Preliminary version

REFINEMENTS OF L_p BRUNN-MINKOWSKI TYPE INEQUALITIES

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ABSTRACT. We investigate improvements of the L_p Brunn-Minkowski inequality for convex bodies containing the origin, under the additional assumption that they have a common projection onto a hyperplane. We provide some positive answers as well as counterexamples to what would seem to be the expected result. Moreover, its connection with the suitable improvements of the L_p first Minkowski inequality is also shown.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the family of all convex bodies, i.e., non-empty compact and convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$, and $\mathcal{K}^n_o$ the subfamily of $\mathcal{K}^n$ consisting of all convex bodies containing the origin. We write B_n for the n-dimensional Euclidean unit ball and $\mathbb{S}^{n-1}$ for its boundary, the unit sphere. For any $u \in \mathbb{S}^{n-1}$, the vector hyperplane with normal vector u is denoted by $u^\perp$, whereas the orthogonal projection of a set $A \subset \mathbb{R}^n$ onto $u^\perp$ is represented by $P_{u^\perp}(A)$. Finally, the n-dimensional volume of a measurable set $A \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}_n(A)$.

Relating the volume of the Minkowski addition of two sets in terms of their volumes, one is led to the famous *Brunn-Minkowski inequality*. One form of it reads as follows (see, e.g., [29, Theorem 7.1.1]).

Theorem A. *Let $K, L \in \mathcal{K}^n$ be two convex bodies. The inequality*

$$\mathrm{vol}_n\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}_n(K)^{1/n} + \lambda \mathrm{vol}_n(L)^{1/n} \tag{1.1}$$

holds for all $\lambda \in (0,1)$.

Here $+$ is used for the Minkowski sum, i.e., $A + B := \{x + y : \ x \in A, y \in B\}$ for any non-empty subsets $A, B \subset \mathbb{R}^n$, whereas $\lambda A := \{\lambda x : \ x \in A\}$, for some $\lambda > 0$, denotes the corresponding dilatate of A.

The Brunn-Minkowski inequality is one of the most well-known and powerful inequalities in Convex Geometry and beyond. We refer to [15] for an exhaustive survey on the Brunn-Minkowski inequality and [29, Chapter 7] for a thorough analysis of this result. Moreover, there are several extensions and generalizations of this inequality (see [15] and [29, Chapter 9]). Among all of them, here we are interested in the L_p version of (1.1), for $p \geq 1$, which is the cornerstone of the so-called L_p *Brunn-Minkowski theory*.

For the statement of such an extension of the Brunn-Minkowski inequality, we need first to recall the notion of *p-sum*: for any pair of convex bodies containing the origin $K, L \in \mathcal{K}^n_o$, and $p \geq 1$ fixed, the p-sum $K +_p L$ of K and L is the unique convex body $K +_p L \in \mathcal{K}^n_o$ such that

$$h(K +_p L, x) := \big(h^p(K, x) + h^p(L, x)\big)^{1/p} \tag{1.2}$$

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for all $x \in \mathbb{R}^n$, where $h(A,\cdot)\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ denotes the *support function* of a non-empty bounded set A (see Section 2 for its precise definition and main properties). Although the case $p=\infty$ can be interpreted as its limit case, i.e., $h(K+_\infty L,x)=\max\{h(K,x),h(L,x)\}$, thus yielding the convex hull of the union of the bodies K and L, we will omit this case along the manuscript, since for such a value of p all the inequalities (both the classical ones and those we aim to present here) trivially coincide. So, throughout the rest of the paper, when writing $p\geq 1$, we will refer to a real number $p\geq 1$. Further, for any $\lambda>0$, the p-scalar multiplication defined by

$$\lambda \cdot_p K := \lambda^{1/p} K \tag{1.3}$$

is also considered, which, in terms of its support function is expressed by $h^p(\lambda\cdot_p K,\cdot)=\lambda h^p(K,\cdot)$. Though this notion clearly depends on p, we use the notation $\cdot$ (instead of $\cdot_p$) throughout this paper whenever such p-scalar multiplication is employed jointly with the p-sum $+_p$. Furthermore, given $\lambda,\mu>0$, we shall write $\lambda\cdot K+_p\mu\cdot L$ for $(\lambda\cdot K)+_p(\mu\cdot L)$. Notice also that the case $p=1$ recovers the classical linear combination $\lambda K+\mu L$ (due to the well-known fact that the support function is Minkowski additive).

Altogether, we get to the following result, originally shown by Firey [12] (cf. also [29, Corollary 9.1.5]), which establishes the L_p Brunn-Minkowski inequality (later proven by Lutwak, Yang and Zhang [25] in the case of arbitrary non-empty compact sets, by using a more general definition of p-sum that coincides with the original one by Firey when considering convex bodies containing the origin).

Theorem B. *Let $p\geq 1$, and let $K,L\in\mathcal{K}^n_o$ be two convex bodies containing the origin. The inequality*

$$\mathrm{vol}_n\big((1-\lambda)\cdot K+_p\lambda\cdot L\big)^{p/n}\geq(1-\lambda)\mathrm{vol}_n(K)^{p/n}+\lambda\mathrm{vol}_n(L)^{p/n} \tag{1.4}$$

holds for all $\lambda\in(0,1)$.

Around three decades after the definition by Firey of the p-sum of convex bodies containing the origin, Lutwak [23, 24] initiated a systematic and thorough study of p-additions and their implications. This novel and remarkable extension of the classical Brunn-Minkowski theory, in the literature commonly referred to as the L_p *Brunn-Minkowski theory*, is not only a very fruitful area of research nowadays, but it has supposed to be the starting point for new generalizations and contributions. We refer the reader to [29, Section 9.1] for further information on the L_p *Brunn-Minkowski theory* and its aftermath.

Our main goal in this paper is the investigation of possible *refinements* of inequality (1.4). To this aim notice first that, since

$$(1-\bar\lambda)\cdot K+_p\bar\lambda\cdot L=(1-\lambda)\cdot\overline{K}+_p\lambda\cdot\overline{L} \tag{1.5}$$

for any $\lambda,\lambda_1,\lambda_2\in(0,1)$, where $\bar\lambda:=(1-\lambda)\lambda_1+\lambda\lambda_2$, $\overline{K}:=(1-\lambda_1)\cdot K+_p\lambda_1\cdot L$ and $\overline{L}:=(1-\lambda_2)\cdot K+_p\lambda_2\cdot L$ (cf. (1.2) and (1.3)), inequality (1.4) ensures that the function

$$\lambda\mapsto\mathrm{vol}_n\big((1-\lambda)\cdot K+_p\lambda\cdot L\big)$$

is (p/n)-concave on the whole interval $(0,1)$ (recall that a non-negative function f is said to be α-concave, for some $\alpha>0$, if f^α is concave on its domain, which is assumed to be convex). So, considering Jensen's inequality for means, it is clear that if a certain non-negative function f is α-concave for some $\alpha>0$, and $0<\beta<\alpha$, then it is β-concave. Therefore, here we aim to enhance the degree of concavity p/n provided by (1.4), when dealing with a suitable subfamily of convex bodies (since, in general, the exponent p/n cannot be improved, due to the non-trivial equality conditions of Theorem B).

In the classical setting of the Minkowski addition (i.e., when $p = 1$), and in the spirit of finding assumptions that allow us to improve the concavity of the volume functional, the following well-known result (which can be found in [2, 5]) asserts that it is enough to consider convex bodies with a common projection onto a hyperplane.

Theorem C. *Let $K, L \in \mathcal{K}^n$ be two convex bodies such that $P_{u^\perp}(K) = P_{u^\perp}(L)$, for some $u \in \mathbb{S}^{n-1}$. The inequality*

$$\mathrm{vol}_n\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}_n(K) + \lambda \mathrm{vol}_n(L) \tag{1.6}$$

holds for all $\lambda \in (0,1)$.

This result, sometimes stated in terms of a condition on a maximal volume section by a hyperplane, instead of a common projection onto it, goes back to Bonnesen [4]. We also refer to [26] for further extensions and details about (1.6), as well to [29, Section 7.7], where several engaging results in that direction are presented when one deals with a so-called *canal class* (namely, a family of convex bodies having a common projection onto a hyperplane). Recent developments about linear refinements of Brunn-Minkowski type inequalities in the geometric and functional setting can be found in [10, 11, 19, 20, 28, 30], whereas further recent applications of canal classes to other convex geometric inequalities and problems stated in [1, 13, 14, 16] can be found in [21, 22].

Taking into account the above-mentioned linear refinement of the Brunn-Minkowski inequality, it is natural to wonder about a possible extension of such a result to the L_p setting. The point at issue in this regard is trying to figure out the suitable "degree of concavity" of the function $\lambda \mapsto \mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)$. To be more precise, we pose the following question:

Question 1.1. *Given $p \geq 1$, for which value of $\alpha = \alpha(n,p) > 0$ does the inequality*

$$\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^\alpha \geq (1-\lambda)\mathrm{vol}_n(K)^\alpha + \lambda \mathrm{vol}_n(L)^\alpha \tag{1.7}$$

hold for all $\lambda \in (0,1)$, provided that $K, L \in \mathcal{K}^n_o$ are convex bodies containing the origin for which $P_{u^\perp}(K) = P_{u^\perp}(L)$ for some $u \in \mathbb{S}^{n-1}$?

For sure, in order that such a value of α provides us with something significant, it must be not smaller than p/n, because of Theorem B, and moreover not less than 1, due to Theorem C jointly with the well-known inclusion

$$(1-\lambda)\cdot K +_p \lambda \cdot L \supset (1-\lambda)K + \lambda L$$

(see e.g. [12]). Furthermore, in Section 4 first we show that α cannot be bigger than p. Thus, altogether, we already know that we are searching for α's lying in the range $\big[\max\{1, p/n\}, p\big]$.

There are various indications to figure out the "right choice" for α in the previous question (we will discuss this in further detail at the beginning of Section 4). Amongst them, we have that one may interpret the statement of Theorem C as that the volume functional "behaves" as the one-dimensional volume (cf. (1.1) for $n = 1$) under the additional assumption, for the bodies there involved, of a common projection onto a hyperplane. Hence, from the case $n = 1$ in Theorem B, it seems natural to conjecture that $\alpha = p$ is the suitable answer to Question 1.1. In fact, if that was the case, such a statement (cf. (1.7) with $\alpha = p$) would be not only the best one can expect but also an extension of Theorem C to the L_p setting (since the value of $p = 1$ recovers the Minkowski sum). Unfortunately, this fact is in general not true:

Theorem 1.2. *Let $p \geq 1$. Then there exist two convex bodies containing the origin $K, L \in \mathcal{K}_o^n$ with $P_{u^\perp}(K) = P_{u^\perp}(L)$, for some $u \in \mathbb{S}^{n-1}$, and such that the inequality*

$$\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^p \geq (1-\lambda)\mathrm{vol}_n(K)^p + \lambda \mathrm{vol}_n(L)^p.$$

does not hold for all $\lambda \in (0,1)$.

Once we know that $\alpha = p$ is not the right answer to Question 1.1, we need to find out which value of α could still provide us with a positive solution to this issue. The following result shows that $\alpha = (n+p-1)/n$ is valid in this regard. At this point, we would like to notice that this inequality gives a refinement of (1.4), since $(n+p-1)/n \in [\max\{1, p/n\}, p)$, and further that it is an extension of Theorem C (because such a value of α is precisely 1 whenever $p = 1$).

Theorem 1.3. *Let $p \geq 1$ and $\alpha = (n+p-1)/n$, and let $K, L \in \mathcal{K}_o^n$ be two convex bodies containing the origin such that $P_{u^\perp}(K) = P_{u^\perp}(L)$, for some $u \in \mathbb{S}^{n-1}$. The inequality*

$$\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^\alpha \geq (1-\lambda)\mathrm{vol}_n(K)^\alpha + \lambda \mathrm{vol}_n(L)^\alpha$$

holds for all $\lambda \in (0,1)$.

The factor $1/n$ appearing in $1 + (p-1)/n$ (namely, the above-mentioned valid value for α) is still a bit mysterious for us. However, as we will show in Section 4, whenever α is a solution to Question 1.1 then $(\alpha - 1)/(p-1)$ is of the order of $1/n$ (as $n \to \infty$). So, at least in this asymptotic sense, the value $1 + (p-1)/n$ seems to be the best possible one can expect for Question 1.1.

Anyway, with the additional cost of adding a suitable multiplicative constant (greater than 1) on the left-hand side of (1.7), one can still get the value $\alpha = p$. This is the content of the following result.

Theorem 1.4. *Let $p \geq 1$, and let $K, L \in \mathcal{K}_o^n$ be two convex bodies containing the origin such that $P_{u^\perp}(K) = P_{u^\perp}(L)$, for some $u \in \mathbb{S}^{n-1}$. The inequality*

$$C(n,p,\lambda)\,\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda \cdot L\big) \geq \big((1-\lambda)\mathrm{vol}_n(K)^p + \lambda \mathrm{vol}_n(L)^p\big)^{1/p}$$

holds for all $\lambda \in (0,1)$, where

$$C(n,p,\lambda) = \min\left\{ n\left[1 - \left(1 - \frac{1}{n}\right)^p\right]^{1/p}, \min\big\{(1-\lambda), \lambda\big\}^{-(n-1)/p} \right\}.$$

Given $\lambda \in (0,1)$, we observe on the one hand that $C(n,1,\lambda) = 1$, which fits well with Theorem C, and on the other hand that $C(n,p,\lambda) \to 1$ as $p \to \infty$. Hence, (for a fixed dimension n and) for values of p either sufficiently close to 1 or large enough, the above result in a sense tells us that $\alpha = p$ is not that far away from being a feasible solution to Question 1.1.

The paper is organized as follows. Section 2 is mainly devoted to recalling some definitions and auxiliary well-known results, whereas in Section 4 we deal (from among other results) with the proofs of Theorems 1.2 and 1.3. To conclude, in Section 5 we discuss and show some other refined versions of certain related inequalities in the L_p setting (such as the corresponding Minkowski first inequality), with the additional aim of finally showing Theorem 1.4.

