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Permanence of moment estimates for p-products of convex bodies

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Abstract. It is shown that two inequalities concerning second and fourth moments of isotropic normed convex bodies in $\mathbb{R}^n$ are permanent under forming p-products. These inequalities are connected with a concentration of mass property as well as with a central limit property. An essential tool are certain monotonicity properties of the Γ-function.

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Introduction

The topic of the present paper originated from and is connected with a version of the central limit theorem in the context of convex bodies in geometry.

A *normed convex body* $K \subseteq \mathbb{R}^n$ is a convex compact set of volume 1 whose centre of mass is at the origin. A normed convex body is *isotropic* if its ellipsoid of inertia is a Euclidean ball, i.e., if

$$L_K^2 := \int_K (x \cdot u)^2 \, dx$$

is independent of $u \in S^{n-1}$, the unit sphere of $\mathbb{R}^n$.

The main objective of the present paper is the validity of two inequalities concerning the moments

$$M_{2,n}(K) := \int_K |x|^2 \, dx,$$
$$M_{4,n}(K) := \int_K |x|^4 \, dx$$

of isotropic normed convex bodies $K \subseteq \mathbb{R}^n$. Here and in the following, the norm $|\cdot|$ always denotes the Euclidean norm. In [9] it has been shown that the inequalities

$$M_{2,n}(K) \leqslant M_{2,n}(\Delta_n) = \frac{n}{(n+1)(n+2)} \frac{n!^{\frac{2}{n}}}{(n+1)^{\frac{1}{n}}}, \tag{0.1}$$

$$\frac{M_{4,n}(K)}{M_{2,n}(K)^2} \leqslant \frac{M_{4,n}(\Delta_n)}{M_{2,n}(\Delta_n)^2} = \frac{(n+1)(n+2)}{(n+3)(n+4)}(1 + \frac{2}{n} + \frac{6}{n+1}) \tag{0.2}$$

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are valid for isotropic normed Euclidean balls, cubes, and cross polytopes, where Δ_n denotes the normed regular simplex in $\mathbb{R}^n$. Moreover, it is shown that these inequalities are persistent under the operations of forming isotropic normed cones, cartesian products, and joins. Here we extend these results and show that (0.1), (0.2) are also persistent under forming p-products of isotropic normed convex bodies. This implies, in particular, that (0.1), (0.2) are valid for p-balls in $\mathbb{R}^n$ (ℓ_p^n-balls).

In order to motivate these investigations we recall the notion of central limit property for isotropic normed convex bodies. We regard an isotropic normed convex body $K \subseteq \mathbb{R}^n$, with the Lebesgue measure λ_n, as a probability space. For a unit vector $u \in S^{n-1}$ we define a random variable $X_{K,u}\colon K \to \mathbb{R}$ by

$$X_{K,u}(x) := x \cdot u.$$

The density of its distribution is given by

$$\varphi_{K,u}(t) = \lambda_{n-1}\big(\{x \in K;\, x \cdot u = t\}\big) \quad (t \in \mathbb{R}).$$

The central limit property of a set of isotropic normed convex bodies roughly means that the distributions of $X_{K,u}$ are close to Gaussian distributions for large n and 'most' directions $u \in S^{n-1}$.

In order to make this more precise we introduce the set $\mathcal{K}_n$ of all isotropic normed convex bodies in $\mathbb{R}^n$, and we define

$$\mathcal{K} := \bigcup_{n=1}^{\infty} \mathcal{K}_n.$$

Following [3], [2], we say that a set $\mathcal{M} \subseteq \mathcal{K}$ has the *central limit property* if

$$\sup_{K \in \mathcal{K}_n \cap \mathcal{M}} \mu_{n-1}\big(\{u \in S^{n-1};\, \|\varphi_{K,u} - g_{L_K^2}\|_1 > \varepsilon\}\big) \to 0 \quad \text{as } n \to \infty, \text{ for all } \varepsilon > 0.$$

Here,

$$g_{\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

denotes the Gaussian density, and μ_{n-1} the surface measure on S^{n-1}, normed to a probability measure. In [2] it is shown that the central limit property can be expressed equivalently using distribution functions or the L_∞-norm instead of the L_1-norm (the latter under an additional asssumption). In [3], the central limit property is shown for the set of cubes and Euclidean balls. In [1] it is shown that, for symmetric isotropic normed convex bodies, the central limit property (in terms of distribution functions) can be derived from a 'concentration hypothesis'. In a more general context, it is shown in [10] that a concentration of mass property implies a central limit property. This concentration of mass property (or concentration hypothesis) means that, for high dimensions, the volume of the bodies under consideration is mainly concentrated in a certain spherical shell. Using [10], and assuming the concentration of mass property for a set $\mathcal{M} \subseteq \mathcal{K}$, it is shown in [2] that $\mathcal{M}$ satisfies the central limit property formulated above.

The role of inequalities (0.1), (0.2) in this context is that they imply the concentration of mass property; cf. [9]. As a consequence, the subset $\mathcal{T}$ of all bodies K in $\mathcal{K}$ satisfying these inequalities satisfies the central limit property. An inequality similar to (but weaker than) (0.2) for ℓ_p^n-balls was shown in [1] in order to conclude the concentration hypothesis in this case. The desire to prove (0.1), (0.2) for ℓ_p^n-balls was the starting point of the present paper. We refer to [4] for a study of the asymptotic behaviour, as $n \to \infty$, of the volume of sections of ℓ_p^n-balls ($0 < p \leqslant \infty$).

As a side remark we mention that for bodies $K \in \mathcal{K}$ satisfying (0.1) one obtains $L_K \leqslant \frac{1}{e}$. It is an open question whether there is a universal bound on L_K for $K \in \mathcal{K}$ (see [6; Sec. 5]). As a second remark we mention that the bound in (0.2) converges to 1 as $n \to \infty$, which is the essential feature for deriving the concentration of mass property (cf. [9; Sec. 1]).

In Section 1 we define the (isotropic normed) p-product of isotropic normed convex bodies, and we compute the moments in terms of the moments of the given bodies.

In Section 2 we show that inequalities (0.1) and (0.2) are permanent under forming p-products.

In Section 3 we give more precise information on the p-balls in $\mathbb{R}^n$.

The most sophisticated material of this paper, in our opinion, is contained in Section 4. Having derived the expressions (1.5) and (1.8), it was not obvious how to handle the abundance of Γ-functions occurring in these expressions. The technical tools concerning the Γ-function presented in this section are essential for Sections 2 and 3 and should also be of independent interest.

Finally, in Section 5 we show that the expression $\frac{M_{4,n}(K)}{M_{2,n}(K)^2}$ considered in (0.2) takes its minimal value for the Euclidean ball. The corresponding property for $M_{2,n}(K)$ is immediate.

