MEAN RADII OF SYMMETRIZATIONS OF A CONVEX BODY

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ABSTRACT. We study the relation between some successive and mean radii of a convex body and its Steiner, Schwarz, and Minkowski symmetral. In particular, we are interested in the mean radii. Based on the convexity of some of the radii of a (particular) parallel chord movement of convex bodies, we prove that the Steiner symmetral does not increase the mean outer radii. Results of the same type hold for the Schwarz and Minkowski symmetrals.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the set of convex bodies in the n-dimensional Euclidean space $\mathbb{R}^n$, i.e., compact and convex sets in $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively.

We denote by B_n the n-dimensional unit ball and by $\mathbb{S}^{n-1}$ its boundary, the unit sphere. The volume of a measurable set $M \subseteq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$, or $\mathrm{vol}_n(M)$ if the distinction of the dimension is needed. In particular, we write $\kappa_n = \mathrm{vol}(B_n)$.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$ and by $L^\perp \in \mathcal{L}_{n-i}^n$ we designate the orthogonal complement of $L \in \mathcal{L}_i^n$. In the case $i = n-1$, we denote $L \in \mathcal{L}_{n-1}^n$ by $L = u^\perp$, for some $u \in \mathbb{S}^{n-1}$.

Moreover, if $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$, we denote the orthogonal projection of K onto L by $K|L$. We denote by $\nu_{n,i}$ the unique Haar probability measure on $\mathcal{L}_i^n$, which is rotation invariant.

The mean inner and mean outer radii of a convex body K, introduced in [1], are geometrical extensions of the classical circumradius and inradius, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, which are well-known notions associated to a convex body. We use the notation $\mathrm{r}(K; A)$ to denote that the calculation of the inradius of K is made with respect to an affine subspace A, which contains K.

Definition 1.1. *For $K \in \mathcal{K}^n$ and $i = 1, \ldots, n$, the i-th mean projection outer and inner radii of K are defined as*

$$(1.1) \qquad \widetilde{\mathrm{R}}_i^\pi(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\nu_{n,i}(L), \qquad \widetilde{\mathrm{r}}_i^\pi(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L; L)\, d\nu_{n,i}(L).$$

In the same manner we define the mean inner and outer radii with respect to sections.

Definition 1.2. *For $K \in \mathcal{K}^n$ and $i = 1, \ldots, n$, the i-th mean section outer and inner radii of K are defined as*

$$(1.2) \quad \widetilde{\mathrm{R}}_i^\sigma(K) = \int_{\mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{R}(K \cap (x+L))\, d\nu_{n,i}(L), \quad \widetilde{\mathrm{r}}_i^\sigma(K) = \int_{\mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}(K \cap (x+L); x+L)\, d\nu_{n,i}(L).$$

From Definitions 1.1 and 1.2 follows that $\widetilde{\mathrm{R}}_n^\pi(K) = \widetilde{\mathrm{R}}_n^\sigma(K) = \mathrm{R}(K)$ and $\widetilde{\mathrm{r}}_n^\pi(K) = \widetilde{\mathrm{r}}_n^\sigma(K) = \mathrm{r}(K)$, and thus, we can see the mean radii as generalizations of the classical inradius and circumradius. Indeed, they belong to a larger family containing inradius and circumradius called *successive radii*.

Let $K \in \mathcal{K}^n$, and $i \in \{1, \ldots, n\}$, then the following eight families of successive radii can be found in the literature.

$$(1.3) \qquad \mathrm{R}_\pi^i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \mathrm{r}_i^\pi(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L);$$

and

$$(1.4) \qquad \mathrm{R}_\sigma^i(K) = \min_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{R}\big(K \cap (x + L)\big), \qquad \mathrm{r}_i^\sigma(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big).$$

When replacing the min-condition over $\mathcal{L}_i^n$ by a max-condition in the definition of R_π^i and R_σ^i, and the max-condition over $\mathcal{L}_i^n$ by a min-condition in the definition of r_i^π and r_i^σ, one obtains four more series of successive outer and inner radii:

$$(1.5) \qquad \mathrm{R}_i^\pi(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \mathrm{r}_\pi^i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L);$$

and

$$(1.6) \qquad \mathrm{R}_i^\sigma(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{R}\big(K \cap (x + L)\big), \qquad \mathrm{r}_\sigma^i(K) = \min_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big).$$

Again, directly from the definition follows that
$$(1.7)$$
$$\mathrm{R}_\pi^n(K) = \mathrm{R}_n^\pi(K) = \mathrm{R}_\sigma^n(K) = \mathrm{R}_n^\sigma(K) = \mathrm{R}(K), \quad \text{and} \quad \mathrm{r}_\pi^n(K) = \mathrm{r}_n^\pi(K) = \mathrm{r}_n^\sigma(K) = \mathrm{r}_\sigma^n(K) = \mathrm{r}(K).$$

We observe that the mean radii are natural extensions of the above introduced radii. Indeed, considering the functions $L \mapsto \mathrm{R}(K|L)$, $L \mapsto \mathrm{r}(K|L; L)$, $L \mapsto \max_{x \in L^\perp} \mathrm{R}\big(K \cap (x + L)\big)$, and $L \mapsto \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big)$, defined on $\mathcal{L}_i^n$, the i-th successive radii are defined by taking the maximum and the minimum over $\mathcal{L}_i^n$ of those. From this point of view, the mean radii are a *natural* average over $\mathcal{L}_i^n$.

We refer to [2, 3, 4, 10, 14, 15, 16, 17, 18, 19, 20, 24, 25, 27] and the references inside for more informations and applications on the topic, including generalizations of the classical Jung's and Steinhagen's inequalities; their interplay with different sums of convex bodies; inequalities relating them to other geometrical magnitudes, like the intrinsic volumes; connections to the theory of random polytopes; or, beyond geometry, their connection to finite dimensional Banach space theory and approximation theory via the Gelfand and Kolmogorov numbers; their connection to dimension reduction for "big data"; and computational complexity, within the realm of computer science and linear programming.

The main purpose of this paper is to analyse certain geometrical behaviour of the mean radii, and some successive radii, especially with respect to symmetrizations and, as a necessary byproduct, to parallel chord movement. In particular, we will consider Steiner, Schwarz and Minkowski symmetrizations, along with parallel chord movement. Although we put a special focus on mean radii, our target lies on the interplay of radii and various geometrical issues.

In Section 4 we prove the following result.

Theorem A (Theorem 4.8). *Let $\mathcal{K}_{[0,1]} := \{K(t) : t \in [0, 1]\} \subset \mathcal{K}^n$ be a family of convex bodies.*

(i) *If $\mathcal{K}_{[0,1]}$ is a convex family of convex bodies, then $t \mapsto \widetilde{\mathrm{R}}_i^\pi(K(t))$ and $t \mapsto \mathrm{R}_i^\pi(K(t))$ are convex functions for $t \in [0, 1]$.*

(ii) *If $\mathcal{K}_{[0,1]}$ is a concave family of convex bodies, then $t \mapsto \widetilde{\mathrm{r}}_i^\pi(K(t))$, $t \mapsto \widetilde{\mathrm{r}}_i^\sigma(K(t))$ and $t \mapsto \mathrm{r}_\sigma^i(K(t))$ and $t \mapsto \mathrm{r}_\pi^i(K(t))$ are concave functions for $t \in [0, 1]$.*

The main goal of Section 5 is to understand different aspects of the relation between the radii and geometric symmetrizations. First, we establish the following result in relation to Steiner symmetrization.

Theorem B (Theorem 5.6). *Let $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$, and let $1 \le i \le n$. We denote by $S_{u^\perp}(K)$ the Steiner symmetrization of K in the direction u. Then,*

(i) $\widetilde{\mathrm{R}}_i^\pi(S_{u^\perp}(K)) \le \widetilde{\mathrm{R}}_i^\pi(K)$,

(ii) $\mathrm{R}_i^\pi(S_{u^\perp}(K)) \le \mathrm{R}_i^\pi(K)$,

with equality for euclidean balls.

For the Minkowski symmetral, we prove the following.

Theorem C (Theorem 5.9). *Let $K \in \mathcal{K}^n$, $1 \leq k \leq n$, and let $L \in \mathcal{L}_k^n$. If we denote by $M_L(K)$ the Minkowski symmetrization of K with respect to the subspace L, then for all $1 \leq i \leq n$, we have*

$$\widetilde{\mathrm{R}}_i^\pi(M_L(K)) \leq \widetilde{\mathrm{R}}_i^\pi(K), \qquad \widetilde{\mathrm{r}}_i^\pi(K) \leq \widetilde{\mathrm{r}}_i^\pi(M_L(K)),$$

and

$$\widetilde{\mathrm{r}}_i^\sigma(K) \leq \widetilde{\mathrm{r}}_i^\sigma(M_L(K)),$$

with equality for euclidean balls.

The paper is organized as follows. In Section 2 we state some basic notions, mostly within Convex Geometry. In Section 3 we collect some known results on successive and mean radii. Sections 4 and 5 are devoted to our main results, containing, in particular, the proofs of Theorems A, B and C.

2. Preliminaries

The support function of a convex body $K \in \mathcal{K}^n$ in the direction $u \in \mathbb{S}^{n-1}$ is defined as $h(K, u) = \max\{\langle x, u \rangle : x \in K\}$ (see [29, Section 1.7]). The width of K in the direction $u \in \mathbb{S}^{n-1}$ is the sum of the support function of K in the directions u and $-u$, i.e., $\omega(K, u) = h(K, u) + h(K, -u)$. If we denote by $\mathrm{D}(K)$ the diameter of K, then the maximum of the widths of K coincides with the diameter itself, i.e, $\mathrm{D}(K) = \max_{u \in S^{n-1}} w(K, u)$, and the minimal width of K, denoted by $\omega(K)$, is given by $\omega(K) = \min_{u \in S^{n-1}} w(K, u)$. By considering the widths of K in all directions, the so-called *mean width* is introduced:

$$\mathrm{b}(K) = \frac{1}{n\kappa_n} \int_{\mathbb{S}^{n-1}} \omega(K, u) \, \mathrm{d}u = \frac{2}{n\kappa_n} \int_{\mathbb{S}^{n-1}} h(K, u) \, \mathrm{d}u,$$

where $\mathrm{d}u$ stands for the usual Lebesgue measure on the sphere $\mathbb{S}^{n-1}$, and $n\kappa_n$ is the surface area of the sphere. The radial function of a convex body K containing the origin is defined as

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

Then, the average length of chords (see e.g. [8]) of a convex body K (containing the origin) through the origin is given by

$$\ell(K) = \frac{2}{n\kappa_n} \int_{\mathbb{S}^{n-1}} \rho(K, u) \, \mathrm{d}u.$$

The Minkowski sum of $K, L \in \mathcal{K}^n$ is defined as $K + L := \{x + y : x \in K, y \in L\} \in \mathcal{K}^n$, and the difference body of $K \in \mathcal{K}^n$ is defined as $K - K := K + (-K)$, where $-K = \{-x : x \in K\}$ is the reflection of K on the origin.

For $K \in \mathcal{K}^n$, its dimension is the dimension of its affine hull, i.e., $\dim K = \dim \mathrm{aff}(K)$. We will denote by $\mathcal{K}_n^n$ the set of all convex bodies with dimension n, which we will refer to as full-dimensional convex bodies.

