INCIDENCE COMBINATORICS OF RESOLUTIONS

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ABSTRACT. We introduce notions of combinatorial blowups, building sets, and nested sets for arbitrary meet-semilattices. This gives a common abstract framework for the incidence combinatorics occurring in the context of De Concini-Procesi models of subspace arrangements and resolutions of singularities in toric varieties. Our main theorem states that a sequence of combinatorial blowups, prescribed by a building set in linear extension compatible order, gives the face poset of the corresponding simplicial complex of nested sets. As applications we trace the incidence combinatorics through every step of the De Concini-Procesi model construction, and we introduce the notions of building sets and nested sets to the context of toric varieties.

There are several other instances, such as models of stratified manifolds and certain graded algebras associated with finite lattices, where our combinatorial framework has been put to work; we present an outline in the end of this paper.

1. INTRODUCTION

For an arbitrary meet-semilattice we introduce notions of *combinatorial blowups, building sets*, and *nested sets*. The definitions are given on a purely order-theoretic level without any reference to geometry. This provides a common abstract framework for the incidence combinatorics occurring in at least two different situations in algebraic geometry: the construction of De Concini-Procesi models of subspace arrangements [7], and the resolution of singularities in toric varieties.

The various parts of this abstract framework have received different emphasis within different situations: while the notion of combinatorial blowups clearly specializes to stellar subdivisions of defining fans in the context of toric varieties, building sets and nested sets were introduced in the context of model constructions by De Concini & Procesi [7] (earlier and in a more special setting by Fulton & MacPherson [11]), from where we adopt our terminology. This correspondence however is not complete: the building sets in [7, 11] are not canonical, they depend on the geometry, while ours do not. See Section 4.1 for further details.

It was proved in [7] that a sequence of blowups within an arrangement of complex linear subspaces leads from the intersection stratification of complex space given by the maximal subspaces of the arrangement to an arrangement model stratified by divisors with normal crossings. In the context of toric varieties, there exist many different procedures for stellar subdivisions of a defining fan that result in a simplicial fan, so-called simplicial resolutions.

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The purpose of our Main Theorem 3.4 is to unify these two situations on the combinatorial level: a sequence of combinatorial blowups, performed on a (combinatorial) building set in linear extension compatible order, transforms the initial semilattice to a semilattice where all intervals are boolean algebras, more precisely to the face poset of the corresponding simplicial complex of nested sets. In particular, the structure of the resulting semilattice can be fully described by the initial data of nested sets. Both the formulation and the proof of our main theorem are purely combinatorial.

We sketch the content of this article:

Section 2. After providing some basic poset terminology, we define building sets and nested sets for meet-semilattices in purely order-theoretic terms and develop general structure theory for these notions.

Section 3. We define combinatorial blowups of meet-semilattices, and study their effect on building sets and nested sets. The section contains our Main Theorem 3.4 which describes the result of blowing up the elements of a building set in terms of the initial nested set complex.

Section 4. This section is devoted to relating our abstract framework to two different contexts in algebraic geometry: In 4.1 we briefly review the construction of De Concini-Procesi models for subspace arrangements. We show that the change of the incidence combinatorics of the stratification in a single construction step is described by a combinatorial blowup of the semilattice of strata. In 4.2 we draw the connection to simplicial resolutions of toric varieties: we recognize stellar subdivisions as combinatorial blowups of the face posets of defining fans and discuss the notions of building and nested sets in this context.

Section 5. Since a first version of this paper has been written and has circulated in the fall of 2000, our combinatorial framework for the incidence combinatorics of resolutions has been taken up in various contexts. We outline the model construction for real subspace and halfspace arrangements and for real stratified manifolds by G. Gaiffi [14]. Moreover, we give a short account of the study of a graded algebra associated with any finite lattice in [10], where our combinatorial generalization of originally geometric notions leads to the construction of an, at first sight, unrelated geometric counterpart for wonderful models of hyperplane arrangements. We also note that, more recently, our combinatorial resolutions were studied in the context of log resolutions of arrangement ideals [1].

2. Building sets and nested sets of meet-semilattices

2.1. **Poset Terminology.**

We recall some notions from the theory of partially ordered sets, and refer to [21, Ch. 3] for further details. All posets discussed in this paper will be finite. A poset $\mathcal{L}$ is called a *meet-semilattice* if any two elements $x, y \in \mathcal{L}$ have a greatest lower bound, i.e., the set $\{z \in \mathcal{L} \mid z \leq x, z \leq y\}$ has a maximal element, called the *meet*, $x \wedge y$, of x and y. Greatest lower bounds of subsets $A = \{a_1, \dots, a_t\}$ in $\mathcal{L}$ we denote with $\bigwedge A = a_1 \wedge \dots \wedge a_t$. In particular, meet-semilattices have a unique minimal element denoted $\hat{0}$. Minimal elements in $\mathcal{L} \setminus \{\hat{0}\}$ are called the *atoms* in $\mathcal{L}$. Meet-semilattices share the following property: for any subset $A = \{a_1, \dots, a_t\} \subseteq \mathcal{L}$ the set $\{x \in \mathcal{L} \mid x \geq a$ for all $a \in A\}$ is either empty or it has a unique minimal element, called the *join*, $\bigvee A = a_1 \vee \dots \vee a_t$, of A. If the meet-semilattice needs to be specified, we write $(\bigvee A)_{\mathcal{L}} = (a_1 \vee \ldots \vee a_t)_{\mathcal{L}}$ for the join of A in $\mathcal{L}$. For brevity, we talk about semilattices throughout the paper, meaning meet-semilattices.

Let P be an arbitrary poset. For $x \in P$ set: $P_{\leq x} = \{y \in P \mid y \leq x\}$; $P_{<x}$, and $P_{\geq x}$, $P_{>x}$ are defined analogously. For subsets $\mathcal{G} \subseteq P$ with the induced order, we define $\mathcal{G}_{\leq x} = \{y \in P \mid y \in \mathcal{G}, y \leq x\}$, and $\mathcal{G}_{<x}$ again analogously. For intervals in P we use the standard notations $[x,y] := \{z \in P \mid x \leq z \leq y\}$, $[x,y) := \{z \in P \mid x \leq z < y\}$, etc.

A poset is called *irreducible* if it is not a direct product of two other posets, both consisting of at least two elements. For a poset P with a unique minimal element $\hat{0}$, we call $I(P) = \{x \in P \mid [\hat{0}, x] \text{ is irreducible}\}$ the *set of irreducible elements* in P. In particular, the minimal element $\hat{0}$ and all atoms of P are irreducible elements in P. For $x \in P$, we call $D(x) := \max(I(P)_{\leq x})$ the *set of elementary divisors* of x – a term which is explained by the following proposition:

Proposition 2.1. *Let P be a poset with a unique minimal element $\hat{0}$. For $x \in P$ there exists a unique finest decomposition of the interval $[\hat{0}, x]$ in P as a direct product, which is given by an isomorphism $\varphi_x^{\mathrm{el}} : \prod_{j=1}^{l} [\hat{0}, y_j] \xrightarrow{\cong} [\hat{0}, x]$, with $\varphi_x^{\mathrm{el}}(\hat{0}, \ldots, y_j, \ldots, \hat{0}) = y_j$ for $j = 1, \ldots, l$. The factors of this decomposition are the intervals below the elementary divisors of x: $\{y_1, \ldots, y_l\} = D(x)$.*

Proof. Whenever a poset with a minimal element $\hat{0}$ is represented as a direct product, all elements which have more than one coordinate different from $\hat{0}$ are reducible. Hence, if $\prod_{j=1}^{l} [\hat{0}, y_j] \cong [\hat{0}, x]$, and the y_j are irreducible for $j = 1, \ldots, l$, then $\{y_1, \ldots, y_l\} = D(x)$. □

2.2. **Building Sets.**
In this subsection we define the notion of building sets of a semilattice and develop their structure theory.

Definition 2.2. *Let $\mathcal{L}$ be a semilattice. A subset $\mathcal{G}$ in $\mathcal{L} \setminus \{\hat{0}\}$ is called a* **building set** *of $\mathcal{L}$ if for any $x \in \mathcal{L} \setminus \{\hat{0}\}$ and $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_k\}$ there is an isomorphism of posets*

$$\varphi_x : \prod_{j=1}^{k} [\hat{0}, x_j] \xrightarrow{\cong} [\hat{0}, x] \tag{2.1}$$

with $\varphi_x(\hat{0}, \ldots, x_j, \ldots, \hat{0}) = x_j$ for $j = 1, \ldots, k$. We call $F(x) := \max \mathcal{G}_{\leq x}$ the **set of factors** *of x in $\mathcal{G}$.*

The next proposition provides several equivalent conditions for a subset of $\mathcal{L} \setminus \{\hat{0}\}$ to be a building set.

Proposition 2.3. *For a semilattice $\mathcal{L}$ and a subset $\mathcal{G}$ of $\mathcal{L} \setminus \{\hat{0}\}$ the following are equivalent:*

(1) *$\mathcal{G}$ is a building set of $\mathcal{L}$;*
(2) *$\mathcal{G} \supseteq I(\mathcal{L}) \setminus \{\hat{0}\}$, and for every $x \in \mathcal{L} \setminus \{\hat{0}\}$ with $D(x) = \{y_1, \ldots, y_l\}$ the elementary divisors of x, there exists a partition $\pi_x = \pi_1 | \ldots | \pi_k$ of $[l]$ with blocks $\pi_t = \{i_1, \ldots, i_{|\pi_t|}\}$ for $t = 1, \ldots, k$, such that the elements in $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_k\}$ are of the form $x_t = \varphi_x^{\mathrm{el}}(\hat{0}, \ldots, \hat{0}, y_{i_1}, \hat{0}, \ldots, \hat{0}, y_{i_2}, \hat{0}, \ldots, \hat{0}, y_{i_{|\pi_t|}}, \hat{0})$.*
Informally speaking, the factors of x in $\mathcal{G}$ are products of disjoint sets of elementary divisors of x.

