\lambda)A$ and λB. Taking into account that $\det(A_i) = \det\left(A|e_i^\perp\right) = \det\left(B|e_i^\perp\right) = \det(B_i)$, we obtain

$$\begin{aligned}\frac{\det((1-\lambda)A+\lambda B)}{\det((1-\lambda)A_i+\lambda B_i)} &\geq \frac{\det((1-\lambda)A)}{\det((1-\lambda)A_i)} + \frac{\det(\lambda B)}{\det(\lambda B_i)}\\ &= (1-\lambda)\frac{\det(A)}{\det(A_i)} + \lambda\frac{\det(B)}{\det(B_i)}\\ &= \frac{(1-\lambda)\det(A)+\lambda\det(B)}{\det(A_i)}.\end{aligned}$$

Therefore, we obtain the inequality

$$\frac{\det(A_i)}{\det((1-\lambda)A_i+\lambda B_i)}\det((1-\lambda)A+\lambda B) \geq (1-\lambda)\det(A)+\lambda\det(B). \tag{5.1}$$

The Brunn-Minkowski inequality (Theorem B) applied to $\det((1-\lambda)A_i+\lambda B_i)$ yields

$$\begin{aligned}\det((1-\lambda)A_i+\lambda B_i) &\geq \left((1-\lambda)\det(A_i)^{\frac{1}{n-1}} + \lambda\det(B_i)^{\frac{1}{n-1}}\right)^{n-1}\\ &= \left((1-\lambda)\det(A_i)^{\frac{1}{n-1}} + \lambda\det(A_i)^{\frac{1}{n-1}}\right)^{n-1}\\ &= \det(A_i).\end{aligned}$$

Thus,

$$\frac{\det(A_i)}{\det((1-\lambda)A_i+\lambda B_i)} \leq 1.$$

The latter, together with (5.1) yields the result. □

Let $A, B \in \mathcal{S}_+^n$. Next, we prove that we can assume equal determinant of the projections of A and B onto a generic hyperplane $u^\perp$, $u \in \mathbb{S}^{n-1}$, to obtain the linearized version of the Brunn-Minkowski inequality for the determinant in Proposition 5.6, using Remark 3.8.

Theorem 5.7. *Let $A, B \in \mathcal{S}_+^n$ be positive semidefinite matrices, and let $u \in \mathbb{S}^{n-1}$. Assume that $\det\left(A|u^\perp\right) = \det\left(B|u^\perp\right)$. Then*

$$\det((1-\lambda)A+\lambda B) \geq (1-\lambda)\det(A)+\lambda\det(B),$$

for every $\lambda \in [0,1]$. Moreover, we have equality if and only if there exists a matrix $R \in \mathcal{S}_+^n$ of rank at most 1, such that $B = A + R$.

Proof. Let O be an orthogonal matrix such that $e_n = Ou$. By Remark 3.8, we have

$$\det(OAO^T|e_n^\perp) = \det(A|u^\perp) = \det(B|u^\perp) = \det(OBO^T|e_n^\perp).$$

Applying Proposition 5.6 to OAO^T and OBO^T, we have,

$$\det((1-\lambda)OAO^T+\lambda OBO^T) \geq (1-\lambda)\det(OAO^T)+\lambda\det(OBO^T).$$

Using that the matrix O is orthogonal we get the desired inequality. The equality follows from Theorem 5.3. □

It is natural to ask whether the assumption that the projections of two matrices onto a k-dimensional linear subspace L, with $k \in \{1,\dots,n\}$, are equal allows establishing a linearized version of the Brunn-Minkowski inequality. The following lemma shows that such an assumption is not enough for that purpose.

