ASYMPTOTIC DISTRIBUTION OF LIKELIHOOD RATIO-STATISTICS

Theorem (Thm. 12.2.3 in Lehmann / Romano (2005)):

Let $(\mathbb{P}_\vartheta)_{\vartheta \in \Theta}$ be differentiable in quadratic mean at ϑ_0, where Θ is an open subset of $\mathbb{R}^d$. Suppose that $I(\vartheta_0)$ is positive definite.

Define, for an iid. sample $X = (X_1, \dots, X_n)^T$ and a fixed $h \in \mathbb{R}^d$

$$L_{n,h} = \frac{\ell(\vartheta_0 + h\, n^{-1/2}, X)}{\ell(\vartheta_0, X)}$$

$$= \prod_{i=1}^{n} \frac{\ell(\vartheta_0 + h n^{-1/2}, X_i)}{\ell(\vartheta_0, X_i)}$$

Then it holds:

(i) As $n \to \infty$,

$$\log(L_{n,h}) - \left[\langle h, Z_n \rangle - \frac{1}{2} \langle h, I(\vartheta_0) h \rangle \right] = o_{\mathbb{P}^n_{\vartheta_0}}(1), \quad \text{where}$$

$$Z_n \equiv Z_{n,\vartheta_0} = n^{-1/2} \sum_{i=1}^{n} \dot{\ell}(X_i, \vartheta_0).$$

(ii) Under ϑ_0, $Z_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, I(\vartheta_0))$ and, hence,

$$\log(L_{n,h}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(-\frac{1}{2}\langle h, I(\vartheta_0) h\rangle, \langle h, I(\vartheta_0) h \rangle\right).$$

Proof: Let $Y_{n,1}, \dots, Y_{n,n}$ be defined by

$$Y_{n,i} = \frac{\sqrt{\ell(\vartheta_0 + h n^{-1/2}, X_i)}}{\sqrt{\ell(\vartheta_0, X_i)}} - 1.$$

Then it holds $\forall n \; \forall 1 \le i \le n$: $\mathbb{E}_{\vartheta_0}[Y_{n,i}^2] < \infty$ and, moreover,

$$\log(L_{n,h}) = 2 \sum_{i=1}^{n} \log(1 + Y_{n,i}).$$

Taylor expansion of $\log(\cdot)$:

$$\log(1+y) = y - \frac{1}{2} y^2 + y^2 r(y), \text{ where } r(y) \to 0 \text{ as } y \to 0.$$

$$\Rightarrow \log(L_{n,h}) = 2 \sum_{i=1}^{n} Y_{n,i} - \sum_{i=1}^{n} Y_{n,i}^2 + 2 \sum_{i=1}^{n} Y_{n,i}^2 \, r(Y_{n,i}).$$

We will now show four convergence results: $(n \to \infty)$

[R1] $\displaystyle\sum_{i=1}^{n} \mathbb{E}_{\vartheta_0}[Y_{n,i}] \longrightarrow -\frac{1}{8} \langle h, I(\vartheta_0) h \rangle$

[R2] $\displaystyle\sum_{i=1}^{n} \{ Y_{n,i} - \mathbb{E}_{\vartheta_0}[Y_{n,i}] \} - \frac{1}{2} \langle h, Z_n \rangle \xrightarrow{\mathbb{P}^n_{\vartheta_0}} 0$

[R3] $\displaystyle\sum_{i=1}^{n} Y_{n,i}^2 \xrightarrow{\mathbb{P}^n_{\vartheta_0}} \frac{1}{4} \langle h, I(\vartheta_0) h \rangle$

[R4] $\displaystyle\sum_{i=1}^{n} Y_{n,i}^2 \, r(Y_{n,i}) \xrightarrow{\mathbb{P}^n_{\vartheta_0}} 0$

Proof of [R1]:

$$\sum_{i=1}^{n} \mathbb{E}_{\vartheta_0}[Y_{n,i}] = n \int_\Omega \left\{ \frac{\sqrt{\ell(\vartheta_0 + h n^{-1/2}, x)}}{\sqrt{\ell(\vartheta_0, x)}} - 1 \right\} \ell(\vartheta_0, x) \, d\mu(x)$$

$$= -\frac{n}{2} \int_\Omega \left[\sqrt{\ell(\vartheta_0 + h n^{-1/2}, x)} - \sqrt{\ell(\vartheta_0, x)} \right]^2 d\mu(x)$$

$$\longrightarrow -\frac{1}{2} \int_\Omega \left| \left\langle \frac{\partial}{\partial \vartheta} \sqrt{\ell(\vartheta, x)} \Big|_{\vartheta = \vartheta_0}, h \right\rangle \right|^2 d\mu(x)$$

That the latter expression equals $-\frac{1}{8} \langle h, I(\vartheta_0) h \rangle$ is left as an exercise.

Proof of [R2]:

We write $$Y_{n,i} = \frac{1}{2} n^{-1/2} \langle h, \dot{\ell}(x_i, \vartheta_0) \rangle + n^{-1/2} \frac{R_n(x_i)}{\sqrt{\ell(\vartheta_0, x_i)}}, \qquad [****]$$

where the remainder term R_n satisfies

$\int_\Omega R_n^2(x) \, d\mu(x) \to 0$ due to differentiability in quadratic mean.

