# Implicitly adaptive FDR control based on the asymptotically optimal rejection curve

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# Outline

Multiple Testing and the False Discovery Rate

Explicitly adaptive FDR control

Asymptotically Optimal Rejection Curve (AORC)

# Multiple statistical decision problems

- Multiple comparisons (multiple tests)
- Simultaneous confidence regions
- Multiple power / sample size calculation
- Selection problems
- Ranking problems
- Partitioning problems

# Prologue

We assume a statistical model (statistical experiment)
$(\Omega, \mathcal{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta})$

More concrete scenario (chosen for exemplary purposes):

Balanced ANOVA1 model:

$X = (X_{ij})_{i=1,\ldots,k,\ j=1,\ldots,n}, \quad X_{ij} \sim \mathcal{N}(\mu_i, \sigma^2),$
$X_{ij}$ stochastically independent random variables on $\mathbb{R}$,
$\mu_i \in \mathbb{R}\ \forall 1 \leq i \leq k, \quad \sigma^2 > 0$ (known or unknown) variance
$k \geq 3, n \geq 2, \quad \nu = k(n-1)$ (degrees of freedom)

$$\Omega = \mathbb{R}^{k \cdot n},\ \mathcal{F} = \mathbb{B}^{k \cdot n}$$
$$\vartheta = (\mu_1, \ldots, \mu_k, \sigma^2) \in \mathbb{R}^k \times [0, \infty) = \Theta$$

# Multiple comparisons (multiple tests)

| | |
|---|---|
| $(\Omega, \mathcal{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta})$ | Statistical model |
| $\mathcal{H}_m = (H_i)_{i=1,\ldots,m}$ | Family of null hypotheses with $\emptyset \neq H_i \subset \Theta$ and alternatives $K_i = \Theta \setminus H_i$ |
| $(\Omega, \mathcal{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta}, \mathcal{H}_m)$ | Multiple testing problem |
| $\varphi = (\varphi_i : i = 1, \ldots, m)$ | multiple test for $\mathcal{H}_m$ |

| | Test decision | | |
|---|---|---|---|
| Hypotheses | 0 | 1 | |
| true | $U_m$ | $V_m$ | $m_0$ |
| false | $T_m$ | $S_m$ | $m_1$ |
| | $m - R_m$ | $R_m$ | $m$ |

# Multiple tests (II)

Wanted features of $\varphi$:

No (or only minor) contradictions among the individual test decisions

Control of the probability of erroneous decisions

(Type I) error measures / concepts:

$$\text{FWER}_m(\varphi) = \mathbb{P}_\vartheta(V_m > 0) \overset{!}{\leq} \alpha \quad \forall \vartheta \in \Theta$$

(Strong) control of the Family-Wise Error Rate (FWER)

$$\text{FDR}_m(\varphi) = \mathbb{E}_\vartheta\left[\frac{V_m}{R_m \vee 1}\right] \overset{!}{\leq} \alpha \quad \forall \vartheta \in \Theta$$

Control of the False Discovery Rate (FDR)

# FWER control

FWER-controlling multiple testing principles:

- Historical single-step tests
  (Bonferroni, Sidak, Tukey, Scheffé, Dunnett)
- Holm's (1979) stepwise rejective procedure
- Closed test principle (Marcus, Peritz, Gabriel (1969))
- Intersection-union principle (generalized closed testing)
- Partitioning principle
  (Finner and Straßburger 2001, Hsu 1996)
- Projection methods under asymptotic normality
  (Bretz, Hothorn and Westfall)
- Resampling-based multiple testing
  (Westfall / Young 1992, Dudoit / van der Laan 2008)

# Multiplicity of current applications

Due to rapid technical developments in many scientific fields, the number $m$ of hypotheses to be tested simultaneously can nowadays become **almost arbitrary large:**

- Genetics, microarrays: $m \sim 30000$ genes / hypotheses
- Genetics, SNPs: $m \sim 500000$ SNPs / hypotheses
- Proteomics: $m \sim 5000$ proteine spots per gele sheet
- Cosmology: Signal detection, $m \sim 10^6$ pixels / hypotheses
- Neurology: Identification of active brain loci, $m \sim 10^3$ voxels
- Biometry: pairwise comparisons of many means,
  tests for correlations

Analyses have (in a first step) typically **explorative character.**

$\Longrightarrow$ Control of the FWER much too conservative goal!