1 The p-product of convex bodies and its moments

For $j = 1, 2$, we assume $n_j \in \mathbb{N}$, and we take an isotropic normed convex body $K_j \subseteq \mathbb{R}^{n_j}$. Let $|\cdot|_j$ denote the gauge functional of K_j. (In general, $|\cdot|_j$ will be sublinear; it will be a norm if and only if K_j is symmetric.) Moreover, define $n := n_1 + n_2$, and let $1 \leqslant p \leqslant \infty$. On $\mathbb{R}^n$ we define the sublinear functional

$$|(x_1, x_2)|_p := \begin{cases} \left(|x_1|_1^p + |x_2|_2^p\right)^{1/p} & \text{for } 1 \leqslant p < \infty, \\ \max\left(|x_1|_1, |x_2|_2\right) & \text{for } p = \infty, \end{cases}$$

where $x_j \in \mathbb{R}^{n_j}$ $(j = 1, 2)$. We define the p-product $K_1 \times_p K_2$ (which is not a normed convex body) by

$$K_1 \times_p K_2 := \left\{(x_1, x_2) \in \mathbb{R}^n;\ |(x_1, x_2)|_p \leqslant 1\right\}.$$

We note that $K_1 \times_\infty K_2$ is just the ordinary cartesian product, whereas

$$K_1 \times_1 K_2 = \left\{(tx_1, (1-t)x_2);\ x_j \in K_j\ (j = 1, 2),\ 0 \leqslant t \leqslant 1\right\}$$

is the join of K_1 and K_2. These cases were studied in [9].

Examples of p-products

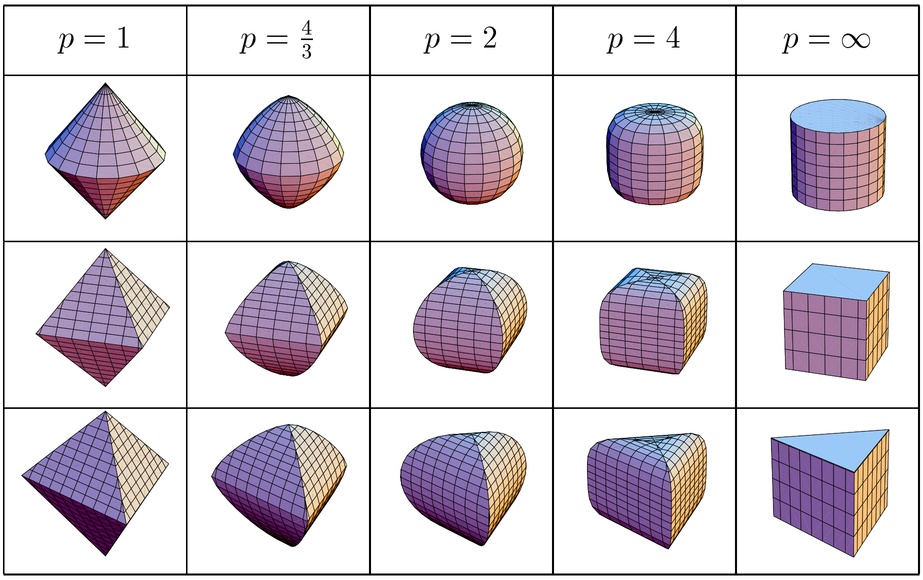

Figure 1

Note that $\int_{K_1 \times_p K_2} (x \cdot u)(x \cdot v)\, dx = 0$ if $u \in \mathbb{R}^{n_1} \times \{0\},\ v \in \{0\} \times \mathbb{R}^{n_2}$, because the centre of mass of K_1 and K_2, respectively, is the origin. Moreover, $\int_{K_1 \times_p K_2} (x \cdot u)^2\, dx$ is constant on $S^{n_1-1} \times \{0\}$ and on $\{0\} \times S^{n_2-1}$ because K_1 and K_2 are isotropic. Therefore, the isotropic normed body associated with $K_1 \times_p K_2$ is of the form $(\alpha_1 K_1) \times_p (\alpha_2 K_2)$, where α_1, α_2 are suitable scaling factors. In Figure 1 we show some examples for the case $n_1 = 2,\ n_2 = 1$, with different p, where K_1 is a circle, a square and an equilateral triangle, respectively (and $K_2 = [-\frac{1}{2}, \frac{1}{2}]$).

In order to find the correct scaling factors we first compute the volume of $K_1 \times_p K_2$. The method we use for doing so is inspired by the computation of the volume of p-balls given in [7; p.11].

First we note

$$\int_{\mathbb{R}^n} e^{-|(x_1,x_2)|_p^p}\, dx = \int_{\mathbb{R}^{n_1}} e^{-|x_1|_1^p}\, dx_1 \int_{\mathbb{R}^{n_2}} e^{-|x_2|_2^p}\, dx_2. \tag{1.1}$$

Now,

$$\begin{aligned}
\int_{\mathbb{R}^{n_1}} e^{-|x_1|_1^p}\, dx_1 &= \int_{\mathbb{R}^{n_1}} \left(-\int_{|x_1|_1^p}^{\infty} \frac{d}{dt} e^{-t}\, dt \right) dx_1 \\
&= \int_0^\infty \int_{|x_1|_1^p \leqslant t} dx_1\, e^{-t}\, dt \\
&= \int_0^\infty t^{\frac{n_1}{p}} e^{-t}\, dt\, V_1 = \Gamma(\tfrac{n_1}{p} + 1),
\end{aligned}$$

with $V_1 := \mathrm{vol}_{n_1}(K_1) = 1$. Computing the other integrals in (1.1) in the same way, we obtain

$$\Gamma(\tfrac{n}{p} + 1)\, \mathrm{vol}_n\big((\alpha_1 K_1) \times_p (\alpha_2 K_2)\big) = \alpha_1^{n_1} \Gamma(\tfrac{n_1}{p} + 1) \alpha_2^{n_2} \Gamma(\tfrac{n_2}{p} + 1),$$

and therefore the first requirement for the scaling factors $\alpha_1, \alpha_2 > 0$ amounts to

$$\alpha_1^{n_1}\alpha_2^{n_2} = \frac{\Gamma(\frac{n}{p}+1)}{\Gamma(\frac{n_1}{p}+1)\Gamma(\frac{n_2}{p}+1)}. \tag{1.2}$$

In order to obtain isotropy we compute second moments. (Recall that $|\cdot|$ denotes the Euclidean norm.)