The space of convex bodies $\mathcal{K}^n$ is endowed with the Hausdorff metric [29, Section 1.8], which makes it a complete metric space. From now on, any topological notion in $\mathcal{K}^n$ is implicitly considered with respect to the Hausdorff metric.

As stated in the introduction, the mean inner and mean outer radii of a convex body K, given by (1.1), and (1.2), happen to be geometrical extensions of the classical circumradius and inradius, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, i.e.,

$$\mathrm{R}(K) := \inf\{R \geq 0 : \exists\, x \in \mathbb{R}^n \text{ with } K \subseteq x + RB_n\}$$

and

$$\mathrm{r}(K) := \sup\{r \geq 0 : \exists\, x \in \mathbb{R}^n \text{ with } x + rB_n \subseteq K\}.$$

From the definition of inradius follows that $r(K) > 0$ if and only if K is full-dimensional. This justifies the advantage of the notation $r(K; A)$ we have introduced for an affine subspace A containing K. Indeed, if $L \in \mathcal{L}_i^n$, $1 \le i \le n$, and $x \in L^\perp$ is such that $A = x + L$, then $r(K; x + L)$ is the inradius of K relative to the affine subspace $x + L$, i.e.,

$$r(K; x + L) := \sup\{r \ge 0 : \exists y \in L \text{ with } y + x + rB_{L,i} \subseteq K\},$$

where $B_{L,i}$ denotes the unit ball in L, that is $B_{L,i} = B_n \cap L$. Since the classical circumradius does not depend on the space where the body is embedded, we do not need to use the notation depending on a subspace for $R(K)$.

As an immediate consequence of the definitions of inradius and circumradius, we have that r and R are (positively) 1-homogeneous, i.e., $r(\lambda K) = \lambda r(K)$ and $R(\lambda K) = \lambda R(K)$, for every $K \in \mathcal{K}^n$ and $\lambda \ge 0$. Indeed, it follows from the latter that all successive and mean radii are 1-homogeneous. Moreover, as the inradius and the circumradius are also invariant with respect to translations, so are all the successive and mean radii too.

In the following, for completeness, we recall very briefly some aspects of the Haar (probability) measure, as it is an essential part in the definition of the mean radii. We refer to [26, Section 5.1], and the references therein, for a detailed study.

For $0 \le i \le n \in \mathbb{N}$, there is a natural measure on the space of $\mathcal{L}_i^n$, of i-dimensional linear subspaces of $\mathbb{R}^n$, endowed with a suitable topology via the operation of the topological group $SO(n)$.

Let ν be the unique Haar probability measure, invariant with respect to translation, in $SO(n)$ [26, Lemma 5.1]. If $L_0 \in \mathcal{L}_i^n$ is a i-dimensional subspace of $\mathbb{R}^n$, using the map $\beta_i : SO(n) \longrightarrow \mathcal{L}_i^n$, $\rho \mapsto \rho L_0$, the space $\mathcal{L}_i^n$ becomes a topological space, which happens to be compact, and the operation $SO(n) \times \mathcal{L}_i^n \longrightarrow \mathcal{L}_i^n$, $(\rho, L) \mapsto \rho L$ is continuous and transitive. Although there is a general construction, we have the following in $\mathcal{L}_i^n$. Let $G \subseteq SO(n)$, we denote by $\mathbf{1}_G : SO(n) \to \{0, 1\}$ the indicator function of the set G.

Proposition 2.1. [26, Corollary 5.1] *Let $i \in \{0, \dots, n\}$ and $L_0 \in \mathcal{L}_i^n$. Then*

$$\nu_{n,i}(\cdot) := \int_{SO(n)} \mathbf{1}_{\{\rho \in SO(n) : \rho L_0 \in \cdot\}} \, \nu(d\rho)$$

is the uniquely determined $SO(n)$-invariant Haar probability measure on $\mathcal{L}_i^n$. In particular, the definition is independent of the choice of the subspace $L_0 \in \mathcal{L}_i^n$.

In the following, we will consider some subsets of $\mathcal{L}_i^n$ having measure zero, mostly to deal with lower dimensional convex bodies, i.e., not full-dimensional ones. In order to do so, we will introduce the following notion.

Definition 2.2. [29, Section 4.4] *Two linear subspaces $L_1, L_2 \subseteq \mathbb{R}^n$ are said to be in Special Position (S. P.) if*

$$\text{lin}(L_1 \cup L_2) = L_1 + L_2 \ne \mathbb{R}^n$$

and

$$L_1 \cap L_2 \ne \{0\}.$$

We observe that two linear subspaces L_1 and L_2 are not in S. P. if and only if either the linear hull of the union of L_1 and L_2 is $\mathbb{R}^n$, i.e., $\text{lin}(L_1 \cup L_2) = \mathbb{R}^n$, or $L_1 \cap L_2 = \{0\}$.

Lemma 2.3. [29, Lemma 4.4.1] *Let ν be the unique Haar measure on $SO(n)$, and let $L_1, L_2 \subseteq \mathbb{R}^n$ be linear subspaces. Then,*

$$\nu(\{g \in SO(n) : gL_1, L_2 \text{ are in S. P.}\}) = 0.$$

Remark 2.4. *Let $n \ge 2$, $1 \le i \le j \le n - 1$, and let $L_j \in \mathcal{L}_j^n$. Then,*

$$\nu_{n,i}(\{L \in \mathcal{L}_i^n : L, L_j \text{ are in S. P.}\}) = 0.$$

We observe that the latter remark is a direct consequence of Lemma 2.3. Indeed, for $L_j \in \mathcal{L}_j^n$ and $L \in \mathcal{L}_i^n$, $\nu_{n,i}(\{L \in \mathcal{L}_i^n : L, L_j \text{ are in S. P.}\}) = \nu(\{g \in SO(n) : g(L), L_j \text{ are in S. P.}\}) = 0$ holds.

If $\{0\} \neq L \subseteq L_j$, with $L, L_j \in \mathcal{L}_j^n$, then L and L_j are in S. P., i.e., in Special Position. Thus, we have

$$\nu_{n,i}(\{L \in \mathcal{L}_i^n : L \subseteq L_j\}) \leq \nu_{n,i}(\{L \in \mathcal{L}_i^n : L, L_j \text{ are in S. P.}\}) = 0.$$

Thus, the following remark follows.

Remark 2.5. *Let $n \geq 2$, $1 \leq i \leq j \leq n-1$. Let $L_j \in \mathcal{L}_j^n$. Then $\nu_{n,i}(\{L \in \mathcal{L}_i^n : L \subseteq L_j\}) = 0$.*

We continue this section with the definition of the Schwarz and Steiner symmetrizations. We define first the Schwarz symmetrization, and introduce the Steiner symmetrization as a particular case of the latter. Indeed, the Steiner symmetrization is also a particular case of the so-called shadow system of convex bodies. The shadow system plays a crucial role in our proof of Theorem B, and will be treated in Section 5.

Next, we recall the definition of Schwarz symmetrization and collect several properties of the Schwarz symmetral of a convex body.

Definition 2.6. [21, Section 9.3] (Schwarz symmetrization). *Let $K \in \mathcal{K}^n$, $1 \leq k \leq n-1$, and $L \in \mathcal{L}_k^n$. For any $y \in K|L$, let $B_k(y, r_k) \subseteq y + L^\perp$ be the k-dimensional ball centered at y with radius r_k such that*

$$\text{vol}_{n-k}(B_k(y, r_k)) = \text{vol}_{n-k}(K \cap (y + L^\perp)).$$

Then, the Schwarz symmetral of K, with respect to L, is defined as

$$S_L(K) = \bigcup_{y \in K|L^\perp} B_k(y, r_k).$$

Lemma 2.7. [5] *Let $K, K_1, K_2 \in \mathcal{K}^n$, $L \in \mathcal{L}_k^n$, and let $1 \leq k \leq n-1$. Then,*

   (i) *$S_L(K)$ is a convex body.*
   (ii) *If $K_1 \subseteq K_2$, then $S_L(K_1) \subseteq S_L(K_2)$.*
   (iii) *$K|L \subseteq S_L(K)$.*

The following basic remark is necessary to introduce the Steiner symmetrization.

Remark 2.8. *Let $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$. There exist two functions f_K, $g_K : K|u^\perp \to \mathbb{R}$, f_K concave, and g_K convex, such that*

$$(2.1) \qquad\qquad K = \{x + \lambda u : x \in K|u^\perp, \ g_K(x) \leq \lambda \leq f_K(x)\}.$$

The defining functions f_K and g_K can be explicitly provided via the section $K \cap (x + u^\perp)$.

From now on, we refer to f_K, g_K as the defining functions associated to a convex body K in the direction u.

Definition 2.9. [29] *Let $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$ and let $f_K, g_K : K|u^\perp \to \mathbb{R}$, be the defining functions associated to K, namely, $K = \{x + \lambda u \mid x \in K|u^\perp, \ g_K(x) \leq \lambda \leq f_K(x)\}$. Then, the Steiner symmetrization $S_{u^\perp}(K)$ of K, or Steiner symmetral of K, in the direction u, is defined as follows:*

$$(2.2) \qquad S_{u^\perp}(K) = \left\{ x + \lambda u : \ x \in K|u^\perp, \ -\frac{f_K(x) - g_K(x)}{2} \leq \lambda \leq \frac{f_K(x) - g_K(x)}{2} \right\}.$$

We observe that

$$S_{u^\perp}(K) = \bigcup_{x \in K|u^\perp} \left[x + \frac{g_K(x) - f_K(x)}{2} u, x + \frac{f_K(x) - g_K(x)}{2} u \right]$$

and thus, $S_{u^\perp}(K) = S_L(K)$ for $L = u^\perp$, $u \in \mathbb{S}^{n-1}$, i.e., the Schwarz symmetrization coincides with the Steiner symmetrization when L is a hyperplane.

If $M \subseteq K \in \mathcal{K}^n$, and $u \in \mathbb{S}^{n-1}$, it follows from Lemma 2.7 (ii), and the previous observation, that $S_{u^\perp}(M) \subseteq S_{u^\perp}(K)$. Further, Lemma 2.7 (i) ensures that the Steiner symmetral of $K \in \mathcal{K}^n$ in any direction $u \in \mathbb{S}^{n-1}$ is again a convex body (see also [21]), i.e., $S_{u^\perp}(K) \in \mathcal{K}^n$, and direct application of Fubini's theorem yield $\text{vol}_n(K) = \text{vol}_n(S_{u^\perp}(K))$.

Indeed, the connection between the Schwarz and Steiner symmetrization goes beyond the above mentioned observation. Refining the argument that led to the "Sphericity Theorem of Gross", see

[21, Corollary 9.1], i.e., for every $K \in \mathcal{K}^n$ there is a sequence of iterations of Steiner symmetrals converging to a ball, shows that we can extrapolate a sequence of Steiner symmetrals that converges to the Schwarz symmetral of the body.