# Definition of the False Discovery Rate

| | |
|---|---|
| $\Theta$ | Parameter space |
| $H_1, \ldots, H_m$ | Null hypotheses |
| $\varphi = (\varphi_1, \ldots, \varphi_m)$ | Multiple test procedure |
| $V_m = \lvert\{i : \varphi_i = 1 \text{ and } H_i \text{ true }\}\rvert$ | Number of falsely rejected, true nulls |
| $R_m = \lvert\{i : \varphi_i = 1\}\rvert$ | Total number of rejections |

$$\mathrm{FDR}_\vartheta(\varphi) = \mathbb{E}_\vartheta[\frac{V_m}{R_m \vee 1}]$$

False Discovery Rate (FDR) given $\vartheta \in \Theta$

**Definition:** Let $\alpha \in (0, 1)$ fixed.

The multiple test $\varphi$ controls the FDR at level $\alpha$ if

$$\mathrm{FDR}(\varphi) = \sup_{\vartheta \in \Theta} \mathrm{FDR}_\vartheta(\varphi) \leq \alpha.$$

# The classical FDR theorem

## Benjamini and Hochberg (1995)

$p_1, \ldots, p_m$ (marginal) $p$-values for hypotheses $H_1, \ldots, H_m$

$H_i$ true for $i \in I_{m,0}$, $H_i$ false for $i \in I_{m,1}$

$I_{m,0} + I_{m,1} = \mathbb{N}_m = \{1, \ldots, m\}$, $m_0 = |I_{m,0}|$

$p_i \sim$ UNI$[0,1]$, $i \in I_{m,0}$ stochastically independent (I1)

$(p_i : i \in I_{m,0}), (p_i : i \in I_{m,1})$ stoch. independent vectors (I2)

$p_{1:m} \leq \cdots \leq p_{m:m}$ ordered $p$-values

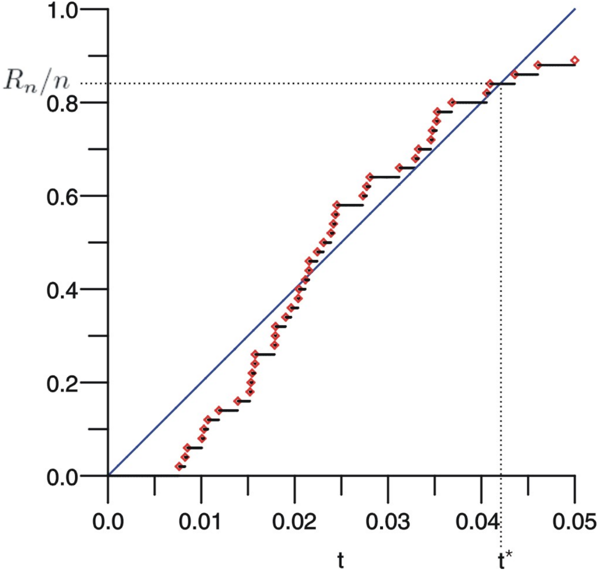
Linear step-up procedure $\varphi^{\text{LSU}}$ with Simes' crit. values $\alpha_{i:m} = i\alpha/m$:

Reject all $H_i$ with $p_i \leq \alpha_{\overline{m}:m}$, where $\overline{m} = \max\{j : p_{j:m} \leq j\alpha/m\}$.

Then it holds:

$$\text{FDR}_\vartheta(\varphi^{\text{LSU}}) = \mathbb{E}_\vartheta\left[\frac{V_m}{R_m \vee 1}\right] = \frac{m_0}{m}\alpha \quad \forall \vartheta \in \Theta.$$

# Linear step-up in terms of Simes' rejection line



# FDR control under positive dependency

**Benjamini, Y. & Yekutieli, D. (2001) / Sarkar, S. K. (2002):**

Proofs for FDR control in presence of special dependency structures:

$$\mathrm{FDR}_{\vartheta}(\varphi) \leq \frac{m_0}{m}\alpha \ \ \forall \vartheta \in \Theta$$

for stepwise test procedures employing Simes' critical values.

