## Proof of Theorem 4.15:

First, we prove the result for $r=0$, i.e., in the case of testing a point hypothesis $\Theta_0 = \{\vartheta_0\}$.

Due to Theorem 3.22, the MLE $\hat{\vartheta}_n$ admits an asymptotic expansion of the form

$$\sqrt{n}\,(\hat{\vartheta}_n - \vartheta_0) = I^{-1}(\vartheta_0)\, Z_n + o_{\mathbb{P}^n_{\vartheta_0}}(1), \qquad [*]$$

where $Z_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, I(\vartheta_0))$.

We define (for given $h$) $\quad L_{n,h} = \dfrac{\ell(\vartheta_0 + h n^{-1/2}, X)}{\ell(\vartheta_0, X)}$

and $\hat{h}_n = \sqrt{n}\,(\hat{\vartheta}_n - \vartheta_0)$, so that

$$\Lambda_n(X) = L_{n,\hat{h}_n}.$$

The preceding theorem, in connection with Remark 12.2.2. in Lehmann and Romano (2005), yields that under $\vartheta_0$

$$\varepsilon_{n,c} := \sup_{|h| \le c} \left| \log(L_{n,h}) - [\langle h, Z_n\rangle - \tfrac{1}{2}\langle h, I(\vartheta_0) h\rangle] \right|$$

tends to $0$ in probability for any fixed $c > 0$.

By the triangle inequality,

$$2 \log L_{n,\hat{h}_n} \le 2\{\langle \hat{h}_n, Z_n\rangle - \tfrac{1}{2}\langle \hat{h}_n, I(\vartheta_0)\hat{h}_n\rangle + \varepsilon_{n,c}\}$$

as soon as $|\hat{h}_n| \le c$.

But, using [*] , we get

$$2\left[\langle \hat{h}_n, Z_n\rangle - \frac{1}{2}\langle \hat{h}_n, I(\vartheta_0)\hat{h}_n\rangle\right]$$
$$= Z_n^T I^{-1}(\vartheta_0) Z_n + o_{\mathbb{P}^n_{\vartheta_0}}(1).$$

$$\Rightarrow \quad 2\log(L_n, \hat{h}_n) \le Z_n^T I^{-1}(\vartheta_0) Z_n + \tilde{\varepsilon}_{n,c}$$
if $|\hat{h}_n| \le c$, where $\tilde{\varepsilon}_{n,c} \xrightarrow{\mathbb{P}^n_{\vartheta_0}} 0$ for any $c>0$.

$$\Rightarrow \mathbb{P}^n_{\vartheta_0}\left(2\log(L_n, \hat{h}_n) \geq x\right)$$
$$\le \mathbb{P}^n_{\vartheta_0}\left(Z_n^T I^{-1}(\vartheta_0) Z_n + \tilde{\varepsilon}_{n,c} \geq x,\ |\hat{h}_n| \le c\right) + \mathbb{P}^n_{\vartheta_0}\left(|\hat{h}_n| > c\right)$$
$$\le \mathbb{P}^n_{\vartheta_0}\left(Z_n^T I^{-1}(\vartheta_0) Z_n + \tilde{\varepsilon}_{n,c} \geq x\right) + \mathbb{P}^n_{\vartheta_0}\left(|\hat{h}_n| > c\right). \quad [**]$$

Now, notice that under $\vartheta_0$ $Z_n^T I^{-1}(\vartheta_0) Z_n \xrightarrow{\mathcal{D}} \chi^2_k$

and $\hat{h}_n \xrightarrow{\mathcal{D}} Z \sim \mathcal{N}_k(0, I^{-1}(\vartheta_0))$, $n \to \infty$.

So, [**] tends to

$$\mathbb{P}(\chi^2_k \geq x) + \mathbb{P}(|Z| > c).$$

Let $c \to \infty$ to conclude
$$\limsup_{n\to\infty} \mathbb{P}^n_{\vartheta_0}\left(2\log(\Lambda_n(X)) \geq x\right) \le \mathbb{P}(\chi^2_k \geq x).$$

Similarly, deduce that

$$\liminf_{n\to\infty} \mathbb{P}^n_{\vartheta_0}\left(2\log\left(\Delta_n(X)\right) \geq x\right) \geq \mathbb{P}\left(\chi^2_k \geq x\right),$$

proving the case $r = 0$.

The proof for general $r$ relies on the same argumentation and the following additional considerations for normally distributed random vectors:

Suppose $X = (X_1, \dots, X_k)^T$ is multivariate normal with unknown mean vector $\vartheta$, but known, positive definite covariance matrix $\Sigma$. The likelihood function of $X$ is given by

$$\ell(\vartheta, X) = \frac{|\Sigma|^{-1/2}}{(2\pi)^{k/2}} \exp\left[-\frac{1}{2}(X-\vartheta)^T \Sigma^{-1} (X-\vartheta)\right].$$

$$\Rightarrow 2\log\left(\Delta_1(X)\right)$$

$$= -\inf_{\vartheta\in\mathbb{R}^k} (X-\vartheta)^T \Sigma^{-1} (X-\vartheta)$$

$$+ \inf_{\vartheta\in\Theta_0} (X-\vartheta)^T \Sigma^{-1} (X-\vartheta)$$

$$= \inf_{\vartheta\in\Theta_0} (X-\vartheta)^T \Sigma^{-1} (X-\vartheta).$$

Letting $z = \Sigma^{-1/2} x$, we have, equivalently,

$$2 \log(\Lambda_n(x)) = \inf_{\vartheta \in \Theta_0} | z - \Sigma^{-1/2} \vartheta |^2 .$$

As $\vartheta$ varies in $\Theta_0$, $\Sigma^{-1/2} \vartheta$ varies in an $r$-dimensional subspace of $\mathbb{R}^k$, say $\mathcal{M}$. If $P$ is the projection matrix onto $\mathcal{M}$, then $2 \log(\Lambda_n(x)) = |(I_k - P) z|^2$. Theorem 4.32.2. completes the argumentation. ∎