$$\begin{aligned}\int_{\mathbb{R}^n} |x_1|^2 e^{-|(x_1,x_2)|_p^p}\,dx &= \int_{\mathbb{R}^{n_1}} |x_1|^2 \left(-\int_{|x_1|_1^p}^{\infty} \frac{d}{dt}e^{-t}\,dt\right)\,dx_1 \int_{\mathbb{R}^{n_2}} e^{-|x_2|_2^p}\,dx_2\\ &= \int_0^\infty \int_{|x_1|_1^p\leqslant t} |x_1|^2\,dx_1\,e^{-t}\,dt\;\Gamma(\tfrac{n_2}{p}+1)\\ &= \int_0^\infty t^{\frac{n_1+2}{p}} M_{2,1}e^{-t}\,dt\;\Gamma(\tfrac{n_2}{p}+1)\\ &= \Gamma(\tfrac{n_1+2}{p}+1)\Gamma(\tfrac{n_2}{p}+1)M_{2,1},\end{aligned}$$

where, for abbreviation, $M_{2,j} := M_{2,n_j}(K_j)$ $(j=1,2)$. The first expression in the last chain of equalities can also be computed as

$$\begin{aligned}\int |x_1|^2 e^{-|(x_1,x_2)|_p^p}\,dx &= \int |x_1|^2 \left(-\int_{|x|_p^p}^{\infty} \frac{d}{dt}e^{-t}\,dt\right)\,dx\\ &= \int_0^\infty \int_{|x|_p^p\leqslant t} |x_1|^2\,dx\,e^{-t}\,dt\\ &= \int_0^\infty t^{\frac{n+2}{p}} e^{-t}\,dt \int_{K_1\times_p K_2} |x_1|^2\,dx\\ &= \Gamma(\tfrac{n+2}{p}+1)\int_{K_1\times_p K_2} |x_1|^2\,dx,\end{aligned}$$

and therefore

$$\int_{(\alpha_1K_1)\times_p(\alpha_2K_2)} |x_1|^2\,dx = \alpha_1^{n_1+2}\alpha_2^{n_2}\frac{\Gamma(\frac{n_1+2}{p}+1)\Gamma(\frac{n_2}{p}+1)}{\Gamma(\frac{n+2}{p}+1)}M_{2,1}. \tag{1.3}$$

Now, isotropy requires

$$\frac{1}{n_1}\int_{(\alpha_1K_1)\times_p(\alpha_2K_2)} |x_1|^2\,dx = \frac{1}{n_2}\int_{(\alpha_1K_1)\times_p(\alpha_2K_2)} |x_2|^2\,dx,$$

which can be expressed as

$$\alpha_1^2\Gamma(\tfrac{n_1+2}{p}+1)\Gamma(\tfrac{n_2}{p}+1)\frac{M_{2,1}}{n_1} = \alpha_2^2\Gamma(\tfrac{n_1}{p}+1)\Gamma(\tfrac{n_2+2}{p}+1)\frac{M_{2,2}}{n_2}. \tag{1.4}$$

With α_1, α_2 obtained from (1.2), (1.4) we define

$$K_1 *_p K_2 := (\alpha_1K_1)\times_p(\alpha_2K_2)$$

as the isotropic normed p-product of K_1 and K_2; we obtain

$$
\begin{aligned}
M_{2,n}(K_1 *_p K_2) &= n \left(\frac{1}{n_1}\int_{K_1 *_p K_2} |x_1|^2\,dx\right)^{\frac{n_1}{n}} \left(\frac{1}{n_2}\int_{K_1 *_p K_2} |x_2|^2\,dx\right)^{\frac{n_2}{n}} \\
&= n\left(\alpha_1^{n_1+2}\alpha_2^{n_2}\frac{\Gamma(\frac{n_1+2}{p}+1)\Gamma(\frac{n_2}{p}+1)}{\Gamma(\frac{n+2}{p}+1)}\frac{M_{2,1}}{n_1}\right)^{\frac{n_1}{n}} \times \\
&\quad \left(\alpha_1^{n_1}\alpha_2^{n_2+2}\frac{\Gamma(\frac{n_1}{p}+1)\Gamma(\frac{n_2+2}{p}+1)}{\Gamma(\frac{n+2}{p}+1)}\frac{M_{2,2}}{n_2}\right)^{\frac{n_2}{n}} \\
&= n\left[\frac{(\alpha_1^{n_1}\alpha_2^{n_2})^{n+2}}{\Gamma(\frac{n+2}{p}+1)^n}\left(\Gamma(\tfrac{n_1+2}{p}+1)\Gamma(\tfrac{n_2}{p}+1)\frac{M_{2,1}}{n_1}\right)^{n_1} \times \right. \\
&\qquad \left.\left(\Gamma(\tfrac{n_1}{p}+1)\Gamma(\tfrac{n_2+2}{p}+1)\frac{M_{2,2}}{n_2}\right)^{n_2}\right]^{\frac{1}{n}}.
\end{aligned}
$$

By (1.2), this can be transformed into

$$
M_{2,n}(K_1 *_p K_2) = n\left(\frac{f(n_1,\frac{1}{p})f(n_2,\frac{1}{p})}{f(n,\frac{1}{p})}\left(\frac{M_{2,n_1}(K_1)}{n_1}\right)^{n_1}\left(\frac{M_{2,n_2}(K_2)}{n_2}\right)^{n_2}\right)^{\frac{1}{n}}, \tag{1.5}
$$

with the function

$$
f(n,\alpha) := \frac{\Gamma(\alpha(n+2)+1)^n}{\Gamma(\alpha n+1)^{n+2}}. \tag{1.6}
$$

The derivation of this formula is only valid for $p<\infty$, but the final expression is also true for $p=\infty$ (cartesian product); cf. [9; Sec. 3].

In order to compute $\frac{M_{4,n}}{M_{2,n}^2}(K_1 *_p K_2)$ we note

$$
\int_{K_1\times_p K_2} |x|^4\,dx = \int_{K_1\times_p K_2}\left(|x_1|^4 + 2|x_1|^2|x_2|^2 + |x_2|^4\right)dx.
$$

In the same way as above we compute

$$
\begin{aligned}
\int_{K_1\times_p K_2} |x_1|^4\,dx &= \frac{\Gamma(\frac{n_1+4}{p}+1)\Gamma(\frac{n_2}{p}+1)}{\Gamma(\frac{n+4}{p}+1)}M_{4,1}, \\
\int_{K_1\times_p K_2} |x_1|^2|x_2|^2\,dx &= \frac{\Gamma(\frac{n_1+2}{p}+1)\Gamma(\frac{n_2+2}{p}+1)}{\Gamma(\frac{n+4}{p}+1)}M_{2,1}M_{2,2}.
\end{aligned}
$$