Model assumptions: **MTP$_2$ oder PRDS**

Examples:
Multivariate normal distributions with non-negative correlations, multivariate (absolute) $t$-distributions

(Finner, Dickhaus, Roters (2007), The Annals of Statistics **35**, 1432-1455)

# Explicit adaptation

Since under positive dependency the FDR of the LSU-procedure is bounded by

$$\frac{m_0}{m}\alpha$$

for any $m > 1$ and given $\alpha \in (0, 1)$, $\varphi^{\text{LSU}}$ does not exhaust the FDR level $\alpha$ in case of $m_0 < m$.

Many modern FDR-controlling methods:

Pre-estimation of $m_0$ or $\pi_0 = m_0/m$ aiming at

tighter $\alpha$-exhaustion and gain of power

**(Explicit adaption)**

# Empirical stochastic processes

Interpret the number of rejections of (true / false) null hypotheses and the FDR of a single-step test with threshold $t \in [0, 1]$ for the $p$-values as empirical processes in $t$:

$$
\begin{aligned}
V(t) &= \sum_{i \in I_{m,0}} \mathbf{1}_{[0,t]}(p_i), \\
S(t) &= \sum_{i \in I_{m,1}} \mathbf{1}_{[0,t]}(p_i), \\
R(t) &= V(t) + S(t) = \sum_{i=1}^{m} \mathbf{1}_{[0,t]}(p_i), \\
\text{FDR}(t) &= \mathbb{E}\left[\frac{V(t)}{R(t) \vee 1}\right].
\end{aligned}
$$

# The adaptive procedure of Storey, Taylor and Siegmund (2004)

For a tuning parameter $\lambda \in (0, 1)$ is the following estimator $\hat{\pi}_0$ of $\pi_0$ reasonable (Schweder and Spjøtvoll (1982)):

$$\hat{\pi}_0 \equiv \hat{\pi}_0(\lambda) = \frac{m - R(\lambda) + 1}{m(1-\lambda)} = \frac{1 - \hat{F}_m(\lambda) + 1/m}{1-\lambda}$$

Under (I1) and (I2), we additionally obtain an estimator for the FDR of a single-step test with threshold $t \in [0, 1]$:

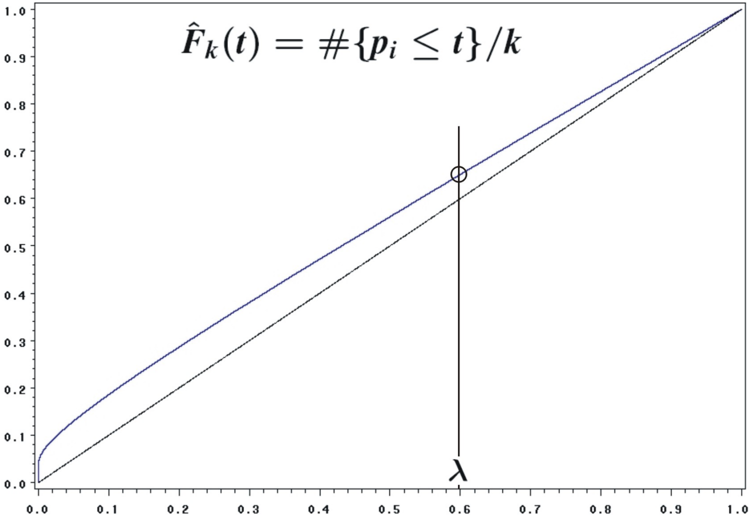
$$\widehat{\mathrm{FDR}}_\lambda(t) = \frac{\hat{\pi}_0(\lambda)t}{(R(t) \vee 1)/m}$$