2. Background and Preliminary Results

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, endowed with the usual euclidean norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. We denote by $\{e_1,\cdots,e_n\}$ the standard canonical orthonormal basis in $\mathbb{R}^n$, and by $\mathrm{span}\{X\}$ the smallest linear subspace of $\mathbb{R}^n$ containing the subset $X\subset\mathbb{R}^n$.

Let $1\le p\le+\infty$, we denote by $1\le q\le+\infty$ the Hölder's conjugate of p, i.e., the real number (we set $q:=+\infty$ if $p=1$) such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

We recall that $\mathcal{K}^n_o$ stands the subfamily of $\mathcal{K}^n$ of all convex bodies containing the origin. Let $C\in\mathcal{K}^n$ is a convex body contained in a linear hyperplane $u^\perp$, for some direction $u\in\mathbb{S}^{n-1}$. The family of convex bodies having the same projection $C\subset u^\perp$ onto a given hyperplane $u^\perp$, $u\in\mathbb{S}^{n-1}$, is known as *canal class* of C, see [29, Section 7.7]. For our purposes in this paper, we will consider only convex bodies in the canal class containing the origin, setting thus $\mathcal{K}_C:=\{K\in\mathcal{K}^n_o:\ P_{u^\perp}(K)=C\}$ to be the canal class with respect to $u\in\mathbb{S}^{n-1}$ and $C\subset u^\perp$.
For $K\in\mathcal{K}^n$, its dimension is the dimension of its affine hull, i.e., $\dim K=\dim\mathrm{aff}(K)$. We denote by $\mathrm{vol}_n(K)$ the n-dimensional volume of $K\in\mathcal{K}^n$, i.e., its n-dimensional Lebesgue measure, and when $\dim(K)=i<n$, the i-dimensional Lebesgue measure is denoted also by $\mathrm{vol}_i(K)$.
We denote by $C^{2,+}$ the set of all convex bodies K with ∂K of class C^2 and the Gauss curvature strictly positive at every $x\in\partial K$, see [29, Section 2.5]. The space of convex bodies $\mathcal{K}^n$ is endowed with the Hausdorff metric [29, Section 1.8], which makes it a complete metric space. From now on, any topological notion in $\mathcal{K}^n$ is implicitly considered with respect to the Hausdorff metric.

2.1. Background in the L_p Brunn-Minkowski theory.

Let $K\in\mathcal{K}^n$, then the function $h(K,\cdot)\colon\mathbb{S}^{n-1}\to\mathbb{R}$ defined as $h(K,u)=\max\{\langle x,u\rangle:\ x\in K\}$ is called the support function of K.
We extend the support function to $\mathbb{R}^n$, $h(K,\cdot)\colon\mathbb{R}^n\to\mathbb{R}$, as a positively 1-homogeneous function, i.e.,

$$h(K,\lambda x)=\lambda h(K,x),$$

for every $\lambda\ge0$ and $x\in\mathbb{R}^n$. Once extended in this way the support function is subadditive; hence $h(K,\cdot)$ is convex in $\mathbb{R}^n$, see [29, Section 1.7.1]. We will denote with the same notation both the support function defined on $\mathbb{S}^{n-1}$ and its extension in $\mathbb{R}^n$.
The subadditivity and the 1-homogeneity properties characterize support functions of convex bodies, as the following theorem states.

Theorem 2.1. [29, Theorem 1.7.1] *If $f\colon\mathbb{R}^n\to\mathbb{R}$ is a positively 1-homogenoeus and subadditive function in $\mathbb{R}^n$, then there exists a unique convex body $K\in\mathcal{K}^n$, such that $f(x)=h(K,x)$, for every $x\in\mathbb{R}^n$.*

It is important to remark here that the support function of a convex body K is non-negative if and only if K contains the origin, and its restriction to the unit sphere is strictly positive if and only if the origin is an interior point opf K.

The p-sum of two convex bodies in $\mathcal{K}^n_0$ is defined as follows.

Definition 2.2. *Let $K, L \in \mathcal{K}_o^n$ be two convex bodies containing the origin and $p \geq 1$. We define the p-sum between K and L as the unique convex body $K +_p L \in \mathcal{K}_o^n$ such that*

$$h(K +_p L, x)^p := h^p(K, x) + h^p(L, x),$$

for all $x \in \mathbb{R}^n$. For $\lambda \geq 0$, the body $\lambda \cdot K \in \mathcal{K}_o^n$ is defined as $\lambda \cdot K := \lambda^{1/p} K$, i.e.

$$h^p(\lambda \cdot K, x) = \lambda h^p(K, x),$$

for all $x \in \mathbb{R}^n$.

We remark that Definition 2.2 is well posed since $(h^p(K, \cdot) + h^p(L, \cdot))^{1/p}$ is a non-negative support function by condition $p \geq 1$. As remarked in the introduction, the case $p = +\infty$ is omitted since for such a value of p all the inequalities (both the classical ones and those we aim to present here) trivially coincide. We recall that we use the notation $\cdot$ (instead of $\cdot_p$) throughout this paper whenever such p-scalar multiplication is employed jointly with the p-sum $+_p$.

In this framework, a natural way to explore possible refinements is considering the α-mean of two non-negative real numbers.

Definition 2.3. *Let $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, $\lambda \in [0, 1]$ and a, b non-negative real numbers. For $ab \neq 0$, we define the α-mean of a and b with weight λ as*

$$\mathcal{M}_\alpha^\lambda(a, b) = \begin{cases} ((1-\lambda)a^\alpha + \lambda b^\alpha)^{\frac{1}{\alpha}}, & if \ \alpha \in \mathbb{R} \setminus \{0\}, \\ a^{1-\lambda} b^\lambda, & if \ \alpha = 0, \\ \max\{a, b\}, & if \ \alpha = +\infty, \\ \min\{a, b\}, & if \ \alpha = -\infty. \end{cases}$$

If $ab = 0$, then $M_\alpha(a, b; \lambda) = 0$, for every α.

Notice that $\alpha = 1$ provides us with the so-called weighted arithmetic mean, the case $\alpha = 0$, with the weighted geometric mean and $\alpha = \frac{1}{n}$, with the weighted harmonic mean.

Using the α-mean, allows us to write the Brunn-Minkowski and the L_p Brunn-Minkowski inequalities as follows.

Theorem 2.4. *If $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, then the L_p Brunn-Minkowski inequality reads as*

$$\mathrm{vol}_n\big((1-\lambda) \cdot K +_p \lambda \cdot L\big) \geq \mathcal{M}_{\frac{p}{n}}^\lambda(\mathrm{vol}_n(K), \mathrm{vol}_n(L)). \tag{2.1}$$

Note that the case $p = 1$ covers the standard Brunn-Minkowski inequality (1.1) for convex bodies $K, L \in \mathcal{K}_o^n$.

We remark that we are going to use along the paper the notation of the α-mean, i.e., $\mathcal{M}_\alpha^\lambda(a, b)$, whenever it occurs for practice and space reason, but we are going to state and discuss refinements of the L^p Brunn-Minkowski ineqauility with and without the notation of the α-mean.

The following standard properties of means of non-negative real numbers happen to play a crucial role when looking for refinements of the Brunn-Minkowski and L_p Brunn-Minkowski inequalities.

Lemma 2.5. [18, Section 2.9] *Let $-\infty \leq \alpha < \beta \leq +\infty$. For every $a, b \in \mathbb{R}$, we have*

$$\mathcal{M}_\alpha^\lambda(a, b) \leq \mathcal{M}_\beta^\lambda(a, b), \tag{2.2}$$

with equality if and only if either $a = b$ or $ab = 0$.

Inequality (2.2) provides us with monotonicity of means, in particular, it recovers the classical arithmetic-geometric mean inequality.

The next inequality is a product property of means. I am not sure that we need this lemma

Lemma 2.6. [18, Section 2.9] *Let a, b be non-negative real numbers and $\lambda \in [0,1]$. In the next we set $\mathcal{M}_\alpha := \mathcal{M}_\alpha^\lambda(a,b)$. The following inequality*

$$\mathcal{M}_\beta^\beta \leq (\mathcal{M}_\alpha)^{\alpha\frac{\gamma-\beta}{\gamma-\alpha}} (\mathcal{M}_\gamma)^{\gamma\frac{\beta-\alpha}{\gamma-\alpha}}, \tag{2.3}$$

holds true for every $0 < \alpha < \beta < \gamma$, with equality if and only if either $a = b$ or $ab = 0$.

By the monotonicity property of the α-means we have also the following.

Lemma 2.7. *If $K, L \in \mathcal{K}_o^n$ are convex bodies containing the origin and $1 \leq p \leq q$, then we have*

$$(1-\lambda)\cdot K +_p \lambda \cdot L \subseteq (1-\lambda)\cdot K +_q \lambda \cdot L,$$

for every $0 \leq \lambda \leq 1$. At the same time we have also

However we have also the following property regarding the p-sum of two convex bodies.

Lemma 2.8. *If $K, L \in \mathcal{K}_o^n$ are convex bodies containing the origin and $1 \leq p \leq q$, then we have*

$$K +_q L \subseteq K +_p L.$$

In [25], the authors extended the notion of p-sum to compact sets of $\mathbb{R}^n$.

Definition 2.9. *Let $p > 1$ and $K, L \subseteq \mathbb{R}^n$ be compact sets. We define*

$$K +_p L := \{(1-t)^{\frac{1}{q}} x + t^{\frac{1}{q}} y : \ x \in K, \ y \in L, \ t \in [0,1]\},$$

where $q \geq 1$ stands for the Hölder conjugate of p.

In [25], the authors proved that Definition 2.9 coincides with Definition 2.2 in the case of $K, L \in \mathcal{K}_o^n$. We remark also that the authors in [25] extended the L_p Brunn-Minkowski inequality (1.4) to compact sets (see [25, Theorem 4].

the next remark concerns the behavior of p-sum and the orthogonal projection. In particular, it shows that the p-convex combination of two convex bodies in $\mathcal{K}_o^n$ which are in the same canal class, is in the same canal class.

Remark 2.10. *Let $u \in \mathbb{S}^{n-1}$. We have*

$$P_{u^\perp}(K +_p L) = P_{u^\perp}(K) +_p P_{u^\perp}(L)$$

for every $K, L \in \mathcal{K}_o^n$.

Proof. We use Definition 2.9. Let $z \in P_{u^\perp}(K) +_p P_{u^\perp}(L)$, hence there exist $x \in P_{u^\perp}(K)$, $y \in P_{u^\perp}(L)$ and $t \in [0,1]$ such that $z = (1-t)^{\frac{1}{q}} x + t^{\frac{1}{q}} y$. Moreover, there are $x' \in K$, $y' \in L$ and $\mu, \omega \in \mathbb{R}$ such that $x' = x + \omega u$ and $y' = y + \mu u$. Hence, we have

$$\begin{aligned} z &= (1-t)^{\frac{1}{q}}(x' - \omega u) + t^{\frac{1}{q}}(y' - \mu u) \\ &= (1-t)^{\frac{1}{q}} x' + t^{\frac{1}{q}} y' - u((1-t)^{\frac{1}{q}}\omega + t^{\frac{1}{q}}\mu), \end{aligned}$$

which tells us that $z \in P_{u^\perp}(K +_p L)$.

Vice versa, let $z \in P_{u^\perp}(K +_p L)$, then there are $z' \in K +_p L$, and $s \in \mathbb{R}$ such that $z' = z + su$. Moreover, we have $z' = (1-t)^{\frac{1}{q}} x' + t^{\frac{1}{q}} y'$, for some $t \in [0,1]$, $x' \in K$ and $y' \in L$, i.e., $z = (1-t)^{\frac{1}{q}} x' + t^{\frac{1}{q}} y' - su$.

There also exist $\omega, \mu \in \mathbb{R}$ such that $x = x' - \omega u \in P_{u^\perp}(K)$ and $y = y' - \mu u \in P_{u^\perp}(L)$, hence

$$z = (1-t)^{\frac{1}{q}}(x' - \omega u) + t^{\frac{1}{q}}(y' - \mu u),$$

since $s = (1-t)^{\frac{1}{q}}\omega + t^{\frac{1}{q}}\mu$.
Hence, we have $z \in P_{u^\perp}(K) +_p P_{u^\perp}(L)$. □

Corollary 2.11. *Let $p \geq 1$. Let $K, L \in \mathcal{K}_{C,o}$, with $C \subseteq u^\perp$ a convex body containing the origin, and $u \in \mathbb{S}^{n-1}$. The following*

$$(1-\lambda) \cdot K +_p \lambda \cdot L \in \mathcal{K}_{C,o}$$

holds for every $\lambda \in (0,1)$.

In [14] the authors stated a formula contained in the following result, for the volume of a p-sum of convex bodies whose addends lie in orthogonal linear subspaces.

Theorem 2.12. [14] *Let $1 \leq n_1, n_2 \leq n-1$ be such that $n_1 + n_2 \leq n$. Let K and L be convex bodies containing the origin such that $K \subseteq \operatorname{span}\{e_1, \cdots, e_{n_1}\}$ and $L \subseteq \operatorname{span}\{e_{n_1+1}, \cdots, e_{n_1+n_2}\}$. We have*

$$\operatorname{vol}_{n_1+n_2}(K +_p L) = \frac{\Gamma\left(\frac{n_1}{q}+1\right)\Gamma\left(\frac{n_2}{q}+1\right)}{\Gamma\left(\frac{n_1+n_2}{q}+1\right)} \operatorname{vol}_{n_1}(K)\operatorname{vol}_{n_2}(L), \tag{2.4}$$

where $\Gamma(\cdot)$ stands for the Gamma function.

The following result is an application of formula (2.4), that will be useful later.