In the following computations we use the fact that $M_2 := M_{2,n}(K_1 *_p K_2)$ can be expressed by means of (1.3) as well as the symmetric expression. Thus,

$$\begin{aligned}
\frac{\int_{K_1 *_p K_2} |x_1|^4\,dx}{M_2^2} &= \frac{\alpha_1^{n_1+4}\alpha_2^{n_2}\frac{\Gamma(\frac{n_1+4}{p}+1)\Gamma(\frac{n_2}{p}+1)}{\Gamma(\frac{n+4}{p}+1)}M_{4,1}}{\frac{n^2}{n_1^2}\alpha_1^{2(n_1+2)}\alpha_2^{2n_2}\frac{\Gamma(\frac{n_1+2}{p}+1)^2\Gamma(\frac{n_2}{p}+1)^2}{\Gamma(\frac{n+2}{p}+1)^2}M_{2,1}^2} \\
&= \frac{n_1^2}{n^2}\frac{1}{\alpha_1^{n_1}\alpha_2^{n_2}}\frac{\Gamma(\frac{n+2}{p}+1)^2}{\Gamma(\frac{n+4}{p}+1)}\frac{\Gamma(\frac{n_1+4}{p}+1)}{\Gamma(\frac{n_1+2}{p}+1)^2}\frac{1}{\Gamma(\frac{n_2}{p}+1)}\frac{M_{4,1}}{M_{2,1}^2} \\
&= \frac{n_1^2}{n^2}\frac{g(n_1,\frac{1}{p})}{g(n,\frac{1}{p})}\frac{M_{4,1}}{M_{2,1}^2},
\end{aligned}$$

with the function

$$g(n,\alpha) := \frac{\Gamma(\alpha n+1)\Gamma(\alpha(n+4)+1)}{\Gamma(\alpha(n+2)+1)^2}, \tag{1.7}$$

and

$$\begin{aligned}
&\frac{\int_{K_1 *_p K_2} |x_1|^2|x_2|^2\,dx}{M_2^2} \\
&= \frac{\alpha_1^{n_1+2}\alpha_2^{n_2+2}\frac{\Gamma(\frac{n_1+2}{p}+1)\Gamma(\frac{n_2+2}{p}+1)}{\Gamma(\frac{n+4}{p}+1)}M_{2,1}M_{2,2}}{\frac{n}{n_1}\alpha_1^{n_1+2}\alpha_2^{n_2}\frac{\Gamma(\frac{n_1+2}{p}+1)\Gamma(\frac{n_2}{p}+1)}{\Gamma(\frac{n+2}{p}+1)}M_{2,1}\frac{n}{n_2}\alpha_1^{n_1}\alpha_2^{n_2+2}\frac{\Gamma(\frac{n_1}{p}+1)\Gamma(\frac{n_2+2}{p}+1)}{\Gamma(\frac{n+2}{p}+1)}M_{2,2}} \\
&= \frac{n_1 n_2}{n^2}\frac{1}{\alpha_1^{n_1}\alpha_2^{n_2}}\frac{\Gamma(\frac{n+2}{p}+1)^2}{\Gamma(\frac{n+4}{p}+1)}\frac{1}{\Gamma(\frac{n_1}{p}+1)\Gamma(\frac{n_2}{p}+1)} \\
&= \frac{n_1 n_2}{n^2}\frac{1}{g(n,\frac{1}{p})}.
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
&\frac{M_{4,n}}{M_{2,n}^2}(K_1 *_p K_2) \\
&\quad = \frac{1}{n^2 g(n,\frac{1}{p})}\left(n_1^2 g(n_1,\tfrac{1}{p})\frac{M_{4,n_1}}{M_{2,n_1}^2}(K_1) + 2n_1n_2 + n_2^2 g(n_2,\tfrac{1}{p})\frac{M_{4,n_2}}{M_{2,n_2}^2}(K_2)\right).
\end{aligned} \tag{1.8}$$

2 Permanence of inequalities for the second and fourth moments

We assume n_j, K_j given as in Section 1, for $j = 1, 2$.

Theorem 2.1. *The function*

$$p \longmapsto M_{2,n}(K_1 *_p K_2)$$

is strictly decreasing on $[1,2]$, strictly increasing on $[2,\infty]$; in particular, it attains its minimum for $p = 2$. Moreover, $M_{2,n}(K_1 *_p K_2)$ attains its maximum for $p = \infty$. If K_1 and K_2 satisfy (0.1), then $K_1 *_p K_2$ also satisfies (0.1) for all $p \in [1,\infty]$.

Proof. The function $\frac{f(n_1,\frac{1}{p})f(n_2,\frac{1}{p})}{f(n,\frac{1}{p})}$ occurring in the expression for $M_{2,n}(K_1 *_p K_2)$, see (1.5), is treated by computing the logarithmic derivative

$$\partial_\alpha \ln \frac{f(n_1,\alpha)f(n_2,\alpha)}{f(n,\alpha)} = \partial_\alpha \ln f(n,\alpha) \left(\frac{\partial_\alpha \ln f(n_1,\alpha)}{\partial_\alpha \ln f(n,\alpha)} + \frac{\partial_\alpha \ln f(n_2,\alpha)}{\partial_\alpha \ln f(n,\alpha)} - 1 \right).$$

By Corollary 4.3, the second factor is strictly increasing and vanishes at $\alpha = \frac{1}{2}$, whereas the first factor is positive. This implies the first assertion.

It is also clear that $M_{2,n}(K_1 *_p K_2)$ attains its maximum for $p = 1$ or $p = \infty$. In order to exclude $p = 1$ it is therefore sufficient to show

$$\frac{f(n_1,1)f(n_2,1)}{f(n,1)} \leqslant 1 \left(= \frac{f(n_1,0)f(n_2,0)}{f(n,0)} \right).$$

This is shown subsequently in Lemma 2.2.

In order to prove the last statement it is therefore sufficient to check (0.1) for $p = \infty$, i.e., the isotropic normed cartesian product. This case, however, has been treated in [9]. □

Lemma 2.2. *(a) The function*

$$k \longmapsto \frac{f(k-1,1)}{f(k,1)}$$

(where $f(k,1) = \frac{\Gamma((k+2)+1)^k}{\Gamma(k+1)^{k+2}} = \frac{(k+2)!^k}{k!^{k+2}} = \frac{((k+1)(k+2))^k}{k!^2}$*, see* (1.6)*) is monotone decreasing in* $k \in \mathbb{N}$.