Resulting adaptive thresholding:

$$t_\alpha^\lambda = \sup\{0 \le t \le \lambda : \widehat{\mathrm{FDR}}_\lambda(t) \le \alpha\}$$

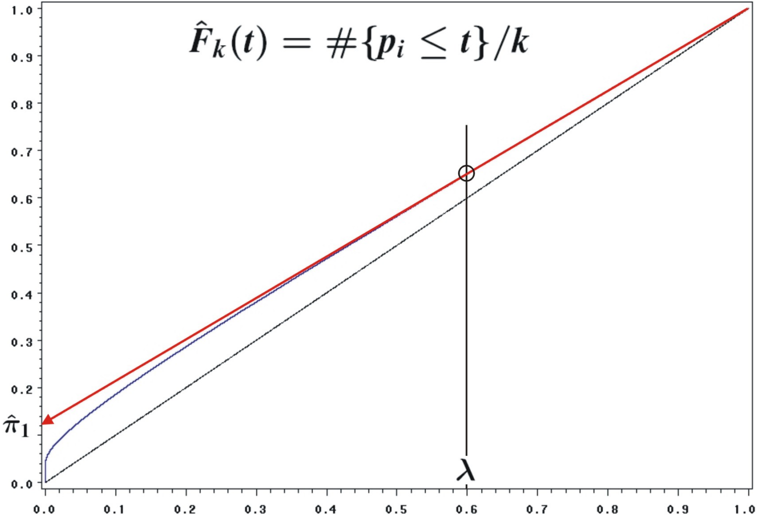
# The estimator $\hat{\pi}_0$

## Schweder and Spjøtvoll (1982)



# The estimator $\hat{\pi}_0$

## Schweder and Spjøtvoll (1982)



# Martingal- and stopping time properties

## Lemma: (Storey, Taylor and Siegmund (2004))

Assuming stochastically independent $p$-values under the $m_0$ null hypotheses, $V(t)/t$ is for $0 \leq t < 1$ a reverse martingal with respect to the filtration $\mathcal{F}_t = \sigma(\mathbf{1}_{[0,s]}(p_i), t \leq s \leq 1, i = 1, \ldots, m)$, i. e., for $s \leq t$ it holds $\mathbb{E}[V(s)/s|\mathcal{F}_t] = V(t)/t$.

The random threshold $t_\alpha^\lambda$ is a stopping time with respect to $\mathcal{F}_{t \wedge \lambda}$.

$\Longrightarrow$ FDR-proofs utilizing theory of optimal stopping

# Proof of FDR-control of the adaptive test procedure by Storey et al.

Assumption: $\lambda$ chosen such that $\widehat{\text{FDR}}_\lambda(\lambda) \geq \alpha$.

It follows: $\widehat{\text{FDR}}_\lambda(t_\alpha^\lambda) = \alpha \Leftrightarrow R(t_\alpha^\lambda) = m\hat{\pi}_0(\lambda)t_\alpha^\lambda/\alpha$.

Moreover, the process $V(t)/t$ stoppt at $t_\alpha^\lambda$ is bounded and

$$
\begin{aligned}
\text{FDR}(t_\alpha^\lambda) &= \mathbb{E}\left[\frac{V(t_\alpha^\lambda)}{R(t_\alpha^\lambda)}\right] = \mathbb{E}\left[\frac{\alpha}{m\hat{\pi}_0(\lambda)}\frac{V(t_\alpha^\lambda)}{t_\alpha^\lambda}\right] \\
&= \mathbb{E}\left[\alpha\frac{1-\lambda}{m-R(\lambda)+1}\frac{V(t_\alpha^\lambda)}{t_\alpha^\lambda}\right] \\
&= \mathbb{E}\left[\alpha\frac{1-\lambda}{m-R(\lambda)+1}\,\mathbb{E}\left[\frac{V(t_\alpha^\lambda)}{t_\alpha^\lambda}|\mathcal{F}_\lambda\right]\right] \\
&= \mathbb{E}\left[\alpha\frac{1-\lambda}{m-R(\lambda)+1}\frac{V(\lambda)}{\lambda}\right].
\end{aligned}
$$

