Mathematisches Forschungsinstitut Oberwolfach

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Mini-Workshop: Combinatorial Stratifications in Geometry and Topology

November 2nd – November 8th, 2003

Geometry and Topology have been strongly intertwined with Combinatorics over various periods of their development. The explicitly combinatorial beginnings of algebraic topology and the discrete-geometric origin of Schubert calculus are just particularly prominent examples. Recent developments again put the fields in close interaction. A major issue is to capture essential information about topological spaces or algebraic varieties in combinatorial terms, e.g., to build combinatorial models in order to compute algebraic, topological, or geometric invariants of the spaces in question.

With the Mini-Workshop presented in this report, we undertook a first effort to bring together experts from various, rather different fields who share *combinatorial stratifications* as a common point of their research interests. As an initiating event, the workshop was focused on familiarizing the participants with recent developments in the respective areas.

The informal setting of an Oberwolfach Mini-Workshop allowed ample time for survey type expositions and for in-detail discussions that naturally evolved due to the different viewpoints represented in the audience. The range of topics was as broad as we had envisioned beforehand. Algebraic issues were addressed such as the recent consolidation of the theory of cluster algebras, or the role of sheaf theory on posets for geometric and topological problems. Topological matters were discussed including the recent progress in the theory of combinatorial differential manifolds, characteristic classes as obstructions to the existence of combinatorial structures, and the Schwarz genus of combinatorially constructed spaces. Geometry was omnipresent as well: just to mention a few topics we name here combinatorial constructions in geometric group theory, De Concini-Procesi compactifications, and stratifications of spaces of polynomials.

We expect that the state-of-the-art picture on combinatorial stratifications laid out during this intense week at Oberwolfach will bear fruits within various research projects and collaborations in the future.

We are most grateful to the Institute, its director, scientific board, and staff, to provide us with the excellent research environment which is so unique at Oberwolfach and which was outmost essential for the success of the meeting.

Eva-Maria Feichtner
Dmitry Kozlov

Abstracts

Arrangements via Minkowski decompositions of lattice polytopes

Klaus Altmann

Let Q be a lattice polytope with primitive edges. Taking the cone over it, it leads to an affine toric variety which is Gorenstein and has isolated singularities. Calculating the base space of its versal deformation, we obtain a variety S whose reduced structure is an arrangement of linear spaces reflecting the decomposition behavior of Q into a Minkowski sum of lattice polyhedra. Moreover, its non-reduced structure reflects decompositions into Minkowski sums of non-lattice polyhedra. Eventually, we show that S is the largest variety being contained in a certain toric variety and invariant under a certain group action of G_a.

What is a smooth map of combinatorial differential manifolds?

Laura Anderson

Combinatorial differential manifolds and their associated bundle theory, matroid bundles, were introduced by Gelfand and MacPherson in 1994. Matroid bundles have since seen great progress: we now know that the theory of matroid bundles is equivalent to the theory of real vector bundles over triangulable base spaces. Combinatorial differential manifolds have lagged behind, in part due to combinatorial difficulties in defining foundational ideas such as smooth maps. This talk surveyed both progress and obstacles.

Topological obstructions to graph colorings

Eric Babson

For any two graphs G and H Lovász has defined a cell complex $Hom(G, H)$ having in mind the general program that the algebraic invariants of these complexes should provide obstructions to graph colorings. Here we announce the proof of a conjecture of Lovász concerning these complexes with G a cycle of odd length.

More specifically, we show that if $Hom(C_{2r+1}, G)$ is k-connected, then $\chi(G) \geq k + 4$.

Our actual statement is somewhat sharper, as we find obstructions already in the non-vanishing of powers of certain Stiefel-Whitney characteristic classes.

This is joint work with Dmitry Kozlov.

Special matchings and Kazhdan-Lusztig polynomials

Francesco Brenti

Let W be a Coxeter group and $v \in W$. In this talk we proved that the Kazhdan-Lusztig polynomials $P_{x,y}(q)_{x,y \leq v}$ depend only on the Bruhat interval $[e, v]$ as an abstract poset. This implies, in particular, that the intersection cohomology of Schubert varieties is determined by the inclusion relations between the closures of its Schubert cells, and also confirms a conjecture of Lusztig. The proof is constructive, and gives an explicit algorithm for the computation of the polynomials starting from $[e, v]$ as an abstract poset.