Corollary 2.13. *Let $p \geq 1$. Let $u \in \mathbb{S}^{n-1}$ a direction, $C \subseteq u^\perp$ a convex body containing the origin such that $C \subseteq u^\perp$. We have*

$$\operatorname{vol}_n(C +_p s \cdot [-u, u]) = 2s^{\frac{1}{p}} D(n,p)\operatorname{vol}_{n-1}(C) \tag{2.5}$$

for every $s > 0$, where $D(n,p) = \frac{\Gamma\left(\frac{n-1}{q}+1\right)\Gamma\left(\frac{1}{q}+1\right)}{\Gamma\left(\frac{n}{q}+1\right)}$.

We introduce the notion of Steiner symmetrization, which will be used in the last lemma of this section. The notion of Schwarz and Steiner symmetrizations are also related to the extension of refinements of Brunn-Minkowski type inequalities under the assumption of equal $(n-1)$ Lebesgue without - ? measure of the projections, see Section 4.5.

We start defining the Schwarz symmetrization, and introduce the Steiner symmetrization as a particular case of the latter.

Definition 2.14. [17, Section 9.3] *Let $K \in \mathcal{K}^n$ and $1 \leq k \leq n-1$. Let H be a $(n-k)$-dimensional subspace of $\mathbb{R}^n$, with orthogonal complement $H^\perp$. For any $y \in P_H(K)$, let $B_k(y, r_k) \subseteq y + H^\perp$ be the k-dimensional ball centered at y with radius r_k such that*

$$\operatorname{vol}_k(B_k(y, r_k)) = \operatorname{vol}_k(K \cap (y + H^\perp)).$$

The Schwarz symmetral of K, with respect to H, is defined as

$$S_H(K) = \bigcup_{y \in P_H(K)} B_k(y, r_k).$$

Lemma 2.15. [17] *Let $K, L \in \mathcal{K}^n$ be two convex bodies and H a $(n-k)$-dimensional subspace of $\mathbb{R}^n$, with $1 \leq k \leq n-1$. The following statements hold.*

i) *$S_H(K)$ is a convex body.*

ii) *If* $L \subseteq K$, *then* $S_H(L) \subseteq S_H(K)$.
iii) $\mathrm{vol}(K) = \mathrm{vol}(S_H(K))$.
iv) $P_H(K) \subseteq S_H(K)$.
v) $S_H(K) + S_H(L) \subseteq S_H(K+L)$.

We recall also the Steiner symmetrization, which is the Schwarz symmetrization taking $k=1$ in Definition 2.14.

Definition 2.16 (Steiner Symmetrization)**.** [29] *Let* $K \in \mathcal{K}^n$ *be a convex body and* $u \in \mathbb{S}^{n-1}$ *a direction. The Steiner symmetral of* K *with respect to* $u^\perp$ *is defined as*

$$S_{u^\perp}(K) = \bigcup_{x \in P_{u^\perp}(K)} \left[x - \frac{1}{2}\mathrm{vol}_1(K \cap (x + L_u))u, x + \frac{1}{2}\mathrm{vol}_1(K \cap (x+L_u))u \right], \tag{2.6}$$

where L_u *stands for the linear* 1*-dimensional subspace of* $\mathbb{R}^n$ *spanned by* u.

We present now the behaviour of the Steiner and the Schwarz symmetrizations with respect to the p-sum.

Theorem 2.17. *Let* $K, L \in \mathcal{K}_o^n$, $u \in \mathbb{S}^{n-1}$ *and* $p > 1$. *We have*

$$S_{u^\perp}(K) +_p S_{u^\perp}(L) \subseteq S_{u^\perp}(K +_p L). \tag{2.7}$$

Proof. Let $z \in S_{u^\perp}(K) +_p S_{u^\perp}(L)$, then there exist $t \in (0,1)$, $x \in S_{u^\perp}(K)$ and $y \in S_{u^\perp}(L)$ such that $z = (1-t)^{\frac{1}{q}}x + t^{\frac{1}{q}}y$.

There exist $x' \in P_{u^\perp}(K)$ and $y' \in P_{u^\perp}(L)$ such that $x = x' + l_x$ and $y = y' + l_y$, where

$$|l_x| \leq \frac{1}{2}\mathrm{vol}_1(K \cap (x' + L_u)), \qquad |l_y| \leq \frac{1}{2}\mathrm{vol}_1(L \cap (y' + L_u))$$

and $L_u = lin(u)$.
Hence we have

$$\begin{aligned} z &= (1-t)^{\frac{1}{q}}x + t^{\frac{1}{q}}y \\ &= (1-t)^{\frac{1}{q}}x' + t^{\frac{1}{q}}y' + (1-t)^{\frac{1}{q}}l_x + t^{\frac{1}{q}}l_y. \end{aligned}$$

Now we notice that $z' := (1-t)^{\frac{1}{q}}x' + t^{\frac{1}{q}}y' \in P_{u^\perp}(K) +_p P_{u^\perp}(L) = P_{u^\perp}(K +_p L)$.
Moreover if we take $l_z := (1-t)^{\frac{1}{q}}l_x + t^{\frac{1}{q}}l_y$, then we have $|l_z| \leq \frac{1}{2}\mathrm{vol}_1((K +_p L) \cap (z' + L_u))$.
Indeed

$$\begin{aligned} |l_z| &\leq (1-t)^{\frac{1}{q}}|l_x| + t^{\frac{1}{q}}|l_y| \\ &\leq \frac{(1-t)^{\frac{1}{q}}}{2}\mathrm{vol}_1(K \cap (x' + L_u)) + \frac{t^{\frac{1}{q}}}{2}\mathrm{vol}_1(L \cap (y' + L_u)) \\ &= \frac{1}{2}\mathrm{vol}_1((1-t)^{\frac{1}{q}}(K \cap (x' + L_u))) + \frac{1}{2}\mathrm{vol}_1(t^{\frac{1}{q}}(L \cap (y' + L_u))). \end{aligned}$$

We prove now that $t^{\frac{1}{q}}(L \cap (y' + L_u)) \subseteq t^{\frac{1}{q}}L \cap (t^{\frac{1}{q}}y' + L_u)$. Let $s \in t^{\frac{1}{q}}(K_2 \cap (y' + L_u))$, i.e., $s = t^{\frac{1}{q}}s'$, with $s' \in L \cap (y' + L_u)$.
Since $s' \in L$, we have that $s \in t^{\frac{1}{q}}L$. Moreover $s' \in y' + L_u$, i.e., $s' = \lambda u + y'$ for some $\lambda \in \mathbb{R}$, hence $s' = t^{\frac{1}{q}}y' + \lambda t^{\frac{1}{q}}u \in t^{\frac{1}{q}}y' + L_u$.

Analog for $(1-t)^{\frac{1}{q}}(K \cap (x' + L_u)) \subseteq (1-t)^{\frac{1}{q}}K \cap ((1-t)^{\frac{1}{q}}x' + L_u)$.

Hence we have

$$|l_z| \leq \frac{1}{2}\mathrm{vol}_1((1-t)^{\frac{1}{q}}K \cap ((1-t)^{\frac{1}{q}}x' + L_u)) + \frac{1}{2}\mathrm{vol}_1(t^{\frac{1}{q}}L \cap (t^{\frac{1}{q}}y' + L_u)).$$

Moreover, we know, with the approach followed by the authors in [25], that

$$((1-t)^{\frac{1}{q}}K \cap ((1-t)^{\frac{1}{q}}x' + L_u)) + (t^{\frac{1}{q}}L \cap (t^{\frac{1}{q}}y' + L_u)) \subseteq (K +_p L) \cap (z' + L_u),$$

i.e.,

$$|l_z| \leq \frac{1}{2}\mathrm{vol}_1((K +_p L) \cap (z' + L_u)),$$

which means that $z \in S_{u^\perp}(K +_p L)$.

□

Refining the argument that led to the "Sphericity Theorem of Gross", see [17, Corollary 9.1] we can extrapolate a sequence of Steiner symmetrals that converges to the Schwarz symmetral of the body.

Theorem 2.18. [17] *Let $1 \leq k \leq n-1$ and $H \subseteq \mathbb{R}^n$ be a $(n-k)$-dimensional subspace of $\mathbb{R}^n$. There exists a sequence $(u_j)_{j\in\mathbb{N}} \subseteq \mathbb{S}^{n-1} \cap H$ of directions in H, such that for any $K \in \mathcal{K}^n_{(0)}$ the sequence $S_{u_j^\perp}(\cdots(S_{u_1^\perp}(K)))$ converges to $S_H(K)$, as $j \to \infty$, with respect to Hausdorff metric, i.e., $S_H(K)$, the Schwarz symmetral of K with respect to H, is the limit of a sequence of Steiner symmetrizations of K.*

The previous convergence result provides us with the same result of Theorem 2.17 for the Schwarz symmetrization.

Theorem 2.19. *Let $K, L \in \mathcal{K}^n_o$, $1 \leq k \leq n-1$ and $p > 1$. Let $H \subseteq \mathbb{R}^n$ be a $(n-k)$-dimensional subspace of $\mathbb{R}^n$. We have*

$$S_H(K) +_p S_H(L) \subseteq S_H(K +_p L). \tag{2.8}$$

Proof. By Theorem 2.18 there is a sequence of directions in H, i.e., $(u_j)_{j\in\mathbb{N}} \subseteq \mathbb{S}^{n-1} \cap H$ such that

$$\lim_{j\to\infty} S_{u_j^\perp}(\cdots(S_{u_1^\perp}(K))) = S_H(K), \quad \lim_{j\to\infty} S_{u_j^\perp}(\cdots(S_{u_1^\perp}(L))) = S_H(L)$$

and

$$\lim_{j\to\infty} S_{u_j^\perp}(\cdots(S_{u_1^\perp}(K +_p L))) = S_H(K +_p L)$$

where the convergence is considered with respect to the Hausdorff metric.

By Theorem 2.17 we have

$$S_{u_j^\perp}(K) +_p S_{u_j^\perp}(L) \subseteq S_{u_j^\perp}(K +_p L),$$

for every $j \in \mathbb{N}$. Passing to the limit we have the desired inclusion. □

We state the following result, concerning the approximation of convex bodies belonging to the same canal class by $C^{2,+}$ convex bodies.

Lemma 2.20. *Let $u \in \mathbb{S}^{n-1}$ be a direction and $C \in \mathcal{K}^n$ an $(n-1)$-dimensional convex body containing the origin, such that $C \subset u^\perp$. Let $K, L \in \mathcal{K}_{C,o}$.*

There exist two sequences $\{K_i\}_{i\in\mathbb{N}}$ and $\{L_i\}_{i\in\mathbb{N}}$ of convex bodies such that:

i) $K_i, L_i \in \mathcal{K}^n_o \cap C^{2,+}$ *and* $\lim_{i\to+\infty} K_i = K$, $\lim_{i\to+\infty} L_i = L$;

ii) K_i *and* L_i *belong to the same canal class, for every* $i \in \mathbb{N}$.

Proof. Note that

$$h(K,w) = h(L,w) \quad \forall\, w \in u^{\perp} \cap \mathbb{S}^{n-1}. \tag{2.9}$$

In the first part of the proof will show that we may reduce to the case when $h(K,\cdot) = h(L,\cdot)$ in an open neighborhood of $u^{\perp} \cap \mathbb{S}^{n-1}$.

Let $r > 0$ and consider

$$K_r = \mathrm{conv}(K \cup (1+r)C),$$

where "conv" denotes the convex hull. Then (see [29])

$$h(K_r,\cdot) = \max\{h(K,\cdot),(1+r)h(C,\cdot)\}.$$

In particular, by (2.9)

$$(1+r)h(C,w) > h(K,w) \quad \forall\, w \in u^{\perp} \cap \mathbb{S}^{n-1}.$$

By the continuity of the support functions, we get that the previous inequality holds in an open subset Ω of $\mathbb{S}^{n-1}$ containing $u^{\perp} \cap \mathbb{S}^{n-1}$. Thus

$$h(K_r,w) = (1+r)h(C,w) \quad \forall\, w \in \Omega.$$

Repeating the previous argument for L, we obtain that there exists some open subset Ω' of $\mathbb{S}^{n-1}$, containing $u^{\perp} \cap \mathbb{S}^{n-1}$, such that

$$h(K_r,w) = h(L_r,w) \quad \forall\, w \in \Omega'.$$

As the families of convex bodies K_r and L_r converges to K and L, respectively, as $r \to 0^+$, by a diagonal argument it is sufficient to show that K_r and L_r can be approximated as indicated in the statement.

Therefore we may assume that

$$h(K,w) = h(L,w) \quad \forall\, w \in \Omega'$$

where Ω' is an open subset of $\mathbb{S}^{n-1}$ containing $u^{\perp} \cap \mathbb{S}^{n-1}$. In particular, there exists $\epsilon > 0$ such that $h(K,w) = h(L,w)$ for every $w \in \mathbb{S}^{n-1}$ such that

$$|(w,u)| \le \epsilon.$$

We now approximate K and L using a standard convolution argument, which can be found for instance in [8]. Following the notations in [8, Appendix A], let $\mathbf{O}(n)$ be the group of rotations of $\mathbb{R}^n$, endowed with the Haar probability measure ν. We define

$$h(K_i,v) = \int_{\mathbf{O}(n)} h(K,\rho v)\omega_i(\rho)\nu(d\rho), \quad \forall\, v \in \mathbb{S}^{n-1}$$

where $\omega_i \colon \mathbf{O}(n) \to [0,+\infty)$ is a sequence of mollifiers over $\mathbf{O}(n)$. As observed in [8, Appendix A], for every $i \in \mathbb{N}$, $h(K_i,\cdot)$, $i \in \mathbb{N}$, is the support function of a convex body K_i of class $C^{2,+}$, and the sequence K_i converges to K. A similar construction can be repeated for L, obtaining a sequence L_i.