Notice: $$\dot{\ell}(x_i, \vartheta_0) = \frac{2 \, \partial/\partial\vartheta \sqrt{\ell(\vartheta, x_i)} \,\big|_{\vartheta = \vartheta_0}}{\sqrt{\ell(\vartheta_0, x_i)}}.$$

Proof of [R2]
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$$\mathring{\ell}(x;\vartheta_0) = \frac{\partial}{\partial\vartheta}\log\left(\ell(\vartheta,x)\right)\Big|_{\vartheta=\vartheta_0}$$

$$= 2\,\frac{\partial}{\partial\vartheta}\log\left(\sqrt{\ell(\vartheta,x)}\right)\Big|_{\vartheta=\vartheta_0}$$

$$= 2\,\frac{\partial/\partial\vartheta\,\sqrt{\ell(\vartheta,x)}\,\big|_{\vartheta=\vartheta_0}}{\sqrt{\ell(\vartheta_0,x)}}.$$

$$Y_{n,i} = \frac{\sqrt{\ell(\vartheta_0+h n^{-1/2},x_i)}-\sqrt{\ell(\vartheta_0,x_i)}}{\sqrt{\ell(\vartheta_0,x_i)}}$$

$$= \frac{h n^{-1/2}}{\sqrt{\ell(\vartheta_0,x_i)}}\;\frac{\sqrt{\ell(\vartheta_0+h n^{-1/2},x_i)}-\sqrt{\ell(\vartheta_0,x_i)}}{h n^{-1/2}}$$

$$\underset{(n\to\infty)}{\approx} \frac{h n^{-1/2}}{\sqrt{\ell(\vartheta_0,x_i)}}\cdot\frac{\partial}{\partial\vartheta}\sqrt{\ell(\vartheta,x_i)}\,\Big|_{\vartheta=\vartheta_0}$$

$$= \frac{h n^{-1/2}}{\sqrt{\ell(\vartheta_0,x_i)}}\left[\mathring{\ell}(x_i,\vartheta_0)\sqrt{\ell(\vartheta_0,x_i)}\,/2\right]$$

$$= \frac{n^{-1/2}}{2}\langle h,\mathring{\ell}(x_i,\vartheta_0)\rangle.$$

$\Rightarrow$

$$\sum_{i=1}^{n} \{ Y_{ni} - \mathbb{E}_{\vartheta_0}[Y_{ni}] \} = \frac{1}{2} \langle h, Z_n \rangle + n^{-1/2} \sum_{i=1}^{n} \left\{ \frac{R_n(X_i)}{\sqrt{\ell(\vartheta_0, X_i)}} - \mathbb{E}_{\vartheta_0}\left[\frac{R_n(X_i)}{\sqrt{\ell(\vartheta_0, X_i)}} \right] \right\}$$

Every summand in the latter sum is centered and the variance is bounded by

$$\mathbb{E}_{\vartheta_0}\left[\frac{R_n^2(X_i)}{\ell(\vartheta_0, X_i)} \right] = \int_\Omega R_n^2(x) \, d\mu(x) \longrightarrow 0,$$

so [R2] follows.

Proof of [R3]:

By the law of large numbers,

$$\frac{1}{n} \sum_{i=1}^{n} [\langle h, \dot{\ell}(X_i, \vartheta_0) \rangle]^2 \xrightarrow{\mathbb{P}^n_{\vartheta_0}} \mathbb{E}_{\vartheta_0}[(\langle h, \dot{\ell}(X_i, \vartheta_0) \rangle)^2] = \langle h, I(\vartheta_0) h \rangle.$$

Using [****], we get

$$\sum_{i=1}^{n} Y_{ni}^2 = \frac{1}{4n} \sum_{i=1}^{n} (\langle h, \dot{\ell}(X_i, \vartheta_0) \rangle)^2 + \frac{1}{n} \sum_{i=1}^{n} (\langle h, \dot{\ell}(X_i, \vartheta_0) \rangle) \sum_{j=1}^{n} \frac{R_n(X_j)}{\sqrt{\ell(\vartheta_0, X_j)}} + \frac{1}{n} \sum_{i=1}^{n} \frac{R_n^2(X_i)}{\ell(\vartheta_0, X_i)}.$$

The first summand converges (under ϑ_0) in probability to $1/4 \langle h, I(\vartheta_0) h \rangle$. The third summand is non-negative and has expectation under ϑ_0 equal to $\int_\Omega R_m^2(x) \, \mu(dx) \longrightarrow 0$. Hence, Markov's inequality yields that it tends to 0 in probability under ϑ_0. To see that the second summand tends to zero in probability under ϑ_0, apply the Cauchy-Schwarz inequality.

<u>Proof of [R4]:</u>

We note that $\left| \sum_{i=1}^{n} Y_{n,i}^2 \, r(Y_{n,i}) \right|$

$$\leq \max_{1 \leq i \leq n} |r(Y_{n,i})| \sum_{i=1}^{n} Y_{n,i}^2 .$$

Therefore, it suffices to show that

$\max_{1 \leq i \leq n} |r(Y_{n,i})| \longrightarrow 0$ in probability under ϑ_0,

which is in turn implied by $\max_{1 \leq i \leq n} |Y_{n,i}| \xrightarrow{\mathbb{P}^n_{\vartheta_0}} 0$.

Since $\sum_{i=1}^{n} \{ Y_{n,i} - \mathbb{E}_{\vartheta_0}[Y_{n,i}] \}$ is asymptotically normal by [R2], we can argue that

$\max_{1 \leq i \leq n} |Y_{n,i}| \xrightarrow{\mathbb{P}^n_{\vartheta_0}} 0$ by making use of Lindeberg's condition and the fact that $\mathbb{E}_{\vartheta_0}[Y_{n,i}] \longrightarrow 0$, $n \to \infty$. $\blacksquare$