(3) *$\mathcal{G}$ generates $\mathcal{L}\setminus\{\hat{0}\}$ by $\vee$, and for any $x\in\mathcal{L}$, any $\{y,y_1,\dots,y_t\}\subseteq\max\mathcal{G}_{\leq x}$, and $z\in\mathcal{L}$ with $z<y$, we have $\mathcal{G}_{\leq y}\cap\mathcal{G}_{\leq z\vee y_1\vee\cdots\vee y_t}=\mathcal{G}_{\leq z}$.*

(4) *$\mathcal{G}$ generates $\mathcal{L}\setminus\{\hat{0}\}$ by $\vee$, and for any $x\in\mathcal{L}$, any $\{y,y_1,\dots,y_t\}\subseteq\max\mathcal{G}_{\leq x}$, and $z\in\mathcal{L}$ with $z<y$, the following two conditions are satisfied:*

i)	$\mathcal{G}_{\leq y}\cap\mathcal{G}_{\leq y_1\vee\cdots\vee y_t}=\emptyset$	*"disjointness,"*
ii)	$z\vee y_1\vee\cdots\vee y_t<y\vee y_1\vee\cdots\vee y_t$	*"necessity."*

Proof.
(1)⇒(2): That $\mathcal{G}$ contains $I(\mathcal{L})\setminus\{\hat{0}\}$ follows directly from the definition of building sets. We have the following isomorphisms: $\varphi_x:\prod_{j=1}^k[\hat{0},x_j]\longrightarrow[\hat{0},x]$ by the building set property, and $\varphi_{x_j}^{\mathrm{el}}:\prod_{y\in D(x_j)}[\hat{0},y]\longrightarrow[\hat{0},x_j]$ for $j=1,\dots,k$ by Proposition 2.1. The composition $\varphi_x\circ(\prod_{j=1}^k\varphi_{x_j}^{\mathrm{el}})$ yields the finest decomposition φ_x^{el} of $[\hat{0},x]$. Thus, $D(x)=\biguplus_{j=1}^k D(x_j)$, which gives the partition described in (2).
(2)⇒(1): The decomposition of $[\hat{0},x]$ into intervals below the elements in $\max\mathcal{G}_{\leq x}$ follows from Proposition 2.1 by assembling factors $[\hat{0},y_j]$ with maximal elements indexed by elements from the same block of the partition π_x into one factor.
(1)⇒(3): (3) is a direct consequence of $[\hat{0},x]$ decomposing into a direct product of the form described in the definition of building sets.
(3)⇒(4): *i)* follows by setting $z=\hat{0}$ in (3). Equality in *ii)* implies with (3) that $\mathcal{G}_{\leq y}=\mathcal{G}_{\leq z}$, in particular, $y\in\mathcal{G}_{\leq z}$ – a contradiction to $z<y$.
(4)⇒(1): For $x\in\mathcal{L}\setminus\{\hat{0}\}$ and $\max\mathcal{G}_{\leq x}=\{x_1,\dots,x_k\}$ consider the poset map

$$\phi:\ \prod_{j=1}^{k}[\hat{0},x_j]\ \longrightarrow\ [\hat{0},x]\,,\quad(\alpha_1,\dots,\alpha_k)\ \longmapsto\ \alpha_1\vee\ldots\vee\alpha_k\,.$$

i) ϕ *is surjective*: For $\hat{0}\neq y\leq x$, let $\max\mathcal{G}_{\leq y}=\{y_1,\dots,y_t\}$. First, $\bigvee_{i=1}^t y_i=y$, since $\mathcal{G}$ generates $\mathcal{L}$ by $\vee$. Second, define $\gamma_j:=\bigvee_{y_i\in S_j}y_i$ with $S_j:=(\max\mathcal{G}_{\leq y})\cap\mathcal{G}_{\leq x_j}$ for $j=1,\dots,k$. Clearly, $\gamma_j\in[\hat{0},x_j]$, and $\cup_{j=1}^k S_j=\max\mathcal{G}_{\leq y}$, since $\mathcal{G}_{\leq y}\subseteq\mathcal{G}_{\leq x}$. Hence, $\phi(\gamma_1,\dots,\gamma_k)=\bigvee_{i=1}^t y_i=y$.
ii) ϕ *is injective*: a) Assume $\phi(\alpha_1,\dots,\alpha_k)=\phi(\beta_1,\dots,\beta_k)=y\neq x$. Let $\max\mathcal{G}_{\leq y}=\{y_1,\dots,y_t\}$. By induction on the number of elements in $[\hat{0},x]$ we can assume that $[\hat{0},y]$ decomposes as a direct product $[\hat{0},y]\cong\prod_{i=1}^t[\hat{0},y_i]$. Moreover, the subsets S_j of $\max\mathcal{G}_{\leq y}$ defined in i) actually partition $\max\mathcal{G}_{\leq y}$ as follows from the disjointness property applied to pairwise intersections of the $\mathcal{G}_{\leq x_j}$. Thus, $[\hat{0},y]\cong\prod_{j=1}^k[\hat{0},\gamma_j]$, with elements $\gamma_j\in[\hat{0},x_j]$ as above, and it follows that $\alpha_j=\beta_j=\gamma_j$ for $j=1,\dots,k$.
b) Assume that $\phi(\alpha_1,\dots,\alpha_k)=\phi(\beta_1,\dots,\beta_k)=x$. By the necessity property it follows that $\alpha_j=\beta_j=x_j$ for $j=1,\dots,k$. □

Remark 2.4. The definition of building sets and of irreducible elements, as well as the characterization of building sets in Proposition 2.3 (2), are independent of the existence of a join operation and can be formulated for any poset with a unique minimal element.

We gather a few important properties of building sets.

Proposition 2.5. *For a building set $\mathcal{G}$ of $\mathcal{L}$, the following holds:*

(1) *Let $x \in \mathcal{L}$, $F(x) = \{x_1, \ldots, x_k\}$ the set of factors of x in $\mathcal{G}$, and $\hat{0} \neq y \in \mathcal{G}$ with $y \leq x$. Then there exists a unique $j \in \{1, \ldots, k\}$ such that $y \leq x_j$; i.e., $F(x) = \max \mathcal{G}_{\leq x}$ induces a partition of $\mathcal{G}_{\leq x}$.*

(2) *For $x \in \mathcal{L}$ and $x_0 \in F(x)$,*
$$\bigvee (F(x) \setminus \{x_0\}) \;<\; \bigvee F(x) \;=\; x\,,$$
i.e., each factor of x in $\mathcal{G}$ is needed to generate x.

(3) *If $h_1, \ldots, h_k$ in $\mathcal{G}$ are such that $(h_i, \bigvee_{j=1}^k h_j] \cap \mathcal{G} = \emptyset$ for $i = 1, \ldots, k$, then $F(\bigvee_{j=1}^k h_j) \;=\; \{h_1, \ldots, h_k\}$.*

Proof. (1) is a consequence of Proposition 2.3 (4)*i*), as was noted already in the proof of (4)⇒(1), part ii) a), in the previous proposition. Taking the full set of factors and setting $z = \hat{0}$ in Proposition 2.3 (4)*ii*), yields (2). For (3) note that $\{h_1, \ldots, h_k\} \subseteq F(\bigvee_{j=1}^k h_j)$ by assumption. If $\{h_1, \ldots, h_k\}$ were not the complete set of factors, we would obtain a contradiction to (2). □

Example 2.6. (1) For the boolean lattice $\mathcal{B}_n$ of rank n, its atoms form the minimal building set. As with any other semilattice, the full poset without its minimal element gives the maximal building set.

In the smallest interesting example, the rank 3 boolean lattice $\mathcal{B}_3$, we see that there are other building sets between these extremal choices: The atoms can be combined with any other rank 2 element to form a building set. Moreover, atoms can be combined with the top element to form a building set, and any other subset of $\mathcal{B}_3$ containing the latter is in fact a building set.

(2) For the partition lattice Π_n, the minimal building set is given by the 1-block partitions. Again, the maximal building set is given by the full lattice without its minimal element. Looking at Π_4, we see that we can add any 2-block partition to the minimal building set, e.g., (12)(34), to obtain building sets other than the extreme ones.

(3) The lattice D_n of positive integral divisors of a natural number $n > 0$ ordered by division relation has the prime powers dividing n as its minimal building set. Note that this example includes the boolean lattice for any n having no square divisors, hence there are ample building sets between the extreme choices.

2.3. **Nested Sets.**

In this subsection we define the notion of nested subsets of a building set of a semilattice and prove some of their properties.

Definition 2.7. *Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$. A subset N in $\mathcal{G}$ is called* **nested** *if, for any set of incomparable elements $x_1, \ldots, x_t$ in N of cardinality at least two, the join $x_1 \vee \cdots \vee x_t$ exists and does not belong to $\mathcal{G}$. The nested sets in $\mathcal{G}$ form an abstract simplicial complex, denoted $\mathcal{N}(\mathcal{G})$.*

Note that the elements of $\mathcal{G}$ are the vertices of the complex of nested sets $\mathcal{N}(\mathcal{G})$. Moreover, the order complex of $\mathcal{G}$ is a subcomplex of $\mathcal{N}(\mathcal{G})$, since linearly ordered subsets of $\mathcal{G}$ are nested.