## Noticing $V(\lambda) \sim \text{Bin}(n_0, \lambda)$, yields:

$$
\begin{aligned}
\text{FDR}(t_\alpha^\lambda) &= \mathbb{E}\left[\alpha \frac{1-\lambda}{m - R(\lambda) + 1} \frac{V(\lambda)}{\lambda}\right] \\
&\leq \mathbb{E}\left[\alpha \frac{1-\lambda}{m_0 - V(\lambda) + 1} \frac{V(\lambda)}{\lambda}\right] \\
&= \sum_{k=0}^{m_0} \alpha \frac{1-\lambda}{m_0 - k + 1} \frac{k}{\lambda} \cdot \mathbb{P}(V(\lambda) = k) \\
&= \alpha \frac{1-\lambda}{\lambda} \sum_{k=0}^{m_0} \frac{k}{m_0 - k + 1} \binom{m_0}{k} \lambda^k (1-\lambda)^{m_0 - k} \\
&= \alpha \frac{1-\lambda}{\lambda} \cdot \frac{\lambda - \lambda^{m_0+1}}{1-\lambda} = \alpha(1 - \lambda^{m_0}) \\
&\leq \alpha \ \text{ for all } \ \lambda \in (0,1) \ \text{ and } \ m_0 \geq 0. \quad \blacksquare
\end{aligned}
$$

# Implicit adaptation

Alternatively to explicit adaptation, it may be asked:

**Is it possible to derive a better rejection curve circumventing the factor $m_0/m$ appearing in the FDR of $\varphi^{\text{LSU}}$ ?**

First step:

Identification of least favorable parameter configurations (LFCs) for the FDR.

# Dirac-uniform models as LFCs

## Theorem: (Benjamini & Yekutieli (2001))

If $p_i \sim U([0,1])$, $i \in I_{m,0}$, stochastically independent (I1) and $(p_i : i \in I_{m,0}), (p_i : i \in I_{m,1})$ stoch. independent vectors (I2), then a step-up procedure $\varphi^{\mathrm{SU}}_{(m)}$ with critical values $\alpha_{1:m} \leq \cdots \leq \alpha_{m:m}$ has the following properties: If

$$\alpha_{i:m}/i \text{ is increasing (decreasing) in } i$$

and the distribution of $(p_i : i \in I_{m,1})$ decreases stochastically, then the FDR of $\varphi^{\mathrm{SU}}_{(m)}$ increases (decreases).

If $\alpha_{i:m}/i$ is increasing in $i$, it follows that the FDR becomes largest for $p_i \sim \delta_0 \; \forall i \in I_{m,1}$ (Dirac-uniform model).

In DU-models, **analytic calculations** are possible!

# Asymptotic Dirac-uniform model: DU$(\zeta)$

## Assumptions:

Independent $p$-values $p_1, \ldots, p_m$;

$m_0 = m_0(m)$ null hypotheses true with

$$\lim_{m \to \infty} \frac{m_0(m)}{m} = \zeta \in (0, 1],$$

$m_0$ $p$-values UNI$([0, 1])$-distributed (corresp. hypotheses true)

$m_1 = m - m_0$ $p$-values $\delta_0$-distributed (corresp. hypotheses false)

Then the ecdf of the $p$-values $F_m$ (say) converges (Glivenko-Cantelli) for $m \to \infty$ to

$$G_\zeta(x) = (1 - \zeta) + \zeta x \text{ for all } x \in [0, 1].$$

# Heuristic for an asymptotically optimal rejection curve

Assume we reject all $H_i$ with $p_i \leq x$ for some $x \in (0, 1)$.