Theorem 2.10. [5, 21] *Let $K \in \mathcal{K}^n$, $1 \leq k \leq n-1$, $L \in \mathcal{L}_k^n$, and let $S_L(K)$ be the Schwarz symmetral of K with respect to L. Then, there exists a sequence $(u_j)_{j \in \mathbb{N}} \subseteq \mathbb{S}^{n-1} \cap L$ of directions in L, such that the sequence $S_{u_j^\perp}(\cdots(S_{u_1^\perp}(K)))$ converges to $S_L(K)$, as $j \to \infty$, with respect to Hausdorff metric, i.e., $S_L(K)$ is the limit of a sequence of Steiner symmetrizations of K.*

Since our aim is to understand the relation between the successive and mean radii of a convex body and its Steiner and Schwarz symmetrals, we first address known results about the behaviour of the mean width, diameter, inradius and circumradius with respect to them.

Lemma 2.11. [21, Chapter 9], *Let $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$, $1 \leq k \leq n-1$, and let $L \in \mathcal{L}_k^n$. Then,*

  (i) $\mathrm{b}(S_{u^\perp}(K)) \leq \mathrm{b}(K)$ *and* $\mathrm{b}(S_L(K)) \leq \mathrm{b}(K)$.

  (ii) $\mathrm{D}(S_{u^\perp}(K)) \leq \mathrm{D}(K)$ *and* $\mathrm{D}(S_L(K)) \leq \mathrm{D}(K)$.

  (iii) $\mathrm{r}(K) \leq \mathrm{r}(S_{u^\perp}(K))$ *and* $\mathrm{r}(K) \leq \mathrm{r}(S_L(K))$.

  (iv) $\mathrm{R}(S_{u^\perp}(K)) \leq \mathrm{R}(K)$, *and* $\mathrm{R}(S_L(K)) \leq \mathrm{R}(K)$.

For the minimal width, we also remark that it is known that the minimal width of a set can be both increased and reduced after performing a Steiner symmetrization on it, see [12, p. 90-92].

Next we recall the definition and basic properties of the Minkowski symmetrization (see e.g. [5], [7]). We need to deal first with some issues about reflections of convex bodies on k-planes. Following [29, Section 10.3] for the case $k = n-1$, we denote by $\sigma_L(K)$ the reflection of $K \in \mathcal{K}^n$ on L, that is the image of K under the linear map $x \to 2(x|L) - x$, where now $L \in \mathcal{L}_k^n$, $1 \leq k \leq n$. For $k = 0$, we have $L = \{0\}$ and then $\sigma_L(K) = -K$.

The following remark includes some straightforward properties of the reflection body of a convex body $K \in \mathcal{K}^n$ with respect to $L \in \mathcal{L}_k^n$, $0 \leq k \leq n$. We denote indistinctly by σ_L the map from $\mathbb{R}^n$ on itself, and its natural extension to subsets of $\mathbb{R}^n$.

Remark 2.12. *Let $K \in \mathcal{K}^n$, $0 \leq k \leq n$, $L \in \mathcal{L}_k^n$, $L_i \in \mathcal{L}_i^n$ for $1 \leq i \leq n-1$, and $x \in L_i^\perp$. Then,*

  (i) $\sigma_L(K)|L_i = \sigma_L(K|\sigma_L(L_i))$,

  (ii) $\sigma_L(K) \cap (x + L_i) = \sigma_L(K \cap \sigma_L(x + L_i))$.

We observe now that the invariance of balls under reflections implies that the inradius and circumradius of a convex body and its reflection onto a subspace coincide. More precisely, we have

$$\mathrm{R}(\sigma_L(K)) = \mathrm{R}(K), \quad \mathrm{r}(\sigma_L(K)) = \mathrm{r}(K), \quad \mathrm{R}(\sigma_L(K)|L_i) = \mathrm{R}(\sigma_L(K|\sigma_L(L_i))),$$
$$\mathrm{r}(\sigma_L(K)|L_i; L_i) = \mathrm{r}(\sigma_L(K|\sigma_L(L_i)); L_i), \quad \mathrm{R}\left(\sigma_L(K) \cap (x + L_i)\right) = \mathrm{R}\left(\sigma_L(K \cap \sigma_L(x + L_i))\right),$$
$$\mathrm{r}\left(\sigma_L(K) \cap (x + L_i); x + L_i\right) = \mathrm{r}\left(\sigma_L(K \cap \sigma_L(x + L_i)); x + L_i\right),$$

for $K \in \mathcal{K}^n$, $L \in \mathcal{L}_k^n$, $L_i \in \mathcal{L}_i^n$ and $0 \leq k \leq n$, $1 \leq i \leq n-1$.

The Minkowski symmetrization of a convex body $K \in \mathcal{K}^n$ is, up to a constant, the Minkowski sum of K and its reflection onto a linear subspace.

Definition 2.13. *Let $K \in \mathcal{K}^n$, $0 \leq k \leq n$, and let $L \in \mathcal{L}_k^n$. The Minkoswki symmetral of K, with respect to L, is defined as*

$$M_L(K) = \frac{1}{2}(K + \sigma_L(K)).$$

From the definition follows directly that $M_L(K)$ is a convex body, which is symmetric with respect to L, and it satisfies $K|L \subset M_L(K)$. Further, if $k = 0$, i.e., when $L = \{0\}$, the Minkowski symmetral of K is, up to $1/2$, the difference body of K, as $M_L(K) = \frac{1}{2}(K + (-K)) = \frac{1}{2}(K - K)$. If $k = n-1$, then $L = u^\perp$, for some $u \in \mathbb{S}^{n-1}$ and in this case, we remark that the Minkowski symmetral contains the Steiner symmetral, i.e.,

(2.3) $$S_{u^\perp}(K) \subseteq M_{u^\perp}(K).$$

For the interplay between the inradius and cirmuradius and the Minkowski symmetral we refer to Sections 4 and 5, where we will deal with the behaviour of the radii with respect to the Minkowski sum.

3. KNOWN RESULTS FOR SUCCESSIVE AND MEAN RADII

In this section we collect some known results about successive and mean radii. Further, we aim to settle some open cases for the successive radii and/or establish them for some of the mean radii. We start recalling the cases $i = n$ and $i = 1$ for all inner and outer succesive and mean radii [1, Lemma 2.1], namely:

$$\mathrm{R}_\pi^n(K) = \mathrm{R}_n^\pi(K) = \mathrm{R}_\sigma^n(K) = \mathrm{R}_n^\sigma(K) = \widetilde{\mathrm{R}}_n^\pi(K) = \widetilde{\mathrm{R}}_n^\sigma(K) = \mathrm{R}(K),$$
$$\mathrm{r}_\pi^n(K) = \mathrm{r}_n^\pi(K) = \mathrm{r}_n^\sigma(K) = \mathrm{r}_\sigma^n(K) = \widetilde{\mathrm{r}}_n^\pi(K) = \widetilde{\mathrm{r}}_n^\sigma(K) = \mathrm{r}(K),$$
$$\mathrm{R}_1^\pi(K) = \mathrm{r}_1^\pi(K) = \mathrm{R}_1^\sigma(K) = \mathrm{r}_1^\sigma(K) = \mathrm{D}(K)/2,$$
$$\mathrm{R}_\pi^1(K) = \mathrm{r}_\pi^1(K) = \mathrm{R}_\sigma^1(K) = \mathrm{r}_\sigma^1(K) = \omega(K)/2,$$
$$\widetilde{\mathrm{R}}_1^\pi(K) = \frac{1}{2}\mathrm{b}(K) = \widetilde{\mathrm{r}}_1^\pi(K) \quad \text{and} \quad \widetilde{\mathrm{R}}_1^\sigma(K) = \frac{1}{4}\ell(K - K) = \widetilde{\mathrm{r}}_1^\sigma(K).$$

For the latter, note that $\mathrm{r}(K|L; L) = \mathrm{R}(K|L)$ for any one-dimensional linear subspace $L \in \mathcal{L}_1^n$. The following result establishes the equality of the maximum outer radii, both section and projection.

Theorem 3.1. [10, Theorem 2.9] *The maximal outer projection and the maximal outer section radii are equal for any $i = 1, \cdots, n$, and any convex body $K \in \mathcal{K}^n$*

$$(3.1) \qquad\qquad\qquad \mathrm{R}_i^\pi(K) = \mathrm{R}_i^\sigma(K).$$

We consider now the monotonicity of the radii in the index i, $1 \leq i \leq n$.

Proposition 3.2. [1, 14] *Let $K \in \mathcal{K}^n$. Then, all the introduced families of outer radii are increasing in $1 \leq i \leq n$, whereas the inner radii are decreasing, in $1 \leq i \leq n$. In particular,*

$$(3.2) \qquad \begin{aligned} \mathrm{r}(K) = \widetilde{\mathrm{r}}_n^\pi(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1^\pi(K) = \frac{1}{2}\mathrm{b}(K) = \widetilde{\mathrm{R}}_1^\pi(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n^\pi(K) = \mathrm{R}(K), \\ \mathrm{r}(K) = \widetilde{\mathrm{r}}_n^\sigma(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1^\sigma(K) = \frac{1}{4}\,\ell(K - K) = \widetilde{\mathrm{R}}_1^\sigma(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n^\sigma(K) = \mathrm{R}(K). \end{aligned}$$

Furthermore,

$$\mathrm{R}_\sigma^i(K) \leq \widetilde{\mathrm{R}}_i^\sigma(K) \leq \mathrm{R}_i^\sigma(K) = \mathrm{R}_i^\pi(K), \quad and \quad \mathrm{r}_\sigma^i(K) \leq \mathrm{r}_\pi^i(K) \leq \widetilde{\mathrm{r}}_i^\pi(K) \leq \mathrm{r}_i^\pi(K).$$

We observe also the straightforward relations $\mathrm{R}_\sigma^i(K) \leq \mathrm{R}_\pi^i(K) \leq \widetilde{\mathrm{R}}_i^\pi(K) \leq \mathrm{R}_i^\pi(K) = \mathrm{R}_i^\sigma(K)$, and their analog for the inner radii $\mathrm{r}_\sigma^i(K) \leq \widetilde{\mathrm{r}}_i^\sigma(K) \leq \mathrm{r}_i^\sigma(K) \leq \mathrm{r}_i^\pi(K)$, which directly enlarge the above chains of inequalities due to monotonicity.

Thus, all types of considered radii are monotonic with respect to the parameter i, $1 \leq i \leq n$. The inner radii are decreasing in i, meanwhile the outer ones are increasing, and this is independent of whether their definition is given by projection or section, maximum or minimum.

Now we focus on the continuity of the successive radii with respect to the Hausdorff metric.

Proposition 3.3. [14] *All the successive radii are continuous in $\mathcal{K}_n^n$. Moreover, they are all continuous in $\mathcal{K}^n$ except for r_i^σ, for all $2 \leq i \leq n - 1$. The convergence is with respect to the Hausdorff metric in the appropriate space.*

The proof of the continuity of all the successive radii in $\mathcal{K}_n^n$ can be found in [14, Proposition 1.2.1], as a consequence of the monotonicity of the inradius and circumradius and their 1-homogeneity. The continuity of all the successive radii, except for r_i^σ, in $\mathcal{K}^n$ can be found in [14, Proposition 1.2.2]. In [14, Remark 4.3.3], the author established a counterexample for the continuity of r_i^σ in $\mathcal{K}^n$.

The next result provides us with inequalities relating the radii of a convex body K, and its difference body.