We show that the bodies K_i and L_i belong to the same canal class, for every i sufficiently large. The support of ω_i is contained in a neighborhood of radius $1/i$ of the identity (see [8, Appendix A]). Hence there exists i_0 such that for every $i \ge i_0$ and for every $v \in u^{\perp}$,

$$|\langle(v,\rho(v)\rangle| > \epsilon \quad \text{implies} \quad \rho(v) = 0.$$

Therefore, for $v \in u^\perp$ and $i \geq i_0$, we have:

$$\begin{aligned} h(P_{u^\perp}(K_i), v) &= \int_{\mathbf{O}(n)} h(K, \rho v)\omega_i(\rho)\nu(d\rho) \\ &= \int_{\mathbf{O}(n)} h(L, \rho u)\omega_i(\rho)\nu(d\rho) \\ &= h(P_{u^\perp}(L_i), v). \end{aligned}$$

□

3. Functional expression of the volume and its derivatives

This section follows the line and the notations of [3, 6, 7, 8, 9]. Let $u \in \mathbb{S}^{n-1}$ and $K \in \mathcal{K}^n_o$. We focus on the construction of a family of convex bodies via K, belonging to the same canal class $\mathcal{K}_C$, with $C = P_{u^\perp}(K)$. In the following, referring to the support function of K, we write either $h(K, \cdot)$ or h, when there is no ambiguity on K, both defined on $\mathbb{S}^{n-1}$.

We recall that an orthonormal frame on the sphere is a map which associates to every $x \in \mathbb{S}^{n-1}$ an orthonormal basis of the tangent space to $\mathbb{S}^{n-1}$ at x. Let $\phi \in C^2(\mathbb{S}^{n-1})$. We denote by $\phi_i(v)$ and $\phi_{ij}(v)$, $i, j \in \{1, \cdots, n-1\}$, the first and second covariant derivatives of ϕ at $v \in \mathbb{S}^{n-1}$, with respect to a fixed local orthonormal frame on an open subset of $\mathbb{S}^{n-1}$. The following definition will be crucial in the sequel. We set, for $v \in \mathbb{S}^{n-1}$,

$$Q(\phi; v) = (q_{ij})_{i,j=1,\cdots,n-1} = (\phi_{ij}(v) + \phi(v)\delta_{ij})_{i,j=1,\cdots,n-1}; \tag{3.1}$$

here the δ_{ij}'s are the usual Kronecker symbols. Hence $Q(\phi, v)$ is a square matrix of order (n-1) *or* $n-1$*?*. On an occasion, instead of $Q(\phi; v)$ we write $Q(\phi)$. Note that $Q(\phi; v)$ is symmetric by standard properties of covariant derivatives.

We also set

$$C^{2,+}(\mathbb{S}^{n-1}) := \{h \in C^2(\mathbb{S}^{n-1}) : \ Q(h; \cdot) > 0 \text{ on } \mathbb{S}^{n-1}\},$$

where $Q(h; \cdot) > 0$ means that the matrix is positive definite.

Recall that $C^{2,+}$ stands for the set of all convex bodies with boundary of class C^2 with strictly positive Gaussian curvature. The following proposition is a special case of Proposition A.1 in [8].

Proposition 3.1. *$h \in C^{2,+}(\mathbb{S}^{n-1})$ if and only if h is the support function of a convex body $K \in C^{2,+}$.*

We work with specific family of differentiable functions.

Definition 3.2. *Let $p \geq 1$, $s \in \mathbb{R}$. Let $K \in C^{2,+}$ and h be the support function of K, and $\phi \in C^2(\mathbb{S}^{n-1})$. We define the following family of functions depending on s*

$$h_s \colon \mathbb{S}^{n-1} \to \mathbb{R}$$

defined as

$$h_s(v) = (h^p(K, v) + s\ \phi(v))^{\frac{1}{p}}, \tag{3.2}$$

for every $v \in \mathbb{S}^{n-1}$ and for every $s \in \mathbb{R}$ such that (3.2) *is well-defined.*

In a similar manner as in [8, Proposition A.1], see also [29, Section 2.5], we have that, for s small enough, h_s is a support function.

Proposition 3.3. *Let $p \geq 1$. Let $K \in C^{2,+}$ and assume that the origin is an interior point of K. There exists $a > 0$, so that for every $s \in \mathbb{R}$ with $0 < |s| < a$, there exists a unique $L_s \in C^{2,+}$ such that*
$$h_s(v) = h(L_s, v),$$
for every $v \in \mathbb{S}^{n-1}$. Moreover, the origin is an interior point of L_s.

Proof. Let h be the support function of K and. Then $h > 0$ on $\mathbb{S}^{n-1}$ (as the origin is an interior point of K). This implies in particular that there exists $a_1 > 0$ such that $h_s > 0$ on $\mathbb{S}^{n-1}$ if $|s| < a_1$. By the definition (3.2) of h_s, as $h, \phi \in C^2(\mathbb{S}^{n-1})$, in the same range we have that $h_s \in C^2(\mathbb{S}^{n-1})$.

Following the same argument in [8, Proposition A.1] we have that $Q(h_s; \cdot) > 0$ on $\mathbb{S}^{n-1}$ for every $s \in (-a, a)$, for a suitable $a > 0$. Indeed, since $Q(h; \cdot) > 0$ on $\mathbb{S}^{n-1}$, by the compactness of $\mathbb{S}^{n-1}$ we have that there exists $\gamma > 0$ such that
$$Q(h; v) \geq \gamma I_{n-1}, \tag{3.3}$$
for every $v \in \mathbb{S}^{n-1}$, where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Here the inequality $Q(h; \cdot) \geq \gamma I_{n-1}$ means that the matrix $Q(h; \cdot) - \gamma I_{n-1}$ is positive semi-definite or semidefinite?. By the regularity assumptions of h and ϕ, and by (3.3), we have that $Q(h_s; \cdot) > 0$ on $\mathbb{S}^{n-1}$ for every $s \in (-a, a)$.

By Proposition 3.1, for every $s \in (-a, a)$, there exists $L_s \in C^{2,+}$ such that $h_s(\cdot) = h(L_s, \cdot)$ on $\mathbb{S}^{n-1}$. Note that the origin is in the interior of each L_s since $h_s(v) > 0$ for every $v \in \mathbb{S}^{n-1}$. □

The case $p = 1$ is a well-known case, studied for instance in [6, 7, 8, 9]. We remark that for every $p \geq 1$ we have $h_0 = h = h(K, \cdot)$.

3.1. The volume functional and its derivatives.

Let $F\colon C^{2,+}(\mathbb{S}^{n-1}) \to [0, +\infty)$ be defined as
$$F(g) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} g(v) \det(Q(g; v)) dv,$$
for $g \in C^{2,+}(\mathbb{S}^{n-1})$. Note that if $g = h$ is the support function of a convex body K, then $F(h)$ is the volume of K (see, for instance [6]).

Given $p \geq 1$, $K \in C^{2,+}$ and $\phi \in C^2(\mathbb{S}^{n-1})$, we want to compute the first and second derivatives of the family of real-valued functionals $F(h_s)$, as function of s, i.e., we consider the following volume function $f\colon (-a, a) \to \mathbb{R}_+$ defined as
$$f(s) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_s \ \det(Q(h_s)) \ dv, \tag{3.4}$$
for some $a > 0$.

We will need some preparation. For simplicity we will write $\det(Q(h))$ for $\det(Q(h; \cdot))$ for a given function $h \in C^{2,+}(\mathbb{S}^{n-1})$. Given a square matrix $A = (a_{ij})_{i,j=1,\ldots,N}$ of order N, we set
$$S^{ij}(A) = \frac{\partial \det}{\partial a_{ij}}(A), \quad i, j = 1, \ldots, N.$$
The matrix
$$(S^{ij}(A))_{i,j=1,\ldots,N}$$
is the *co-factor* matrix of A.

Given a function $h \in C^{2,+}(\mathbb{S}^{n-1})$, we define the linear functional
$$L(h)\colon C^2(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{n-1})$$

as follows:

$$L(h)\phi := \sum_{i,j=1}^{n-1} S^{ij}(Q(h))(\phi_{ij} + \phi\delta_{ij}), \quad \forall\ \phi \in C^2(\mathbb{S}^{n-1}). \tag{3.5}$$

A special case is given by $K = B_n$ the unit ball, which will be useful later in the paper. Since in this case $Q(h) = I_{n-1}$, and since the co-factor matrix of the identity matrix is the identity matrix itself, we have

$$L(h)\phi = \sum_{i,j=1}^{n-1} \delta_{ij}(\phi_{ij} + \phi\delta_{ij}) = \sum_{i=1}^{n-1} \phi_{ii} + (n-1)\phi = \Delta_{\mathbb{S}^{n-1}}\phi + (n-1)\phi \tag{3.6}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the spherical Laplace operator.

The following result is a differentiable result for the volume function.

Lemma 3.4. [3, Proposition 3.1] *We have*

$$\frac{d}{ds}F(h_s) = \int_{\mathbb{S}^{n-1}} \left(\frac{d}{ds}h_s\right)\ \det(Q(h_s))\ dv$$

and

$$\frac{d^2}{ds^2}F(h_s) = \int_{\mathbb{S}^{n-1}} \left(\frac{d^2}{ds^2}h_s\right)\ \det(Q(h_s))\ dv + \int_{\mathbb{S}^{n-1}} \left(\frac{d}{ds}h_s\right)\ L(h_s)\frac{d}{ds}h_s\ dv.$$

Note that in [3, Proposition 3.1], the authors established the derivative results of Lemma 3.4 for any differentiable path h_s in $C^{2,+}(\mathbb{S}^{n-1})$, passing through h. We focus on the family h_s given by Definition 3.2.

Remark 3.5. *The derivatives of h_s as given by Definition 3.2, with respect to s have the following expressions:*

$$\begin{aligned} &h_s = (h^p + s\phi)^{\frac{1}{p}}\,, \quad \frac{d}{ds}h_s = \frac{1}{p}\,(h^p + s\phi)^{\frac{1-p}{p}}\,\phi = \frac{\phi}{p}h_s^{1-p}, \\ &\frac{d^2}{ds^2}h_s = \frac{1-p}{p^2}\,(h^p + s\phi)^{\frac{1-2p}{p}}\,\phi^2 = \frac{1-p}{p^2}\phi^2 h_s^{1-2p}. \end{aligned} \tag{3.7}$$

Taking into account Lemma 3.4 and (3.7), we deduce the following result.

Lemma 3.6. *Let $p \geq 1$, $K \in C^{2,+}$ and $\phi \in C^2(\mathbb{S}^{n-1})$. We have*

$$\frac{d}{ds}f(s) = \frac{1}{p}\int_{\mathbb{S}^{n-1}} \phi\ h_s^{1-p}\ \det(Q(h_s))\ dv$$

and

$$\frac{d^2}{ds^2}f(s) = \frac{1}{p^2}\left[(1-p)\int_{\mathbb{S}^{n-1}} \phi^2\ h_s^{1-2p}\det(Q(h_s))\ dv + \int_{\mathbb{S}^{n-1}} \phi\ h_s^{1-p}L(h_s)\phi\ h_s^{1-p}\ dv\right].$$

3.2. **The case $p = 1$.** We conclude this section focusing on the case $p = 1$, and connecting the differentiation formulae established above to the linear Brunn-Minkowski inequality (1.6).

The main result of this part is the following proposition.

Proposition 3.7. *Let $u \in \mathbb{S}^{n-1}$. For every convex body $K \in C^{2,+}$ and $\phi \in C^2(\mathbb{S}^{n-1})$ such that $\phi|_{u^\perp} = 0$, the inequality*

$$\int_{\mathbb{S}^{n-1}} \phi L(h)\phi\ dv \leq 0 \tag{3.8}$$

holds, where h is the support function of K.

This result will be used in the proof of Theorem 1.3, and, as we will see, it follows from the linear form of the Brunn-Minkowski inequality expressed by Theorem C.

Proof of Proposition 3.7. By Proposition 3.1, we know that there exists $a > 0$ such that for every $s \in (-a, a)$,

$$h_s := h + s\phi$$

is the support function of a convex body $L_s \in C^{2,+}$. The condition $\phi|_{u^\perp} = 0$ provides us with the assumption that $K, L_s \in \mathcal{K}_C$, for every $s \in (-a, a)$, i.e., they belong to the same canal class: $P_{u^\perp}(K) = P_{u^\perp}(L_s) = C$, for every $s \in (-a, a)$.

Let $f\colon (-a, a) \to \mathbb{R}$ be defined by

$$f(s) = \mathrm{vol}_n(L_s), \quad s \in (-a, a).$$

Inequality (1.6) implies that f is concave. Indeed, let $s_1, s_2 \in (-a, a)$ and $\lambda \in [0, 1]$. We observe, by the linearity property of the support function, that we have $L_{(1-\lambda)s_1+\lambda s_2} = (1-\lambda)L_{s_1} + \lambda L_{s_2}$. Therefore

$$\begin{aligned} f((1-\lambda)s_1 + \lambda s_2) &= \mathrm{vol}_n(L_{(1-\lambda)s_1+\lambda s_2}) \\ &= \mathrm{vol}_n((1-\lambda)L_{s_1} + \lambda L_{s_2}) \\ &\geq (1-\lambda)\mathrm{vol}_n(L_{s_1}) + \lambda \mathrm{vol}_n(L_{s_2}) = (1-\lambda)f(s_1) + \lambda f(s_2), \end{aligned}$$

where the inequality holds because $L_{s_1}, L_{s_2} \in \mathcal{K}_C$, and this proves the concavity of f.

Since f is concave, we have in particular that

$$\frac{d^2}{ds^2} f(0) \leq 0.$$

Applying Lemma 3.6, with $h_s = h + s\phi$ and $p = 1$, we deduce

$$\frac{d^2}{ds^2} f(s) = \int_{\mathbb{S}^{n-1}} \phi L(h_s)\phi \, dv. \tag{3.9}$$

We conclude that

$$\int_{\mathbb{S}^{n-1}} \phi L(h)\phi \, dv \leq 0.$$

□

4. Proofs of Theorems 1.2 and 1.3

Suggestion: leave the content of subsection 4.1 without being inside a proper subsection, but just as the introductory content of this section. In this way Subsection 4.2 will become the first subsection

4.1. **Preliminary considerations.** Let $u \in \mathbb{S}^{n-1}$. As before, given a convex body $C \subseteq u^\perp$ containing the origin, with $u \in \mathbb{S}^{n-1}$. We recall that we denote by $\mathcal{K}_C$ the set of all convex bodies K containing the origin and such that $P_{u^\perp}(K) = C$.