Proposition 2.8. *For a given semilattice $\mathcal{L}$ and a subset N of a building set $\mathcal{G}$ of $\mathcal{L}$, the following are equivalent:*

(1) *N is nested.*
(2) *Whenever $x_1, \dots, x_t$ are noncomparable elements in N, the join $x_1 \vee \dots \vee x_t$ exists, and $F(x_1 \vee \dots \vee x_t) = \{x_1, \dots, x_t\}$.*
(3) *There exists a chain $C \subseteq \mathcal{L}$, such that $N = \bigcup_{x \in C} F(x)$.*
(4) *$N \in \Lambda$, where Λ is the maximal subset of $2^{\mathcal{G}}$, for which the following three conditions are satisfied:*
 (o) *$\emptyset \in \Lambda$, and $\{g\} \in \Lambda$, for $g \in \mathcal{G}$;*
 (i) *if $N \in \Lambda$ and $x \in \max N$, then $N_{<x} \in \Lambda$;*
 (ii) *if $N \in \Lambda$, then $\max N = F(\bigvee \max N)$.*

Proof.
(1)⇒(2): Let N be a nested set, and $M = \{x_1, \dots, x_t\} \subseteq N$ a set of incomparable elements with $\bigvee_{i=1}^t x_i \notin \mathcal{G}$. We can assume that for some x_j: $(x_j, \bigvee_{i=1}^t x_i] \cap \mathcal{G} \neq \emptyset$, otherwise the claim follows by Proposition 2.5 (3). Without loss of generality, we assume that there exists an element $y \in (x_1, \bigvee_{i=1}^t x_i] \cap \mathcal{G}$ and that $y \in \max \mathcal{G}_{\leq \bigvee M}$. Define $M' := \{x_1, \dots, x_t\} \cap \mathcal{G}_{\leq y} = \{x_1 = x_{j_0}, x_{j_1}, \dots, x_{j_k}\}$ and $z := \bigvee_{l=0}^k x_{j_l}$. Since $M' = \{x_{j_0}, x_{j_1}, \dots, x_{j_k}\}$ is nested (it is a subset of N), we have the strict inequality $z < y$. Furthermore,

$$\bigvee_{i=1}^t x_i \,=\, z \vee \bigvee (M \setminus M') \,\leq\, z \vee \bigvee (\max G_{\leq \bigvee M} \setminus \{y\}) \,<\, \bigvee_{i=1}^t x_i \,,$$

where the first inequality follows from Proposition 2.5 (1) and the second inequality from Proposition 2.5 (2). We thus arrive to a contradiction, which finishes the proof.
(2)⇒(1): Obvious.
(2)⇒(3): Let N be a set satisfying condition (2). Fix a particular linear extension $\{x_1, \dots, x_k\}$ on the partial order of N, and define $\alpha_j := x_1 \vee \ldots \vee x_j$, for $j = 1, \dots, k$. By (2) we have $F(\alpha_j) = \max\{x_1, \dots, x_j\}$, and therefore $x_j \in F(\alpha_j)$ and $x_{j+1} \notin F(\alpha_j)$ for $j = 1, \dots, k$. Hence, the α_j's are different and form a chain $C = \alpha_1 < \alpha_2 < \dots < \alpha_k$. By construction, $N = \bigcup_{x \in C} F(x)$.
(1),(2)⇒(4): Let N be a nested set, we shall prove that $N \in \Lambda$ by induction on the size of N:

(1) if $|N| = 0$, then $N \in \Lambda$ by condition (o);
(2) if $|N| \geq 1$, then $\max N = F(\bigvee \max N)$ by condition (2). Furthermore, since $|N_{<x}| < |N|$, and $N_{<x}$ is nested (it is a subset of N), $N_{<x} \in \Lambda$ by induction. Hence $N \in \Lambda$.

(3)⇒(1): Let $C = (\alpha_1 < \ldots < \alpha_k)$ be a chain in $\mathcal{L}$ and $N = \bigcup_{x \in C} F(x)$. Let $N' = \{x_1, \dots, x_t\} \subseteq N$, $t \geq 2$, be an antichain in N, and s the maximal index in C such that $N' \cap F(\alpha_s) \neq \emptyset$. In particular, $N' \cap F(\alpha_s) \neq \{\alpha_s\}$ due to $|N'| > 1$ and N' being an antichain.

Let $y \in N' \cap F(\alpha_s)$. If $|N' \cap F(\alpha_s)| > 1$,

$$y \,<\, \bigvee (N' \cap F(\alpha_s)) \,\leq\, \bigvee N' \,\leq\, \alpha_s \,,$$

where the strict inequality is a consequence of the necessity property for building sets. Thus, $\bigvee N' \notin \mathcal{G}$. If $|N' \cap F(\alpha_s)| = 1$, we have $y < \bigvee N' \leq \alpha_s$, due to N' being an antichain with $|N'| > 1$, and again $\bigvee N' \notin \mathcal{G}$.
(4)⇒(3): We need the following fact:

Fact. *If there are elements $x_1,\dots,x_t$ and $y_1,\dots,y_k$ in $\mathcal{L}$, such that $x_t>y_j$ for $j=1,\dots,k$, and $F(\bigvee_{i=1}^t x_i)=\{x_1,\dots,x_t\}$, and $F(\bigvee_{j=1}^k y_j)=\{y_1,\dots,y_k\}$, then $F(x_1\vee\dots\vee x_{t-1}\vee y_1\vee\dots\vee y_k)=\{x_1,\dots,x_{t-1},y_1,\dots,y_k\}$.*

Once the fact above is proved, one can derive (3) as follows: For $N\in\Lambda$ we shall form a chain $C=(\alpha_1<\dots<\alpha_{|N|})$ such that $N=\bigcup_{i=1}^{|N|}F(\alpha_i)$. Choose a linear extension $\{x_1,\dots,x_t\}$ of N. Set $\alpha_t=\bigvee\max N$, $\alpha_{t-1}=\bigvee\max(N\setminus\{x_t\})$, $\alpha_{t-2}=\bigvee\max(N\setminus\{x_t,x_{t-1}\})$, and so on. By (4)(ii), $F(\alpha_t)=\max N$. Applying (4)(i) to $x_t\in\max N$, and (4)(ii) to $N_{<x_t}$, we obtain $F(\bigvee\max N_{<x_t})=\max N_{<x_t}$. With the fact above, we conclude that $F(\alpha_{t-1})=\max(N\setminus\{x_t\})$, and, using the same argument iteratively, we arrive to $N=\bigcup_{i=1}^t F(\alpha_i)$.

Proof of the fact. Set $\alpha:=x_1\vee\dots\vee x_{t-1}\vee y_1\vee\dots\vee y_k$. Since $\alpha\le\bigvee_{i=1}^t x_i$, the factors of α can be partitioned into groups of elements below the x_i for $i=1,\dots,t$, by Proposition 2.5 (1). Since $x_i\le\alpha$ for $i=1,\dots,t-1$, we obtain $F(\alpha)=\{x_1,\dots,x_{t-1},\gamma_1,\dots,\gamma_m\}$ with $\gamma_j\le x_t$ for $j=1,\dots,m$.

Again using Proposition 2.5 (1), the $y_1,\dots,y_k$ can be partitioned into groups below the factors γ_j for $j=1,\dots,m$. The occurrence of one strict inequality $\bigvee\{y_l\,|\,y_l\le\gamma_j\}<\gamma_j$ for some $j\in\{1,\dots,m\}$ yields a contradiction to $\alpha=\bigvee_{i=1}^{t-1}x_i\vee\bigvee_{j=1}^k y_j=\bigvee_{i=1}^{t-1}x_i\vee\bigvee_{j=1}^m\gamma_j$, due to the necessity property of building sets. Moreover, since the y_i are factors themselves, joins of more than two of the y_i's are not elements of $\mathcal{G}$. Thus, $y_i=\gamma_i$, for $i=1,\dots,k{=}m$, as claimed. □

Example 2.9. (1) For the boolean lattice B_n with its minimal building set, any subset of atoms is nested. The nested set complex hence is a simplex on n vertices. As for any other semilattice with maximal building set, the nested sets are the totally ordered subsets of the poset, hence the nested set complex is the order complex of the poset. In the particular case of $\mathcal{B}_n$ it is the barycentric subdivision of a simplex on n vertices. For $\mathcal{B}_3$ with building set $\mathcal{G}=\{1,2,3,23\}$ the nested set complex consists of two triangles, namely $\{1,2,23\}$ and $\{1,3,23\}$.

(2) For the partition lattice Π_n with its minimal building set of 1-block partitions, a subset of such partitions is nested if and only if any two non-trivial blocks are either contained one in another or disjoint. This is the example which has suggested the terminology of *nested* sets in the first place, it appeared as the central combinatorial structure in the paper of Fulton & MacPherson [11] on models for configuration spaces of smooth complex varieties.

3. Sequences of Combinatorial Blowups

We introduce the notion of a combinatorial blowup of an element in a semilattice and prove that the set of semilattices is closed under this operation.

3.1. Combinatorial Blowups.

Definition 3.1. *For a semilattice $\mathcal{L}$ and an element $\alpha\in\mathcal{L}$ we define a poset $\mathrm{Bl}_\alpha\mathcal{L}$, the* **combinatorial blowup of $\mathcal{L}$ at** *α, as follows:*

- *elements of $\mathrm{Bl}_\alpha\mathcal{L}$:*
  (1) *$y\in\mathcal{L}$, such that $y\not\ge\alpha$;*
  (2) *$[\alpha,y]$, for $y\in\mathcal{L}$, such that $y\not\ge\alpha$ and $(y\vee\alpha)_{\mathcal{L}}$ exists*
  (in particular, $[\alpha,\hat{0}]$ can be thought of as the result of blowing up α);

- *order relations in* $\mathrm{Bl}_\alpha\mathcal{L}$*:*
  (1) $y > z$ *in* $\mathrm{Bl}_\alpha\mathcal{L}$ *if* $y > z$ *in* $\mathcal{L}$*;*
  (2) $[\alpha, y] > [\alpha, z]$ *in* $\mathrm{Bl}_\alpha\mathcal{L}$ *if* $y > z$ *in* $\mathcal{L}$*;*
  (3) $[\alpha, y] > z$ *in* $\mathrm{Bl}_\alpha\mathcal{L}$ *if* $y \geq z$ *in* $\mathcal{L}$*;*
  where in all three cases $y, z \not\geq \alpha$.

Note that the atoms in $\mathrm{Bl}_\alpha\mathcal{L}$ are the atoms of $\mathcal{L}$ together with the element $[\alpha, \hat{0}]$. It is easy, albeit tedious, to check that the class of (meet-)semilattices is closed under combinatorial blowups.

Lemma 3.2. *Let* $\mathcal{L}$ *be a semilattice and* $\alpha \in \mathcal{L}$*, then* $\mathrm{Bl}_\alpha\mathcal{L}$ *is a semilattice.*

Proof. The joins in $\mathrm{Bl}_\alpha\mathcal{L}$ are defined by the rule

$$\begin{aligned}([\alpha, y] \vee [\alpha, z])_{\mathrm{Bl}_\alpha\mathcal{L}} &= [\alpha, (y \vee z)_{\mathcal{L}}],\\ ([\alpha, y] \vee z)_{\mathrm{Bl}_\alpha\mathcal{L}} &= [\alpha, (y \vee z)_{\mathcal{L}}],\\ (y \vee z)_{\mathrm{Bl}_\alpha\mathcal{L}} &= (y \vee z)_{\mathcal{L}},\end{aligned}$$

which is applicable only if $(y \vee z)_{\mathcal{L}}$ exists, otherwise the corresponding joins in $\mathrm{Bl}_\alpha\mathcal{L}$ do not exist. Also, the first and second formulae are applicable only in the case $(y \vee z)_{\mathcal{L}} \not\geq \alpha$, otherwise the corresponding joins do not exist. The check of this is straightforward and is left to the reader.