Lemma 3.4. [1, 14] *Let* $K \in \mathcal{K}^n$ *and* $1 \leq i \leq n$. *Then*

$$(3.3) \quad \sqrt{\frac{2(i+1)}{i}} \mathrm{R}_\pi^i(K) \leq \mathrm{R}_\pi^i(K - K) \leq 2\mathrm{R}_\pi^i(K) \quad and \quad 2\mathrm{r}_i^\sigma(K) \leq \mathrm{r}_i^\sigma(K - K) < 2(i+1)\mathrm{r}_i^\sigma(K)$$

$$(3.4) \quad \sqrt{\frac{2(i+1)}{i}} \widetilde{\mathrm{R}}_i^\pi(K) \leq \widetilde{\mathrm{R}}_i^\pi(K - K) \leq 2\widetilde{\mathrm{R}}_i^\pi(K) \quad and \quad 2\widetilde{\mathrm{r}}_i^\sigma(K) \leq \widetilde{\mathrm{r}}_i^\sigma(K - K) \leq c(i)\widetilde{\mathrm{r}}_i^\sigma(K)$$

where

$$c(i) = \begin{cases} 2\sqrt{i}, & \text{if } i \text{ is odd}, \\ \frac{2(i+1)}{\sqrt{i+2}}, & \text{if } i \text{ is even}. \end{cases}$$

Before we deal with the behaviour of some of the radii of convex bodies without interior points, we observe that for any $K \in \mathcal{K}^n$, $\dim(K) \geq 1$, and $1 \leq i \leq n$, we have

$$\widetilde{\mathrm{R}}_i^\pi(K) \geq \widetilde{\mathrm{R}}_1^\pi(K) = \widetilde{\mathrm{r}}_1^\pi(K) = \frac{\mathrm{b}(K)}{2} > 0,$$

hence, $\widetilde{\mathrm{R}}_i^\pi$ is strictly positive for any i. The next proposition, which follows directly from the definitions, provides us with the fact that several inner radii vanish for lower dimensional convex bodies.

Proposition 3.5. *Let* $K \in \mathcal{K}^n$ *be a convex body such that* $\dim(K) = j < n$. *Then, for every* $j < i \leq n$, *we have* $\mathrm{r}_\pi^i(K) = \widetilde{\mathrm{r}}_i^\pi(K) = \mathrm{r}_i^\pi(K) = 0$ *and* $\mathrm{r}_i^\sigma(K) = 0$.

We remark that by $\mathrm{r}_i^\sigma(K) = 0$ follows also $\widetilde{\mathrm{r}}_i^\sigma(K) = \mathrm{r}_\sigma^i(K) = 0$ for a convex body $K \in \mathcal{K}^n$ such that $\dim(K) = j < n$.

Using (3.2) we can establish the same statement for all $1 \leq i \leq n$, in the case of the mean section inner radii, and the minimum section inner radii.

Proposition 3.6. *Let* $K \in \mathcal{K}^n$ *be a convex body such that* $\dim(K) = j < n$. *Then, for every* $i \in \{1, \cdots, n\}$

$$\widetilde{\mathrm{r}}_i^\sigma(K) = \mathrm{r}_\sigma^i(K) = 0.$$

Proof. Let $K \in \mathcal{K}^n$ be a convex body such that $\dim(K) = j < n$. Then, there exists $L_j \in \mathcal{L}_j^n$, so that $K - K \subseteq L_j \in \mathcal{L}_j^n$, and thus, $\dim(K - K) = j$. By (3.2), we have

$$\widetilde{\mathrm{r}}_i^\sigma(K) \leq \widetilde{\mathrm{r}}_1^\sigma(K) = \frac{1}{4}\ell(K - K) = \frac{1}{2n\kappa_n}\int_{\mathbb{S}^{n-1}} \rho(K - K, u)\,\mathrm{d}u.$$

Hence, we have

$$\frac{1}{4}\ell(K - K) = \frac{1}{2n\kappa_n}\int_{\mathbb{S}^{n-1}} \rho(K - K, u)\,\mathrm{d}u = \frac{1}{2n\kappa_n}\int_{\mathbb{S}^{n-1} \cap L_j} \rho(K - K, u)\,\mathrm{d}u = 0.$$

$\square$

A similar result is established for $\widetilde{\mathrm{R}}_i^\sigma$, as follows.

Proposition 3.7. *Let* $K \in \mathcal{K}^n$, *with* $1 \leq \dim(K) = j < n$. *Let* $1 \leq i \leq j$ *be such that* $i + j \leq n$. *Then,* $\widetilde{\mathrm{R}}_i^\sigma(K) = 0$.

Proof. Let $K \in \mathcal{K}^n$ with $1 \leq \dim(K) = j < n$, and let $1 \leq i \leq j$ with $i + j \leq n$. W.l.o.g. we assume that $K \subseteq L_j$ for some $L_j \in \mathcal{L}_j^n$. By Remark 2.4, we obtain

$$\widetilde{\mathrm{R}}_i^\sigma(K) = \int_{\{L \in \mathcal{L}_i^n:\ L, L_j \text{ are not in S.P.}\}} \max_{x \in L^\perp} \mathrm{R}(K \cap (x + L))d\nu_{n,i}(L).$$

Let now $L \in \mathcal{L}_i^n$ be such that $L \cap L_j = \{0\}$. Then, $\dim((L + x) \cap L_j) = 0$, for all $x \in L^\perp$. Hence, $\max_{x \in L^\perp} \mathrm{R}(K \cap (x + L)) = 0$, and we have

$$\widetilde{\mathrm{R}}_i^\sigma(K) = \int_{\{L \in \mathcal{L}_i^n:\ L + L_j = \mathbb{R}^n,\ 1 \leq \dim(L \cap L_j) \leq i\}} \max_{x \in L^\perp} \mathrm{R}(K \cap (x + L))d\nu_{n,i}(L).$$

If $\{L \in \mathcal{L}_i^n : \ L + L_j = \mathbb{R}^n \ \text{and} \ 1 \leq \dim(L \cap L_j) \leq i\}$ is not the empty set, then

$$n = \dim(L + L_j) = \dim(L) + \dim(L_j) - \dim(L \cap L_j) = i + j - \dim(L \cap L_j),$$

and thus, $1 \leq i + j - n \leq i$. This implies that $n + 1 \leq i + j$, which yields a contradiction, providing us with $\{L \in \mathcal{L}_i^n : \ L + L_j = \mathbb{R}^n \ 1 \leq \dim(L \cap L_j) \leq i\} = \emptyset$ and, finally, with $\widetilde{\mathrm{R}}_i^\sigma(K) = 0$. $\square$

We finish this section with the following remark on convex bodies of lower dimension.

Remark 3.8. *Let $K \in \mathcal{K}^n$ such that $1 \leq \dim(K) = j < n$. Let $1 \leq i \leq j$ be such that $i + j \leq n$. From Proposition 3.7, together with $\mathrm{R}_\sigma^i(K) \leq \widetilde{\mathrm{R}}_i^\sigma(K)$, we directly obtain that $\mathrm{R}_\sigma^i(K) = 0$.*

4. Minkowski sums, continuity and convexity & concavity issues of radii

In this section we will analyse the behaviour of the inner and outer radii with respect to the Minkowski sum, and address different aspects of the continuity, and concavity and convexity of those.

We start recalling the following classical inequalities for the inradius, the circumradius, the minimal width and the diameter of two convex bodies $K, M \in \mathcal{K}^n$:

$$(4.1) \qquad \begin{array}{cc} \mathrm{r}(K + M) \geq \mathrm{r}(K) + \mathrm{r}(M), & \mathrm{R}(K + M) \leq \mathrm{R}(K) + \mathrm{R}(M), \\ \omega(K + M) \geq \omega(K) + \omega(M), & \mathrm{D}(K + M) \leq \mathrm{D}(K) + \mathrm{D}(M). \end{array}$$

We remark that from (4.1) the behaviour of the inradius, circumradius, minimal width and diameter with respect to the Minkowski symmetrizations follows immediately, more precisely $\mathrm{r}(K) \leq \mathrm{r}(M_L(K))$, $\mathrm{R}(M_L(K)) \leq \mathrm{R}(K)$, $\omega(K) \leq \omega(M_L(K))$, and $\mathrm{D}(M_L(K)) \leq \mathrm{D}(K)$. Moroever, the *linear nature* of the Minkowski symmetrization and the mean width with respect to the Minkowski addition yield

$$(4.2) \qquad \qquad \mathrm{b}(M_L(K)) = \mathrm{b}(K),$$

which also provides us immediately, together with inequality (2.3), with a proof of the inequality $\mathrm{b}(S_{u^\perp}(K)) \leq \mathrm{b}(K)$.

In the next, we address the analogous relations for some of the other radii. We beginn with known results.

Proposition 4.1. [1, 14] *Let $K, M \in \mathcal{K}^n$ be convex bodies. Then,*

    (i) $\widetilde{\mathrm{r}}_i^\pi(K) + \widetilde{\mathrm{r}}_i^\pi(M) \leq \widetilde{\mathrm{r}}_i^\pi(K + M)$ *if $2 \leq i \leq n$. For $i = 1$ this is an equality.*

    (ii) $\frac{1}{\sqrt{2}}(\mathrm{r}_i^\sigma(K) + \mathrm{r}_i^\sigma(M)) \leq \mathrm{r}_i^\sigma(K + M)$. *The inequality is best possible.*

    (iii) $\frac{\sqrt{2}}{2}\left(\widetilde{\mathrm{R}}_i^\pi(K) + \widetilde{\mathrm{R}}_i^\pi(M)\right) \leq \widetilde{\mathrm{R}}_i^\pi(K + M) \leq \widetilde{\mathrm{R}}_i^\pi(K) + \widetilde{\mathrm{R}}_i^\pi(M)$, *if $2 \leq i \leq n$.*
       For $i = 1$, inequality in the right hand-side is an equality.

    (iv) $\mathrm{R}_\pi^i(K) + \mathrm{R}_\pi^i(M) \leq \sqrt{2}\,\mathrm{R}_\pi^i(K + M)$, *if $2 \leq i \leq n$.*
       For $i = 1$ we have $\mathrm{R}_\pi^1(K + M) \geq \mathrm{R}_\pi^1(K) + \mathrm{R}_\pi^1(M)$.

The inequalities (i) and (iii) come from [1, Proposition 4.3]. We would like to remark, that the upper bound for $\widetilde{\mathrm{r}}_i^\pi(K + M)$ stated in [1] is clearly not true (it is an erratum). The inequalities in (ii) and (iv) can be found in [14, Theorem 4.2.1 and Theorem 4.1.1]. Notice that when $i = 1$, as $2\mathrm{R}_\pi^1(K) = \omega(K)$, the last inequality follows from (4.1).

In the spirit of the result just stated, we analyse the behavior of further radii with respect to the Minkowski sum.

Proposition 4.2. *Let $K, M \in \mathcal{K}^n$ be convex bodies, and $1 \leq i \leq n$. Then,*

    (i) $\mathrm{r}_\pi^i(K) + \mathrm{r}_\pi^i(M) \leq \mathrm{r}_\pi^i(K + M)$.

    (ii) $\frac{1}{2}(\mathrm{r}_i^\pi(K) + \mathrm{r}_i^\pi(M)) \leq \mathrm{r}_i^\pi(K + M)$.