Let $p \geq 1$. We are interested in finding $\alpha \in \mathbb{R}$ such that the inequality

$$\mathrm{vol}_n\big((1-\lambda) \cdot K +_p \lambda \cdot L\big) \geq \big((1-\lambda)\mathrm{vol}_n(K)^\alpha + \lambda \mathrm{vol}_n(L)^\alpha\big)^{\frac{1}{\alpha}} \tag{4.1}$$

holds true for every $K, L \in \mathcal{K}_o^n$ such that $P_{u^\perp}(K) = P_{u^\perp}(L)$.

Remark 4.1. From the inclusion

$$(1-\lambda)K + \lambda L \subseteq (1-\lambda)\cdot K +_p \lambda \cdot L,$$

and from Theorem C, it follows that

$$\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda\cdot L\big) \geq \mathrm{vol}_n\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}_n(K) + \lambda \mathrm{vol}_n(L),$$

that is, (4.1) holds for $\alpha = 1$.

Remark 4.2. Let $\alpha = p$. Let $u \in \mathbb{S}^{n-1}$ and $C \subseteq u^\perp$ a convex body containing the origin. Taking $K = C$ and $L = C +_p [-u,u]$ inequality (4.1) becomes an equality. By (2.5) we have

$$\mathrm{vol}_n((1-\lambda)\cdot K +_p \lambda \cdot L)^p = \mathrm{vol}_n(C +_p \lambda\cdot[-u,u])^p = 2^p D^p(n,p)\mathrm{vol}_{n-1}(C)^p\lambda,$$

for every $\lambda \in [0,1]$. On the other hand

$$(1-\lambda)\mathrm{vol}_n(C)^p + \lambda \mathrm{vol}_n(C +_p [-u,u])^p = \lambda \mathrm{vol}_n(C +_p [-u,u])^p = \lambda 2^p D^p(n,p)\mathrm{vol}_{n-1}(C)^p.$$

The next two results aim to prove that (4.1) does not hold for any $\alpha > p$.

Proposition 4.3. *Let $p, \alpha > 1$. If there exist $u \in \mathbb{S}^{n-1}$ and $C \subseteq u^\perp$ a convex body containing the origin with $C \subseteq u^\perp$ such that inequality (4.1) holds for every pair of convex bodies in $\mathcal{K}_C$, then for every choice of $K \in \mathcal{K}_C$ the function $f\colon [0,+\infty) \to \mathbb{R}$, defined by $f(s) := \mathrm{vol}_n(K +_p s\cdot[-u,u])^\alpha$, is concave.*

Proof. The concavity of f follows from the same argument of the proof of Proposition 3.7, we just remark that in the case of the p-sum, with $p > 1$, we have $P_{u^\perp}(K) = P_{u^\perp}(K +_p s \cdot [-u,u])$ for every $s > 0$ by Remark 2.10, hence $K +_p s\cdot[-u,u] \in \mathcal{K}_C$ for every $s > 0$ and we can apply inequality (4.1). □

Proposition 4.4. *Let $p > 1$. Let $C \subseteq u^\perp$ be a convex body containing the origin, with $u \in \mathbb{S}^{n-1}$. If $\alpha \geq 1$ is such that inequality (4.1) holds for every $K, L \in \mathcal{K}_C$, then $\alpha \leq p$.*

Proof. We set $f\colon [0,+\infty) \to \mathbb{R}$ as $f(s) = \mathrm{vol}_n(C +_p s\cdot[-u,u])^\alpha$. By (2.5) we have

$$f(s) = \mathrm{vol}_n(C +_p s\cdot[-u,u])^\alpha = 2^\alpha D^\alpha(n,p) s^{\frac{\alpha}{p}} \mathrm{vol}_{n-1}(C)^\alpha.$$

By Proposition 4.3, f is concave, then we have

$$f(s) \geq (1-s)f(0) + sf(1),$$

for $s \in [0,1]$, whence

$$\mathrm{vol}_n(C +_p s\cdot[-u,u])^\alpha \geq s\mathrm{vol}_n(C +_p [-u,u]) = s2^\alpha D^\alpha(n,p)\mathrm{vol}_{n-1}(C)^\alpha.$$

This means that we have $s^{\frac{\alpha}{p}} \geq s$, for every $s \in [0,1]$, that is, $\alpha \leq p$. □

4.2. **The main tools for the proof of Theorems 1.2 and 1.3.** Let $p \geq 1$. Let $K \in C^{2,+} \cap \mathcal{K}^n_o$ and $\phi \in C^2(\mathbb{S}^{n-1})$. We recall (see Proposition 3.3) that there exists $\mathbb{R} \ni a > 0$ such that

$$h(K,\cdot)^p + s\phi(\cdot)$$

is the support function of a uniquely determined convex body $L_s \in C^{2,+} \cap \mathcal{K}^n_o$, for every $s \in (-a,a)$.

Proposition 4.5. *Let $\alpha, p > 1$. Inequality (4.1) holds true for every pair of convex bodies $K, L \in \mathcal{K}_C$ if and only the following condition holds: for every $K \in C^{2,+} \cap \mathcal{K}^n_o$ and $\phi \in C^2(\mathbb{S}^{n-1})$ such that $\phi|_{u^\perp} = 0$, and $a > 0$ such that $h(K,\cdot)^p + s\phi(\cdot)$ is a support function for every $s \in (-a, a)$, the function $f\colon (-a,a) \to \mathbb{R}$ defined by*

$$f(s) = \mathrm{vol}_n(L_s)^\alpha, \tag{4.2}$$

is concave. Here L_s is the convex body such that $h(L_s,\cdot)^p = h(K,\cdot)^p + s\phi(\cdot)$.

Proof. Assume that (4.1) holds. Let $K \in C^{2,+} \cap \mathcal{K}^n_o$, $\phi \in C^2(\mathbb{S}^{n-1})$, $a > 0$, and L_s and f defined as in the statement of the proposition. We observe that, if $s_1, s_2 \in (-a, a)$ and $\lambda \in [0, 1]$, then we have

$$h(L_{(1-\lambda)s_1+\lambda s_2}, v)^p = (1-\lambda)h(L_{s_1}, v)^p + \lambda h(L_{s_2}, v)^p,$$

for every $v \in \mathbb{S}^{n-1}$, that is, $f((1-\lambda)s_1 + \lambda s_2) = \mathrm{vol}_n((1-\lambda)\cdot L_{s_1} +_p \lambda \cdot L_{s_2})^\alpha$. Hence we can proceed as in the proof of Proposition 3.7, since L_{s_1} and L_{s_2} belong to the same canal class, and conclude that f is concave.

Vice versa let $K, L \in \mathcal{K}_C$ be a pair of convex bodies belonging to the same canal class and containing the origin. We assume first that $K, L \in \mathcal{K}^n_o \cap C^{2,+}$. We define a function $\phi\colon \mathbb{S}^{n-1} \to \mathbb{R}$ as $\phi(v) = h(L, v) - h(K, v)$, $v \in \mathbb{S}^{n-1}$, hence $\phi \in C^2$ by the smoothness property of K, L, and $\phi|_{u^\perp} = 0$ by the canal class assumption. We have a family of functions $h_s = (h^p + s\phi)^{1/p}$, where h stands for the support function of K, and there exists $a > 0$ such that h_s is the support function of a body $L_s \in C^{2,+}$ for every $s \in (-a, a)$. We observe that $h_0 = h = h(K,\cdot)$, meanwhile $h_1 = h + \phi = h(L,\cdot)$. We have also that $h_s > 0$ on $\mathbb{S}^{n-1}$ for every $s \in (0, 1)$, since we have $h_s = (1-s)h(K,\cdot) + sh(L,\cdot)$. This allows us to claim that $a > 1$.
We consider now the function f defined via K and the ϕ above. By the concavity assumption of f we must have that

$$f(\lambda) \geq (1-\lambda)f(0) + \lambda f(1),$$

for every $\lambda \in [0, 1]$, hence we have

$$\begin{aligned}
\mathrm{vol}_n((1-\lambda)\cdot K +_p \lambda \cdot L) &= \mathrm{vol}_n((1-\lambda)\cdot L_0 +_p \lambda \cdot L_1) = \mathrm{vol}_n(L_{(1-\lambda)0+\lambda 1}) \\
&= \mathrm{vol}_n(L_\lambda) = f(\lambda) \\
&\geq (1-\lambda)f(0) + \lambda f(1) = (1-\lambda)\mathrm{vol}_n(K) + \lambda \mathrm{vol}_n(L),
\end{aligned}$$

for every $\lambda \in [0, 1]$. Now we can extend the inequality above to every convex bodies $K, L \in \mathcal{K}_C$ by the approximation result established in Lemma 2.20. ☐

The operator L appearing in the next result is the one introduced in section 3.

Proposition 4.6. *Let $p > 1$ and $1 < \alpha \leq p$. Let $u \in \mathbb{S}^{n-1}$ and C be a convex body containing the origin, such that $C \subseteq u^\perp$. Inequality (4.1) holds for every $K, L \in \mathcal{K}_C$ if and only if the inequality*

$$\begin{aligned}
&\frac{1}{n}\left(\int_{\mathbb{S}^{n-1}} h\ \det(Q(h))\ dv\right)\left[(1-p)\int_{\mathbb{S}^{n-1}} h^{1-2p}\phi^2 \det(Q(h))\ dv + \int_{\mathbb{S}^{n-1}} h^{1-p}\phi L(h) h^{1-p}\phi\ dv\right] \\
&\leq (1-\alpha)\left(\int_{\mathbb{S}^{n-1}} h^{1-p}\phi \det(Q(h))\ dv\right)^2
\end{aligned} \tag{4.3}$$

holds for every $K \in C^{2,+}$, with support function h, and for every $\phi \in C^2(\mathbb{S}^{n-1})$ such that $\phi|_{u^\perp} = 0$.

Proof. Let assume that (4.1) holds for every $K, L \in \mathcal{K}_C$. Let $K \in C^{2,+}$ and $\phi \in C^2(\mathbb{S}^{n-1})$ such that $\phi|_{u^\perp} = 0$. We consider the function $f\colon (-a,a) \to \mathbb{R}_+$ defined by $f(s) := \mathrm{vol}_n(L_s)^\alpha$, where L_s is the convex body with support function $h(K,\cdot)^p + s\phi$ and $a > 0$ is sufficiently small. By Proposition 4.5, we know that f is concave. We set $g(s) := \mathrm{vol}_n(L_s)$, so that $f(s) = g(s)^\alpha$. Therefore

$$
\begin{aligned}
\frac{d^2}{ds^2} f(0) &= \alpha(\alpha-1) g(0)^{\alpha-2}(g'(0))^2 + \alpha g(0)^{\alpha-1} g''(0)\\
&= \alpha(\alpha-1)\mathrm{vol}_n(K)^{\alpha-2}\left(\frac{d}{ds}|_{s=0}\ \mathrm{vol}_n(L_s)\right)^2\\
&+ \alpha \mathrm{vol}_n(K)^{\alpha-1}\left(\frac{d^2}{ds^2}|_{s=0}\ \mathrm{vol}_n(L_s)\right)\\
&\le 0.
\end{aligned}
$$

Equivalently

$$
\mathrm{vol}_n(K)\frac{d^2}{ds^2}|_{s=0}\ \mathrm{vol}_n(L_s) \le (1-\alpha)\left(\frac{d}{ds}|_{s=0}\ \mathrm{vol}_n(L_s)\right)^2.
$$

By Lemma 3.4 we obtain

$$
\begin{aligned}
\frac{1}{np^2}\left(\int_{\mathbb{S}^{n-1}} h\ \det(Q(h))\ dv\right)&\left[(1-p)\int_{\mathbb{S}^{n-1}} h^{1-2p}\phi^{2p}\det(Q(h))\ dv + \int_{\mathbb{S}^{n-1}} h^{1-p}\phi^p L(h) h^{1-p}\phi^p\ dv\right]\\
&\le \frac{1-\alpha}{p^2}\left(\int_{\mathbb{S}^{n-1}} h^{1-p}\phi^p\det(Q(h))\ dv\right)^2,
\end{aligned}
$$

which provides us with inequality (4.3).

Vice versa, let $K, L \in \mathcal{K}_C$ be a pair of convex bodies belonging to the same canal class and containing the origin. We assume first that $K, L \in C^{2,+}$. As in the proof of Proposition 4.5 we define $\phi\colon \mathbb{S}^{n-1} \to \mathbb{R}$ as $\phi(v) = h(L,v) - h(K,v)$, $v \in \mathbb{S}^{n-1}$, so there exists $a > 0$ such that $h_s = (h^p + s\phi)^{1/p}$ is the support function of a body $L_s \in C^{2,+}$ for every $s \in (-a,a)$, where h stands for the support function of K. Taking into account the function $f\colon (-a,a) \to \mathbb{R}_+$ defined by $f(s) := \mathrm{vol}_n(L_s)^\alpha$, we notice, as above, that inequality (4.3) is equivalent to

$$
\frac{d^2}{ds^2} f(0) \le 0.
$$

Arguing in the same manner as in [29, Theorem 7.4.5] we have that $\frac{d^2}{ds^2} f(s) \le 0$, for every $s \in (-a,a)$, that is, f is concave in $(-a,a)$. The conclusion follows from Proposition 4.5. Moreover, we can extend the inequality to every convex bodies $K, L \in \mathcal{K}_C$ by the approximation result established in Lemma 2.20.