□

Observe that it is possible that $(x \vee y)_{\mathcal{L}}$ exists, while $(x \vee y)_{\mathrm{Bl}_\alpha\mathcal{L}}$ does not.

3.2. **Blowing Up Building Sets.**

In this subsection we prove that if one combinatorially blows up a building set of a semilattice in any chosen linear extension order, then one ends up with the face poset of the simplicial complex of nested sets of this building set. The following proposition provides the essential step for the proof.

Proposition 3.3. *Let* $\mathcal{L}$ *be a semilattice,* $\mathcal{G}$ *a building set of* $\mathcal{L}$*, and* $\alpha \in \max \mathcal{G}$*. Then,* $\widetilde{\mathcal{G}} = (\mathcal{G} \setminus \{\alpha\}) \cup \{[\alpha, \hat{0}]\}$ *is a building set of* $\mathrm{Bl}_\alpha\mathcal{L}$*. Furthermore, the nested subsets of* $\widetilde{\mathcal{G}}$ *are precisely the nested subsets of* $\mathcal{G}$ *with* α *replaced by* $[\alpha, \hat{0}]$.

Proof. It is easy to see that $\widetilde{\mathcal{G}}$ is a building set of $\mathrm{Bl}_\alpha\mathcal{L}$. Indeed, given $x \in \mathcal{L} \setminus \mathcal{L}_{\geq \alpha}$, (2.1) is obvious for $x \in \mathrm{Bl}_\alpha\mathcal{L}$, and, if $(x \vee \alpha)_{\mathcal{L}}$ exists, it follows for $[\alpha, x] \in \mathrm{Bl}_\alpha\mathcal{L}$ from the identity

$$[\hat{0}, [\alpha, x]]_{\mathrm{Bl}_\alpha\mathcal{L}} \;=\; [\hat{0}, x]_{\mathrm{Bl}_\alpha\mathcal{L}} \times B_1\,,$$

where B_1 is the subposet consisting of the two comparable elements $\hat{0} < [\alpha, \hat{0}]$.

Let us now see that the sets of nested subsets of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are the same when replacing α by $[\alpha, \hat{0}]$:

Let N be a nested set in $\mathcal{G}$, not containing α. For incomparable elements $x_1, \dots, x_t$ in N, $\bigvee_{i=1}^t x_i \not\geq \alpha$, since otherwise we had $\alpha \in \max \mathcal{G}_{\leq \bigvee x_i} = F(\bigvee_{i=1}^t x_i) = \{x_1, \dots, x_t\}$ by Proposition 2.8(2). Thus, $\bigvee_{i=1}^t x_i$ exists in $\mathrm{Bl}_\alpha\mathcal{L}$ and $\bigvee_{i=1}^t x_i \notin \widetilde{\mathcal{G}}$. Hence, N is nested in $\widetilde{\mathcal{G}}$. A nested subset in $\widetilde{\mathcal{G}}$ not containing $[\alpha, \hat{0}]$ is obviously nested in $\mathcal{G}$.

Let now N be nested in $\mathcal{G}$ containing α, and set $\widetilde{N} = (N \setminus \{\alpha\}) \cup \{[\alpha, \hat{0}]\}$. Subsets of incomparable elements in $\widetilde{N}$ not containing $[\alpha, \hat{0}]$ can be dealt with as above. Thus assume that $[\alpha, \hat{0}], x_1, \dots, x_t$ are incomparable in $\widetilde{N}$. Then, $x_1, \dots, x_t$ are incomparable in the nested set N, and, as above, we conclude that $\bigvee_{i=1}^t x_i$ exists and $\bigvee_{i=1}^{t} x_i \not\geq \alpha$. Moreover, $\alpha \vee \bigvee_{i=1}^{t} x_i$ exists in $\mathcal{L}$ (joins of nested sets always exist!), thus, $[\alpha, \bigvee_{i=1}^{t} x_i] = [\alpha, \hat{0}] \vee \bigvee_{i=1}^{t} x_i$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$ and is obviously not contained in $\widetilde{\mathcal{G}}$. We conclude that $\widetilde{N}$ is nested in $\widetilde{\mathcal{G}}$.

Vice versa, let $\widetilde{N}$ be nested in $\widetilde{\mathcal{G}}$ containing $[\alpha, \hat{0}]$, and set $N = (\widetilde{N} \setminus \{[\alpha, \hat{0}]\}) \cup \{\alpha\}$. Again it suffices to consider subsets of incomparable elements $\alpha, x_1, \dots, x_t$ in N. With $[\alpha, \hat{0}], x_1, \dots, x_t$ incomparable in $\widetilde{N}$, $[\alpha, \hat{0}] \vee \bigvee_{i=1}^{t} x_i = [\alpha, \bigvee_{i=1}^{t} x_i]$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$, thus $\alpha \vee \bigvee_{i=1}^{t} x_i$ exists in $\mathcal{L}$. Incomparability implies that $\alpha \vee \bigvee_{i=1}^{t} x_i > \alpha$, and thus $\alpha \vee \bigvee_{i=1}^{t} x_i \notin \mathcal{G}$. We conclude that N is nested in $\mathcal{G}$. □

By iterating the combinatorial blowup described in Proposition 3.3 through all of $\mathcal{G}$, we obtain the following theorem, which serves as a motivation for the entire development.

Theorem 3.4. *Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$ with some chosen linear extension: $\mathcal{G} = \{G_1, \dots, G_t\}$, with $G_i > G_j$ implying $i < j$. Let $\mathrm{Bl}_k\mathcal{L}$ denote the result of subsequent blowups $\mathrm{Bl}_{G_k}(\mathrm{Bl}_{G_{k-1}}(\dots \mathrm{Bl}_{G_1}\mathcal{L}))$. Then the final semilattice $\mathrm{Bl}_t\mathcal{L}$ is equal to the face poset of the simplicial complex $\mathcal{N}(\mathcal{G})$.*

Proof. The building set $\mathcal{G}_t$ of $\mathrm{Bl}_t\mathcal{L}$ that results from iterated application of Proposition 3.3 obviously is the set of atoms $\mathfrak{A}$ in $\mathrm{Bl}_t\mathcal{L}$. Every element $x \in \mathrm{Bl}_t\mathcal{L}$ is the join of atoms below it: $x = \bigvee \mathfrak{A}_{\leq x}$. The subset $\mathfrak{A}_{\leq x}$ of $\mathcal{G}_t$ is nested, in particular, it is the set of factors of x in $\mathrm{Bl}_t\mathcal{L}$ with respect to $\mathcal{G}_t$ (Proposition 2.8(2)). Proposition 2.5(2) implies that the interval $[\hat{0}, x]$ in $\mathrm{Bl}_t\mathcal{L}$ is boolean. We conclude that $\mathrm{Bl}_t\mathcal{L}$ is the face poset of a simplicial complex with faces in one-to-one correspondence with the nested sets in $\mathcal{G}_t$, which in turn correspond to the nested sets in $\mathcal{G}$ by Proposition 3.3. □

4. Instances of combinatorial blowups

4.1. De Concini-Procesi Models Of Subspace Arrangements.

Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be an arrangement of linear subspaces in complex space $\mathbb{C}^d$. Much effort has been spent on describing the cohomology of the complement $\mathcal{M}(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup \mathcal{A}$ of such an arrangement and, in particular, on answering the question whether the cohomology algebra is completely determined by the combinatorial data of the arrangement. Here, combinatorial data is understood as the lattice $\mathcal{L}(\mathcal{A})$ of intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion together with the complex codimensions of the intersections. A major step towards the solution of this problem (for a complete answer see [3, 4]) was the construction of smooth models for the complement $\mathcal{M}(\mathcal{A})$ by De Concini & Procesi [7] that allowed for an explicit description of rational models for $\mathcal{M}(\mathcal{A})$ following [17]. The De Concini-Procesi models for arrangements in turn are one instance in a sequence of model constructions reaching from compactifications of symmetric spaces [5, 6], over the Fulton-MacPherson compactifications of configuration spaces [11] to the general framework of wonderful conical compactifications proposed by MacPherson & Procesi [18].

Given a complex subspace arrangement $\mathcal{A}$ in $\mathbb{C}^d$, De Concini & Procesi describe a smooth irreducible variety Y together with a proper map $\pi : Y \longrightarrow \mathbb{C}^d$ such that π is isomorphism over $\mathcal{M}(\mathcal{A})$, and the complement of the preimage of $\mathcal{M}(\mathcal{A})$ is a union of irreducible divisors with normal crossings in Y. The model Y can be constructed by a sequence of blowups of smooth subvarieties that is prescribed by the stratification of complex space induced by the arrangement.

4.1.1. *Building sets for subspace arrangements.* In order to enumerate the strata in the intersection stratification of Y given by the irreducible divisors, De Concini & Procesi introduced the notions of building sets, nested sets and irreducible elements as follows:

Definition 4.1. ([7, §2]) *Let $\mathcal{L}(\mathcal{A})$ be the intersection lattice of an arrangement $\mathcal{A}$ of linear subspaces in a finite dimensional complex vector space. Consider the lattice $\mathcal{L}(\mathcal{A})^*$ formed by the orthogonal complements of intersections ordered by inclusion.*

1. *For $U \in \mathcal{L}(\mathcal{A})^*$, $U = \oplus_{i=1}^k U_i$ with $U_i \in \mathcal{L}(\mathcal{A})^*$, is called a* **decomposition** *of U if for any $V \subseteq U$, $V \in \mathcal{L}(\mathcal{A})^*$, $V = \oplus_{i=1}^k (U_i \cap V)$ and $U_i \cap V \in \mathcal{L}(\mathcal{A})^*$ for $i = 1, \dots, k$.*
2. *Call $U \in \mathcal{L}(\mathcal{A})^* \setminus \{\hat{0}\}$* **irreducible** *if it does not admit a non-trivial decomposition.*
3. *$\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})^* \setminus \{\hat{0}\}$ is called a* **building set** *for $\mathcal{A}$ if for any $U \in \mathcal{L}(\mathcal{A})^*$ and $G_1, \dots, G_k$ maximal in $\mathcal{G}$ below U, $U = \oplus_{i=1}^k G_i$ is a decomposition (the $\mathcal{G}$-decomposition) of U.*
4. *A subset $\mathcal{S} \subseteq \mathcal{G}$ is called* **nested** *if for any set of non-comparable elements $U_1, \dots, U_k$ in $\mathcal{S}$, $U = \oplus_{i=1}^k U_i$ is the $\mathcal{G}$-decomposition of U.*

Note that $\mathcal{L}(\mathcal{A})^*$ coincides with $\mathcal{L}(\mathcal{A})$ as abstract lattices. We will therefore talk about irreducible elements, building sets and nested sets in $\mathcal{L}(\mathcal{A})$ without explicitly referring to the dual setting of the preceding definition.