    (iii) $\widetilde{\mathrm{r}}_i^\sigma(K) + \widetilde{\mathrm{r}}_i^\sigma(M) \leq \widetilde{\mathrm{r}}_i^\sigma(K + M)$.

    (iv) $\mathrm{r}_\sigma^i(K) + \mathrm{r}_\sigma^i(M) \leq \mathrm{r}_\sigma^i(K + M)$.

(v) $\frac{1}{2\sqrt{2}}(\mathrm{R}_i^\pi(K) + \mathrm{R}_i^\pi(M)) \leq \mathrm{R}_i^\pi(K + M) \leq \mathrm{R}_i^\pi(K) + \mathrm{R}_i^\pi(M).$

Proof.

(i) Let $L \in \mathcal{L}_i^n$, then $(K + M)|L = K|L + M|L$. Hence,
$$\mathrm{r}((K + M)|L; L) = \mathrm{r}(K|L + M|L; L) \geq \mathrm{r}(K|L; L) + \mathrm{r}(M|L; L).$$
This yields
$$\min_{L \in \mathcal{L}_i^n} \mathrm{r}((K + M)|L; L) \geq \min_{L \in \mathcal{L}_i^n}(\mathrm{r}(K|L; L) + \mathrm{r}(M|L; L)) \geq \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L) + \min_{L \in \mathcal{L}_i^n} \mathrm{r}(M|L; L),$$
as (i) states.

(ii) Let $L_1, L_2 \in \mathcal{L}_i^n$ be such that $\mathrm{r}_i^\pi(K) = \mathrm{r}(K|L_1; L_1)$ and $\mathrm{r}_i^\pi(M) = \mathrm{r}(M|L_2; L_2)$. Then, (4.1) yields
$$\begin{aligned}
\mathrm{r}(K|L_1; L_1) + \mathrm{r}(M|L_2; L_2) &\leq \mathrm{r}(K|L_1; L_1) + \mathrm{r}(M|L_1; L_1) + \mathrm{r}(K|L_2; L_2) + \mathrm{r}(M|L_2; L_2) \\
&\leq \mathrm{r}((K + M)|L_1; L_1) + \mathrm{r}((K + M)|L_2; L_2) \\
&\leq 2\mathrm{r}_i^\pi(K + M).
\end{aligned}$$

(iii) Let $1 \leq i \leq n$, and let $L \in \mathcal{L}_i^n$. Let x_1 and $x_2 \in L^\perp$ be such that
$$\max_{x \in L^\perp} \mathrm{r}(K \cap (x + L); x + L) = \mathrm{r}(K \cap (x_1 + L); x_1 + L)$$
and
$$\max_{x \in L^\perp} \mathrm{r}(M \cap (x + L); x + L) = \mathrm{r}(M \cap (x_2 + L); x_2 + L).$$
Then, we have
$$\begin{aligned}
\mathrm{r}(K \cap (x_1 + L); x_1 + L) + \mathrm{r}(M \cap (x_2 + L); x_2 + L) \\
\leq \mathrm{r}((K \cap (x_1 + L)) + (M \cap (x_2 + L)); x_1 + x_2 + L) \\
\leq \mathrm{r}((K + M) \cap (x_1 + x_2 + L); x_1 + x_2 + L) \\
\leq \max_{x \in L^\perp} \mathrm{r}((K + M) \cap (x + L); x + L).
\end{aligned}$$
Thus, for every $L \in \mathcal{L}_i^n$,

(4.3)
$$\begin{aligned}
\max_{x \in L^\perp} \mathrm{r}(K \cap (x + L); x + L) + \max_{x \in L^\perp} \mathrm{r}(M \cap (x + L); x + L) \\
\leq \max_{x \in L^\perp} \mathrm{r}((K + M) \cap (x + L); x + L).
\end{aligned}$$

From this, it is enough to integrate on $\mathcal{L}_i^n$ in (4.3) to obtain the result.

(iv) We follow the steps in (iii) up to (4.3). Now, we can apply the minimum over $\mathcal{L}_i^n$ in (4.3) to obtain (iv).

(v) We observe first that, following [14, p. 49], we have
$$\mathrm{R}((K + M)|L) \geq \frac{1}{\sqrt{2}}(\mathrm{R}(K|L) + \mathrm{R}(M|L)).$$
Let $L_1, L_2 \in \mathcal{L}_i^n$ be such that $\mathrm{R}_i^\pi(K) = \mathrm{R}(K|L_1; L_1)$ and $\mathrm{R}_i^\pi(M) = \mathrm{R}(M|L_2; L_2)$. Then,
$$\begin{aligned}
\mathrm{R}_i^\pi(K) + \mathrm{R}_i^\pi(M) &= \mathrm{R}(K|L_1) + \mathrm{R}(M|L_2) \\
&\leq \mathrm{R}(K|L_1) + \mathrm{R}(M|L_1) + \mathrm{R}(M|L_2) + \mathrm{R}(K|L_2) \\
&\leq \sqrt{2}\big(\mathrm{R}((K + M)|L_1) + \mathrm{R}((K + M)|L_2)\big) \\
&\leq 2\sqrt{2}\mathrm{R}_i^\pi(K + M).
\end{aligned}$$
In order to prove the right-hand side inequality, we just need to apply (4.1). We consider $L_i \in \mathcal{L}_i^n$, so that $\mathrm{R}_i^\pi(K + M) = \mathrm{R}((K + M)|L_i)$. Then,
$$\begin{aligned}
\mathrm{R}_i^\pi(K + M) = \mathrm{R}((K + M)|L_i) &\leq \mathrm{R}(K|L_i) + \mathrm{R}(M|L_i) \\
&\leq \mathrm{R}_i^\pi(K) + \mathrm{R}_i^\pi(M).
\end{aligned}$$

$\square$

We observe that euclidean balls provide us with equality in Proposition 4.2 (i), (iii), (iv), and the right-hand-side of (v). Recalling that $\mathrm{r}_1^\pi(K) = \mathrm{D}(K)/2$, taking (4.1) into account, we have $\mathrm{r}_1^\pi(K + M) \leq \mathrm{r}_1^\pi(K) + \mathrm{r}_1^\pi(M)$, which is the reverse inequality of (ii), hence, in general, an improvement of inequality (ii) can not hold.

It is natural to ask, whether there is a constant, such that some of the lower bounds for the radii of the Minkowski sum, as in Proposition 4.2 (i)–(iv), can become upper bounds. The following result in that spirit was established in [14].

Proposition 4.3. [14, Proposition 4.1.1] *Let* $1 < i < n$. *Then, there is no* $c > 0$ *such that* $\mathrm{R}_\pi^i(K + M) \leq c(\mathrm{R}_\pi^i(K) + \mathrm{R}_\pi^i(M))$.

With the same ideas of the proofs of the latter result in [14], we can prove the following proposition.

Proposition 4.4. *Let* $n \geq 3$.

(i) *Let* $1 < i < n$ *be such that* $n + 1 < 2i$. *There is no constant* $c > 0$ *such that any of the following inequalities holds for all* $K, M \in \mathcal{K}^n$:

(a) $\mathrm{r}_\pi^i(K + M) \leq c(\mathrm{r}_\pi^i(K) + \mathrm{r}_\pi^i(M))$.

(b) $\widetilde{\mathrm{r}}_i^\pi(K + M) \leq c(\widetilde{\mathrm{r}}_i^\pi(K) + \widetilde{\mathrm{r}}_i^\pi(M))$.

(c) $\mathrm{r}_i^\pi(K + M) \leq c(\mathrm{r}_i^\pi(K) + \mathrm{r}_i^\pi(M))$.

(ii) *For every* $1 < i < n$, *there is no constant* $c > 0$ *such that any of the following inequalities holds for all* $K, M \in \mathcal{K}^n$:

(a) $\mathrm{r}_\sigma^i(K + M) \leq c(\mathrm{r}_\sigma^i(K) + \mathrm{r}_\sigma^i(M))$,

(b) $\widetilde{\mathrm{r}}_i^\sigma(K + M) \leq c(\widetilde{\mathrm{r}}_i^\sigma(K) + \widetilde{\mathrm{r}}_i^\sigma(M))$,

(c) $\mathrm{r}_i^\sigma(K + M) \leq c(\mathrm{r}_i^\sigma(K) + \mathrm{r}_i^\sigma(M))$.

(iii) *Let* $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. *There is no constant* $c > 0$ *such that any of the following inequalities holds for all* $K, M \in \mathcal{K}^n$:

(a) $\widetilde{\mathrm{R}}_i^\sigma(K + M) \leq c(\widetilde{\mathrm{R}}_i^\sigma(K) + \widetilde{\mathrm{R}}_i^\sigma(M))$.

(b) $\mathrm{R}_\sigma^i(K + M) \leq c(\mathrm{R}_\sigma^i(K) + \mathrm{R}_\sigma^i(M))$.

Proof. The idea of the proof for all inequalities has the same underlying construction, which follows the ideas in [14, Theorem 1.1 and Proposition 1.1]. The construction consists on finding two approppriate coordinate cubes and then, use the properties of the radii applied to bodies of lower dimensions.

In the first case (i), we consider

$$K = \sum_{k=1}^{i-1}[-e_k, e_k], \qquad M = \sum_{k=i}^{n}[-e_k, e_k],$$

where $\{e_i, 1 \leq i \leq n\}$, denote the vectors of the standard orthonormal basis of $\mathbb{R}^n$. Then, from Proposition 3.5 follows that $\mathrm{r}_i^\pi(K) = \mathrm{r}_i^\pi(M) = 0$, since $n - i + 1 < i$. On the other hand, it is clear that

$$K + M = \sum_{k=1}^{n}[-e_k, e_k]$$

and thus, $\mathrm{r}_\pi^i(K + M) > 0$.

For (ii), we take into consideration $K = \sum_{k=1}^{i}[-e_k, e_k]$, and $M = \sum_{k=i+1}^{n}[-e_k, e_k]$. Then, Proposition 3.6 provides us with $\widetilde{\mathrm{r}}_i^\sigma(K) = \widetilde{\mathrm{r}}_i^\sigma(M) = 0$, whilst

$$K + M = \sum_{k=1}^{n}[-e_k, e_k],$$

and thus, $\mathrm{r}_\sigma^i(K + M) > 0$.

For the last part (iii), we take the cubes $K = \sum_{k=1}^{j}[-e_k, e_k]$ and $M = \sum_{k=j+1}^{n}[-e_k, e_k]$, with $i \leq j \leq \lfloor \frac{n}{2} \rfloor$. It is clear that $i + j \leq n$ and $i + n - j \leq n$. Hence, using Proposition 3.7 we obtain $\widetilde{\mathrm{R}}_i^\sigma(K) = \widetilde{\mathrm{R}}_i^\sigma(M) = 0$. In this case it is $K + M = \sum_{k=1}^{n}[-e_k, e_k]$ and so $\mathrm{R}_\sigma^i(K + M) > 0$.   □

We consider in the following the continuity of the radii. By Proposition 3.3 we know that $\mathrm{R}_\pi^i, \mathrm{R}_i^\pi = \mathrm{R}_i^\sigma, \mathrm{R}_\sigma^i, \mathrm{r}_i^\pi, \mathrm{r}_\pi^i, \mathrm{r}_\sigma^i$ are all continuous in $\mathcal{K}^n$, for every $1 \leq i \leq n$, meanwhile r_i^σ is continous only on $\mathcal{K}_n^n$, see [14, Remark 4.3.3].