(b) If $n, n_1, n_2 \in \mathbb{N}_0$, $n = n_1 + n_2$, *then*

$$\frac{f(n_1,1)f(n_2,1)}{f(n,1)} \leqslant 1.$$

Proof. (a) An elementary computation yields

$$\frac{f(k-1,1)}{f(k,1)} = \frac{k}{k+1}\left(\frac{k}{k+2}\right)^k = \frac{k}{k+1}\left(1 - \frac{2}{k+2}\right)^k.$$

For $k = 1, 2$ we obtain

$$\frac{f(0,1)}{f(1,1)} = \frac{f(1,1)}{f(2,1)} = \frac{1}{6}.$$

For the logarithmic derivative we compute and estimate

$$\begin{aligned} \frac{d}{dk} \ln \left(\frac{k}{k+1}\left(1 - \frac{2}{k+2}\right)^k \right) &= \frac{1}{k} - \frac{1}{k+1} + \ln\left(1 - \frac{2}{k+2}\right) + \frac{k}{k} - \frac{k}{k+2} \\ &\leqslant \frac{1}{k} - \frac{1}{k+1} + \frac{2}{k+2} - \left(\frac{2}{k+2} + \frac{2}{(k+2)^2} + \frac{8}{3(k+2)^3} \right) \\ &= \frac{-k^3 - \frac{8}{3}k^2 + \frac{16}{3}k + 8}{k(k+1)(k+2)^3}, \end{aligned}$$

and this is $\leqslant 0$ for $k \geqslant 2$.

(b) For arbitrary $n \in \mathbb{N}_0$ we obviously have $\frac{f(0,1)f(n,1)}{f(n,1)} = 1$. If $n_1, n_2 \in \mathbb{N}_0$, $n_1 < n_2$, $n_1 + n_2 = n$, then (a) implies

$$\frac{f(n_1+1,1)f(n_2-1,1)}{f(n,1)} \leqslant \frac{f(n_1,1)f(n_2,1)}{f(n,1)}$$

which by induction is $\leqslant 1$. □

Theorem 2.3. *There exists $p_0 \in [2,\infty)$ such that*

$$p \longmapsto \frac{M_{4,n}}{M_{2,n}^2}(K_1 *_p K_2)$$

is strictly decreasing on $[1,p_0]$, strictly increasing on $[p_0,\infty]$. Here $p_0 = 2$ if and only if $K_j = B_{n_j,2}$, the normed Euclidean ball in $\mathbb{R}^{n_j}$, for $j = 1,2$. In particular, $\frac{M_{4,n}}{M_{2,n}^2}(K_1 *_p K_2)$ attains its maximum for $p = 1$ or $p = \infty$. If K_1 and K_2 satisfy (0.2), *then $K_1 *_p K_2$ also satisfies* (0.2) *for all $p \in [1,\infty]$.*

As a preparation for the proof, we treat the expression occurring in (1.8) in a more general setting.

Lemma 2.4. *Let $n, n_1, n_2 > 0,\ n > n_1, n_2,\ c_1, d > 0,\ c_2 \geqslant 0$, and define*

$$G(\alpha) := \frac{1}{g(n,\alpha)}\big(c_1 g(n_1,\alpha) + d + c_2 g(n_2,\alpha)\big)$$

for $\alpha \geqslant 0$. Then there exists $\alpha_0 > 0$ such that G is strictly decreasing on $[0,\alpha_0]$, strictly increasing on $[\alpha_0,\infty)$. In particular, G attains its maximum on $[0,1]$ at $\alpha = 0$ or at $\alpha = 1$.

Proof. We compute

$$\begin{aligned} G'(\alpha) &= \frac{1}{g(n,\alpha)}\big(c_1\partial_\alpha g(n_1,\alpha) + c_2\partial_\alpha g(n_2,\alpha)\big) \\ &\quad - \frac{\partial_\alpha g(n,\alpha)}{g(n,\alpha)^2}\big(c_1 g(n_1,\alpha) + d + c_2 g(n_2,\alpha)\big) \\ &= \frac{\partial_\alpha \ln g(n,\alpha)}{g(n,\alpha)}\left[c_1\left(\frac{\partial_\alpha \ln g(n_1,\alpha)}{\partial_\alpha \ln g(n,\alpha)} - 1\right) g(n_1,\alpha) - d\right. \\ &\qquad \left. + c_2\left(\frac{\partial_\alpha \ln g(n_2,\alpha)}{\partial_\alpha \ln g(n,\alpha)} - 1\right) g(n_2,\alpha)\right]. \end{aligned} \tag{2.1}$$

By Corollary 4.3, the second factor is strictly increasing (note that $g(n,\alpha)$ is increasing, by the positivity of $\partial_\alpha \ln g(n,\alpha)$) and negative for small α since $d > 0$. Since the first factor is positive it follows that $G'(\alpha)$ has a unique zero α_0. (It is not difficult to show that, as $\alpha \to \infty$, the second factor tends to infinity because $g(n_1,\alpha) \to \infty$.) □

Proof of Theorem 2.3. Because of formula (1.8), the first statement follows from Lemma 2.4, except for the fact that $p_0 \geqslant 2$. If $K_j = B_{n_j,2}$ for $j = 1, 2$, then Theorem 5.1 implies $p_0 = 2$; in particular, in Lemma 2.4, with $c_j = n_j^2 \frac{M_{4,n_j}}{M_{2,n_j}^2}(B_{n_j,2})$ $(j = 1, 2)$, $d = 2n_1n_2$, one has $G'(\frac{1}{2}) = 0$. Now, Theorem 5.1 implies $G'(\frac{1}{2}) > 0$ in all other cases (cf. formula (2.1)), i.e., $p_0 > 2$.

In order to prove the last statement it is therefore sufficient to check (0.2) for $p = 1$ and $p = \infty$, i.e., the isotropic normed join and the isotropic normed cartesian product. These cases have been treated in [9]. $\square$

3 Second and fourth moments for the p-balls

For $n \in \mathbb{N}$, $p \in [1, \infty]$, we denote by $B_{n,p}$ the normed ball in $\mathbb{R}^n$ corresponding to the p-norm $|\cdot|_p$. The volume of the unit p-ball

$$\check{B}_{n,p} := \left\{ x \in \mathbb{R}^n;\ |x|_p \leqslant 1 \right\}$$

is given by

$$\mathrm{vol}_n(\check{B}_{n,p}) = \frac{\left(2\Gamma(\frac{1}{p}+1)\right)^n}{\Gamma(\frac{n}{p}+1)}$$

(cf. [7; p.11]); therefore

$$B_{n,p} = \frac{\Gamma(\frac{n}{p}+1)^{\frac{1}{n}}}{2\Gamma(\frac{1}{p}+1)} \check{B}_{n,p}.$$

For $n_1, n_2 \in \mathbb{N}$, $n = n_1 + n_2$, one obviously obtains

$$B_{n,p} = B_{n_1,p} *_p B_{n_2,p},$$

and therefore recursion formulas for $M_{2,n}(B_{n,p})$ and $\frac{M_{4,n}}{M_{2,n}^2}(B_{n,p})$ are obtained from (1.5) and (1.8).