The notions of Definition 4.1 are in part based on the earlier notions introduced by Fulton & MacPherson in [11] to study compactifications of configuration spaces. Our terminology is naturally adopted from [11, 7]. Building sets and nested sets in the sense of De Concini & Procesi are building and nested sets for the intersection lattices of subspace arrangements in our combinatorial sense (see Proposition 4.5 (1) below). However, there are differences. The opposite is not true: A combinatorial building set for the intersection lattice of a subspace arrangement is not necessarily a building set for this arrangement in the sense of De Concini & Procesi, neither are irreducible elements in the sense of De Concini & Procesi irreducible in our sense.

Example 4.2. (*Combinatorial versus De Concini-Procesi building sets*)
Consider the following arrangement $\mathcal{A}$ of 3 subspaces in $\mathbb{C}^4$:

$$A_1 : \ z_4 = 0\,, \quad A_2 : \ z_1 = z_2 = 0\,, \quad A_3 : \ z_1 = z_3 = 0\,.$$

The intersection lattice $\mathcal{L}(\mathcal{A})$ is a boolean algebra on 3 elements. Combinatorial building sets of this lattice have been discussed in Example 2.6, in particular, the set of atoms $\{A_1, A_2, A_3\} \subseteq \mathcal{L}(\mathcal{A})$ is the minimal combinatorial building set. However, any building set for $\mathcal{A}$ in the sense of De Concini & Procesi necessarily includes the intersection $A_2 \cap A_3$, since its orthogonal complement does not decompose in $\mathcal{L}(\mathcal{A})^*$. The minimal building set for $\mathcal{A}$, i.e., the set of irreducibles for $\mathcal{A}$, in the sense of De Concini & Procesi is $\{A_1, A_2, A_3, A_2 \cap A_3\}$. Any other building set contains this minimal building set and the total intersection $\bigcap \mathcal{A} = 0$.

The main difference between our combinatorial set-up and the original context of De Concini-Procesi model constructions can be formulated in the following way:

our constructions are order-theoretically canonical for a given semilattice. The set of combinatorial building sets, in particular the set of irreducible elements, depends only on the semilattice itself and not on the geometry of the subspace arrangement which it encodes. See Proposition 4.5 for a complete explanation.

4.1.2. *Local subspace arrangements.* In order to trace the De Concini-Procesi construction step by step we need the more general notion of a local subspace arrangement.

Definition 4.3. *Let M be a smooth complex d-dimensional manifold and $\mathcal{A}$ a union of finitely many smooth complex submanifolds of M such that all non-empty intersections of submanifolds in $\mathcal{A}$ are connected smooth complex submanifolds. $\mathcal{A}$ is called a* **local subspace arrangement** *if for any $x \in \mathcal{A}$ there exists an open set N in M with $x \in N$, a subspace arrangement $\widetilde{\mathcal{A}}$ in $\mathbb{C}^d$, and a biholomorphic map $\phi : N \to \mathbb{C}^d$, such that $\phi(N \cap \mathcal{A}) = \widetilde{\mathcal{A}}$.*

Given a subspace arrangement $\mathcal{A}$, the initial ambient space $\mathbb{C}^d$ of $\mathcal{M}(\mathcal{A})$ carries a natural stratification by the subspaces of $\mathcal{A}$ and their intersections, the poset of strata being the intersection lattice $\mathcal{L}(\mathcal{A})$ of the arrangement. For a local subspace arrangement $\mathcal{A} = \{A_1, \dots, A_n\}$ in M we again consider the stratification of M by all possible intersections of the A_i's, just like in the global case. The poset of strata is also denoted by $\mathcal{L}(\mathcal{A})$ and is called the intersection semilattice (it is a lattice if the intersection of all maximal strata is nonempty).

Definition 4.4. *Let $\mathcal{A}$ be a local subspace arrangement and $\mathcal{L}(\mathcal{A})$ its intersection semilattice. For $U \in \mathcal{L}(\mathcal{A})$, $U_1, \dots, U_k \in \mathcal{L}(\mathcal{A})$ are said to form a* **decomposition** *of U if for any $x \in U$ there exists an open set N with $x \in N$ and a biholomorphic map $\phi : N \to \mathbb{C}^d$, such that $\phi(N \cap U_1), \dots, \phi(N \cap U_k)$ form a decomposition of $\phi(N \cap U)$ in the sense of* Definition 4.1(1).

As in the global case, $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a **building set** *for $\mathcal{A}$ if for any $U \in \mathcal{L}(\mathcal{A})$, the set of strata* $\max \mathcal{G}_{\leq U}$ *gives a decomposition of U.*

We shall refer to these building sets as *geometric* building sets. The difference between combinatorial building sets and geometric ones is contained in the dimension function as is explained in the following proposition.

Proposition 4.5. *Let $\mathcal{A}$ be a local subspace arrangement with intersection semilattice $\mathcal{L}(\mathcal{A})$.*

(1) *If $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a geometric building set of $\mathcal{A}$, then it is a combinatorial building set.*
(2) *If $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a combinatorial building set of $\mathcal{L}(\mathcal{A})$, and for any $x \in \mathcal{L}(\mathcal{A})$ the sum of codimensions of its factors is equal to the codimension of x, then $\mathcal{G}$ is a geometric building set.*

Proof. In both cases it is enough to consider the case when $\mathcal{A}$ is a subspace arrangement.

(1) Consider $\mathcal{G}$ as a subset of $\mathcal{L}(\mathcal{A})^*$, then, for $U \in \mathcal{G}$, the isomorphism φ_U requested in Definition 2.2 is given by taking direct sums:

$$\varphi_U : \prod_{j=1}^{k} [\hat{0}, G_j] \xrightarrow{\oplus_{j=1}^{k}} [\hat{0}, U],$$

where $G_1, \dots, G_k$ are maximal in $\mathcal{G}$ below U.

(2) For $U \in \mathcal{L}(\mathcal{A})^*$, the set $\{U_1, \dots, U_k\} = \max \mathcal{G}_{\leq U}$ gives a decomposition of U because:

a) By the definition of $\mathcal{L}(\mathcal{A})^*$ and the definition of combinatorial building sets, we have $U = \operatorname{span}(U_1, \dots, U_k)$, and, since $\sum_{i=1}^k \dim U_i = \dim U$, we have $U = \bigoplus_{i=1}^k U_i$;

b) for any $V \subseteq U$, $\bigoplus_{i=1}^k (U_i \cap V) \subseteq V = \operatorname{span}(U_1 \wedge V, \dots, U_k \wedge V) \subseteq \bigoplus_{i=1}^k (U_i \cap V)$, where "$\wedge$" denotes the meet operation in $\mathcal{L}(\mathcal{A})^*$, hence $V = \bigoplus_{i=1}^k (U_i \cap V)$. □

4.1.3. *Intersection stratification of local arrangements after blowup.* Let a space X be given with an intersection stratification induced by a local subspace arrangement, and let G be a stratum in X. In the blowup of X at G, $\mathrm{Bl}_G X$, we find the following maximal strata:

- maximal strata in X that do not intersect with G,
- blowups of maximal strata V at $G \cap V$, $\mathrm{Bl}_{G\cap V} V$, where V is maximal in X and intersects G,
- the exceptional divisor $\widetilde{G}$ replacing G.

We consider the intersection stratification of $\mathrm{Bl}_G X$ induced by these maximal strata. We will later see (proof of Proposition 4.7) that in case G is maximal in a building set for the local arrangement in X, then the union of maximal strata in $\mathrm{Bl}_G X$ is again a local arrangement with induced intersection stratification. In general, this is not the case, see Example 4.6

For ease of notation, let us agree here that formally blowing up an empty (non-existing) stratum has no effect on the space. We think about a stratum Y in X, intersection of all maximal strata $V_1, \dots, V_t$ that contain Y, as being replaced by the intersection of corresponding maximal strata in $\mathrm{Bl}_G X$:

$$\mathrm{Bl}_{G\cap V_1} V_1 \ \cap \ \dots \ \cap \ \mathrm{Bl}_{G\cap V_t} V_t \,, \tag{4.1}$$

(recall that $\mathrm{Bl}_{G\cap V_j} V_j = V_j$ for $G \cap V_j = \emptyset$). The intersection (4.1) being empty means that the stratum Y vanishes under blowup of G. For notational convenience, we most often retain names of strata under blowups, thereby referring to the replacement of strata described above.

Example 4.6. (*Local subspace arrangements are not closed under blowup*)
We give an example which shows that blowing up a stratum in a local subspace arrangement does not necessarily result in a local subspace arrangement again. Consider the following arrangement of 2 planes and 1 line in $\mathbb{C}^3$:

$$A_1 :\ y - z = 0\,, \quad A_2 :\ y + z = 0\,, \quad L :\ x = y = 0\,.$$

After blowing up L, the planes A_1 and A_2 are replaced by complex line bundles over $\mathbb{C}\mathrm{P}^1$, which have in common their zero section Z and a complex line Y; L is replaced by a direct product of $\mathbb{C}$ and $\mathbb{C}\mathrm{P}^1$, which intersects both line bundles in Z. The new maximal strata fail to form a local subspace arrangement in the point $Z \cap Y$.

4.1.4. *Tracing incidence structure during arrangement model construction.* We now give a more detailed description of the model construction by De Concini & Procesi via successive blowups, and then proceed with linking our notion of combinatorial blowups to the context of arrangement models.

