Compatible Confidence Regions

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Outline

Review

In classic test theory, we have:

- A model $(\mathscr{X}, \mathscr{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta})$
- A confidence level $\alpha \in (0, 1)$

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Then a confidence region is a set $\mathscr{C} = (C(x) : x \in \mathscr{X})$ with

- $C(x) \subseteq \Theta$
- $\{x : C(x) \ni \vartheta\}$ measurable for all $\vartheta \in \Theta$
- $\mathbb{P}_\vartheta(\{x : C(x) \ni \vartheta\}) \geq 1 - \alpha$

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How can one translate this idea to a multiple testing framework in a way that makes sense?

Say that $\varphi \in \Phi_\alpha(\mathscr{H})$ and $C \in \mathscr{C}_{1-\alpha}$, then we per definition have

$$\forall \vartheta \in \Theta : \mathbb{P}_\vartheta\Big(\bigcup_{i \in I_0(\vartheta)} \{\varphi_i(x) = 1\}\Big) \leq \alpha$$

and

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Then a natural requirement on a confidence region $C(x)$ is that

$$\forall \vartheta \in \Theta : \mathbb{P}_\vartheta\big(\bigcup_{i \in I_0(\vartheta)} \{\varphi_i(x) = 1\} \cup \{x : C(x) \not\ni \vartheta\}\big) \leq \alpha$$

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Intuition: $\bigcup_{i \in I_0(\vartheta)}\{x : \varphi_i(x) = 1\}$ and $\{x : C(x) \not\ni \vartheta\}$ should more or less overlap

Compatible

Thus we get the following definition

Definition

A confidence region $C \in \mathscr{C}_{1-\alpha}$ and a multiple test $\varphi \in \Phi_\alpha(\mathscr{H})$ are said to be **compatable** if $C \subseteq C(\varphi)$, i.e. $C(x) \subseteq \bigcap_{i:\varphi(x)=1} K_i$ for all $x \in \mathscr{X}$.

Notice that compatiblity implies

$$\forall \vartheta \in \Theta : \mathbb{P}_\vartheta\Big(\bigcup_{i \in I_0(\vartheta)} \{x : \varphi_i(x) = 1\} \cup \{x : C(x) \not\ni \vartheta\}\Big) \leq \alpha,$$

the conditon from the motivational slide.

Example

Example for a non-compatable set

Extended Correspondence Theorem

With this theorem we can easily construct confidence regions for single step procedures:

Theorem

Let $\mathscr{H} = \{H_i : i \in I\}$ *be a family of hypothesis. If*

- $\varphi \in \Phi_\alpha(\mathscr{H})$
- $C(x) = \bigcap_{i:\varphi_i(x)=1} K_i \quad \forall x \in \mathscr{X}$ *where the convention* $\bigcap_{i\in\emptyset} K_i = \Theta$ *is used*

then

$$C = (C(x) : x \in \mathscr{X}) \in \mathscr{C}_{1-\alpha}.$$

and with this, C *and* φ *are compatable.*

Model for Tukey Test

Let $X_{i1}, \ldots, X_{in} \sim N(\mu_i, \sigma^2)$ be iid observations, $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{R}^k$ unknown. We want find confidence regions for $\vartheta_{ij} := \mu_i - \mu_j$, $1 \leq i < j \leq k$. Let $\bar{Y}_i := \bar{X}_{i.} - \mu_i$, $1 \leq i \leq k$, then it is clear that

$$\bar{X}_{i.} - \bar{X}_{j.} - \vartheta_{ij} = \bar{Y}_i - \bar{Y}_j \qquad \forall 1 \leq i \leq k$$

and that for the statistic

$$\tilde{T}_{ij}(x) = \sqrt{n}\frac{|\bar{Y}_i - \bar{Y}_j|}{S(x)} \quad \text{with} \quad S(x) = \frac{1}{k(n-1)} \sum_{i=1}^{k} \sum_{l=1}^{n} (x_{il} - \bar{x}_{i.})^2$$

$\max_{1 \leq i < j \leq k} \tilde{T}_{ij}(x)$ is $q_{k,k(n-1)}$ distributed.