Then the FDR (depending on $\zeta$ and $x$) under DU($\zeta$) is asymptotically given by

$$\mathrm{FDR}_\zeta(x) = \frac{\zeta x}{(1-\zeta) + \zeta x}.$$

Aim: Find an optimal threshold $x_\zeta$ (say), such that

$$\mathrm{FDR} \equiv \alpha \text{ for all } \zeta \in (\alpha, 1).$$

We obtain:

$$\mathrm{FDR}_\zeta(x_\zeta) = \alpha \iff x_\zeta = \frac{\alpha(1-\zeta)}{\zeta(1-\alpha)}.$$

# Asymptotically optimal rejection curve

**Ansatz:** Rejection curve $f_\alpha$ and $G_\zeta$ shall cross each other

in $x_\zeta$, i.e., $f_\alpha(x_\zeta) = G_\zeta(x_\zeta)$.

Plugging in $x_\zeta$ derived above yields

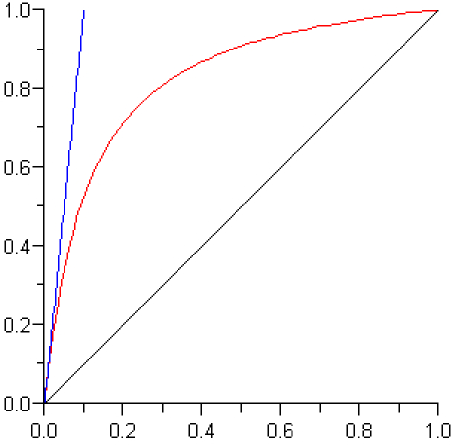
$$f_\alpha\left(\frac{\alpha(1-\zeta)}{\zeta(1-\alpha)}\right) = \frac{1-\zeta}{1-\alpha}.$$

Substituting $t = \dfrac{\alpha(1-\zeta)}{\zeta(1-\alpha)} \Longleftrightarrow \zeta = \dfrac{\alpha}{(1-\alpha)t+\alpha}$,

we get that $f_\alpha(t) := \dfrac{t}{(1-\alpha)t+\alpha}$, $t \in [0,1]$,

is the curve solving the problem!

# Asymptotically optimal rejection curve for $\alpha = 0.1$



# Critical values, step-up-down procedure

The critical values induced by $f_\alpha$ are given by

$$\alpha_{i:m} = f_\alpha^{-1}(i/m) = \frac{i\alpha}{m - i(1-\alpha)}, \quad i = 1, \dots, m. \qquad (1)$$

Due to $\alpha_{m:m} = 1$, a **step-up** procedure based on $f_\alpha$ cannot work.

One possible solution:

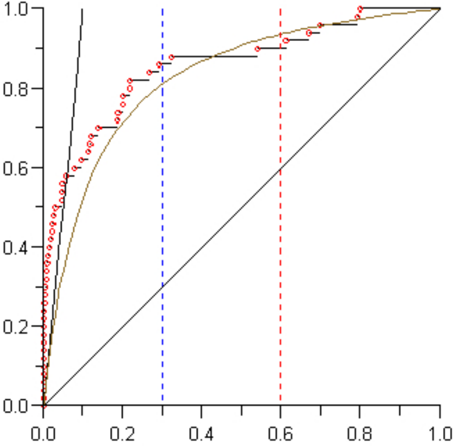
Step-up-down procedure with parameter $\lambda \in (0,1)$:

$F_m(\lambda) \geq f_\alpha(\lambda) \Rightarrow t^* = \inf\{p_i > \lambda : F_m(p_i) < f_\alpha(p_i)\}$ (SD-branch),

$F_m(\lambda) < f_\alpha(\lambda) \Rightarrow t^{**} = \sup\{p_i < \lambda : F_m(p_i) \geq f_\alpha(p_i)\}$ (SU-branch).

Reject all $H_i$ with $p_i < t^*$ or $p_i \leq t^{**}$, respectively.

# SUD-procedure for $\lambda = 0.3,\ 0.6$ $(m = 50, \alpha = 0.1)$



# Refined results for SUD tests

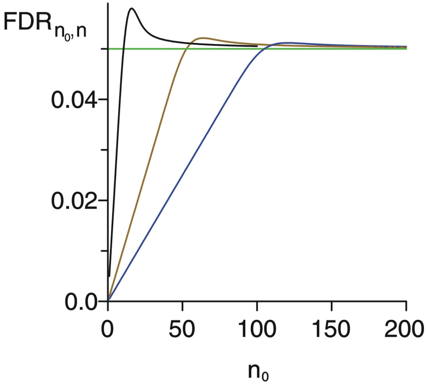
Finner, Dickhaus, Roters (2009), Annals of Statistics 37, 596-618

- AORC-based stepwise test procedures asymptotically keep the FDR level under (I1) und (I2).
- In the class of stepwise test procedures with fixed rejection curve asymptotically keeping the FDR level, AORC-based tests have largest power for $\zeta \in (\alpha, 1)$.