Taking into account

$$M_{2,1}(B_{1,p}) = \frac{1}{12}, \quad \frac{M_{4,1}}{M_{2,1}^2}(B_{1,p}) = \frac{9}{5}$$

one obtains

$$\begin{aligned} M_{2,n}(B_{n,p}) &= \frac{n}{12} \frac{f(1, \frac{1}{p})}{f(n, \frac{1}{p})^{\frac{1}{n}}}, \\ \frac{M_{4,n}}{M_{2,n}^2}(B_{n,p}) &= \frac{1}{g(n, \frac{1}{p})} \left(\frac{9}{5n} g(1, \tfrac{1}{p}) + 1 - \frac{1}{n} \right). \end{aligned}$$

Theorem 3.1. *The functions*

$$p \longmapsto M_{2,n}(B_{n,p}), \quad p \longmapsto \frac{M_{4,n}}{M_{2,n}^2}(B_{n,p})$$

are strictly decreasing on $[1, 2]$, *strictly increasing on* $[2, \infty]$.

Proof. For the first function we compute the logarithmic derivative

$$\partial_\alpha \ln \frac{f(1,\alpha)}{f(n,\alpha)^{\frac{1}{n}}} = \partial_\alpha \ln f(n,\alpha) \left(\frac{\partial_\alpha \ln f(1,\alpha)}{\partial_\alpha \ln f(n,\alpha)} - \frac{1}{n} \right).$$

By Corollary 4.3, this implies the assertion for the first function as in the proof of Theorem 2.1.

For the second function, we apply Lemma 2.4 (with $c_2 = 0$); note that $\alpha_0 = \frac{1}{2}$ follows from Theorem 5.1 (or can be obtained by direct computation). □

4 Monotonicity properties of the Γ-function

In this section we study properties of functions of the type

$$F(\alpha) := \prod_{j=1}^{N} \Gamma(\alpha n_j + 1)^{\gamma_j} \quad (\alpha \geqslant 0),$$

where $n_j \geqslant 0$, $\gamma_j \in \mathbb{R}$ ($j = 1, \ldots, N$). We will make use of the following integral representation of the logarithmic derivative of the Γ-function,

$$\Psi(x) := \frac{d}{dx} \ln \Gamma(x+1) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{e^t - 1} \right) dt$$

(cf. [5; §1, formula (60)] or [8; 6.351]). If $\sum_{j=1}^N \gamma_j n_j = 0$ then the logarithmic derivative of F can be expressed as

$$\begin{aligned} \frac{d}{d\alpha} \ln F(\alpha) &= \sum_j \gamma_j n_j \Psi(\alpha n_j) \\ &= -\int_0^\infty \frac{1}{e^t - 1} \sum_j \gamma_j n_j e^{-\alpha n_j t} dt \\ &= \frac{1}{4\alpha} \int_0^\infty \frac{1}{\sinh^2(\frac{t}{2})} \sum_j \gamma_j \big(e^{-\alpha n_j t} - 1 \big) dt, \end{aligned} \tag{4.1}$$

where in the last step we used partial integration and $\frac{d}{dt}\left(\frac{1}{e^t-1}\right) = -\frac{e^t}{(e^t-1)^2} = -\frac{1}{4\sinh^2(\frac{t}{2})}$. Note that the functions f and g needed in Sections 2 and 3 are of the above type, with $\sum_j \gamma_j n_j = 0$.

Proposition 4.1. *Let $k_j \geqslant 0$, $m_j \in \mathbb{R}$ ($j = 1, \ldots, N$) be such that the function*

$$\eta(x) := \sum_j m_j x^{k_j}$$

has no zeros on $(0,1)$.

(a) Define $\varphi(n,\alpha) := \prod_j \Gamma(\alpha(n+k_j)+1)^{\frac{m_j}{n+k_j}}$ ($n > 0$, $\alpha \geqslant 0$) and assume $\eta(1) = 0$. Then

$$\partial_\alpha \ln \varphi(n,\alpha) = -\frac{1}{\alpha} \int_0^\infty \frac{\eta(e^{-t})}{e^{t/\alpha} - 1} e^{-nt} \, dt \quad (\alpha > 0)$$

and, for $n > n_1 > 0$, the function

$$Q(\alpha) := \frac{\partial_\alpha \ln \varphi(n_1, \alpha)}{\partial_\alpha \ln \varphi(n, \alpha)}$$

is strictly increasing on $(0, \infty)$. If, in addition, $\eta'(1) \neq 0$ then $Q(\alpha) \to 1$ as $\alpha \to 0$.

(b) Define $\psi(n, \alpha) := \prod_j \Gamma(\alpha(n + k_j) + 1)^{m_j}$ $(n, \alpha \geqslant 0)$ and assume $\eta(1) = \eta'(1) = 0$. Then

$$\partial_\alpha \ln \psi(n, \alpha) = \frac{1}{4\alpha^2} \int_0^\infty \frac{\eta(e^{-t})}{\sinh^2(\frac{t}{2\alpha})} e^{-nt}\, dt \quad (\alpha > 0)$$

and, for $n > n_1 \geqslant 0$, the function

$$Q(\alpha) := \frac{\partial_\alpha \ln \psi(n_1, \alpha)}{\partial_\alpha \ln \psi(n, \alpha)}$$

is strictly increasing on $(1, \infty)$. If, in addition, $\eta''(1) \neq 0$ then $Q(\alpha) \to 1$ as $\alpha \to 0$.

For the proof we need the following elementary tool.

Lemma 4.2. *Let $f_1, f_2, h\colon (0, \infty) \to (0, \infty)$ be continuous functions, $\frac{f_2}{f_1}$ strictly increasing, h strictly increasing. Then*

$$\frac{\int_0^\infty f_1(t) h(t)\, dt}{\int_0^\infty f_1(t)\, dt} < \frac{\int_0^\infty f_2(t) h(t)\, dt}{\int_0^\infty f_2(t)\, dt}.$$

(All the integrals are assumed to exist.)