Lemma 5.8. *There exist matrices $A, B \in \mathcal{S}_+^n$, and an $(n-2)$-linear subspace $L \subset \mathbb{R}^n$ with $\det(A|L) = \det(B|L)$, so that*

$$\det\left((1-\lambda)A+\lambda B\right) < (1-\lambda)\det(A)+\lambda\det(B),$$

Proof. Let $a_1,\dots,a_n,b_1,\dots,b_n \geq 0$, and let $A = diag\{a_1,\dots,a_n\}$ and $B = diag\{b_1,\dots,b_n\}$ be diagonal positive semidefinite matrices satisfying that $\prod_{i=1}^{n-2} a_i = \prod_{i=1}^{n-2} b_i$. Then, from Lemma 3.4, $A|L = A_{(n-2)}$ and $B|L = B_{(n-2)}$, thus $\det(A|L) = \det(B|L) = \prod_{i=1}^{n-2} a_i$ for L being the $(n-2)$-dimensional linear subspace generated by the first $n-2$ vectors of the canonical basis $\{e_1,\dots,e_{n-2}\}$ of $\mathbb{R}^n$.

We consider now the matrices $A' = diag\{a_1,\dots,a_{n-1}\}$ and $B' = diag\{b_1,\dots,b_{n-1}\}$, which are clearly positive semidefinite. From the assumption follows that the determinants of their projections onto the first $n-2$ vectors of the standard basis of $\mathbb{R}^{n-1}$ coincide. Using Theorem 5.7, we have

$$\begin{aligned}\det((1-\lambda)A'+\lambda B') &= \prod_{i=1}^{n-1}((1-\lambda)a_i+\lambda b_i) = ((1-\lambda)a_{n-1}+\lambda b_{n-1})\prod_{i=1}^{n-2}((1-\lambda)a_i+\lambda b_i)\\ &\geq (1-\lambda)\det(A')+\lambda\det(B') = (1-\lambda)\prod_{i=1}^{n-1}a_i+\lambda\prod_{i=1}^{n-1}b_i\\ &= ((1-\lambda)a_{n-1}+\lambda b_{n-1})\prod_{i=1}^{n-2}a_i.\end{aligned}$$

Hence, $\prod_{i=1}^{n-2}(1-\lambda)a_i+\lambda b_i \geq \prod_{i=1}^{n-2} a_i$, for every $\lambda\in[0,1]$. Now, if we assume that $a_i = b_i$ for $1\leq i\leq n-2$, we have not only $\det(A|L) = \det(B|L)$, but $A|L = B|L$, which leads for $\lambda\in[0,1]$ to:

$$\det((1-\lambda)A+\lambda B) = \prod_{i=1}^{n}((1-\lambda)a_i+\lambda b_i) = ((1-\lambda)a_n+\lambda b_n)\prod_{i=1}^{n-2}a_i,$$

while

$$(1-\lambda)\det(A)+\lambda\det(B) = ((1-\lambda)a_n a_{n-1}+\lambda b_n b_{n-1})\prod_{i=1}^{n-2}a_i.$$

Thus, it is enough to consider

$$((1-\lambda)a_{n-1}+\lambda b_{n-1})((1-\lambda)a_n+\lambda b_n) \geq ((1-\lambda)a_n a_{n-1}+\lambda b_n b_{n-1}).$$

The latter is equivalent to $(a_n-b_n)(a_{n-1}-b_{n-1})\leq 0$, which does not depend on $\lambda\in[0,1]$. Now it is easy to provide non-negative reals a_{n-1},a_n,b_{n-1},b_n, for which $(a_n-b_n)(a_{n-1}-b_{n-1})\leq 0$ does not hold, as, e.g., $a_n > b_n \geq 0$, and $a_{n-1} > b_{n-1}\geq 0$. ☐

The previous result, along with Theorems 5.6 and 5.7 allows us to argue that the assumption $\det(A|L) = \det(B|L)$, and even $A|L = B|L$, for an $(n-2)$-dimensional is not enough to ensure a linearized version of the Brunn-Minkowski inequality.