Let $\mathcal{A}$ be a complex subspace arrangement, $\mathcal{G}\subseteq\mathcal{L}(\mathcal{A})$ a geometric building set for $\mathcal{A}$, and $\{G_1,\dots,G_t\}$ some linear extension of the partial containment order on associated strata in $\mathbb{C}^d$ such that $G_k\supset G_l$ implies $l<k$. The De Concini-Procesi model $Y=Y_{\mathcal{G}}$ of $\mathcal{M}(\mathcal{A})$ is the result of blowing up the strata indexed by elements of $\mathcal{G}$ in the given order. Note that the linear order was chosen so that at each step the stratum which is to be blown up does *not* contain any other stratum indexed by an element of $\mathcal{G}$. At each step we consider intersection stratifications as described above, and we denote the poset of strata after blowup of G_i with $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$. For the case of a stratum G_i being empty after previous blowups remember our agreement of considering blowups of $\emptyset$ as having no effect on a space. The later Proposition 4.7 however shows that strata indexed by elements in $\mathcal{G}$ do not disappear during the sequence of blowups.

Let us remark that the combinatorial data of the initial stratification, i.e., of the arrangement, prescribes much of the geometry of $Y_{\mathcal{G}}$: the complement $Y_{\mathcal{G}}\setminus\mathcal{M}(\mathcal{A})$ is a union of smooth irreducible divisors indexed by elements of $\mathcal{G}$, and these divisors intersect if and only if the set of indices is nested in $\mathcal{G}$ [7, Thm 3.2].

Proposition 4.7. *Let $\mathcal{A}$ be an arrangement of complex subspaces, $\mathcal{G}$ a building set for $\mathcal{A}$ in the sense of De Concini & Procesi, and $\{G_1,\dots,G_t\}$ some linear extension of the partial containment order on associated strata as described above. Let $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ denote the geometric result of successively blowing up strata $G_1,\dots,G_i$, for $1\le i\le t$. Then,*

1. *The poset of strata $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ of $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ can be described as the result of a sequence of combinatorial blowups of the intersection lattice $\mathcal{L}=\mathcal{L}(\mathcal{A})$:*

$$\mathcal{L}_i^{\mathcal{G}}(\mathcal{A}) \;=\; \mathrm{Bl}_i(\mathcal{L})\,, \qquad \textit{for } 1\le i\le t\,.$$

   (Recall that $\mathrm{Bl}_i(\mathcal{L})=\mathrm{Bl}_{G_i}(\mathrm{Bl}_{G_{i-1}}(\dots\mathrm{Bl}_{G_1}\mathcal{L}))$ for $1\le i\le t$.)
2. *The union of maximal strata $\mathcal{A}_i^{\mathcal{G}}$ in $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ is a local subspace arrangement, with $\mathcal{G}$ in $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ being a building set for $\mathcal{A}_i^{\mathcal{G}}$ in the sense of* Definition 4.4. *(Recall that $\mathcal{G}$ here refers to the preimages of the original strata in $\mathcal{G}\subseteq\mathcal{L}(\mathcal{A})$ under the sequence of blowups.)*

Proof. We proceed by induction on the number of blowups. The induction start is obvious, since the lattice of strata $\mathcal{L}_0^{\mathcal{G}}(\mathcal{A})$ of the initial stratification of $\mathbb{C}^d$ coincides with the intersection lattice $\mathcal{L}(\mathcal{A})=\mathrm{Bl}_0(\mathcal{L})$ of the arrangement $\mathcal{A}$. The union of maximal strata is the arrangement $\mathcal{A}$ itself with its given building set $\mathcal{G}$.

Assume that $\mathcal{L}_{i-1}^{\mathcal{G}}(\mathcal{A})=\mathrm{Bl}_{i-1}(\mathcal{L})$ for some $1\le i\le t$, the union of maximal strata $\mathcal{A}_{i-1}^{\mathcal{G}}$ in $\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$ being a local arrangement, and $\mathcal{G}$ a building set for $\mathcal{L}_{i-1}^{\mathcal{G}}(\mathcal{A})$. Let $G=G_i$ be the next stratum to be blown up. First, we proceed in 4 steps to show that $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})=\mathrm{Bl}_i(\mathcal{L})$. In 2 further steps we then verify the claims in (2).

Step 1: *Assign strata of $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ to elements in $\mathrm{Bl}_i(\mathcal{L})$.*

We distinguish two types of elements in $\mathrm{Bl}_i(\mathcal{L})$:

$$\begin{array}{lrl} \text{Type I}: & Y & \text{with } Y\in\mathrm{Bl}_{i-1}(\mathcal{L}) \text{ and } Y\not\geq G\,,\\ \text{Type II}: & [G,Y] & \text{with } Y\in\mathrm{Bl}_{i-1}(\mathcal{L})\,,\ Y\not\geq G\,,\\ & & \text{and } Y\vee G \text{ exists in } \mathrm{Bl}_{i-1}(\mathcal{L})\,. \end{array}$$

To $Y\in\mathrm{Bl}_i(\mathcal{L})$ of type I, assign $\mathrm{Bl}_{G\cap Y}Y$ (recall that blowing up an empty stratum does not change the space). Note that $\dim\mathrm{Bl}_{G\cap Y}Y=\dim Y$.

To $[G,Y]\in \mathrm{Bl}_i(\mathcal{L})$ of type II, assign $(\mathrm{Bl}_{G\cap Y}Y)\cap \widetilde{G}$, where $\widetilde{G}$ denotes the exceptional divisor that replaces G in $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$. This description comprises $\widetilde{G}$ being assigned to $[G,\hat{0}]$. Note that $\dim(\mathrm{Bl}_{G\cap Y}Y)\,\cap\,\widetilde{G}=\dim Y-1$.

Step 2: *Reverse inclusion order on the assigned spaces coincides with the partial order on* $\mathrm{Bl}_i(\mathcal{L})$.

(1) $X,Y\in\mathrm{Bl}_i(\mathcal{L})$, both of type I:

$$X\leq_{\mathrm{Bl}_i(\mathcal{L})}Y\;\Leftrightarrow\;X\leq_{\mathrm{Bl}_{i-1}(\mathcal{L})}Y\;\Leftrightarrow\;X\supseteq_{\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})}Y\;\Leftrightarrow\;\mathrm{Bl}_{G\cap X}X\supseteq\mathrm{Bl}_{G\cap Y}Y\,,$$

where "$\Leftarrow$" in the last equivalence can be seen by first noting that $Y\setminus(G\cap Y)\subseteq X\setminus(G\cap X)$, and then comparing points in the exceptional divisors.

(2) $X,[G,Y]\in\mathrm{Bl}_i(\mathcal{L})$, X of type I, $[G,Y]$ of type II:

As above we conclude

$$\begin{aligned}X\leq_{\mathrm{Bl}_i(\mathcal{L})}[G,Y]\quad&\Leftrightarrow\quad X\leq_{\mathrm{Bl}_{i-1}(\mathcal{L})}Y\\&\Leftrightarrow\quad X\supseteq_{\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})}Y\;\Rightarrow\;\mathrm{Bl}_{G\cap X}X\supseteq\mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G}\,.\end{aligned}$$

To prove the converse is rather subtle. Note first that $G\cap Y\subseteq G\cap X$. Assume that G strictly contains $G\cap X$, then both $G\cap X$ and $G\cap Y$ are not in the building set due to the linear order chosen on $\mathcal{G}$, and G is a factor of both $G\cap X$ and $G\cap Y$. Let $F(G\cap X)=\{G,G_1,\dots,G_k\}$, $F(G\cap Y)=\{G,H_1,\dots,H_t\}$. X written as a join of elements in $\mathrm{Bl}_{i-1}(\mathcal{L})$ below the factors of $G\cap X$ reads

$$X\;=\;g_X\vee Z_1\vee\ldots\vee Z_k$$

for some $g_X\in[\hat{0},G]$, $Z_i\in[\hat{0},G_i]$ for $i=1,\dots,k$. If $Z_i<G_i$ for some $i\in\{1,\dots,k\}$, we have

$$\begin{aligned}G\vee X\;&=\;G\vee(g_X\vee Z_1\vee\ldots\vee Z_i\vee\ldots\vee Z_k)\\&\leq\;G\vee(g_X\vee G_1\vee\ldots\vee Z_i\vee\ldots\vee G_k)\\&<\;G\vee G_1\vee\ldots\vee G_k\;=\;G\vee X\,,\end{aligned}$$

by the "necessity" property of Proposition 2.3(4), yielding a contradiction. Hence,

$$X\;=\;g_X\vee G_1\vee\ldots\vee G_k\,,$$

and similarly, $Y=g_Y\vee H_1\vee\ldots\vee H_t$ for some $g_Y\in[\hat{0},G]$.

For each $j\in\{1,\dots,k\}$ there exists a unique $i_j\in\{1,\dots,t\}$ such that $G_j\leq H_{i_j}$ by Proposition 2.5(1). Thus, $\bigvee G_i<\bigvee H_j$, and, for showing that $X\leq Y$, it is enough to see that $g_X\leq g_Y$.

We show that in an open neighborhood of any point $y\in G\cap Y$, $g_Y\subseteq g_X$. This yields our claim since strata in $\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})$ have pairwise transversal intersections: if they coincide locally, they must coincide globally. With $\mathcal{A}^{\mathcal{G}}_{i-1}$ being a local arrangement, there exists an open neighborhood of $y\in G\cap Y$ where the stratification is biholomorphic to a stratification induced by a subspace arrangement. We tacitly work in the arrangement setting, using that $(\mathrm{Bl}_{i-1}(\mathcal{L}))_{\leq G\vee Y}$ is the intersection lattice of a product arrangement. The $\mathcal{G}$-decomposition of $(G\vee Y)^{\perp}$ described in Definition 4.4 yields (when transferred to the primal setting):

$$g_Y\;=\;\mathrm{span}(G,Y)\,.$$

Analogously, $g_X=\mathrm{span}(G,X)$.

In the linear setting we are concerned with, we interpret points in the exceptional divisor of a blowup as follows:

$$\mathrm{Bl}_{G\cap Y}Y \cap \widetilde{G} \;=\; \{(a, \mathrm{span}(p, G\cap Y))\,|\, a\in G\cap Y,\, p\in Y\setminus (G\cap Y)\}\,. \tag{4.2}$$

In terms of this description, the inclusion map $\mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G} \hookrightarrow \mathrm{Bl}_G(\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A}))$ reads

$$(a, \mathrm{span}(p, G\cap Y)) \longmapsto (a, \mathrm{span}(p, G))\,.$$

Therefore, $\mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G}$ being contained in $\mathrm{Bl}_{G\cap X}X \subseteq \mathrm{Bl}_G(\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A}))$ means that for $(a, \mathrm{span}(p, G\cap Y)) \in \mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G}$ there exists $q\in X\setminus(G\cap X)$ such that $\mathrm{span}(p,G) = \mathrm{span}(q,G)$. In particular, $\mathrm{span}(Y,G)\subseteq\mathrm{span}(X,G)$, which by our previous arguments implies that $Y\subseteq X$.