We observe that with the same technique as in [14, Proposition 1.2.1] it is possible to prove that also the mean section and projection, inner and outer radii ones, are continuous in the space of convex bodies with non-empty interior. Next, we prove that the mean projection inner and outer radii are indeed continuous on the whole $\mathcal{K}^n$.

Proposition 4.5. *Let $1 \leq i \leq n$. Then, the radii $\widetilde{\mathrm{R}}_i^\pi$ and $\widetilde{\mathrm{r}}_i^\pi$ are continuous on $\mathcal{K}^n$ with respect to the Hausdorff metric.*

Proof. Let $1 \leq i \leq n$, and let $(K_j)_{j \in \mathbb{N}}$, $K_j \in \mathcal{K}^n$ for all $j \in \mathbb{N}$, be a sequence of convex bodies converging to the convex body $K \in \mathcal{K}^n$. For every $L \in \mathcal{L}_i^n$, the convergence $K_j|L \to K|L$ holds, as orthogonal projections are linear maps, thus, continuous. Indeed, also the following inequalities hold:

$$\mathrm{r}(K_j|L; L) \leq \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L) = \mathrm{r}_i^\pi(K), \quad \mathrm{R}(K_j|L) \leq \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L; L) = \mathrm{R}_i^\pi(K).$$

Now, since r and R are also continuous on $\mathcal{K}^n$, the sequences $(\mathrm{r}(K_j|L; L))_{j \in \mathbb{N}}$, and $(\mathrm{R}(K_j|L))_{j \in \mathbb{N}}$ converge in $\mathbb{R}$, and further,

$$\mathrm{r}(K_j|L; L) \longrightarrow \mathrm{r}(K|L; L), \quad \mathrm{R}(K_j|L) \longrightarrow \mathrm{R}(K|L).$$

Thus, $\mathrm{r}(K_j|L; L)$ and $\mathrm{R}(K_j|L)$ are bounded sequences. Finally, the bounded convergence Theorem on $(\mathcal{L}_i^n; \nu_{n,i})$ (see e.g. [9, Corollary A.18]) provides us with the convergence:

$$\int_{\mathcal{L}_i^n} \mathrm{r}(K_j|L; L) d\nu_{n,i} \longrightarrow \int_{\mathcal{L}_i^n} \mathrm{r}(K|L; L) d\nu_{n,i} \quad \text{and} \quad \int_{\mathcal{L}_i^n} \mathrm{R}(K_j|L) d\nu_{n,i} \longrightarrow \int_{\mathcal{L}_i^n} \mathrm{R}(K|L) d\nu_{n,i}.$$

□

For the case of section inner radii, we obtain:

Proposition 4.6. *Let $1 \leq i \leq n$. Then, with respect to the Hausdorff metric, the i-th mean section inner radii $\widetilde{\mathrm{r}}_i^\sigma$ is continous on $\mathcal{K}^n$ and $\widetilde{\mathrm{R}}_i^\sigma$ is continuous on the subspace of convex bodies origin-symmetric.*

Proof. The continuity of $\widetilde{\mathrm{r}}_i^\sigma$ in $\mathcal{K}^n$ follows from the same steps of the proof of [14, Proposition 1.2.2].
To prove the continuity of $\widetilde{\mathrm{R}}_i^\sigma$ in the subspace of convex bodies origin-symmetric, let K be an origin-symmetric convex body, and let $(K_j)_{j \in \mathbb{N}}$ be a sequence of origin-symmetric convex bodies converging to K. Let $L \in \mathcal{L}_i^n$. Since K and K_j are origin-symmetric

$$\max_{x \in L^\perp} \mathrm{R}(K \cap (x + L)) = \mathrm{R}(K \cap L)$$

and

$$\max_{x \in L^\perp} \mathrm{R}(K_j \cap (x + L)) = \mathrm{R}(K_j \cap L).$$

Using that $K_j \to K$, we have $K_j \cap L \to K \cap L$ (see [29, Theorem 1.8.10]) and $\mathrm{R}(K_j \cap L) \to \mathrm{R}(K \cap L)$. The continuity of the circumradius provides us with the fact that $(\mathrm{R}(K_j \cap L))_{j \in \mathbb{N}}$ is a bounded sequence. Thefore, again by the bounded convergence Theorem on $(\mathcal{L}_i^n; \nu_{n,i})$, see e.g. [9, Corollary A.18], we have the required convergence for the continuity.   □

We will focus now on concavity and convexity properties of successive radii on approppriate families of convex bodies. We first recall the notions of a convex and a concave family of convex bodies [23, § 23].

Definition 4.7. *Let $0 \le t \le 1$, and let $\mathcal{K}_{[0,1]} := \{K(t) : t \in [0,1]\} \subset \mathcal{K}^n$. If for all $t_1, t_2 \in [0,1]$, and $\lambda \in [0,1]$,*

(i) $K((1-\lambda)t_1 + \lambda t_2) \subset (1-\lambda)K(t_1) + \lambda K(t_2)$, *then the family $\mathcal{K}_{[0,1]}$ is said to be a convex family.*

(ii) $(1-\lambda)K(t_1) + \lambda K(t_2) \subset K((1-\lambda)t_1 + \lambda t_2)$, *then the family $\mathcal{K}_{[0,1]}$ is said to be a concave family.*

Next we deal with concavity and convexity aspects of radii. Whenever we deal with convexity or concavity issues of any radii f here, we refer to convexity or concavity of the real valued function $f(K(t))$. We prove now that the radii $\widetilde{\mathrm{R}}_i^\pi, \mathrm{R}_i^\pi$ and $\widetilde{\mathrm{R}}_i^\sigma$ are convex functions when applied on a convex family of convex bodies. On the other hand, $\widetilde{\mathrm{r}}_i^\pi, \widetilde{\mathrm{r}}_i^\sigma, \mathrm{r}_\sigma^i$ and r_π^i are, in the same sense, concave, when applied on a concave family of convex bodies. This corresponds to Theorem A in the Introduction.

Theorem 4.8 (Theorem A). *Let $\mathcal{K}_{[0,1]} := \{K(t) : t \in [0,1]\} \subset \mathcal{K}^n$ be a family of convex bodies.*

(i) *If $\mathcal{K}_{[0,1]}$ is a convex family of convex bodies, then the three functions $t \mapsto \widetilde{\mathrm{R}}_i^\pi(K(t))$ and $t \mapsto \mathrm{R}_i^\pi(K(t)) = \mathrm{R}_i^\sigma(K(t))$ are convex functions for $t \in [0,1]$.*

(ii) *If $\mathcal{K}_{[0,1]}$ is a concave family of convex bodies, then the four functions $t \mapsto \widetilde{\mathrm{r}}_i^\pi(K(t))$, $t \mapsto \widetilde{\mathrm{r}}_i^\sigma(K(t))$, $t \mapsto \mathrm{r}_\sigma^i(K(t))$ and $t \mapsto \mathrm{r}_\pi^i(K(t))$ are concave functions for $t \in [0,1]$.*

Proof of Theorem 4.8. Let $\{K(t) : t \in [0,1]\} \subset \mathcal{K}^n$ be a convex family of convex bodies, let $t_1, t_2 \in (0,1)$, $\lambda \in [0,1]$, and let $\bar{t} = (1-\lambda)t_1 + \lambda t_2$. The convexity of the family $K(t)$ yields

$$K(\bar{t}) \subseteq (1-\lambda)K(t_1) + \lambda K(t_2).$$

Let further $L \in \mathcal{L}_i^n$. From

$$K(\bar{t})|L \subseteq (1-\lambda)K(t_1)|L + \lambda K(t_2)|L,$$

the 1-homogeneity of the circumradius and (4.1) follows

$$\mathrm{R}(K(\bar{t})|L) \le \mathrm{R}((1-\lambda)K(t_1)|L + \lambda K(t_2)|L) \le (1-\lambda)\mathrm{R}(K(t_1)|L) + \lambda \mathrm{R}(K(t_2)|L).$$

To show the convexity of R_i^π, it is enough to take the maximum over $\mathcal{L}_i^n$, then we have

$$\begin{aligned} \mathrm{R}_i^\pi(K(\bar{t})) = \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K(\bar{t})|L) &\le \max_{L \in \mathcal{L}_i^n}((1-\lambda)\mathrm{R}(K(t_1)|L) + \lambda \mathrm{R}(K(t_2)|L)) \\ &\le (1-\lambda)\max_{L \in \mathcal{L}_i^n} \mathrm{R}(K(t_1)|L) + \lambda \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K(t_2)|L) \\ &= (1-\lambda)\mathrm{R}_i^\pi(K(t_1)) + \lambda \mathrm{R}_i^\pi(K(t_2)). \end{aligned}$$

Analogously, passing to the integral over $\mathcal{L}_i^n$, the convexity of $\widetilde{\mathrm{R}}_i^\pi$ is obtained. As an immediate consequence of Theorem 3.1, i.e., $\mathrm{R}_i^\sigma(K(t)) = \mathrm{R}_i^\pi(K(t))$, the convexity of R_i^σ is obtained for every i and t.

To prove (ii), let again $K(t)$, $t \in [0,1]$, be a concave family of convex bodies, $t_1, t_2 \in (0,1)$, $\lambda \in [0,1]$, and $\bar{t} = (1-\lambda)t_1 + \lambda t_2$. Then, $(1-\lambda)K(t_1) + \lambda K(t_2) \subseteq K(\bar{t})$.

The concavity of $\widetilde{\mathrm{r}}_i^\pi$ and r_π^i follows from the same argument as above, where now the inequality (reverse to the previous case) arises from the super-additivity of the inradius, (4.1), namely,

$$\mathrm{r}(K(\bar{t})|L); L) \ge \mathrm{r}((1-\lambda)K(t_1); L) + \mathrm{r}(\lambda K(t_2); L) = (1-\lambda)\mathrm{r}(K(t_1)|L; L) + \lambda \mathrm{r}(K(t_2)|L; L),$$

for all $L \in \mathcal{L}_i^n$. Thus, just as above, passing to the integral over $\mathcal{L}_i^n$ yields the concavity of $\widetilde{\mathrm{r}}_i^\pi$, while taking the minimum over $\mathcal{L}_i^n$ provides us with the concavity of r_π^i.

Finally, the concavity of $\widetilde{\mathrm{r}}_i^\sigma$ and r_σ^i are consequences of their super-additivity, i.e., of Proposition 4.2, (iii) and (iv). $\qquad\square$

With essentially the same proof as in the previous theorem, the following result is also obtained.