□

4.3. **Proof of Theorem 1.2.** Given $p \ge 1$ and α such that (4.1) holds, we establish an upper bound for α, strictly smaller than n. This in particular will provide the proof of Theorem 1.2.

Theorem 4.7. *Let $u \in \mathbb{S}^{n-1}$ and C be a convex body containing the origin, such that $C \subseteq u^\perp$. Let $p \ge 1$ and α be such that (4.1) holds for every $K, L \in \mathcal{K}_C$. Then*

$$
\alpha - 1 \le c(n)\ (p-1),
$$

where

$$c(n) = \begin{cases} \dfrac{\pi^2}{4}\left[\dfrac{(2k-1)!!}{(2k)!!}\right]^2 & \textit{if } n = 2k,\ k \in \mathbb{N}, \\ \\ \left[\dfrac{(2k)!!}{(2k+1)!!}\right]^2 & \textit{if } n = 2k+1,\ k \in \mathbb{N}. \end{cases} \tag{4.4}$$

Proof. By Proposition 4.6, inequality (4.3) holds. For $K = B_n$, the unit ball in $\mathbb{R}^n$, we have in particular

$$h \equiv 1, \quad \det(Q(h)) \equiv 1,$$

and, by (3.6),

$$L(h)\phi = \sum_{i,i=1}^{n-1} \delta_{ij}(\phi_{ij} + \phi\delta_{ij}) = (n-1)\phi + \sum_{i}^{n-1} \phi_{ii} = (n-1)\phi + \Delta_{\mathbb{S}^{n-1}}\phi$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the spherical Laplacian. Hence inequality 4.3 becomes

$$\kappa_n \left[(1-p)\int_{\mathbb{S}^{n-1}} \phi^2 dv + \int_{\mathbb{S}^{n-1}} \phi((n-1)\phi + \Delta_{\mathbb{S}^{n-1}}\phi)dv\right] \le (1-\alpha)\left(\int_{\mathbb{S}^{n-1}} \phi\ dv\right)^2. \tag{4.5}$$

After an integration by parts we get

$$\kappa_n \left[(1-p)\int_{\mathbb{S}^{n-1}} \phi^2 dv + \int_{\mathbb{S}^{n-1}} [(n-1)\phi^2 - |\nabla_{\mathbb{S}^{n-1}}\phi|^2]dv\right] \le (1-\alpha)\left(\int_{\mathbb{S}^{n-1}} \phi\ dv\right)^2 \tag{4.6}$$

where $\nabla_{\mathbb{S}^{n-1}}$ stands for the spherical gradient. This inequality must hold for every $\phi \in C^2(\mathbb{S}^{n-1})$, such that $\phi \equiv 0$ on $u^\perp$. By a standard approximation argument, it follows that the same inequality has to hold for every Lipschitz function ϕ on $\mathbb{S}^{n-1}$, such that $\phi \equiv 0$ on $u^\perp$.

We now fix an orthonormal frame $\{e_1, \dots, e_n\}$ in $\mathbb{R}^n$ such that $u = e_n$, and define $\phi\colon \mathbb{S}^{n-1} \to \mathbb{R}$ by

$$\phi(x) = \phi(x_1, \dots, x_n) = |(x, u)| = |x_n|.$$

As ϕ is even, inequality (4.6) is equivalent to

$$\frac{\kappa_n}{2}\left[(1-p)\int_{\mathbb{S}^{n-1}_+} \phi^2 dv + \int_{\mathbb{S}^{n-1}_+} [(n-1)\phi^2 - |\nabla_{\mathbb{S}^{n-1}_+}\phi|^2]dv\right] \le (1-\alpha)\left(\int_{\mathbb{S}^{n-1}_+} \phi\ dv\right)^2 \tag{4.7}$$

where

$$\mathbb{S}^{n-1}_+ = \{(x_1, \dots, x_n) \in \mathbb{S}^{n-1} \colon x_n \ge 0\}.$$

By the divergence theorem

$$\int_{\mathbb{S}^{n-1}_+} |\nabla_{\mathbb{S}^{n-1}_+}\phi|^2 dv = -\int_{\mathbb{S}^{n-1}_+} \phi\Delta_{\mathbb{S}^{n-1}}\phi dv;$$

(no boundary terms appear, as ϕ vanishes on the boundary of $\mathbb{S}^{n-1}_+$ relative to $\mathbb{S}^{n-1}$). Hence we must have

$$\frac{\kappa_n}{2}\left[(1-p)\int_{\mathbb{S}^{n-1}_+} \phi^2 dv + \int_{\mathbb{S}^{n-1}_+} \phi((n-1)\phi + \Delta_{\mathbb{S}^{n-1}}\phi)dv\right] \le (1-\alpha)\left(\int_{\mathbb{S}^{n-1}_+} \phi\ dv\right)^2. \tag{4.8}$$

On the other hand, on $\mathbb{S}^{n-1}_+$ the function ϕ coincides with the restriction of a linear function to the unit sphere, that is, a spherical harmonic of degree 1. Therefore it solves the equation

$$(n-1)\phi + \Delta_{\mathbb{S}^{n-1}}\phi = 0.$$

We deduce

$$\frac{\kappa_n(1-p)}{2}\int_{\mathbb{S}^{n-1}_+}\phi^2 dv \le (1-\alpha)\left(\int_{\mathbb{S}^{n-1}_+}\phi\ dv\right)^2,$$

or, equivalently,

$$\alpha - 1 \le (p-1)\frac{\kappa_n\displaystyle\int_{\mathbb{S}^{n-1}_+}\phi^2 dv}{2\left(\displaystyle\int_{\mathbb{S}^{n-1}_+}\phi\ dv\right)^2}.$$

The two integrals in the right hand side of the last inequality can be explicitly computed, exploiting the equality $\phi(x_1,\dots,x_n) = x_n$ on $\mathbb{S}^{n-1}_+$. In particular

$$\int_{\mathbb{S}^{n-1}_+}\phi dv = \kappa_{n-1}, \quad \text{and} \quad \int_{\mathbb{S}^{n-1}_+}\phi^2 dv = \kappa_{n-1}I_n,$$

where

$$I_n = \int_0^{\pi/2}(\sin(t))^n dt.$$

[The computations to prove the previous formulas are standard, but not straightforward - should we show at least a part of them?]. Hence

$$\alpha - 1 \le (p-1)\frac{\kappa_n I_n}{2\kappa_{n-1}}.$$

Let

$$c(n) = \frac{\kappa_n I_n}{2\kappa_{n-1}}.$$

Using the recursion formula

$$I_n = \frac{n-1}{n}I_{n-2}$$

(which can be obtained by an integration by parts) and the equalities

$$I_0 = \frac{\pi}{2} \quad \text{and} \quad I_1 = 1,$$

we obtain:

$$I_n = \begin{cases} \dfrac{\pi}{2}\dfrac{(2k-1)!!}{(2k)!!} & \text{if } n = 2k,\ k\in\mathbb{N}, \\[2ex] \dfrac{(2k)!!}{(2k+1)!!} & \text{if } n = 2k+1,\ k\in\mathbb{N}, \end{cases} \tag{4.9}$$

for $n \ge 2$. On the other hand

$$\frac{\kappa_n}{2\kappa_{n-1}} = \frac{\sqrt{\pi}\ \Gamma((n+1)/2)}{2\ \Gamma(1+n/2)}. \tag{4.10}$$

In particular,

$$c(2k) = \frac{\pi^{3/2}}{4}\cdot\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\cdot\frac{(2k-1)!!}{(2k)!!}.$$

Using the identity

$$\Gamma\left(k+\frac{1}{2}\right)=\sqrt{\pi}\ \frac{(2k-1)!!}{2^k}$$

we get

$$c(2k)=\frac{\pi^2}{4}\left[\frac{(2k-1)!!}{(2k)!!}\right]^2.$$

The value of $c(2k+1)$

$$c(2k+1)=\left[\frac{(2k)!!}{(2k+1)!!}\right]^2$$

can be obtained from (4.9) and (4.10) in a similar way. □

Proof of theorem 1.2. It is straightforward to check that

$$c(2k)\le\frac{\pi^2}{16}<1 \quad\text{and}\quad c(2k+1)\le\frac{4}{9}<1.$$

This implies that inequality (4.1) can be true only if $(\alpha-1)<(p-1)$, that is, $\alpha<p$. □

4.4. **Proof of Theorem 1.3.**

Proof of Theorem 1.3. By Proposition 4.6, it is enough to prove inequality (4.3) with

$$\alpha=\frac{n+p-1}{n}.$$

Let $K\in C^{2,+}$ and $\phi\in C^2(\mathbb{S}^{n-1})$ be such that $\phi\equiv 0$ on $u^\perp$. Let $f\colon(-a,a)\to\mathbb{R}_+$ be defined as

$$f(s):=\mathrm{vol}_n(L_s)^\alpha,$$

where $L_s\in\mathcal{K}^n_o\cap\mathcal{C}^{2,+}$ such that $h(L_s,\cdot):=(h(K,\cdot)^p+s\phi(\cdot))^{\frac{1}{p}}$, and $a>0$ is sufficiently small (see Proposition 3.3). We first remark that by Proposition 3.7 we have

$$\int_{\mathbb{S}^{n-1}}\psi L(h)\psi\ dv\le 0$$

for every $\psi\in C^2(\mathbb{S}^{n-1})$ such that $\psi|_{u^\perp}=0$. Since $h^{1-p}\phi$ is still a function of class C^2, vanishing on $u^\perp=0$, we obtain

$$\int_{\mathbb{S}^{n-1}}h^{1-p}\phi L(h)h^{1-p}\phi\ dv\le 0.$$

Therefore it suffices to prove

$$\begin{aligned}&\frac{1-p}{n}\left(\int_{\mathbb{S}^{n-1}}h\ \det(Q(h))\ dv\right)\left(\int_{\mathbb{S}^{n-1}}h^{1-2p}\phi^2\det(Q(h))\ dv\right)\\&\le(1-\alpha)\left(\int_{\mathbb{S}^{n-1}}h^{1-p}\phi\det(Q(h))\ dv\right)^2=\frac{1-p}{n}\left(\int_{\mathbb{S}^{n-1}}h^{1-p}\phi\det(Q(h))\ dv\right)^2.\end{aligned}$$

As

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}}h\ \det(Q(h))\ dv=\mathrm{vol}_n(K),$$

we are reduced to prove

$$n\mathrm{vol}_n(K)\ \int_{\mathbb{S}^{n-1}}h^{1-2p}\phi^2\det(Q(h))\ dv\ge\left(\int_{\mathbb{S}^{n-1}}h^{1-p}\phi\det(Q(h))\ dv\right)^2.$$

The previous inequality can be written in the following form:

$$\int_{\mathbb{S}^{n-1}} \left(\frac{\phi}{h}\right)^2 d\mu \geq \left(\int_{\mathbb{S}^{n-1}} \frac{\phi}{h}\, d\mu\right)^2, \tag{4.11}$$

where μ is the probability measure on $\mathbb{S}^{n-1}$ such that

$$d\mu = \frac{1}{n\mathrm{vol}_n(K)} h \det(Q(h)) dv.$$

Finally, (4.11) follows from Jensen's inequality. □

Remark 4.8. We know from Theorem 1.3 that inequality (4.1) holds if $\alpha = \frac{n+p-1}{n}$, that is

$$\alpha - 1 = (p-1) \cdot \frac{1}{n}.$$

We also know from Theorem 4.7 that the same inequality does not hold in general, if

$$\alpha - 1 > c(n)(p-1)$$

where $c(n)$ is given by (4.4). It is therefore natural to compare $c(n)$ with $\frac{1}{n}$. An explicit comparison can be made when n tends to infinity. Indeed, using (4.4) and the Stirling formula the following relation can be proved:

$$\lim_{n\to\infty} nc(n) = \frac{\pi}{2}.$$

4.5. **Steiner Symmetrization and improvements of the L_p Brunn-Minkowski inequality.** This subsection will be moved to the fifth section The section deals with the extension of the result of Theorem 1.3 in the case of equal $(n-1)$-dimensional volume of the projections, instead of equal projections.
In the classical literature, an extension of the linear improvements of the Brunn-Minkowski inequality reads as follows.

Theorem 4.9. [2, Section 1.7] *Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists a direction $u \in \mathbb{S}^{n-1}$ with the condition*

$$\mathrm{vol}_{n-1}(P_{u^\perp}(K)) = \mathrm{vol}_{n-1}(P_{u^\perp}(L)). \tag{4.12}$$

The inequality

$$\mathrm{vol}_n\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}_n(K) + \lambda \mathrm{vol}_n(L), \tag{4.13}$$

holds for all $\lambda \in [0,1]$.

The previous result goes back to [5], see also [29, Section 7.7].
The case $p > 1$ reads as follows and it is an extension of Theorem 1.3.

Theorem 4.10. *Let $u \in \mathbb{S}^{n-1}$, $p > 1$ and $\alpha := \frac{n+p-1}{n} \in (1,p)$. The inequality*

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda \cdot L\big) \geq ((1-\lambda)\mathrm{vol}_n(K)^\alpha + \lambda \mathrm{vol}_n(L)^\alpha)^{1/\alpha} \tag{4.14}$$

holds for every $\lambda \in [0,1]$ and $K, L \in \mathcal{K}_o$ such that $\mathrm{vol}_{n-1}(P_{u^\perp}(K)) = \mathrm{vol}_{n-1}(P_{u^\perp}(L))$.

Proof. Let first apply the Steiner symmetrization of K and L with respect to $u^\perp$, which we recall they are denoted by $S_{u^\perp}(K)$ and $S_{u^\perp}(L)$. Afterwards, we apply the Schwarz symmetrization to $S_{u^\perp}(K)$ and $S_{u^\perp}(L)$ with respect to the 1-dimensional linear subspace spanned by u. We donete the Schwartz symmetrals as follows: $S_u(S_{u^\perp}(K))$ and $S_u(S_{u^\perp}(L))$.