We assumed above that $G\supset G\cap X$. If $G\cap X$ coincides with G, i.e., X contains G, then $g_X = X$ and a similar reasoning applies to see that $Y\subseteq X$. Similarly for $G\cap X = G\cap Y = G$.

(3) $[G,X]$, $[G,Y]\in\mathrm{Bl}_i(\mathcal{L})$, both of type II:

$$\begin{aligned}
[G,X] \le_{\mathrm{Bl}_i(\mathcal{L})} [G,Y] \;&\Leftrightarrow\; X \le_{\mathrm{Bl}_{i-1}(\mathcal{L})} Y\\
&\Leftrightarrow\; X \supseteq_{\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})} Y \;\Leftrightarrow\; \mathrm{Bl}_{G\cap X}X\cap\widetilde{G} \supseteq \mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G}\,,
\end{aligned}$$

where "$\Leftarrow$" follows from (2) and $\mathrm{Bl}_{G\cap X}X\supseteq\mathrm{Bl}_{G\cap X}X\cap\widetilde{G}\supseteq\mathrm{Bl}_{G\cap Y}Y\cap\widetilde{G}$.

Step 3: *Each of the assigned spaces is the intersection of maximal strata in* $\mathrm{Bl}^{\mathcal{L}}_i(\mathcal{A})$. It is enough to show that spaces assigned to elements of type I in $\mathrm{Bl}_i(\mathcal{L})$ are intersections of new maximal strata. Those associated to elements of type II then are intersections as well by definition.

Let $Y\in\mathrm{Bl}_i(\mathcal{L})$, $Y\not\supseteq G$, and $Y = \cap_{i=1}^t V_i$ with $V_1,\ldots,V_t$ the maximal strata in $\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})$ containing Y. We claim that

$$\mathrm{Bl}_{G\cap Y}Y \;=\; \bigcap_{i=1}^{t}\mathrm{Bl}_{G\cap V_i}V_i\,. \tag{4.3}$$

For the inclusion "$\subseteq$" note that $\mathrm{Bl}_{G\cap Y}Y\subseteq\mathrm{Bl}_{G\cap V_i}V_i$ is a direct consequence of $Y\subseteq V_i$ as discussed in Step 2 (1).

For the reverse inclusion we need the following identity:

$$\bigvee_{i=1}^{t}(G\wedge V_i) \;=\; G\wedge Y\,. \tag{4.4}$$

This identity holds in any semilattice without referring to G being an element of the building set.

Let $\alpha\in\cap_{i=1}^t\,\mathrm{Bl}_{G\cap V_i}V_i$. In case $\alpha\in\cap_{i=1}^t\,V_i\setminus(G\cap V_i)$, we conclude that $\alpha\in Y\setminus(G\cap Y)$. We thus assume that α is contained in the intersection of exceptional divisors $\widetilde{G\cap V_i}$, $i=1,\ldots,t$. We again switch to local considerations in the neighborhood of a point $y\in G\cap Y$, using that it carries a stratification biholomorphic to an arrangement stratification.

Using the description (4.2) of points in exceptional divisors that are created by blowups in the arrangement setting, $\alpha\in\cap_{i=1}^t\,\widetilde{G\cap V_i}\subseteq\cap_{i=1}^t\,\mathrm{Bl}_{G\cap V_i}V_i$ means that there exist $a\in\cap_{i=1}^t\,(G\cap V_i)$, and $p_i\in V_i\setminus(G\cap V_i)$ for $i=1,\ldots,t$, with

$$\alpha \;=\; (a, \mathrm{span}(p_i, G\cap V_i)) \;\in\; \mathrm{Bl}_{G\cap V_i}V_i\,.$$

In particular, $\mathrm{span}(p_i, G) = \mathrm{span}(p_j, G)$ for $1 \le i, j \le t$. Thus,

$$\mathrm{span}(p_j, G) \subseteq \bigcap_{i=1}^{t} \mathrm{span}(V_i, G) = \mathrm{span}(Y, G)$$

using the identity (4.4). We conclude that there exists $y \in Y \setminus (G \cap Y)$ such that $\mathrm{span}(y, G) = \mathrm{span}(p_j, G)$ for all $j \in \{1, \ldots, k\}$, hence

$$\alpha = (a, \mathrm{span}(y, G \cap Y)) \in \mathrm{Bl}_{G \cap Y} Y .$$

Though we are for the moment not concerned with the case of $Y \subseteq G$, we note for later reference that (4.3) remains true, with $\mathrm{Bl}_Y Y = \emptyset$ meaning that the intersection on the right-hand side is empty. Following the proof of the inclusion "$\supseteq$" in (4.3) for $G \cap Y = Y$, we first find that the intersection of blowups can only contain points in the exceptional divisors. Assuming $\alpha \in \cap_{i=1}^{t} \widetilde{G \cap V_i}$ we arrive to a contradiction when concluding that $\mathrm{span}(p_j, G) \subseteq \cap_{i=1}^{t} \mathrm{span}(V_i, G) = \mathrm{span}(Y, G) = G$ for $j = 1, \ldots, t$.

Step 4: *Any intersection of maximal strata in* $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ *occurs as an assigned space.* Every intersection involving the exceptional divisor $\widetilde{G}$ occurs if we can show that all other intersections occur (intersections that additionally involve $\widetilde{G}$ then are assigned to corresponding elements of type II).

Consider $W = \bigcap_{i=1}^{t} \mathrm{Bl}_{G \cap V_i} V_i$, where the V_i are maximal strata in $\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$; recall here that a blowup in an empty stratum does not alter the space. We can assume that $\cap_{i=1}^{t} V_i \neq \emptyset$, otherwise the intersection W were empty. With the identity (4.3) in Step 3 we conclude that either $W = \emptyset$ (in case $\cap_{i=1}^{t} V_i \subseteq G$) or $W = \mathrm{Bl}_{G \cap \bigcap_{i=1}^{t} V_i} \cap_{i=1}^{t} V_i$, in which case it is assigned to the element $\cap_{i=1}^{t} V_i$ in $\mathrm{Bl}_i(\mathcal{L})$.

Step 5: $\mathcal{A}_i^{\mathcal{G}}$ *is a local subspace arrangement in* $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$.

It follows from the description (4.3) of strata in $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ that all intersections of maximal strata are connected and smooth. It remains to show that $\mathcal{A}_i^{\mathcal{G}}$ locally looks like a subspace arrangement. Let $y \in \mathcal{A}_i^{\mathcal{G}}$. We can assume that y lies in the exceptional divisor $\widetilde{G}$. Let $x \in G \subseteq \mathcal{A}_{i-1}^{\mathcal{G}}$ be the image of y under the blowdown map.

We first give a local description around x in $\mathcal{A}_{i-1}^{\mathcal{G}}$. By induction hypothesis, there exists a neighborhood N of x, and an arrangement of linear subspaces $\mathcal{B}$ in $\mathbb{C}^n$ such that the pair $(N, \mathcal{A}_{i-1}^{\mathcal{G}} \cap N)$ is biholomorphic to the pair $(\mathbb{C}^n, \mathcal{B})$. We can assume that under this biholomorphic map, x is mapped to the origin. Let $T = \bigcap_{B \in \mathcal{B}} B$ and note that $G \cap N$ is mapped to some subspace Γ in $\mathcal{B}$.

With G being maximal in the building set for $\mathcal{A}_{i-1}^{\mathcal{G}}$, $\mathcal{B}/T$ is a product arrangement with one of the factors being an arrangement in Γ/T. More precisely, there exists a subspace $\Gamma' \subseteq \mathbb{C}^n$, and two subspace arrangements, $\mathcal{C}$ in Γ/T and $\mathcal{C}'$ in Γ'/T, such that

(1) $\Gamma/T \oplus \Gamma'/T \oplus T = \mathbb{C}^n$,
(2) $\mathcal{B} = \{A \oplus \Gamma'/T \oplus T \mid A \in \mathcal{C}\} \cup \{\Gamma/T \oplus A' \oplus T \mid A' \in \mathcal{C}'\}$.

Blowing up G in $\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$ locally corresponds to blowing up Γ in $\mathbb{C}^n$. Let t be the point on the special divisor $\widetilde{\Gamma}$ corresponding to $y \in \widetilde{G}$, thus t maps to the origin in $\mathbb{C}^n$ under the blowdown map. A neighborhood of t in $\mathrm{Bl}_\Gamma \mathbb{C}^n$ is an n-dimensional open ball which can be parameterized as a direct sum

$$M \oplus M' \oplus I \oplus T\,.$$

Here, M is an open ball around 0 in Γ/T, M' is an open ball on the unit sphere in Γ'/T around the point of intersection with the line $\langle p\rangle$ in Γ'/T that defines t as a point in the exceptional divisor, $t = (0, \mathrm{span}(p,\Gamma)) \in \widetilde{\Gamma}$ (compare (4.2)), and I an open unit ball in $\mathbb{C}$.

The maximal strata in this neighborhood are the following:

- the hyperplane $M \oplus M' \oplus \{0\} \oplus T$, as the exceptional divisor,
- $(M \cap A) \oplus M' \oplus I \oplus T$, replacing $A \oplus \Gamma'/T \oplus T$ after blowup,
- $M \oplus (M' \cap A') \oplus I \oplus T$, replacing $\Gamma/T \oplus A' \oplus T$ after blowup for $A' \neq 0$.

This proves that around t in $\mathrm{Bl}_{\Gamma}\mathbb{C}^n$ we have the structure of a local subspace arrangement, which in turn shows the local arrangement property around y in $\mathcal{A}_i^{\mathcal{G}}$.
Step 6: *$\mathcal{G}$ is a building set for $\mathcal{A}_i^{\mathcal{G}}$ in the sense of* Definition 4.4.
$\mathcal{G}$ is a combinatorial building set by Proposition 3.3. Complementing this with the dimension information about the strata, we conclude, by Proposition 4.5(2), that $\mathcal{G}$ is a geometric building set. □

4.2. **Simplicial Resolutions Of Toric Varieties.**

The study of toric varieties has proved to be a field of fruitful interplay between algebraic and convex geometry: toric varieties are determined by rational polyhedral fans, and many of their algebraic geometric properties are reflected by combinatorial properties of their defining fans.