Proof. Without restriction $\int_0^\infty f_1(t)\, dt = \int_0^\infty f_2(t)\, dt = 1$. Then $\int_0^\infty \big(\frac{f_2}{f_1} - 1\big) f_1\, dt = 0$, so there exists $\tau > 0$ such that $\frac{f_2(\tau)}{f_1(\tau)} = 1$. We conclude

$$\begin{aligned} \textstyle\int_0^\infty (f_2 - f_1) h\, dt &= \int_0^\tau \left(\frac{f_2}{f_1} - 1\right) f_1 h\, dt + \int_\tau^\infty \left(\frac{f_2}{f_1} - 1\right) f_1 h\, dt \\ &> h(\tau) \int_0^\infty \left(\frac{f_2}{f_1} - 1\right) f_1\, dt = 0. \end{aligned}$$

□

Proof of Proposition 4.1. Without restriction assume $\eta \geqslant 0$. (Otherwise replace m_j by $-m_j$ for $j = 1, \dots, N$.)

(a) Since $\sum_j \frac{m_j}{n+k_j}(n + k_j) = \eta(1) = 0$ we obtain from (4.1) the formula

$$\partial_\alpha \ln \varphi(n, \alpha) = -\int_0^\infty \frac{1}{e^t - 1} e^{-\alpha n t} \eta(e^{-\alpha t})\, dt = -\frac{1}{\alpha} \int_0^\infty \frac{\eta(e^{-t})}{e^{t/\alpha} - 1} e^{-nt}\, dt. \tag{4.2}$$

Letting $f_\alpha(t) := \frac{\eta(e^{-t})}{e^{t/\alpha} - 1} e^{-nt}$ and $h(t) := e^{(n-n_1)t}$ we get

$$Q(\alpha) = \frac{\int_0^\infty f_\alpha(t) h(t)\, dt}{\int_0^\infty f_\alpha(t)\, dt}.$$

By Lemma 4.2, for the proof of the strict monotonicity of Q it remains to show that, for $0 < \alpha_1 < \alpha_2$, the function

$$\frac{f_{\alpha_2}(t)}{f_{\alpha_1}(t)} = \frac{e^{t/\alpha_1} - 1}{e^{t/\alpha_2} - 1}$$

is strictly increasing (on $(0, \infty)$), or equivalently, that $s \mapsto \frac{e^{ps}-1}{e^s-1}$, where $p > 1$, is strictly increasing. The numerator of the derivative of this function,

$$\frac{p-1}{p} e^{(p+1)s} + \frac{1}{p} e^s - e^{ps},$$

is positive by the convexity of the exponential function.

For the proof of $Q(\alpha) \to 1$ $(\alpha \to 0)$ we note that

$$\partial_\alpha \ln \varphi(n, 0) = 0, \quad \partial_\alpha^2 \ln \varphi(n, 0) = \int_0^\infty \frac{t}{e^t - 1} \eta'(1)\, dt$$

by (4.2) and the assumption $\eta(1) = 0$, so we are done by de l'Hospital's rule.

(b) Since $\sum_j m_j = \eta(1) = 0$ and $\sum_j m_j(n + k_j) = n\eta(1) + \eta'(1) = 0$ we obtain from (4.1) the formula

$$\partial_\alpha \ln \psi(n, \alpha) = \frac{1}{4\alpha} \int_0^\infty \frac{1}{\sinh^2(\frac{t}{2})} e^{-\alpha n t} \eta(e^{-\alpha t})\, dt = \frac{1}{4\alpha^2} \int_0^\infty \frac{\eta(e^{-t})}{\sinh^2(\frac{t}{2\alpha})} e^{-nt}\, dt. \tag{4.3}$$

Letting $f_\alpha(t) := \frac{\eta(e^{-t})}{\sinh^2(\frac{t}{2\alpha})} e^{-nt}$ and $h(t) := e^{(n-n_1)t}$, we get

$$Q(\alpha) = \frac{\int_0^\infty f_\alpha(t) h(t)\, dt}{\int_0^\infty f_\alpha(t)\, dt}.$$

Again by Lemma 4.2, for the proof of the strict monotonicity of Q it remains to show that, for $0 < \alpha_1 < \alpha_2$, the function

$$\frac{f_{\alpha_2}(t)}{f_{\alpha_1}(t)} = \left(\frac{\sinh(\frac{t}{2\alpha_1})}{\sinh(\frac{t}{2\alpha_2})} \right)^2$$

is strictly increasing (on $(0, \infty)$), or equivalently, that $s \mapsto \frac{\sinh(ps)}{\sinh s}$, where $p > 1$, is strictly increasing. Differentiation transforms this into the question whether

$$p \coth(ps) > \coth s.$$

The latter is a consequence of the strict monotonicity of

$$(0, \infty) \ni p \mapsto p \coth p,$$

which in turn follows from

$$\frac{d}{dp}(p \coth p) = \frac{\cosh p \sinh p - p}{(\sinh p)^2} > \frac{1 \cdot p - p}{(\sinh p)^2} = 0.$$

For the proof of $Q(\alpha) \to 1$ ($\alpha \to 0$) we apply de l'Hospital's rule twice to the first representation of $\partial_\alpha \ln \psi(n,\alpha)$ in (4.3):

$$\lim_{\alpha\to 0} Q(\alpha) = \lim_{\alpha\to 0} \frac{\alpha\partial_\alpha \ln\psi(n_1,\alpha)}{\alpha\partial_\alpha \ln\psi(n,\alpha)} = \lim_{\alpha\to 0} \frac{\partial_\alpha^2\big(\alpha\partial_\alpha \ln\psi(n_1,\alpha)\big)}{\partial_\alpha^2\big(\alpha\partial_\alpha \ln\psi(n,\alpha)\big)} = 1$$

since $\partial_\alpha^2\big(\alpha\partial_\alpha \ln\psi(n,\alpha)\big)\big|_{\alpha=0} = \frac{1}{4}\int_0^\infty \frac{t^2}{\sinh^2(\frac{t}{2})}\eta''(1)\,dt$ for all $n \geqslant 0$, by the assumption $\eta(1) = \eta'(1) = 0$. $\square$

In the following corollary we apply Proposition 4.1 to the functions f and g used in Sections 2 and 3.