<u>Technical results:</u>

- New methodology of proof for stepwise test procedures with non-linear rejection curves resp. critical values
- Upper FDR bounds for step-up-down tests (asymptotically and finite)

# SU-test with modified version of $f_\alpha$, finite case ($m = 100, 500, 1000$, $\alpha = 0.05$)




For $m = 100$, maximum DU FDR is $\mathrm{FDR}_{16,100} \approx 0.05801$.

$\Longrightarrow$ **Adjustment** of critical values for finite cases necessary!

# FDR control for step-up implies FDR control for step-up-down

## Theorem:

Consider a SU test $\varphi^m$ and a SUD$(\lambda)$ test $\varphi^\lambda$ for $\lambda \in \{1, \ldots, m-1\}$ with the same set of critical values $(\alpha_{i:m})_{i=1}^m$ belonging to
$\mathcal{M}_m = \{(c_{i:m})_{i=1}^m : 0 \leq c_{1:m} \leq \ldots \leq c_{m:m} \leq 1, c_{i:m}/i \text{ increases in } i\}$.
Then, under (I1) and (I2), it holds

$$\mathrm{FDR}_\vartheta(\varphi^\lambda) \leq \mathrm{FDR}_\vartheta(\varphi^m) \text{ for all } \vartheta \in \Theta.$$

Hence, if the FDR is controlled by the SU test, then the SUD$(\lambda)$ test also controls the FDR.

**Sketch of Proof:** $\{R_m^{\lambda_1} \geq j,\ p_{i_0} \leq \alpha_{j:m}\} \subseteq \{R_m^{\lambda_2} \geq j,\ p_{i_0} \leq \alpha_{j:m}\}$ for any $1 \leq \lambda_1 \leq \lambda_2 \leq m$, which implies that $P_\vartheta(R_m^\lambda \geq j | p_{i_0} \leq \alpha_{j:m})$ is non-decreasing in $\lambda$ for each $j \in \mathbb{N}_m$.

# Exact finite adjustment (for step-up)

**(Slight) modification of $f_\alpha$ or its critical values, e. g.**

$$\alpha_{i:m} = \frac{i\alpha}{m + \beta_m - i(1-\alpha)}, \; i = 1, \ldots, m,$$

for a suitable adjustment constant $\beta_m > 0$.

(Same as: Use $\tilde{f}_\alpha(t) = (1 + \beta_m/m)\, f_\alpha(t)$, $t \in [0, \alpha/(\alpha + \beta/m)]$.)

$m = 100$ leads to $\beta_{100} \approx 1.76$.

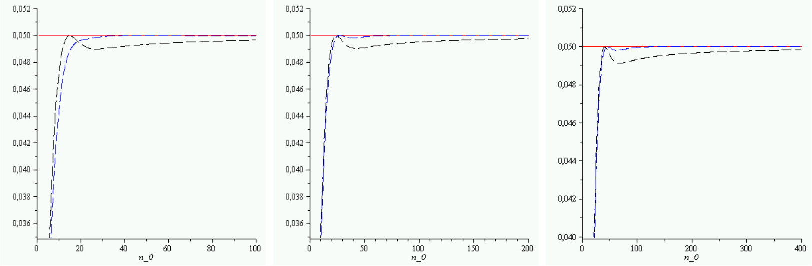
Ray of light:

GAVRILOV, Y., BENJAMINI, Y. AND SARKAR, S. K. (2009).
An adaptive step-down procedure with proven FDR control.
*Annals of Statistics* **37**, 619-629:
SD-procedure with $\beta_m \equiv 1.0$ controls FDR for every $m \in \mathbb{N}$.