We recall one such correspondence – between subdivisions of fans and special toric morphisms – and show that so-called stellar subdivisions are instances of combinatorial blowups. This allows us to apply our Main Theorem in the present context: Given a polyhedral fan, we specify a class of *simplicial* subdivisions, and, interpreting our notions of building sets and nested sets, we describe the incidence combinatorics of the subdivisions in terms of the combinatorics of the initial fan. For background material on toric varieties we refer to the standard sources [2, 20, 12, 9].

Let X_Σ be a toric variety defined by a rational polyhedral fan Σ. Any subdivision of Σ gives rise to a proper, birational toric morphism between the associated toric varieties (cf [2, 5.5.1]). In particular, simplicial subdivisions yield toric morphisms from quasi-smooth toric varieties to the initial variety – so-called *simplicial resolutions*. Quasi-smooth toric varieties being rational homology manifolds, such morphisms can replace smooth resolutions for (co)homological considerations.

We define a particular, elementary, type of subdivisions:

Definition 4.8. *Let $\Sigma = \{\sigma\}_{\sigma\in\Sigma} \subseteq \mathbb{R}^d$ be a polyhedral fan, i.e., a collection of closed polyhedral cones σ in $\mathbb{R}^d$ such that $\sigma \cap \tau$ is a cone in Σ for any $\sigma, \tau \in \Sigma$. Let* $\mathrm{cone}(x)$ *be a ray in $\mathbb{R}^d$ generated by $x \in \mathrm{relint}\,\tau$ for some $\tau \in \Sigma$. The* stellar subdivision $\mathrm{sd}(\Sigma, x)$ *of Σ in x is given by the collection of cones*

$$(\,\Sigma \setminus \mathrm{star}(\tau,\Sigma)\,)\ \cup\ \{\,\mathrm{cone}(x,\rho)\,|\,\rho \subseteq \sigma \ \ \textit{for some}\ \ \sigma \in \mathrm{star}(\tau,\Sigma)\,\}\,,$$

where $\mathrm{star}(\tau,\Sigma) = \{\sigma \in \Sigma \,|\, \tau \subseteq \sigma\}$, *and* $\mathrm{cone}(x,\rho)$ *the closed polyhedral cone spanned by ρ and x. If only concerned with the combinatorics of the subdivided fan, we also talk about stellar subdivision of Σ in τ,* $\mathrm{sd}(\Sigma,\tau)$, *meaning any stellar subdivision in x for $x \in \mathrm{relint}\,\tau$.*

Proposition 4.9. *Let $\mathcal{F}(\Sigma)$ be the face poset of a polyhedral fan Σ, i.e., the set of closed cones in Σ ordered by inclusion, together with the zero cone $\{0\}$ as a minimal element. For $\tau \in \Sigma$, the face poset of the stellar subdivision of Σ in τ can be described as the combinatorial blowup of $\mathcal{F}(\Sigma)$ at τ:*

$$\mathcal{F}(\mathrm{sd}(\Sigma,\tau)) \;=\; \mathrm{Bl}_{\tau}\mathcal{F}(\Sigma)\,.$$

Proof. Removing $\mathrm{star}(\tau,\Sigma)$ from Σ corresponds to removing $\mathcal{F}(\Sigma)_{\geq\tau}$ from $\mathcal{F}(\Sigma)$, adding cones as described in Definition 4.8 corresponds to extending $\mathcal{F}(\Sigma)\setminus\mathcal{F}(\Sigma)_{\geq\tau}$ by elements $[\tau,\rho]$ for $\rho\in\mathcal{F}(\Sigma)$, $\rho\subseteq\sigma$ for some $\sigma\in\mathrm{star}(\tau,\Sigma)$. The comparison of order relations is straightforward. □

We apply our Main Theorem to the present context.

Theorem 4.10. *Let Σ be a polyhedral fan in $\mathbb{R}^d$ with face poset $\mathcal{F}(\Sigma)$. Let $\mathcal{G}\subseteq\mathcal{F}(\Sigma)$ be a building set of $\mathcal{F}(\Sigma)$ in the sense of Definition 2.2, $\mathcal{N}(\mathcal{G})$ the complex of nested sets in $\mathcal{G}$ (cf. Definition 2.7). Then, the consecutive application of stellar subdivisions in every cone $G\in\mathcal{G}$ in a non-increasing order yields a simplicial subdivision of Σ with face poset equal to the face poset of $\mathcal{N}(\mathcal{G})$.*

As examples of building sets for face lattices of polyhedral fans let us mention:

(1) the full set of faces, with the corresponding complex of nested sets being the order complex of $\mathcal{F}(\Sigma)$ (stellar subdivision in all cones results in the barycentric subdivision of the fan);
(2) the set of rays together with the non-simplicial faces of Σ;
(3) the set of irreducible elements in $\mathcal{F}(\Sigma)$: the set of rays together with all faces of Σ that are not products of some of their proper faces.

Remark 4.11. For a smooth toric variety X_Σ, the union of closed codimension 1 torus orbits is a local subspace arrangement, in particular, the codimension 1 orbits form a divisor with normal crossings, [12, p. 100]. The intersection stratification of this local arrangement coincides with the torus orbit stratification of the toric variety. For any face τ in the defining fan Σ, the torus orbit $\mathcal{O}_\tau$ together with all orbits corresponding to rays in Σ form a geometric building set. Our proof in 4.1.4 applies in this context with $\mathcal{O}_\tau$ playing the role of G. We conclude that under blowup of X_Σ in the closed torus orbit $\mathcal{O}_\tau$, the incidence combinatorics of torus orbits changes exactly in the way described by a stellar subdivision of Σ in τ. This is the combinatorial part of the well-known fact that in the smooth case, the blowup of X_Σ in a torus orbit $\mathcal{O}_\tau$ corresponds to a regular stellar subdivision of the fan Σ in τ [19].

5. An outlook

5.1. Models for real subspace arrangements and stratified manifolds.

In the spirit of the De Concini-Procesi wonderful model construction for subspace arrangements, Gaiffi [14] presents a model construction for the complement of arrangements of real linear subspaces modulo $\mathbb{R}^+$: Given a central subspace arrangement $\mathcal{A}$ in some Euclidean vector space V, denote by $\widehat{\mathcal{M}}(\mathcal{A})$ the quotient of its complement by $\mathbb{R}^+$. Denote the unit sphere in V by $S(V)$, and consider, for a given (geometric) building set $\mathcal{G}$ in $\mathcal{L}(\mathcal{A})$, the embedding

$$\rho\,:\;\widehat{\mathcal{M}}(\mathcal{A})\;\longrightarrow\;S(V)\,\times\prod_{G\in\mathcal{G}}G\cap S(V)\,.$$

The map is obtained by composing the natural section $\widehat{\mathcal{M}}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$, $[x] \mapsto \frac{x}{|x|}$, with a projection onto each factor of the right-hand side product. Denote the closure of this map by $Y_{\mathcal{G}}$. $Y_{\mathcal{G}}$ is shown to be a manifold with corners, which enjoys much of the properties familiar from the projective setting: the boundary of $Y_{\mathcal{G}}$ is stratified by codimension 1 manifolds with corners indexed with building set elements and having non-empty intersection whenever the index set is nested with respect to $\mathcal{G}$. The set-up allows for a straightforward generalization to mixed subspace and halfspace arrangements motivated by compactifications of configuration spaces in work of Kontsevich [15]. A step aside from classical (linear) arrangements, our combinatorial framework still applies is this context.

In a second part of his paper, Gaiffi extends the previous construction to conically stratified manifolds with corners. Replacing the explicit construction of taking the closure of an embedding into a product of spheres as above, he describes a sequence of "real blowups" in the sense of Kuperberg & Thurston [16]. The sequence is prescribed by the choice of a subset of strata in the original manifold that is a combinatorial building set in our sense. The resulting space is a manifold with corners with its boundary stratified by codimension 1 manifolds with corners that are indexed by the building set elements, and intersections being non-empty if and only if the corresponding index sets are nested.

5.2. **A graded algebra associated with a finite lattice.**

In a joint paper of Yuzvinsky and the first author [10], we start out from the combinatorial notions of building sets and nested sets given in the present paper and define a commutative graded algebra in purely combinatorial terms:

Definition 5.1. *For a finite lattice $\mathcal{L}$, $\mathfrak{A}$ its set of atoms, and $\mathcal{G}$ a combinatorial building set in $\mathcal{L}$, define the algebra $D(\mathcal{L},\mathcal{G})$ as the quotient of a polynomial algebra over $\mathbb{Z}$ with generators in* 1-1 *correspondence with the elements of $\mathcal{G}$:*

$$D(\mathcal{L},\mathcal{G}) \ := \ \mathbb{Z}\,[\{x_G\}_{G\in\mathcal{G}}]\Big/\,\mathcal{I}\,,$$

where the ideal of relations $\mathcal{I}$ is generated by

$$\prod_{i=1}^{t} x_{G_i}\,, \qquad \textit{for } \{G_1,\ldots,G_t\} \textit{ not nested}\,,$$

$$\sum_{G\geq H} x_G\,, \qquad \textit{for } H\in\mathfrak{A}\,.$$

For $\mathcal{L}$ the intersection lattice of an arrangement of complex hyperplanes $\mathcal{A}$ and $\mathcal{G}$ its minimal building set, this algebra was shown to be isomorphic to the integer cohomology algebra of the compact wonderful arrangement model in [8, 1.1]. We show in [10] that the algebra in fact is isomorphic to the cohomology algebra of the arrangement model for *any* choice of a building set in the intersection lattice.

Going beyond the arrangement context, we can provide yet another geometric interpretation of the algebras $D(\mathcal{L},\mathcal{G})$: For an arbitrary atomic lattice and a given combinatorial building set we construct a smooth, non-compact toric variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$ and show that its Chow ring is isomorphic to the algebra $D(\mathcal{L},\mathcal{G})$.

In a sense, this is a prototype result of what we had hoped for when working on our combinatorial framework: to provide the outset for going beyond the geometric context of resolutions and yet get back to it in a different, elucidating, and, other than via the abstract combinatorial detour, seemingly unrelated way.

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