Nevertheless, we can prove the following result, which follows the same spirit, by means of the Ky-Fan inequality. An *almost linear* version of the Brunn-Minkowski inequality under the assumption of equal determinant of the projections of the matrices onto $(n-2)$-dimensional subspaces, is a particular case of it.

Theorem 5.9. *Let $A,B\in\mathcal{M}^n$ be two positive definite matrices. If $\det(A|L) = \det(B|L)$ for some $(n-k)$-dimensional linear subspace $L\subset\mathbb{R}^n$, then*

$$\det((1-\lambda)A+\lambda B) \geq (1-\lambda)^k\det(A)+\lambda^k\det(B), \tag{5.2}$$

for every $\lambda\in[0,1]$.

Proof. The proof is essentially the same as the proof of Theorem 5.7, except for the fact, that we apply Ky Fan's inequality (2.3) instead of Bergstrom's inequality (2.2).

Using Remark 3.8 we can assume that L is the span of the first $n-k$ vectors of the canonical basis. Then, using Lemma 3.4 the projection onto L of the matrix A, is the matrix $A_{(n-k)}$ in Ky Fan's Theorem, constructed by taking the first k rows and columns of A, i.e., $A_{(n-k)} = A|L$. Now we can apply Ky Fan's inequality to $(1-\lambda)A$ and λB to obtain

$$\left(\frac{\det((1-\lambda)A+\lambda B)}{\det((1-\lambda)A_{(n-k)}+\lambda B_{(n-k)})}\right)^{\frac{1}{k}} \geq \frac{(1-\lambda)^{\frac{n}{k}}\det(A)^{\frac{1}{k}}}{(1-\lambda)^{\frac{n-k}{k}}\det(A_{(n-k)})^{\frac{1}{k}}} + \frac{\lambda^{\frac{n}{k}}\det(B)^{\frac{1}{k}}}{\lambda^{\frac{n-k}{k}}\det(B_{(n-k)})^{\frac{1}{k}}} =$$
$$= (1-\lambda)\frac{\det(A)^{\frac{1}{k}}}{\det(A_{(n-k)})^{\frac{1}{k}}} + \lambda\frac{\det(B)^{\frac{1}{k}}}{\det(B_{(n-k)})^{\frac{1}{k}}} =$$
$$= \frac{(1-\lambda)\det(A)^{\frac{1}{k}} + \lambda\det(B)^{\frac{1}{k}}}{\det(A_{(n-k)})^{\frac{1}{k}}},$$

where the last equality comes from the assoumption $\det(A|L) = \det(B|L)$.

From the Brunn-Minkowski inequality for $(n-k)\times(n-k)$ matrices applied to the matrix $(1-\lambda)A_{(n-k)} + \lambda B_{(n-k)}$ we get

$$\frac{\det((1-\lambda)A+\lambda B)^{\frac{1}{k}}}{\left((1-\lambda)\det(A_{(n-k)})^{\frac{1}{n-k}} + \lambda\det(B_{(n-k)})^{\frac{1}{n-k}}\right)^{\frac{n-k}{k}}} \geq \left(\frac{\det((1-\lambda)A+\lambda B)}{\det((1-\lambda)A_{(n-k)}+\lambda B_{(n-k)})}\right)^{\frac{1}{k}}.$$

By the assoumption $\det(A_{(n-k)}) = \det(B_{(n-k)})$, we have

$$\left((1-\lambda)\det(A_{(n-k)})^{\frac{1}{n-k}} + \lambda\det(B_{(n-k)})^{\frac{1}{n-k}}\right)^{\frac{n-k}{k}} = \det(A_{(n-k)})^{\frac{1}{k}},$$

thus $\det((1-\lambda)A+\lambda B) \geq (1-\lambda)^k\det(A) + \lambda^k\det(B)$. □