This is joint work with F. Caselli and M. Marietti.

As part of the problem session:

Let P be a (graded) poset, and M be a complete matching of the Hasse diagram of P. For $x \in P$ denote by $M(x)$ the match of x. We say that M is a *special matching* if, for all $x, y \in P$, such that $M(x) \neq y$, we have that

$$x \lhd y \Rightarrow M(x) \leq M(y).$$

Note that this implies, in particular, that if $x \lhd y$ and $M(x) \rhd x$ then $M(y) \rhd y$ and $M(y) \rhd M(x)$, and dually that if $x \lhd y$ and $M(y) \lhd y$ then $M(x) \lhd x$ and $M(x) \lhd M(y)$.

Problem. Which posets have special matchings?

For example, the Boolean algebra of rank 3 (also called a 3-crown) has three special matchings, but the 4-crown does not have any special matchings.

On the Schwarz genus of the Coxeter fibration

Corrado De Concini

Consider the space $\mathcal{A} = \{(x_1, \ldots, x_n) \mid \sum x_i = 0,\ x_i \neq x_j \forall i \neq j\}$. The symmetric group $\mathcal{S}_n$ acts freely on $\mathcal{A}$. We set $X := \mathcal{A}/\mathcal{S}_n$ and let $\pi : \mathcal{A} \to X$ denote the obvious projection. The problem we address is the following:

Which is the smallest number g such that we can find open sets $U_1, \ldots, U_m$ such that

(1) $X = \bigcup_{i=1}^{g} U_i$.

(2) For each $i = 1, \ldots, g$, the fibration $\pi|_{\pi^{-1}U_i} : \pi^{-1}U_i \to U_i$ is trivial.

This number is called the Schwarz genus of π. It is easy to see that $g \leq n$. Also, Vassiliev has shown that, if n is a prime power, then $g = n$. In this talk we review some recent results in the direction of the determination of g in general, obtained with Procesi and Salvetti. We introduce a homology class in $H_n(\mathcal{S}_n, H^{n-1}(\mathcal{A}, \mathbb{Z}))$ whose non-vanishing is a necessary and sufficient condition for g to be equal to n. We show how this result, together with some recent result of Arone and Dwyer, can be used to show that, if $n = 2^t 3$ then $g < n$ (if $n = 6$ one has that $g = 5$).

The Hessian arrangement

Michael Falk

We use the Hessian arrangement to illustrate some of the interesting features of resonance varieties of hyperplane complements. This arrangement is fibered, in the sense that its projective image consists of the four singular fibers in a pencil of cubics. As a consequence the complement of the Hessian arrangement is aspherical. This algebro-geometric structure is reflected in the first resonance variety over $\mathbb{C}$: one obtains three (four minus one) linearly independent logarithmic 1-forms on the complement of the arrangement, with poles along all 12 hyperplanes, whose pairwise products are zero.

At the same time, the Hessian arrangement exhibits interesting resonance characteristics over fields of characteristic three. To describe these features, we outline the general structure theory for resonance varieties for general coefficients (e.g., PID's, algebraically closed fields). The first resonance variety lives in the kernel of the line-point incidence matrix of the projective arrangement - here "point" means point of multiplicity at least three. For the Hessian arrangement this kernel has dimension six when the underlying field has characteristic three. The resonance variety is, projectively, the union of the set of lines transversal to a set of linear "directrices" determined by a neighborly partition of the arrangement. For the Hessian arrangement there is such a partition, for which the directrices are four $\mathbb{P}^2$'s in $\mathbb{P}^5$. These four planes lie in a hyperplane, and the three points of intersection in each plane are collinear. The resulting projective resonance variety is a cubic threefold in $\mathbb{P}^4$, singular along the plane containing the six intersection points, with the six lines through the intersection points forming an embedded quadric component.

Abelianizing real diffeomorphic actions of finite groups

Eva-Maria Feichtner

In this talk we provide abelianizations of diffeomorphic actions of finite groups on smooth real manifolds. Wonderful models for (local) subspace arrangements as defined by De Concini and Procesi and a careful analysis of linear actions on real vector spaces are at the core of our construction. In fact, we show that our abelianizations have stabilizers isomorphic to elementary abelian 2-groups.