By Lemma 2.15 we have $\mathrm{vol}_n(S_u(S_{u^\perp}(K))) = \mathrm{vol}_n(S_{u^\perp}(K)) = \mathrm{vol}_n(K)$ and $\mathrm{vol}_n(S_u(S_{u^\perp}(L))) = \mathrm{vol}_n(S_{u^\perp}(L)) = \mathrm{vol}_n(L)$. By Theorems 2.17 and 2.19 we have also

$$S_u(S_{u^\perp}(K +_p L)) \supseteq S_u(S_{u^\perp}(K) +_p S_{u^\perp}(L)) \supseteq S_u(S_{u^\perp}(K)) +_p S_u(S_{u^\perp}(L)).$$

Let $\lambda \in [0,1]$, we observe that we have also

$$S_u(S_{u^\perp}((1-\lambda)\cdot K)) = S_u(S_{u^\perp}((1-\lambda)^{1/p}K)) = (1-\lambda)^{1/p}S_u(S_{u^\perp}(K)) = (1-\lambda)\cdot S_u(S_{u^\perp}(K)),$$

and similar $S_u(S_{u^\perp}(\lambda \cdot L)) = \lambda \cdot S_u(S_{u^\perp}(L))$.
We observe that $S_u(S_{u^\perp}(K))$ and $S_u(S_{u^\perp}(L))$ belongs to the same canal class along the direction u, i.e., $P_{u^\perp}(S_u(S_{u^\perp}(K))) = P_{u^\perp}(S_u(S_{u^\perp}(L)))$.Hence, by Theorem 1.3, we have

$$\begin{aligned}
\mathrm{vol}_n((1-\lambda)\cdot K +_p \lambda \cdot L) &= \mathrm{vol}_n(S_u(S_{u^\perp}((1-\lambda)\cdot K +_p \lambda\cdot L)))\\
&\geq \mathrm{vol}_n(S_u(S_{u^\perp}((1-\lambda)\cdot K)) +_p S_u(S_{u^\perp}(\lambda\cdot L)))\\
&= \mathrm{vol}_n((1-\lambda)\cdot S_u(S_{u^\perp}(K)) +_p \lambda\cdot S_u(S_{u^\perp}(L)))\\
&\geq ((1-\lambda)\mathrm{vol}_n(S_u(S_{u^\perp}(K)))^\alpha + \lambda \mathrm{vol}_n(S_u(S_{u^\perp}(L)))^\alpha)^{1/\alpha}\\
&= ((1-\lambda)\mathrm{vol}_n(K)^\alpha + \lambda\mathrm{vol}_n(L)^\alpha)^{1/\alpha}
\end{aligned}$$

for $\alpha = \frac{n+p-1}{n}$. □

5. Other related inequalities, and proof of Theorem 1.4

We focus our attention in this section on the proof of Theorem 1.4. We split the proof of the theorem into two parts. The first subsection deals with the inequality

$$\min\{(1-\lambda),\lambda\}^{-(n-1)/p}\,\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda\cdot L\big) \geq \big((1-\lambda)\mathrm{vol}_n(K)^p + \lambda\mathrm{vol}_n(L)^p\big)^{1/p},$$

meanwhile the second one deals with

$$n\left[1-\left(1-\frac{1}{n}\right)^p\right]^{1/p}\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda\cdot L\big) \geq \big((1-\lambda)\mathrm{vol}_n(K)^p + \lambda\mathrm{vol}_n(L)^p\big)^{1/p}.$$

There are two separted subsections because the proofs are obtained by two different approaches. The first one comes form the refinement result of the Prékopa-Leindler inequality, see [10], and application of functional results that can be found in [27].
The second approach comes from the linear refinement of the first Minkowski inequality extended to the L_p setting.

5.1. Inequalities in the functional setting.

Questions: Where do we write the definitions of projection of a function? Do we need a definition of α-concave function or later in the theorems we write only the inequality that the function should hold, without giving a name? Do we need the definition of ∞-concave function?

From the homogeneity of the volume, it is well-known that the Brunn-Minkowski inequality (1.1) is equivalent to the fact that

$$\mathrm{vol}_n(tK+sL)^{1/n} \geq t\mathrm{vol}_n(K)^{1/n} + s\mathrm{vol}_n(L)^{1/n} \tag{5.1}$$

holds for any $K, L \in \mathcal{K}^n$ and $t, s > 0$.

Let $K, L \in \mathcal{K}^n$ be convex bodies belonging to the same canal class, i.e. $P_{u^\perp}(K) = P_{u^\perp}(L)$, for some $u \in \mathbb{S}^{n-1}$. The inequality

$$\mathrm{vol}_n\big(tK+sL\big) \geq (t+s)^{n-1}(t\mathrm{vol}_n(K) + s\mathrm{vol}_n(L)), \tag{5.2}$$

holds for every $t,s>0$. Indeed, as a consequence of the homogeneity of the Lebesgue measure, we have

$$\frac{1}{(t+s)^{n-1}}\mathrm{vol}_n\big(tK+sL\big)=(t+s)\mathrm{vol}_n\left(\frac{t}{s+t}K+\frac{s}{s+t}L\right).$$

Taking $\lambda=\frac{s}{s+t}\in(0,1)$ in Theorem C we have

$$\frac{1}{(t+s)^{n-1}}\mathrm{vol}_n\big(tK+sL\big)\geq(t+s)\left[\frac{t}{s+t}\mathrm{vol}_n(K)+\frac{s}{s+t}\mathrm{vol}_n(L)\right]=t\mathrm{vol}_n(K)+s\mathrm{vol}_n(L),$$

as desired.

Remark 5.1. i) *Inequality* (5.2) *is a refinement of the Brunn-Minkowski inequality* (5.1). *Indeed, one has*

$$\begin{aligned}\mathrm{vol}_n\big(tK+sL\big)&\geq(t+s)^{n-1}\big(t\mathrm{vol}_n(K)+s\mathrm{vol}_n(L)\big)=(t+s)^n\left(\frac{t}{t+s}\mathrm{vol}_n(K)+\frac{s}{t+s}\mathrm{vol}_n(L)\right)\\&\geq(t+s)^n\left(\frac{t}{t+s}\mathrm{vol}_n(K)^{\frac1n}+\frac{s}{t+s}\mathrm{vol}_n(L)^{\frac1n}\right)^n=\left(t\mathrm{vol}_n(K)^{\frac1n}+s\mathrm{vol}_n(L)^{\frac1n}\right)^n.\end{aligned}$$

ii) *Inequality* (5.2) *is equivalent to the linear Brunn-Minkowski inequality* (1.6).

iii) *If $t+s\geq1$, then inequality* (5.2) *implies*

$$\mathrm{vol}_n(tK+sL)\geq t\mathrm{vol}_n(K)+s\mathrm{vol}_n(L)$$

under the same projections constraint.

We recall that $\mathcal{M}_\alpha^\lambda(a,b)$ the α-mean of a and b with weight λ. We have the following inequality.

Theorem 5.2. *Let $p\geq1$. Let $K,L\in\mathcal{K}^n_{(o)}$ be two convex bodies with the origin in their interiors and $p\geq1$. If there exists $u\in\mathbb{S}^{n-1}$ and $C\in\mathcal{K}^n$ such that $C\subset u^\perp$ and $K,L\in\mathcal{K}_C$, then*

$$\left(\frac{\mathcal{M}_p^\lambda(a,b)}{\mathcal{M}_{p-1}^\lambda(a,b)}\right)^{(p-1)(n-1)}\mathrm{vol}_n\big((1-\lambda)\cdot K+_p\lambda\cdot L\big)\geq\mathcal{M}_p^\lambda(a,b)\geq\mathcal{M}_{\frac{p}{n}}^\lambda(a,b),\tag{5.3}$$

hold for any $\lambda\in[0,1]$, where $a=\mathrm{vol}_n(K)$ and $b=\mathrm{vol}_n(L)$.

We will show the latter inequality as a consequence of a functional and more general result. To this aim, we first recall the following refinement of the Borell-Brascamp-Lieb inequality:

Theorem 5.3. [10, Theorem 1.6] *Let $\lambda\in(0,1)$. Let $f,g:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq0}\cup\{\infty\}$ be α-concave functions, where $-1/n\leq\alpha\leq\infty$, and let $h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq0}\cup\{\infty\}$ be a measurable function such that*

$$h\big((1-\lambda)x+\lambda y\big)\geq\big((1-\lambda)f(x)^\alpha+\lambda g(y)^\alpha\big)^{1/\alpha}$$

for all x,y such that $f(x)g(y)>0$. If there exists a hyperplane $u^\perp$, for $u\in\mathbb{S}^{n-1}$, such that

$$\int_{u^\perp}\mathrm{proj}_{u^\perp}(f)(x)\,\mathrm{d}x=\int_{u^\perp}\mathrm{proj}_{u^\perp}(g)(x)\,\mathrm{d}x$$

then

$$\int_{\mathbb{R}^n}h(x)\,\mathrm{d}x\geq(1-\lambda)\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x+\lambda\int_{\mathbb{R}^n}g(x)\,\mathrm{d}x.$$

Here $\mathrm{proj}_{u^\perp}(f)$ stands for the projection of the function f onto H.

Definition 5.4. [10, Definition 1.2] *Let $u \in \mathbb{S}^{n-1}$ and $f\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$, the projection of f onto $u^\perp$ is the function $\mathrm{proj}_{u^\perp}(f)\colon u^\perp \to \mathbb{R}_+ \cup \{+\infty\}$ defined by*

$$\mathrm{proj}_{u^\perp}(f)(y) = \sup_{t\in\mathbb{R}} f(y+tu),$$

for $y \in u^\perp$.

Geometrically, Definition 5.4 means that the hypograph of the projection of f onto $u^\perp$ is the projection of the hypograph of f onto $u^\perp$. In particular, the projection of the characteristic function of a convex body K is just the characteristic function of the projection of K onto $u^\perp$.

Moreover, in [27] the following L_p extension of the Borell-Brascamp-Lieb inequality was obtained (we recall that in this setting we write q to denote the Hölder's conjugate of p).

Theorem 5.5. [27, Theorem 2.1] *Let $\lambda \in (0,1)$ and let $p \geq 1$. Let $-1/n \leq \alpha \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be integrable functions such that*

$$\begin{aligned} &h\Big((1-\mu)^{1/q}(1-\lambda)^{1/p}x + \mu^{1/q}\lambda^{1/p}y\Big) \\ &\qquad\geq \Big[(1-\mu)^{1/q}(1-\lambda)^{1/p}f(x)^\alpha + \mu^{1/q}\lambda^{1/p}g(y)^\alpha\Big]^{1/\alpha} \end{aligned}$$

for all x, y such that $f(x)g(y) > 0$ and all $\mu \in [0,1]$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \mathcal{M}^\lambda_{\frac{p\alpha}{n\alpha+1}}\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right).$$

Now we are in conditions to show the following L_p extension of Theorem 5.3:

Theorem 5.6. *Let $\lambda \in (0,1)$ and let $p \geq 1$. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be α-concave integrable functions, where $-1/n \leq \alpha \leq \infty$, I suggest to use a different letter instead of α, since in the last section α is used as the best exponent for the refinement, meanwhile here the exponent is p and later we let α going to infinity and let $h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measurable function such that*

$$\begin{aligned} &h\Big((1-\mu)^{1/q}(1-\lambda)^{1/p}x + \mu^{1/q}\lambda^{1/p}y\Big) \\ &\qquad\geq \Big[(1-\mu)^{1/q}(1-\lambda)^{1/p}f(x)^\alpha + \mu^{1/q}\lambda^{1/p}g(y)^\alpha\Big]^{1/\alpha} \end{aligned} \tag{5.4}$$

for all x, y such that $f(x)g(y) > 0$ and all $\mu \in [0,1]$. If there exists a hyperplane $u^\perp$, for $u \in \mathbb{S}^{n-1}$, such that

$$\int_{u^\perp} \mathrm{proj}_{u^\perp}(f)(x)\,\mathrm{d}x = \int_{u^\perp} \mathrm{proj}_{u^\perp}(g)(x)\,\mathrm{d}x \tag{5.5}$$

then

$$\left(\frac{\mathcal{M}^\lambda_p(a,b)}{\mathcal{M}^\lambda_{p-1}(a,b)}\right)^{((p-1)(n-1)+1/\alpha)} \int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \mathcal{M}^\lambda_p(a,b),$$

where $a = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$ and $b = \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x$.

Notice that the case $p = 1$ of this result (that is, in the classical setting of the Minkowski addition) is precisely the statement of Theorem 5.3.