Corollary 4.3. *As in* (1.6) *and* (1.7)*, let* $f(n,\alpha) := \frac{\Gamma(\alpha(n+2)+1)^n}{\Gamma(\alpha n+1)^{n+2}}$, $g(n,\alpha) := \frac{\Gamma(\alpha n+1)\Gamma(\alpha(n+4)+1)}{\Gamma(\alpha(n+2)+1)^2}$ $(n,\alpha \geqslant 0)$*. Then, for* $n > n_1 > 0$*, the functions*

$$\alpha \longmapsto \frac{\partial_\alpha \ln f(n_1,\alpha)}{\partial_\alpha \ln f(n,\alpha)} \quad \textit{and} \quad \alpha \longmapsto \frac{\partial_\alpha \ln g(n_1,\alpha)}{\partial_\alpha \ln g(n,\alpha)}$$

are strictly increasing, $\partial_\alpha \ln f$ *and* $\partial_\alpha \ln g$ *are positive on* $(0,\infty)\times(0,\infty)$*,* $\partial_\alpha \ln f(n,\frac{1}{2}) = 2n$*, and*

$$\frac{\partial_\alpha \ln g(n_1,\alpha)}{\partial_\alpha \ln g(n,\alpha)} \to 1 \quad \textit{as } \alpha \to 0.$$

Proof. To prove the statements about f we define $\varphi := f^{\frac{1}{n(n+2)}}$. Then φ is of the type considered in Proposition 4.1(a), with $k_1 = 2,\ k_2 = 0,\ m_1 = 1,\ m_2 = -1$. The corresponding function η is $\eta(x) = x^2 - 1$, so it fulfils the conditions of Proposition 4.1(a). Hence

$$\frac{\partial_\alpha \ln f(n_1,\alpha)}{\partial_\alpha \ln f(n,\alpha)} = \frac{n_1(n_1+2)}{n(n+2)}\frac{\partial_\alpha \ln\varphi(n_1,\alpha)}{\partial_\alpha \ln\varphi(n,\alpha)}$$

is strictly increasing and

$$\partial_\alpha \ln\varphi(n,\alpha) = \frac{1}{\alpha}\int_0^\infty \frac{1-e^{-2t}}{e^{t/\alpha}-1}e^{-nt}\,dt > 0 \quad (\alpha > 0).$$

For $\alpha = \frac{1}{2}$ we obtain

$$\partial_\alpha \ln f(n,\frac{1}{2}) = n(n+2)\partial_\alpha \ln\varphi(n,\frac{1}{2}) = 2n(n+2)\int_0^\infty e^{-(n+2)t}\,dt = 2n.$$

The function g is of the type considered in Proposition 4.1(b), with $k_1 = 0,\ k_2 = 4,\ k_3 = 2,\ m_1 = m_2 = 1,\ m_3 = -2$. The corresponding function η is $\eta(x) = 1 + x^4 - 2x^2 = (1-x^2)^2 \geqslant 0$, so an application of Proposition 4.1(b) yields the first and the last statement about g. The positivity of $\partial_\alpha \ln g$ follows from (4.3). $\square$

5 The minimum of $\frac{M_{4,n}M_{0,n}}{M_{2,n}^2}$

The result of this section will be formulated for the more general context of compact subsets of $\mathbb{R}^n$ of non-vanishing volume which are star shaped with respect to the origin. A body K like this is described by its radial function $\rho = \rho_K \colon S^{n-1} \to [0,\infty)$,

$$\rho(u) := \max\{r \geqslant 0;\, ru \in K\}.$$

The moments of K are then given by

$$\begin{aligned} M_{0,n}(K) &= \operatorname{vol}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(u)^n \, dS(u), \\ M_{2,n}(K) &= \frac{1}{n+2}\int_{S^{n-1}} \rho(u)^{n+2} \, dS(u), \\ M_{4,n}(K) &= \frac{1}{n+4}\int_{S^{n-1}} \rho(u)^{n+4} \, dS(u). \end{aligned}$$

Theorem 5.1. *Let $K \subseteq \mathbb{R}^n$ be compact and star shaped with respect to the origin, with the origin contained in the interior of K. Then*

$$\frac{M_{4,n}M_{0,n}}{M_{2,n}^2}(K) \geqslant \frac{M_{4,n}M_{0,n}}{M_{2,n}^2}(B_{n,2}),$$

with equality if and only if K is a multiple of the Euclidean ball $B_{n,2}$.

Proof. In contrast to the remainder of this paper, we shall choose the scaling of K in such a way that the second moment is normed to $M_{2,n}(K) = n$. By ρ_0 we denote the radius of the accordingly scaled Euclidean ball,

$$\rho_0 = \left(\frac{n(n+2)}{\sigma_{n-1}}\right)^{\frac{1}{n+2}} = \left(\frac{n+2}{\omega_n}\right)^{\frac{1}{n+2}}.$$

Denoting by μ_{n-1} the surface measure on S^{n-1}, normed to a probability measure, we then have to show

$$\int \rho_0^{n+4}\, d\mu_{n-1} \int \rho_0^n \, d\mu_{n-1} \leqslant \int \rho(u)^{n+4}\, d\mu_{n-1} \int \rho(u)^n \, d\mu_{n-1},$$

or equivalently,

$$1 \leqslant \int \left(\left(\frac{\rho}{\rho_0}\right)^{n+2}\right)^{\frac{n+4}{n+2}} d\mu_{n-1} \int \left(\left(\frac{\rho}{\rho_0}\right)^{n+2}\right)^{\frac{n}{n+2}} d\mu_{n-1},$$

with equality if and only if $\rho = \rho_0$. Note that for $p := \frac{n+4}{n+2}$, $q := \frac{n}{n+2}$ one has $0 < q < p < 2$, $p + q = 2$, and that $f := \left(\frac{\rho}{\rho_0}\right)^{n+2}$ is strictly positive on S^{n-1}, with $\int f \, d\mu_{n-1} = 1$. Therefore the conclusion is a consequence of the Cauchy-Schwarz inequality,

$$1 = \int f^{\frac{p}{2}+\frac{q}{2}}\, d\mu_{n-1} \leqslant \left(\int f^p \, d\mu_{n-1}\right)^{\frac{1}{2}} \left(\int f^q \, d\mu_{n-1}\right)^{\frac{1}{2}},$$

where equality holds if and only if $f^{\frac{p}{2}} = cf^{\frac{q}{2}}$ for some constant c. Because of $p \neq q$, the latter is equivalent to $f = 1$. $\square$

Remark 5.2. For any compact set K in $\mathbb{R}^n$, one has

$$\left(\int_K |x|^2\,dx\right)^2 \leqslant \int_K |x|^4\,dx \int_K dx,$$

by the Cauchy-Schwarz inequality. Theorem 5.1 tells that, for compact sets $K \subseteq \mathbb{R}^n$ which are star shaped with respect to the origin, this inequality can be improved to

$$\left(\int_K |x|^2\,dx\right)^2 \leqslant c_0 \int_K |x|^4\,dx \int_K dx,$$

with

$$c_0 = \frac{n^2}{\frac{\sigma_{n-1}}{n+4}\rho_0^{n+4}\frac{\sigma_{n-1}}{n}\rho_0^n} = \frac{n(n+4)}{(n+2)^2} < 1.$$

This conclusion makes it clear that the inequality of Theorem 5.1 is not true for arbitrary compact subsets of $\mathbb{R}^n$. (Take, e.g., thin spherical shells!)

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