A lot of inspiration and intuition for our abelianization construction is drawn from studying the permutation action of the symmetric group $\mathcal{S}_n$ on real n-dimensional vector space in detail. We discuss stabilizer distinguishing stratifications for the De Concini-Procesi model of the braid arrangement, and provide an algebro-combinatorial setup for describing stabilizers.

This is joint work with Dmitry Kozlov (KTH Stockholm).

Real structures and combinatorics of models of Coxeter arrangements

Giovanni Gaiffi

We start by comparing two different families of models associated to a real subspace arrangement: the De Concini-Procesi models of its complexification and certain real differentiable models which are smooth manifolds with corners. It turns out that there are differentiable maps from the real models onto the real points of the De Concini-Procesi models, and that the properties of the fibers of these maps can be described in terms of the combinatorics of nested sets and building sets. This allows us to make homological computations.

When we specialize to Coxeter hyperplane arrangements, an extra structure appears. In fact, given any real model with corners of a Coxeter arrangement, it can be extended to a convex set whose face lattice defines a poset which can be described in detail. For instance, in the case of the braid arrangement, we recover the geometry of Kapranov's permutoassociahedron.

Combinatorics and Topology of Resonances

Dmitry Kozlov

I outlined some connections between the topology of spaces of polynomials with roots of fixed multiplicities and combinatorics.

In the case of spaces of complex monic polynomials I explained how to compute their Betti numbers via some combinatorially constructed cell complexes whose cells are enumerated by marked forests. I then used this connection to prove Arnold's Finiteness Theorem and disproved a conjectured generalization of it.

In the case of real hyperbolic polynomials, one can take advantage of a certain combinatorial object which I introduced. This object is a category, which I called resonance category. It reflects the combinatorial structures arising in the canonical stratifications of the symmetric smash products. I then used the current knowledge of the combinatorial structure of this category to perform explicit computations for several spaces of interest.

LS–galleries and Mirkovic-Vilonen cycles

Peter Littelmann

Let G be a complex semisimple algebraic group. We give an interpretation of the path model of a representation in terms of the geometry of the affine Grassmannian for G. In this setting, the paths are replaced by LS–galleries in the affine Coxeter complex associated to the Weyl group of G.

The connection with geometry is obtained as follows: consider a Demazure–Hansen–Bott–Samelson desingularization of the closure of an orbit $G(\mathbb{C}[[t]]).\lambda$ in the affine Grassmannian. The points of this variety can be viewed as galleries of a fixed type in the affine Tits building associated to G. The retraction with center $-\infty$ of the Tits building onto the affine Coxeter complex induces, in this way, a stratification of the $G(\mathbb{C}[[t]])$–orbit (identified with an open subset of $\hat{\Sigma}(\lambda)$), indexed by certain folded galleries in the Coxeter complex.

Each strata can be viewed as an open subset of a Białynicki–Birula cell of $\hat{\Sigma}(\lambda)$. The connection with representation theory is given by the fact that the closures of the strata associated to LS-galleries are the Mirkovic-Vilonen–cycles, which form a basis of the representation $V(\lambda)$ for the Langland's dual group $G^\vee$.

Vector space arrangements and minimal resolutions of fine-graded modules

Markus Perling

The category of coherent sheaves over an affine toric variety U_σ is equivalent to the category of so-called σ-families. Such a σ-family encodes the structure of equivariant sheaves in terms of vector spaces and certain homomorphisms between these vector spaces. In the particular case of a reflexive equivariant sheaf $\mathcal{E}$, its associated σ-family can be represented by a linear subvectorspace arrangement contained in a certain limit vector space. We are interested in the question whether general properties of $\mathcal{E}$ depend only on this arrangement. In the special case, where the toric variety is just the affine space $\mathbf{A}_k^n$ together with its standard diagonal torus action, and $\mathcal{E}$ corresponds to a $\mathbf{Z}^n$-graded module over the polynomial ring $k[x_1, \ldots, x_n]$, we construct a minimal resolution of $\mathcal{E}$. For this, we transport the notion of free resolutions from commutative algebra into the setting of vector space arrangements and construct explicitly resolutions in this setting. Transporting back these resolutions into the setting of sheaves, we obtain a minimal free resolution of $\mathcal{E}$.