Proposition 4.9. *Let $\mathcal{K}_{[0,1]} = \{K(t) : t \in [0,1]\} \subset \mathcal{K}^n$ be a concave family of convex bodies. Let $t_1, t_2 \in (0,1)$, $\lambda \in [0,1]$, and let $\bar{t} = (1-\lambda)t_1 + \lambda t_2$, then we have*

(i) $\sqrt{2}\mathrm{R}_\pi^i(K(\bar{t})) \ge (1-\lambda)\mathrm{R}_\pi^i(K(t_1)) + \lambda \mathrm{R}_\pi^i(K(t_2))$,

(ii) $2\mathrm{r}_i^\pi(K(\bar{t})) \ge (1-\lambda)\mathrm{r}_i^\pi(K(t_1)) + \lambda \mathrm{r}_i^\pi(K(t_2))$,

(iii) $\sqrt{2}\mathrm{r}_i^\sigma(K(\bar{t})) \geq (1-\lambda)\mathrm{r}_i^\sigma(K(t_1)) + \lambda\mathrm{r}_i^\sigma(K(t_2))$.

In the next section we will apply Theorem 4.8 (i) to the special convex family of parallel chord movement. The latter will be further applied to obtain results relating (Steiner) symmetrization and radii.

5. Interplay of radii and symmetrizations

First, we consider briefly the reflection of a convex body with respect to a k-plane, as it will be useful when dealing with the Steiner symmetrization. Next, we deal with the shadow system and the Steiner symmetrization. Subsequently, we deal with the Schwarz symmetrization, and at last, with the Minkowski symmetrization.

Remark 5.1. *Let $0 \leq k \leq n$, $L \in \mathcal{L}_k^n$, $K \in \mathcal{K}^n$, and let $1 \leq i \leq n$. Then:*
  (i) $\widetilde{\mathrm{R}}_i^\pi(\sigma_L(K)) = \widetilde{\mathrm{R}}_i^\pi(K)$, $\widetilde{\mathrm{R}}_i^\sigma(\sigma_L(K)) = \widetilde{\mathrm{R}}_i^\sigma(K)$, $\widetilde{\mathrm{r}}_i^\pi(\sigma_L(K)) = \widetilde{\mathrm{r}}_i^\pi(K)$ *and*
      $\widetilde{\mathrm{r}}_i^\sigma(\sigma_L(K)) = \widetilde{\mathrm{r}}_i^\sigma(K)$.
  (ii) $\mathrm{R}_\pi^i(\sigma_L(K)) = \mathrm{R}_\pi^i(K)$, $\mathrm{R}_i^\pi(\sigma_L(K)) = \mathrm{R}_i^\pi(K)$, $\mathrm{r}_i^\pi(\sigma_L(K)) = \mathrm{r}_i^\pi(K)$ *and*
      $\mathrm{r}_\pi^i(\sigma_L(K)) = \mathrm{r}_\pi^i(K)$.
  (iii) $\mathrm{R}_\sigma^i(\sigma_L(K)) = \mathrm{R}_\sigma^i(K)$, $\mathrm{R}_i^\sigma(\sigma_L(K)) = \mathrm{R}_i^\sigma(K)$, $\mathrm{r}_i^\sigma(\sigma_L(K)) = \mathrm{r}_i^\sigma(K)$ *and*
      $\mathrm{r}_\sigma^i(\sigma_L(K)) = \mathrm{r}_\sigma^i(K)$.

The proofs follow directly from the relation between the inradius and circumradius of a convex body and its reflection onto a subspace and the invariance of the euclidean ball under reflection.

We recall now the notion of *shadow systems*, introduced in [28]. After that, we will concentrate on parallel chord movements of convex bodies, which happen to be particular cases of shadow systems [29, Section 10.4] (see also [11] and the references therein).

Definition 5.2. *For a compact set $A \subset \mathbb{R}^n$, a unit vector $u \in \mathbb{S}^{n-1}$, and a bounded function $\alpha : A \longrightarrow \mathbb{R}$, the one parameter family of convex bodies*

$$(5.1) \qquad K(t) = \mathrm{conv}\{x + \alpha(x)tu : x \in A\}, \qquad t \in [0,1],$$

is called a shadow system -along the direction u.

This definition is equivalent, by a result of Shephard [30] (see also [6]), to the existence of another convex body $\tilde{K} \subset \mathbb{R}^{n+1}$, such that every convex body $K(t)$, $t \in [0,1]$, of the shadow system is the projection of $\tilde{K}$ onto $e_{n+1}^\perp$ along the direction $e_{n+1} - t\,u$.

Thus, a shadow system can be viewed as a continuous transformation depending on the parameter t, which is obtained by providing to every chord in the direction u, at t, the value $\alpha(x)t$.

The analysis of the relations between radii and parallel chord movement is one of the main purposes of this paper, for which we introduce the latter.

In the particular case in which a shadow system $K(t)$ is defined by the continuous function $\beta \colon K|u^\perp \to \mathbb{R}$, such that $\alpha(x) = \beta(x|u^\perp)$ for all $x \in K$, and

$$(5.2) \qquad K(t) = \{x + \beta(x|u^\perp)tu : x \in K\}, \qquad t \in [0,1],$$

the family $K(t)$ is known as *parallel chord movement*. If $K(t)$ is a parallel chord movement, then it is clear that $K(t)|u^\perp = K|u^\perp$ for all $t \in [0,1]$.

It is proven in [28] that the volume of a shadow system is a convex function of the parameter $t \in [0,1]$. Moreover, in [30] it was also proven that other magnitudes, like the diameter or the mean width do also share this convexity.

The following remark provides us with a direct connection between shadow systems and symmetrizations, which has been our main motivation to use shadow systems in this context.

Remark 5.3. [29, Section 10.4] *Let $K \in \mathcal{K}^n$, and $u \in \mathbb{S}^{n-1}$. Further, let $f_K, g_K \colon K|u^\perp \to \mathbb{R}$ be the defining functions associated to K, that is, f_K is concave, g_K is convex, and the convex body K is given by $K = \{x + \lambda u| \ x \in K|u^\perp, \ g_K(x) \leq \lambda \leq f_K(x)\}$, as in Remark 2.8. Then,*

$$(5.3) \quad K(t) = \{x + \lambda u : \ x \in K|u^\perp, \ (1-t)g_K(x) - tf_K(x) \leq \lambda \leq (1-t)f_K(x) - tg_K(x)\}, \quad t \in [0,1],$$

is a parallel chord movement of K given by the continuous function $\beta(x) = -(f_K(x) + g_K(x))$ for $x \in K|u^\perp$. We observe that $K(0) = K$.

In the next theorem we gather some aspects of the just introduced parallel chord movement, given by the defining functions associated to a convex body, which will be called K *defining parallel chord movement*, and its connection to the symmetrization procedures. We refer the reader to [28, 29, 30], and the references therein.

Theorem 5.4. [30] *Let $K \in \mathcal{K}^n$, and let $K(t)$, $0 \leq t \leq 1$, be the K defining parallel chord movement in the direction $u \in \mathbb{S}^{n-1}$. Then,*

    (i) $K(t)$ *is a convex body for every $t \in [0,1]$.*

    (ii) $K(t) = \bigcup_{x \in K|u^\perp} [x + ((1-t)g_K(x) - tf_K(x))u, x + ((1-t)f_K(x) - tg_K(x))u]$.

    (iii) *The length of the segment $[x + ((1-t)g_K(x) - tf_K(x))u, x + ((1-t)f_K(x) - tg_K(x))u]$ does not depend on t, for $t \in [0,1]$. Indeed, it coincides with $f_K(x) - g_K(x) \geq 0$.*

    (iv) $K(\frac{1}{2}) = S_{u^\perp}(K)$,

    (v) $K(1) = \sigma_{u^\perp}(K)$.

The items (iv) and (v) in the previous proposition do provide us with the main motivation to work with shadow systems, as they establish a connection of those with Steiner and (implicitly) Minkowski symmetrization. Further, Theorem 2.9 (iii), and Definition 2.6 allow us to see a connection between the Schwarz symmetrization and shadow systems via Theorem 5.4 (ii).

The following result will be crucial for our purposes. It establishes the convexity on the defining parameter of the family of parallel chord movement for any $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$, in the spirit of the convexity results in [30] for the diameter, or the mean width.

Proposition 5.5. *Let $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$, and let $K(t)$, $0 \leq t \leq 1$, be the K defining parallel chord movement in the direction u. Then, $K(t)$ is convex in $t \in [0,1]$.*

Proof. Let $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$, and let $K(t)$, $0 \leq t \leq 1$, be the K defining parallel chord movement in the direction u. Further, let $t_1, t_2 \in [0,1]$ and $\lambda \in [0,1]$, and denote by $\bar{t} = (1-\lambda)t_1 + \lambda t_2$. We prove that

$$K(\bar{t}) \subseteq (1-\lambda)K(t_1) + \lambda K(t_2).$$

Let f_K, g_K denote the defining functions associated to K, and let $z \in K(\bar{t})$. Then, there exist $x \in K|u^\perp$ and $y = \mu u$, so that $z = x + y$, and

$$(1-\bar{t})g_K(x) - \bar{t}f_K(x) \leq \mu \leq (1-\bar{t})f_K(x) - \bar{t}g_K(x).$$

We construct $z_i \in K(t_i)$, $i \in \{1,2\}$, such that $z = (1-\lambda)z_1 + \lambda z_2$, by means of finding $x_1, x_2 \in K|u^\perp$, and $y_1 = \mu_1 u, y_2 = \mu_2 u$, such that

$$(1-t_i)g_K(x) - t_i f_K(x) \leq \lambda_i \leq (1-t_i)f_K(x) - t_i g_K(x),$$

for $i = 1,2$, and $x + y = (1-\lambda)(x_1 + y_1) + \lambda(x_2 + y_2)$.

Let $x_1 = x_2 = x$, and ,et $d = (1-\bar{t})f_K(x) - \bar{t}g_K(x) - \mu \geq 0$. Furthermore, for $i = 1,2$ let $\mu_i = (1-t_i)f_K(x) - t_i g_K(x) - d$. Then, as $d \geq 0$, we have $\mu_i \leq (1-t_i)f_K(x) - t_i g_K(x)$. Moreover, since $\mu \geq (1-\bar{t})g_K(x) - \bar{t}f_K(x)$, we have

$$\mu_i = (1-t_i)f_K(x) - t_i g_K(x) - (1-\bar{t})f_K(x) + \bar{t}g_K(x) + \mu \geq$$
$$= (1-t_i)f_K(x) - t_i g_K(x) - (1-\bar{t})f_K(x) + \bar{t}g_K(x) + (1-\bar{t})g_K(x) - \bar{t}f_K(x) =$$
$$= (1-t_i)f_K(x) - t_i g_K(x) + g_K(x) - f_K(x) = (1-t_i)g_K(x) - t_i f_K(x).$$

Thus, we have proven that $\mu_i \in [(1-t_i)g_K(x) - t_i f_K(x), (1-t_i)f_K(x) - t_i g_K(x)]$ and therefore, we obtain $z_i = x + \mu_i u \in K(t_i)$.

