Massey products of arrangement groups, by Daniel Matei.

Let $\mathcal{A}$ a complex line arrangement in $\mathbb{C}^2$, with complement X and group $G = \pi_1(X)$. Recall that the rings $H^{*\leq 2}(X, \mathbb{K})$ and $H^{*\leq 2}(G, \mathbb{K})$ are isomorphic for $\mathbb{K}$ a field or $\mathbb{Z}$. Taking advantage of this, we use the cochains of G rather than of X to compute Massey products. By a classical result in rational homotopy theory X is a rationally formal space, therefore all its k-fold Massey products in $H^*(X, \mathbb{Q})$ vanish if $k \geq 3$. We exhibit here arrangements whose complement X, although $\mathbb{Q}$-formal, is not $\mathbb{F}_p$-formal, by presenting non-vanishing Massey products in $H^2(X, \mathbb{F}_p)$.

Theorem: *For every prime p, there exists an arrangement $\mathcal{A}$ such that there are, modulo indeterminacy, non-vanishing Massey products in $H^2(X, \mathbb{F}_p)$.*

For example, for $p = 3$ the arrangement $\mathcal{A}$ may be taken to be either of the two MacLane arrangements of 8 lines in $\mathbb{CP}^2$. Dehomogenizing, we obtain two affine arrangements of 7 lines in $\mathbb{C}^2$, say $\mathcal{A}^+$ and $\mathcal{A}^-$. Their complements $X^\pm$ are of the same homotopy type, say X. In [2] we used the work of Falk [1] to compute the resonance varieties of $\mathcal{A}^\pm$ over $\mathbb{F}_p$ and found that at the prime $p = 3$ they are special: ${}_1(X, \mathbb{F}_p) \subset H^1(X, \mathbb{F}_p) = \mathbb{F}_3^7$ consists of 8 components in general, but for $p = 3$ a ninth arises, say C. That is a plane in $\mathbb{F}_3^7$, consisting of 9 points. Choosing any two of them different from origin, we get linearly independent cohomology classes in $H^1(X, \mathbb{F}_p)$ that cup zero, and so it possible to define their triple products.

Theorem: *Any triple Massey product in $H^2(X, \mathbb{F}_3)$ of the form $\langle \lambda, \lambda, \mu \rangle$ with λ and μ distinct points on $C \subset H^1(X, \mathbb{F}_3)$ does not vanish modulo the indeterminacy $\lambda \cup H^1(X, \mathbb{F}_3) + H^1(X, \mathbb{F}_3) \cup \mu$.*

Interestingly enough, the existence of non-vanishing triple products in $H^2(X, \mathbb{F}_p)$ is related with the occurence of p-torsion in the homology $H_1(Y)$ of some p-fold coverings Y of X. For example, for the decones $\mathcal{A}^\pm$ of the Maclane arrangements we have that the 3-fold cyclic coverings Y_l of $X^\pm$, determined by $\lambda \in C \setminus \{\mathbf{0}\}$ exhibit 3-torsion in H_1, more exactly: $H_1(Y_l, \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_3$.

References

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[2] D. Matei, A. Suciu, *Cohomology rings and nilpotent quotients of real and complex arrangements*, Arrangements–Tokyo 1998, Adv. Stud. in Pure Math., vol. 27, Math. Soc. Japan, Tokyo, 2000.

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