Remark 5.10. *Although we cannot provide a characterization of the equality in the previous result, we provide next an example of two positive semidefinite matrices A, B satisfying $B = A + R$ with* $\operatorname{rank} R \leq 2$*, for which there is no equality in the previous result, showing that the equality in the latter result differs strongly from the equality in the previous refinements of the Brunn-Minkowski inequality for the determinant of positive semidefinite matrices.*

Let $A = diag\{a_1,\dots,a_n\}$, with $a_i > 0$, for all $i = 1,\dots,n$, be a positive definite matrix. Let $R = diag\{0,\dots,0,r_{n-1},r_n\}$, with $r_i > 0$, for $i = n-1, n$, be a positive semidefinite with rank 2. Let $\lambda \in [0,1]$, and $B = A + R$. Then we have

$$\det((1-\lambda)A+\lambda B) = \det(A+\lambda R) = \left(\prod_{i=1}^{n-2} a_i\right)(a_{n-1}+\lambda r_{n-1})(a_n+\lambda r_n).$$

$$(a_{n-1}+\lambda r_{n-1})(a_n+\lambda r_n) = (1-\lambda)^2 a_{n-1}a_n + \lambda^2(a_{n-1}+r_{n-1})(a_n+r_n),$$

which yields

$$\lambda(a_{n-1}r_n + a_n r_{n-1} + 2a_n a_{n-1}) = \lambda^2(2a_{n-1}a_n + a_{n-1}r_n + r_{n-1}a_n).$$

However, the latter is only true for $\lambda \in \{0,1\}$, as long as $2a_{n-1}a_n + a_{n-1}r_n + r_{n-1}a_n \neq 0$.

In the case $\operatorname{rank} R \leq 1$*, it follows from Theorem 5.3, that*

$$\det((1-\lambda)A+\lambda B) = (1-\lambda)\det A + \lambda\det B.$$

Thus, there cannot be equality in (5.2)*, in general.*

As the first Minkowski inequality in Theorem C follows from the Brunn-Minkowski inequality, we state now a linearized version of the first Minkowski inequality for the mixed discriminant, following from the linearized version of the Brunn-Minkowski inequality.

Theorem 5.11. *Let $A, B \in \mathcal{M}^n$ be two positive semidefinite matrices. If $\det\left(A|u^\perp\right) = \det\left(B|u^\perp\right)$ for some $u \in \mathbb{S}^{n-1}$, then we have*

$$n\mathrm{D}(A,\dots,A,B) \geq (n-1)\det(A) + \det(B). \tag{5.3}$$

Moreover equality holds if and only if there exists $L \in \mathcal{M}^n$ such that $\operatorname{rank}(L) \leq 1$ *and* $B = A + L$.

The proof follows the lines of (some of) the classical proofs of the first Minkowski inequality within Convex Geometry, as in [19].

Proof. It follows from (1.4) that

$$n\mathrm{D}(A[n-1],B)=\lim_{\lambda\to 0^+}\frac{\det((1-\lambda)A+\lambda B)-\det((1-\lambda)A)}{\lambda}.$$

Hence, by means of Theorem 5.7, we have

$$\begin{aligned}
n\mathrm{D}(A[n-1],B) &= \lim_{\lambda\to 0^+}\frac{\det((1-\lambda)A+\lambda B)-\det((1-\lambda)A)}{\lambda}\\
&\geq \lim_{\lambda\to 0^+}\frac{(1-\lambda)\det(A)+\lambda\det(B)-\det((1-\lambda)A)}{\lambda}\\
&= \lim_{\lambda\to 0^+}\frac{(1-\lambda)\det(A)+\lambda\det(B)-(1-\lambda)^n\det(A)}{\lambda}\\
&= \lim_{\lambda\to 0^+}\frac{1-\lambda-(1-\lambda)^n}{\lambda}\det(A)+\det(B)\\
&= (n-1)\det(A)+\det(B).
\end{aligned}$$

The equality follows from Theorem 5.3. ☐

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