It remains to prove, that $\mu = (1 - \lambda)\mu_1 + \lambda\mu_2$:

$$
\begin{aligned}
(1 - \lambda)\mu_1 + \lambda\mu_2 &= (1 - \lambda)(1 - t_1)f_K(x) - (1 - \lambda)t_1 g_K(x) - (1 - \lambda)d + \lambda(1 - t_2)f_K(x) - \lambda t_2 g_K(x) - \lambda d \\
&= (1 - \lambda)(1 - t_1)f_K(x) - (1 - \lambda)t_1 g_K(x) + \lambda(1 - t_2)f_K(x) - \lambda t_2 g_K(x) - d \\
&= f_K(x)((1 - \lambda)(1 - t_1) + \lambda(1 - t_2)) - g_K(x)((1 - \lambda)t_1 + \lambda t_2) - (1 - \bar{t})f_K(x) + \bar{t}g_K(x) + \mu \\
&= f_K(x)((1 - \lambda)(1 - t_1) + \lambda(1 - t_2)) - g_K(x)((1 - \lambda)t_1 + \lambda t_2) \\
&\quad - (1 - (1 - \lambda)t_1 - \lambda t_2)f_K(x) + ((1 - \lambda)t_1 + \lambda t_2)g_K(x) + \mu \\
&= \mu
\end{aligned}
$$

Thus, $z = x + y \in (1 - \lambda)K(t_1) + \lambda K(t_2)$ and the statement follows.

$\square$

As a consequence of the latter result we can prove Theorem 5.6, i.e., that the Steiner symmetrization does not increase the mean projection outer radii.

Theorem 5.6 (Theorem B). *Let $K \in \mathcal{K}^n$, $u \in \mathbb{S}^{n-1}$, and let $1 \leq i \leq n$. Then, we have*

(i) $\widetilde{\mathrm{R}}_i^\pi(S_{u^\perp}(K)) \leq \widetilde{\mathrm{R}}_i^\pi(K)$;

(ii) $\mathrm{R}_i^\sigma(S_{u^\perp}(K)) = \mathrm{R}_i^\pi(S_{u^\perp}(K)) \leq \mathrm{R}_i^\pi(K) = \mathrm{R}_i^\sigma(K)$;

and equality holds in all of the inequalities for euclidean balls.

Proof. Let $K \in \mathcal{K}^n$, and $u \in \mathbb{S}^{n-1}$. We consider the one-parameter family of chord movement of K in the direction $u \in \mathbb{S}^{n-1}$, $K(t)$, $t \in [0,1]$. By Proposition 5.5, the family $K(t)$ is convex in $t \in [0,1]$, hence, applying Theorem 4.8, $\widetilde{\mathrm{R}}_i^\pi(K(t))$ and $\mathrm{R}_i^\pi(K(t))$ are convex in $t \in [0,1]$. Thus, recalling that $K(0) = K$, $K(\frac{1}{2}) = S_{u^\perp}(K)$ and $K(1) = \sigma_{u^\perp}(K)$, Remark 5.1 yields

$$
\widetilde{\mathrm{R}}_i^\pi(S_{u^\perp}(K)) \leq \widetilde{\mathrm{R}}_i^\pi(K)
$$

and

$$
\mathrm{R}_i^\pi(S_{u^\perp}(K)) \leq \mathrm{R}_i^\pi(K).
$$

By Theorem 3.1, the same inequality for R_i^π holds also for R_i^σ.

Finally, equality holds in all three inequalities with euclidean balls, since the Steiner symmetrization does not change balls, up to a translation, and thus, all the involved radii equal the radius of the ball. $\square$

Remark 5.7. *We point out, that as $\mathrm{R}_\pi^1(K) = \mathrm{r}_\pi^1(K) = \mathrm{R}_\sigma^1(K) = \mathrm{r}_\sigma^1(K) = \omega(K)/2$, and as it is known that the minimal width of a set can be both increased and reduced after performing a Steiner symmetrization on it, see [12, p. 90], the analog of Proposition 5.6 for $\mathrm{R}_\pi^i, \mathrm{r}_\pi^i, \mathrm{R}_\sigma^i, \mathrm{r}_\sigma^i$ can not hold, in general.*

Now we can use Theorem 5.6 to prove that the Schwarz symmetrization does not increase the mean projection outer radii.

Theorem 5.8. *Let $K \in \mathcal{K}^n$, $1 \leq k \leq n - 1$, and $L \in \mathcal{L}_k^n$, and $1 \leq i \leq n$. Then, we have*

(i) $\widetilde{\mathrm{R}}_i^\pi(S_L(K)) \leq \widetilde{\mathrm{R}}_i^\pi(K)$;

(ii) $\mathrm{R}_i^\sigma(S_L(K)) = \mathrm{R}_i^\pi(S_L(K)) \leq \mathrm{R}_i^\pi(K) = \mathrm{R}_i^\sigma(K)$

and equality holds in all of the inequalities for euclidean balls.

Proof. Let $K \in \mathcal{K}^n$, $1 \leq k \leq n - 1$, and $L \in \mathcal{L}_k^n$. The fundamental step in the proof is Theorem 2.10, i.e., the existence of a sequence of Steiner symmetrizations of K converging to $S_L(K)$. Then, Theorem 5.6 proves the result. More precisely, let $u_j \in \mathbb{S}^{n-1} \cap L$ be the sequence of directions from Theorem 2.10, so that $S_{u_j^\perp}(\cdots(S_{u_1^\perp}(K))) \to S_L(K)$, and let $K_j := S_{u_j^\perp}(\cdots(S_{u_1^\perp}(K)))$. Theorem 5.6 yields

$$
\widetilde{\mathrm{R}}_i^\pi(K_j) \leq \widetilde{\mathrm{R}}_i^\pi(K_{j-1}) \leq \cdots \leq \widetilde{\mathrm{R}}_i^\pi(K_1) \leq \widetilde{\mathrm{R}}_i^\pi(K).
$$

By the continuity of the mean projection outer radii proven in Proposition 4.5, we have

$$\widetilde{R}_i^\pi(S_L(K)) = \widetilde{R}_i^\pi\left(\lim_{j\to+\infty} K_j\right) = \lim_{j\to+\infty} \widetilde{R}_i^\pi(K_j) \le \widetilde{R}_i^\pi(K).$$

By Proposition 3.3 and Theorem 3.1, we have also that $R_i^\sigma(S_L(K)) = R_i^\pi(S_L(K)) \le R_i^\pi(K) = R_i^\sigma(K)$ holds. Equality holds in all three inequalities for the euclidean ball, for the same reason as in Theorem 5.6. $\qquad\square$

To finish, we address the Minkowski symmetrization and its relation to the successive and mean radii. We observe first, that (4.2) does already provide us with a first relation, as $\widetilde{R}_1^\pi(K) = \frac{1}{2}b(K)$, and thus, $\widetilde{R}_1^\pi(K) = \widetilde{R}_1^\pi(M_L(K))$. Now, we can extend this result to some mean radii.

Theorem 5.9 (Theorem C). *Let $K \in \mathcal{K}^n$, $0 \le k \le n$, and let $L \in \mathcal{L}_k^n$. For all $1 \le i \le n$, we have*

$$\widetilde{R}_i^\pi(M_L(K)) \le \widetilde{R}_i^\pi(K), \qquad \widetilde{r}_i^\pi(K) \le \widetilde{r}_i^\pi(M_L(K))$$

and

$$\widetilde{r}_i^\sigma(K) \le \widetilde{r}_i^\sigma(M_L(K)),$$

and equality holds for euclidean balls.

For $i = 1$, we have $\widetilde{R}_1^\pi(M_L(K)) = \widetilde{R}_1^\pi(K)$, and $\widetilde{r}_1^\pi(M_L(K)) = \widetilde{r}_1^\pi(K)$, that is, there is equality in the inequalities involving the projection mean radii, when $i = 1$.

Proof. The two first inequalities involving projection radii, namely, $\widetilde{R}_i^\pi(M_L(K)) \le \widetilde{R}_i^\pi(K)$, and $\widetilde{r}_i^\pi(K) \le \widetilde{r}_i^\pi(M_L(K))$ are consequences of Proposition 4.1 (i) and (iii), while the last inequality, $\widetilde{r}_i^\sigma(K) \le \widetilde{r}_i^\sigma(M_L(K))$, follows from the super-additivity property of the mean section inner radii in Proposition 4.2 (iii).

Equality holds in all three inequalities for euclidean balls as the Minkowski symmetral of a euclidean ball K is again a euclidean ball having the same radius as K.

For $i = 1$, the equalities follow from (4.2), and the equality $\widetilde{r}_1^\pi(K) = \dfrac{1}{2}b(K)$, together with the fact that the mean width is linear with respect to the Minkowski sum, provides us with the result:

$$\widetilde{R}_1^\pi(M_L(K)) = \widetilde{R}_1^\pi(K) = \frac{1}{2}b(K) = \widetilde{r}_1^\pi(K) = \widetilde{r}_1^\pi(M_L(K)),$$

for all $L \in \mathcal{L}_k^n$, $1 \le k \le n$ and $K \in \mathcal{K}^n$. $\qquad\square$

With essentially the same proof as in the previous theorem, the following result is also obtained.

Proposition 5.10. *Let $K \in \mathcal{K}^n$, $0 \le k \le n$, and let $L \in \mathcal{L}_k^n$. For all $1 \le i \le n$, we have*

(i) $R_i^\sigma(M_L(K)) = R_i^\pi(M_L(K)) \le R_i^\pi(K) = R_i^\sigma(K)$;

(ii) $R_\pi^i(K) \le \sqrt{2}R_\pi^i(M_L(K))$;

(iii) $r_i^\pi(K) \le 2r_i^\pi(M_L(K))$ *and* $r_i^\sigma(K) \le \sqrt{2}r_i^\sigma(M_L(K))$;

(iv) $r_\pi^i(K) \le r_\pi^i(M_L(K))$ *and* $r_\sigma^i(K) \le r_\sigma^i(M_L(K))$.

We observe that euclidean balls provide us with equality in Proposition 5.10 (i) and the left-hand-side of (iv). We point out, that recalling the proof of Proposition 4.4 (iii), it is possible to find two convex bodies K and M, not full-dimensional, such that $\widetilde{R}_i^\sigma(K) = \widetilde{R}_i^\sigma(M) = 0$, but $R_\sigma^i(K + M) > 0$. Hence, under some suitable conditions on n, $1 \le i < n$ and $1 \le k \le n$, it is possible to consider $M = \sigma_L(K)$, for a suitable $L \in \mathcal{L}_k^n$, and this allows us to establish that there is no $c \in \mathbb{R}$, such that $\widetilde{R}_i^\sigma(M_L(K)) \le c\widetilde{R}_i^\sigma(K)$ and $R_\sigma^i(M_L(K)) \le cR_\sigma^i(K)$, for all $K \in \mathcal{K}^n$.

Remark 5.11. *We remark, that unlike Theorem 5.9, there can not be equality for the case $i = 1$ in Proposition 5.10, since*

$$R_1^\pi(K) = \frac{D(K)}{2}, \quad r_\pi^1(K) = \frac{\omega(K)}{2} = r_\sigma^1(K),$$

and the diameter and the minimal width are not linear with respect to the Minkowski sum.

We observe that Propositions 4.1 (iii) and 4.2 (v) yield also the inequalities

$$\widetilde{\mathrm{R}}_i^\pi(M_L(K)) \geq \frac{\sqrt{2}}{2}\widetilde{\mathrm{R}}_i^\pi(K), \qquad \mathrm{R}_i^\pi(M_L(K)) \geq \frac{1}{2\sqrt{2}}\mathrm{R}_i^\pi(K),$$

which provide us with a lower and upper bound for the mean projection outer and the maximal projection outer radii.

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