Now we can test a candidate μ' to level α using the Tukey principle, because μ will be rejected if and only if

$$\sqrt{n}\frac{|\bar{y}_i' - \bar{y}_j'|}{S(x)} > q_{k,k(n-1);\alpha}$$

where $\bar{y}_i' = \bar{x}_{i\cdot} - \mu_i'$ If we rewrite this, we see that μ' will be accepted if and only if for all $1 \leq i < j \leq k$:

$$|(\bar{x}_{i\cdot} - \bar{x}_{j\cdot}) - \vartheta_{ij}'| \leq \frac{S(x)}{\sqrt{n}} q_{k,k(n-1);\alpha}$$
$$\Leftrightarrow \vartheta_{ij}' \in \left[(\bar{x}_{i\cdot} - \bar{x}_{j\cdot}) - \frac{S(x)}{\sqrt{n}} q_{k,k(n-1);\alpha}, (\bar{x}_{i\cdot} - \bar{x}_{j\cdot}) + \frac{S(x)}{\sqrt{n}} q_{k,k(n-1);\alpha} \right]$$

Because of the extended correspondence theorem we get a compatable confidence region for our ϑ_{ij}'.

Scheffé test

In the ANOVA model, let $q \le k$, $a_1, \ldots, a_q \in \mathbb{R}^k$ be linear independent and define $\mathscr{L} = \text{span}\,(a_1, \ldots, a_q)$. In the lecture on Multiple Tests we learned that $\forall \mu \in \mathbb{R}^k, \sigma^2 > 0$ we have:

$$\mathbb{P}\left(c^\top \mu \in \left[c^\top \hat{\mu} - K, c^\top \hat{\mu} + K\right] \quad \forall c \in \mathscr{L}\right) = 1 - \alpha$$

where $\hat{\mu} := (\bar{X}_{1\cdot}, \ldots, \bar{X}_{k\cdot})^\top$, and

$$K := \sqrt{q \cdot s^2 \sum_{i=1}^{k} \frac{c_i^2}{n} F_{q,k(n-1);\alpha}}$$

Why Bother??

- Stepwise procedures are typically more powerful
- But in the past single step procedures were used when confidence regions were required (medical experiments. . .)
- So if one wants the best of both worlds, a better theory needs to be developed

General Stratigy

We can construct (in theory) a confidence region $C \in \mathscr{C}_{1-\alpha}$ which is compatible with $\psi \in \Phi_\alpha(\mathscr{H})$ using the following two steps:

1. Determine $\varphi \in \Phi_\alpha(\Theta)$ induced by ψ. Then calculate $\mathbb{P}_\vartheta(\varphi_\vartheta = 1)$ for all $\vartheta \in \Theta$. If $\mathbb{P}_\vartheta(\varphi_\vartheta = 1) < \alpha$ try to find a $\tilde{\varphi}_\vartheta \geq \varphi_\vartheta$ where
   - $\alpha \geq \mathbb{P}_\vartheta(\tilde{\varphi}_\vartheta = 1) > \mathbb{P}_\vartheta(\varphi_\vartheta = 1)$
   - ideally we want $\mathbb{P}_\vartheta(\tilde{\varphi}_\vartheta = 1) = \alpha$
2. Then construct $\tilde{C} := C(\tilde{\varphi})$
   - $\forall x \in \mathscr{X} : \tilde{C}(x) \subseteq C(x)$ where $C = C(\varphi)$.
   - $\tilde{C}$ and ψ are compatable.

Problems with stepwise procedures

There are two main problems with the practical implementation of this strategy:

1. Finding a good $\tilde{\varphi}_\vartheta$ for all $\vartheta \in \Theta$.
2. Inverting the set of $\tilde{\varphi}_\vartheta$ into a multiple confidence region $\tilde{C}$.

Example

- Z_i, $i \in I = \{1, \ldots, k\}$ permutation-symmetric random variables, with Lebesgue density $f(x) > 0 \; \forall x \in \mathbb{R}$
- $X_i := Z_i + \vartheta_i$, $\vartheta_i \in \mathbb{R}$
- $\mathscr{H} = \{H_i : i \in I\}$ with $H_i : \vartheta_i \leq 0$ vs. $K_i : \vartheta_i > 0$

We want to find lower confidence bounds for $\underline{\vartheta_i}$ which are compatible with step-up and step-down procedures. Thus set:

- $\alpha \in (0, 1)$
- $c_1 \leq \cdots \leq c_k$ defined by $\mathbb{P}\left(\max_{1 \leq i \leq j} Z_i \leq c_j\right) = 1 - \alpha$, $j \in I$

Example (cont.)