Proof. Without loss of generality, we assume that $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$, $\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x$ are not both zero. Moreover, we will first assume that $\alpha \neq 0, \infty$. Then, taking

$$\mu := \frac{\lambda\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^p}{(1-\lambda)\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^p + \lambda\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^p}, \tag{5.6}$$

for which we clearly have that $\mu \in [0,1]$, we define

$$t = t(\mu) := (1-\mu)^{1/q}(1-\lambda)^{1/p} \ \text{ and } \ s = s(\mu) := \mu^{1/q}\lambda^{1/p}. \tag{5.7}$$

From (5.4) we have that

$$h(tx+sy) \geq \left(tf(x)^\alpha + sg(y)^\alpha\right)^{1/\alpha} \tag{5.8}$$

for all x, y with $f(x)g(y) > 0$. Now, for any given pair of points $x, y \in \mathbb{R}^n$, we write $x' := (t+s)x$ and $y' := (t+s)y$. Moreover, we define the auxiliary functions $\overline{f}, \overline{g} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\overline{f}(x') := (t+s)^{1/\alpha} f\left(\frac{x'}{t+s}\right) \ \text{ and } \ \overline{g}(y') := (t+s)^{1/\alpha} g\left(\frac{y'}{t+s}\right),$$

respectively. Thus, (5.8) yields

$$h\left(\frac{t}{t+s}x' + \frac{s}{t+s}y'\right) \geq \left[\frac{t}{t+s}\left(\overline{f}(x')\right)^\alpha + \frac{s}{t+s}\left(\overline{g}(y')\right)^\alpha\right]^{1/\alpha}$$

for all x', y' such that $\overline{f}(x')\overline{g}(y') > 0$. Notice that, from the definition of $\overline{f}$ and $\overline{g}$, the functions $\overline{f}$ and $\overline{g}$ are α-concave and, from (5.5), they satisfy that

$$\int_{u^\perp} \mathrm{proj}_H\left(\overline{f}\right)(z)\,\mathrm{d}z = \int_{u^\perp} \mathrm{proj}_H\left(\overline{g}\right)(z)\,\mathrm{d}z.$$

Hence, by Theorem 5.3 applied to the functions $\overline{f}, \overline{g}, h$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x &\geq \frac{t}{t+s}\int_{\mathbb{R}^n} \overline{f}(x')\,\mathrm{d}x' + \frac{s}{t+s}\int_{\mathbb{R}^n} \overline{g}(x')\,\mathrm{d}x' \\
&= t(t+s)^{-1+1/\alpha}\int_{\mathbb{R}^n} f\left(\frac{x'}{t+s}\right)\mathrm{d}x' + s(t+s)^{-1+1/\alpha}\int_{\mathbb{R}^n} g\left(\frac{x'}{t+s}\right)\mathrm{d}x' \\
&= t(t+s)^{n-1+1/\alpha}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x + s(t+s)^{n-1+1/\alpha}\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x,
\end{aligned}$$

where in the last equality we have made the change of variables $x = x'/(t+s)$.

Therefore, using the definition of p-mean as well as that of the parameters t, s and μ given in (5.6) and (5.7) respectively, we obtain

$$\left(\frac{\mathcal{M}_p^\lambda(a,b)}{\mathcal{M}_{p-1}^\lambda(a,b)}\right)^{((p-1)(n-1)+1/\alpha)} \int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq$$ why not equal?

$$\begin{aligned}
\frac{1}{(t+s)^{n-1+1/\alpha}}\int_{\mathbb{R}^n} h(z)\,\mathrm{d}z &\geq t\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x + s\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x \\
&= \mathcal{M}_p^\lambda\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right) = \mathcal{M}_p^\lambda(a,b),
\end{aligned}$$

as desired.

Finally, note that the cases $\alpha = 0$ and $\alpha = \infty$ can be derived from the previous ones by continuity. This concludes the proof. $\square$

Now, the geometric version of the latter result, collected in Theorem 5.2 is a direct consequence of Theorem 5.6. To show it, we need to introduce the following notation: χ_M will refer the characteristic function of a subset $M \subset \mathbb{R}$.

Proof of Theorem 5.2. It is enough to apply Theorem 5.6 with $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)\cdot_p K +_p \lambda \cdot_p L}$, and noticing that f and g are ∞-concave do we need a proper definition?. □

Remark 5.7. *Along the proof of Theorem 5.6, the factor*

$$\left(\frac{\mathcal{M}_p^\lambda(a,b)}{\mathcal{M}_{p-1}^\lambda(a,b)}\right)^{(p-1)\left((n-1)+1/\alpha\right)} \tag{5.9}$$

arose by computing the value

$$\frac{1}{\big(t(\mu)+s(\mu)\big)^{n-1+1/\alpha}}$$

for a suitable choice of μ.

Hence, the factor (5.9) *depending on the integrals of f and g can be bounded from above by the constant*

$$C(p,\lambda,\alpha) := \sup_{\mu\in[0,1]} \frac{1}{\left((1-\mu)^{1/q}(1-\lambda)^{1/p}+\mu^{1/q}\lambda^{1/p}\right)^{n-1+1/\alpha}}. \tag{5.10}$$

Notice that $C(1,\lambda,\alpha) = 1$ whereas

$$C(p,\lambda,\alpha) = \min\big\{(1-\lambda),\lambda\big\}^{-(n-1+1/\alpha)/p}$$

for $p > 1$.

In the geometric case, that is, in Theorem 5.2, the left-hand side may be bounded from above by

$$C(p,\lambda) := C(p,\lambda,\infty) = \min\big\{(1-\lambda),\lambda\big\}^{-(n-1)/p},$$

and hence, when $\lambda = 1/2$ the constant factor on the left-hand side is $2^{(n-1)/p}$ (for $p > 1$). In other words, the following inequality holds:

$$2^{(n-1)/p}\,\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda \cdot L\big) \geq \mathcal{M}_p^\lambda\big(\mathrm{vol}_n(K), \mathrm{vol}_n(L)\big). \tag{5.11}$$

Remark 5.8. *Let $K \in \mathcal{K}_o^n$. Since the projection of the characteristic function of the orthogonal projection of K on a hyperplane $u^\perp$ is the characteristic function of $P_{u^\perp}(K)$, we have*

$$\int_{u^\perp} \mathrm{proj}_{u^\perp}(\chi_K)(x)\,\mathrm{d}x = \mathrm{vol}_{n-1}(P_{u^\perp}(K)).$$

Hence, condition (5.5) *in the geometric setting for two convex bodies $K, L \in \mathcal{K}_o^n$ is equivalent to $\mathrm{vol}_{n-1}(P_{u^\perp}(K)) = \mathrm{vol}_{n-1}(P_{u^\perp}(L))$.*

5.2. **Refinement of the first L_p Minkowski inequality.** Associated to the Brunn-Minkowski inequality there is also the so-called first Minkowski inequality.

Suggestion: remove Definition 5.9 and use Proposition 5.10 only with the limit as a definition for the V_1. In this way there is no need to introduce the surface area measure.

Definition 5.9. *Let $K, L \in \mathcal{K}^n$. The functional $\mathrm{V}_1\colon (\mathcal{K}^n)^2 \to \mathbb{R}_+$ is defined as*

$$\mathrm{V}_1(K, L) := \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L, u)\ dS_{n-1}(K, u), \tag{5.12}$$

where $S_{n-1}(K, \cdot)$ is the $(n-1)$- area measure of K.

It is possible to prove the following.

Proposition 5.10. [29, Section 5.1] *The following limit exists and*

$$\mathrm{V}_1(K, L) = \lim_{\epsilon\to 0^+} \frac{\mathrm{vol}_n(K + \epsilon L) - \mathrm{vol}_n(K)}{\epsilon} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L, u)\ dS_{n-1}(K, u) \tag{5.13}$$

holds for every $K, L \in \mathcal{K}^n$.

We refer to [29] for properties of V_1 and relations with other magnitudes within the theory of convex bodies. The first Minkowski inequality reads as follows.

Theorem 5.11. [29, Theorem 7.2.1] *The following inequality*

$$\mathrm{V}_1(K, L)^n \geq \mathrm{vol}_n(K)^{n-1}\mathrm{vol}_n(L), \tag{5.14}$$

holds for every $K, L \in \mathcal{K}^n$.

The following is the linear refinement of inequality (5.14).

Theorem 5.12. [30, Theorem 1.1] *Let $K, L \in \mathcal{K}^n$ be two convex bodies such that there exists $u \in \mathbb{S}^{n-1}$ with $P_{u^\perp}(K) = P_{u^\perp}(L)$, i.e., they belong to the same canal class along the direction $u \in \mathbb{S}^{n-1}$. Hence, the inequality*

$$n\mathrm{V}_1(K, L) \geq (n-1)\mathrm{vol}_n(K) + \mathrm{vol}_n(L) \tag{5.15}$$

holds.

We remark that, as a consequence of the arithmetic-geometric mean inequality, we have that inequality (5.15) is actually a refinement of the standard first Minkowski inequality.

In [23] we can find a extension of the first Minkowski inequality in the L_p Brunn-Minkowski theory, where the author introduced also the following functional that covers the role of the functional V_1.

Suggestion: remove Definition 5.13 and Proposition 5.14 as stated like now, and consider only the definition of V_p via the limit, i.e.,

$$\mathrm{V}_p(K, L) := \frac{1}{p}\lim_{\epsilon\to 0^+} \frac{\mathrm{vol}_n(K +_p \epsilon \cdot L) - \mathrm{vol}_n(K)}{\epsilon}$$

Definition 5.13. *Let $p \geq 1$, and $K, L \in \mathcal{K}^n_o$. We set*

$$\mathrm{V}_p(K, L) := \frac{1}{n}\int_{\mathbb{S}^{n-1}} h^p(L, u)h^{1-p}(K, u)dS_{n-1}(K, u).$$

We remark that if $p = 1$, then we have $\mathrm{V}_p(K, L) = \mathrm{V}_1(K, L)$.

Proposition 5.14. [23] *Let $p \geq 1$. We have*

$$\mathrm{V}_p(K,L) := \frac{1}{n}\int_{\mathbb{S}^{n-1}} h^p(L,u)h^{1-p}(K,u)dS_{n-1}(K,u)$$

for every $K, L \in \mathcal{K}_o^n$.

The following is the L_p first Minkowski inequality.

Theorem 5.15. [23, Theorem 1.2] *Let $p \geq 1$. The inequality*

$$\mathrm{V}_p(K,L)^n \geq \mathrm{vol}_n(K)^{n-p}\mathrm{vol}_n(L)^p \tag{5.16}$$

holds for every $K, L \in \mathcal{K}_o^n$.

Taking Theorem 5.12 into account, we prove the following.

Theorem 5.16. *Let $p > 1$ and $u \in \mathbb{S}^{n-1}$. The inequality*

$$\mathrm{V}_p(K,L) \geq \left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(K) + \frac{1}{n}\mathrm{vol}_n(L)\right]^p \mathrm{vol}_n^{1-p}(K) \tag{5.17}$$

holds for every $K, L \in \mathcal{K}_C$, where $C \subset u^{\perp}$ is a convex body containing the origin and $u \in \mathbb{S}^{n-1}$.

Proof. Remove this part: By Proposition 5.14, we know that

$$\mathrm{V}_p(K,L) := \frac{1}{n}\int_{\mathbb{S}^{n-1}} h^p(L,u)h^{1-p}(K,u)dS_{n-1}(K,u).$$

Taking into account the integral representation of V_p, see [29, Theorems 9.1.1], by Hölder's inequality with $k = \frac{1}{1-p} < 0$ (see (6.9.3) in [18]), we have

$$\mathrm{V}_p(K,L) \geq \mathrm{V}_1^p(K,L)\mathrm{vol}_n^{1-p}(K),$$

see also [29, Theorem 9.1.2].

By inequality (5.15) we have

$$n\mathrm{V}_1(K,L) \geq (n-1)\mathrm{vol}_n(K) + \mathrm{vol}_n(L), \tag{5.18}$$

which implies

$$\mathrm{V}_p(K,L) \geq \left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(K) + \frac{1}{n}\mathrm{vol}_n(L)\right]^p \mathrm{vol}_n^{1-p}(K).$$

$\square$

We remark that inequality (5.17) is a refinement of the first L_p Minkowksi inequality (5.16) by the arithmetic-geometric mean inequality.

Theorem 5.16 provides us with a different type of improvement of the L_p Brunn-Minkowski inequality under the canal class constraint.

Theorem 5.17. *Let $p \geq 1$, $u \in \mathbb{S}^{n-1}$, and $C \subseteq u^{\perp}$ containing the origin. The inequality*

$$n^p\left[1-\left(1-\frac{1}{n}\right)^p\right]\mathrm{vol}_n\big((1-\lambda)\cdot K +_p \lambda\cdot L\big)^p \geq (1-\lambda)\mathrm{vol}_n(K)^p + \lambda\mathrm{vol}_n(L)^p, \tag{5.19}$$

holds for every $\lambda \in (0,1)$ and $K, L \in \mathcal{K}_C$.

Proof. Let $M \in \mathcal{K}_C$, then by direct computation we have

$$\mathrm{V}_p(M, (1-\lambda)\cdot K +_p \lambda \cdot L) = (1-\lambda)\mathrm{V}_p(M, K) + \lambda \mathrm{V}_p(M, L).$$

By Theorem 5.16, we have

$$\begin{aligned}
&\mathrm{V}_p(M, (1-\lambda)\cdot K +_p \lambda \cdot L)\\
&\geq (1-\lambda)\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(K)\right]^p \mathrm{vol}_n^{1-p}(M)\\
&+ \lambda\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(L)\right]^p \mathrm{vol}_n^{1-p}(M)\\
&= \mathrm{vol}_n^{1-p}(M)\left\{(1-\lambda)\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(K)\right]^p + \lambda\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(L)\right]^p\right\}.
\end{aligned}$$

Now we take $M = (1-\lambda)\cdot_p K_1 +_p \lambda \cdot_p L$, then $\mathrm{V}_p(M, M) = \mathrm{vol}_n(M)$, i.e.,

$$\begin{aligned}
&\mathrm{vol}_n^p(M)\\
&\geq (1-\lambda)\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(K)\right]^p + \lambda\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(L)\right]^p.
\end{aligned}$$

Now we consider the right-hand side of the previous inequality, i.e.,

$$\begin{aligned}
&(1-\lambda)\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(K)\right]^p + \lambda\left[\left(1-\frac{1}{n}\right)\mathrm{vol}_n(M) + \frac{1}{n}\mathrm{vol}_n(L)\right]^p\\
&\geq (1-\lambda)\left[\left(1-\frac{1}{n}\right)^p \mathrm{vol}_n^p(M) + \frac{1}{n^p}\mathrm{vol}_n^p(K)\right] + \lambda\left[\left(1-\frac{1}{n}\right)^p \mathrm{vol}_n^p(M) + \frac{1}{n^p}\mathrm{vol}_n^p(L)\right]\\
&= \frac{(1-\lambda)\mathrm{vol}_n^p(K) + \lambda \mathrm{vol}_n^p(L)}{n^p} + \left(1-\frac{1}{n}\right)^p \mathrm{vol}_n^p(M),
\end{aligned}$$

which implies the thesis.

□

Remark 5.18. *Inequality* (5.19) *is an improvement of the L_p Brunn-Minkowski inequality as a consequence of the improvement inequality*(5.17)*.*

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