Diagram groups

Mark Sapir

Diagram groups are fundamental groups of spaces of "positive loop spaces" in directed 2-complexes. This point of view allows us to consider representations of diagram by homeomorphisms, orderability of diagram groups, integer homology, and other properties of these groups. In particular, we find many new examples of FP_∞ but not finite dimensional groups, and compute their Poincaré series.

This is joint work with Victor Guba.

Topology and combinatorics of stratification of totally positive matrices

MICHAEL SHAPIRO

The starting point for our research is the following famous result.

Theorem. (Björner) *For any Bruhat interval $[u, v] \subset W$, there exists a stratified space such that*

- *its strata are labeled by the elements of $[u, v]$;*
- *adjacency is described by the Bruhat order;*
- *the closure (resp., boundary) of each stratum has the homology of a ball (resp., of a sphere).*

More is true: $[u, v]$ is the face poset of a regular cell complex (i.e., closed strata are balls). Björner's construction was entirely "synthetic" (a succession of cell attachments).

Problem. *Find a natural geometric construction of a stratified space with the desired homological properties.*

We suggest a solution of this problem in the special situation where W is a Weyl group of a semisimple Lie group G, and prove its validity in the case of the symmetric group. Our stratified spaces arise as links in the Bruhat decomposition of the *totally nonnegative part* of the unipotent radical of G.

Graph complexes and discrete Morse Theory

JOHN SHARESHIAN

I will give a brief review of some results and open problems on the homotopy type and homology of complexes whose faces are indexed by graphs on a fixed labeled vertex set [n] whose connectivity is bounded by a fixed number k. These complexes arise in the study of Vassiliev invariants of knots and ornaments. Most of the known results have been obtained by using discrete Morse theory.

From Schubert cells to cluster algebras

ALEK VAINSHEIN

In late 90's we (B. Shapiro, M.Shapiro and myself) have solved the following problem due to Arnold: find the number of connected components in the intersection of two opposite Schubert cells in the space of real complete flags. We proved that the number in question equals $3 \cdot 2^{n-1}$ for $n > 5$. The main ingredient of the proof is a family of maps from $(R \setminus 0)^{n(n-1)/2}$ to this intersection. Given a map from this family, one can get another map by changing only one of its coordinates according to the rule $xx' = uv + zt$; all the other coordinates remain unchanged.

To define a cluster algebra, we replace the above rule by $xx' = M_1 + M_2$, where M_1 and M_2 are coprime monomials in other coordinates. We define the notion of a Poisson structure compatible with the cluster algebra and explain how to obtain the transformations for the exponents of the above monomials from the transformations of the coefficient matrix of the corresponding Poisson structure. We then define the cluster manifold and find the number of its connected components.

Polytopal proofs of unimodality

Volkmar Welker

Unimodal sequences are finite sequences $a_0, \ldots, a_d$ of numbers – in our setting the numbers are natural numbers – such that there is an index i for which $a_n \leq \cdots \leq a_i \geq \cdots \geq a_d$. A prominent example of such a sequence is the h-vector $h(P) = (h_0, \ldots, h_d)$ of the boundary complex of a d-dimensional simplicial polytope P. Here it is known by the g-theorem of Stanley that the sequence is unimodal. In addition it is well known and much easier to prove that $h_i = h_{d-i}$ and $h_0 = 1$. Thus if we have to show that a sequence $a = (a_0, \ldots, a_d)$ satisfying these additional side-constraints is unimodal then one may try to set up a polytope P such that $h(P) = a$. In this talk we give several instances of sequences for which this strategy has been successfully employed, including the Stanley-Neggers Conjecture, Magic Squares and h-polynomials of Pfaffian rings.

Sheaves on posets and applications

Sergey Yuzvinsky

We review the theory of sheaves on finite posets and their cohomology. We emphasize some methods for computation of cohomology, in particular the rank filtrations. Then we discuss certain old and new applications to arrangement theory and toric varieties. For instance, we briefly explain the work of Deligne-Goresky-MacPherson who completed computation of integer cohomology rings of subspace complements using sheaves on the intersection lattices.

Edited by Eva-Maria Feichtner