We define a step-down test with the rule:

$$\psi_i(x) = 1 \Leftrightarrow \exists r \in I : x_i \geq c_r \text{ and } \forall j \in \{r, \ldots, k\} : x_{(j)} \geq c_j$$

Now we must choose a test $\varphi = (\varphi_\vartheta : \vartheta \in \Theta)$ with $C(\varphi) \subset C(\psi)$. To do this, define:

$$\kappa(\vartheta) := \{i \in I : \vartheta_i \leq 0\},\ \vartheta \in \mathbb{R}$$

and consider the following two cases:

Case 1: $\kappa(\vartheta) \neq \emptyset$

Example (cont.)

In this case, define a level-α test by:

$$\varphi_\vartheta(x) = 1 \Leftrightarrow \max_{i \in \kappa(\vartheta)} (x_i - \vartheta_i) \geq c_{|\kappa(\vartheta)|}.$$

This is indeed a level-α test because:

$$\begin{aligned} \mathbb{P}_\vartheta\left(\varphi_\vartheta(X) = 1\right) &= \mathbb{P}_\vartheta\Big(\max_{i \in \kappa(\vartheta)} Z_i \geq c_{|\kappa(\vartheta)|}\Big) \\ &= \mathbb{P}_\vartheta\Big(\max_{1 \leq i \leq |\kappa(\vartheta)|} Z_i \geq c_{|\kappa(\vartheta)|}\Big) \leq \alpha \end{aligned}$$

Case 2: $\kappa(\vartheta) = \emptyset$

Example (cont.)

Here we define a level-α test by:

$$\varphi_\vartheta(x) = 1 \Leftrightarrow \max_{i \in I}(x_i - \vartheta_i) \geq c_k.$$

Again we see that

$$\mathbb{P}_\vartheta(\varphi_\vartheta(X) = 1) = \mathbb{P}_\vartheta(\max_{i \in I} Z_i \geq c_k) = \mathbb{P}_\vartheta(\max_{1 \leq i \leq k} Z_i \geq c_k) \leq \alpha$$

Theorem

For the Bonferroni-Holm test we have the following lower bounds:

$$\underline{\vartheta_i}(x) = \begin{cases} 0 & \psi_i(x) = 1,\ m(x) < k \\ L_i(x) & \psi_i(x) = 0 \\ K_i(x) & m(x) = k \end{cases}$$

where $m(x) := |\{i \in I : \psi_i(x) = 1\}|$ *and* $K_i(x)$ *is such that:*

$$\forall \vartheta \in \Theta : \quad \mathbb{P}_\vartheta\left(\vartheta_i \geq K_i(x)\right) \geq 1 - \alpha_k$$

and $L_i(x)$ *such that:*

$$\forall \vartheta \in \Theta : \quad \mathbb{P}_\vartheta\left(\vartheta_i \geq L_i(x)\right) \geq 1 - \alpha_{k-\kappa(\vartheta)+1}$$

Example (cont.)

Returning to our example, we can define

$$K_i(x) = x_i - c_k \quad \text{and} \quad L_i(x) = x_i - c_{k-m(x)}.$$

This gives us then

$$\underline{\vartheta_i}(x) = \begin{cases} 0 & \psi_i(x) = 1,\ m(x) < k \\ x_i - c_{k-m(x)} & \psi_i(x) = 0 \\ x_i - c_k & m(x) = k \end{cases}$$

or equivalently

$$\underline{\vartheta_i}(x) = \begin{cases} \min\{0, x_i - c_{k-m(x)}\} & m(x) < k \\ \max\{0, x_i - c_k\} & m(x) = k. \end{cases}$$

Fazit

- Compatable confidence regions are easy to create for single step procedures
- For multiple step procedures things are a good bit harder, but not impossible
- But all told it is an interesting question deserving more consideration.

Vielen Dank für eure Aufmerksamkeit.