# Adjustment with three parameters

For $i = 1, \ldots, m$, utilize adjusted critical values of the form

$$\alpha_{i:m} = \frac{i\alpha}{m + \beta_{1,m} + \beta_{2,m}\left(\frac{i}{m}\right)^{\beta_{3,m}} - i(1-\alpha)}$$




Resulting FDRs for adjusted critical values with one (black) and three parameters, respectively, for $m = 100, 200, 400$, depending on $m_0$.

# Iterative method

Let $J \in \mathbb{N}$ be fixed and $\alpha_{1:m}^{(0)}, \ldots, \alpha_{m:m}^{(0)} \in \mathcal{M}_m$ be start critical values, for instance adjusted AORC-based critical values, fulfilling that $\text{FDR}_{m,m_0} \approx \alpha$ for all $m_0 \geq k$.

Now, try to iteratively modify crit. values most influencing $\text{FDR}_{m,m_0}$ for $k \leq m_0 \leq m$ to reduce $d(m_0, m) = \alpha - FDR_{m,m_0}$.

<u>One possible iteration scheme:</u>

For $j$ from $1$ to $J$ do:

For $i$ from $1$ to $i^*(k)$ do:

1. Determine $\alpha_i^{(j-1)}$ from $\alpha_{i:m}^{(j-1)} = i\alpha_i^{(j-1)}/(m - i(1 - \alpha_i^{(j-1)}))$.
2. Put $\alpha_i^j = \alpha\alpha_i^{(j-1)}/FDR_{m,m_0(i)}(\alpha_i^{(j-1)})$.

<u>Motivation:</u> Fixed-point iteration for

$$f(\alpha_i) = \alpha\alpha_i/\text{FDR}_{m,m_0(i)}(\alpha_i).$$

# Stepwise finding

Subsequently solve (for $\alpha_{m_1+1:m}$) the target equations

$$FDR_{m,m_0}(\alpha_{m_1+1:m}, \ldots, \alpha_{m:m}) = \alpha, \quad m_0 \in \mathbb{N}_m. \tag{2}$$

$m_0 = 1 \Rightarrow \alpha_{m:m} = m\alpha$ for a fixed $\alpha$, because $FDR_{m,1} = \alpha_{m:m}/m$. It follows that $\alpha_{m:m} \geq 1$ for each $m \geq 1/\alpha$, which is unacceptable. We can try to fix that by replacing (2) by

$$FDR_{m,m_0}(\alpha_{m_1+1:m}, \ldots, \alpha_{m:m}) = min\left(\frac{m_0}{m}, \alpha\right), \quad m_0 \in \mathbb{N}_m. \tag{3}$$

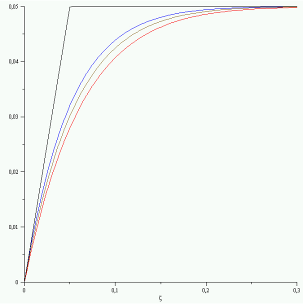
If even (3) cannot be solved in $\mathcal{M}_m$, we further generalize the right-hand-side and solve

$$FDR_{m,m_0}(\alpha_{m_1+1:m}, \ldots, \alpha_{m:m}) = g(\alpha, \zeta_m), \quad m_0 \in \mathbb{N}_m.$$

# Flexible FDR-control

## Candidates for $g(\alpha, \zeta_m)$:

$$g(\zeta|\gamma,\eta) = \begin{cases} \alpha(1-(1-\zeta/\gamma)^{\eta}), & 0 \leq \zeta < \gamma, \\ \alpha, & \gamma \leq \zeta \leq 1. \end{cases}$$




$g(\zeta|\gamma,\eta)$ for $\alpha = 0.05$ and $\gamma = 1, \eta = 20.0, 18.0, 16.0$ together with $\min(\alpha,\zeta)$ for $\zeta$ ranging from 0 to 0.3.

## Acknowledgments:


- Helmut Finner
- Veronika Gontscharuk
- Markus Roters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