L_p-properties of second order elliptic differential operators

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Introduction

The aim of this thesis is to study the L_p-theory of (formal) second order elliptic differential operators with singular measurable coefficients. The main tool of our study is the theory of strongly continuous semigroups (C_0-semigroups) of bounded linear operators on Banach spaces. It is well-known that, if A_p is an operator realisation of a formal differential expression in L_p with $\rho(A_p) \neq \varnothing$, then the abstract Cauchy problem

$$u'(t) + A_p u(t) = 0 \quad (t > 0), \qquad u(0) = f$$

has a unique solution for all $f \in D(A_p)$ if and only if $-A_p$ is the generator of a C_0-semigroup T_p on L_p. In this case, the solution u is given by $u(t) = T_p(t)f$.

Another motivation to study L_p-theory is that it often yields significant information for weak L_2-solutions of the corresponding elliptic and parabolic equations, such as integrability or smoothness properties of eigenfunctions and solutions.

We confine ourselves to the case of real-valued coefficients so that the corresponding semigroups will be positive. Our main interest lies in the case of singular coefficients where the first problem one faces is constructing a C_0-semigroup on a suitable L_p-space, associated with the differential expression.

In the case of a uniformly elliptic principal part of the differential expression, and bounded coefficients in the lower order terms, the associated semigroup exists in the whole L_p-scale, which follows, for instance, from classical estimates on the fundamental solution of the corresponding parabolic equation [Aro67]. If the coefficients of the lower order terms are allowed to have strong singularities then a semigroup associated with the differential expression can be constructed in L_p for p from a proper subinterval of $[1, \infty)$ only. This phenomenon was first observed in the study of Schrödinger operators with singular negative potentials ([HeSl78], [KPS81]), later also for the operator $-\Delta + b \cdot \nabla$ in [KoSe90].

In this thesis we study general second order elliptic expressions in divergence form with both first and zero order perturbations, namely

$$\mathcal{L} := -\nabla \cdot (a\nabla) + b_1 \cdot \nabla + \nabla \cdot b_2 + V,$$

on an open set $\Omega \subseteq \mathbb{R}^N$, for a wide class of boundary conditions. Generalising results from [BeSe90], [Lis96], we establish a precise condition controlling the interval of those $p \in [1, \infty)$ for which $\mathcal{L}$ gives rise to a quasi-contractive C_0-semigroup $T_p = (e^{-tA_p}; t \geqslant 0)$ on $L_p(\Omega)$.

By the Lumer-Phillips theorem, T_p is quasi-contractive, i.e., there exists $\omega \in \mathbb{R}$ such that the operators $e^{-\omega t}T_p(t)$ are contractive, if and only if A_p is quasi-accretive. Thus, it is very natural to expect that the condition controlling the interval of quasi-contractivity involves only the expression $\langle \mathcal{L}u, u^{p-1} \rangle$ occurring in the definition of quasi-accretivity, computed in a suitable sense (see page 59). We have to overcome several technical problems in order to show that this natural condition is indeed sufficient.

In general, the set of all $p \in [1,\infty)$ such that $\mathcal{L}$ is associated with a (not necessarily quasi-contractive) C_0-semigroup on L_p is strictly larger. It was already observed in [KPS81] that the Schrödinger semigroup with negative $L_{\frac{N}{2},\mathrm{weak}}$-potential can be defined on L_p for p from an interval strictly larger than the interval of quasi-contractivity. In [Sem00] this result was extended to uniformly elliptic second order divergence type operators on $\mathbb{R}^N$ perturbed by a form small potential. Here we show that this behaviour is typical for rather general second order uniformly elliptic operators.

A traditional way of constructing a semigroup associated with the differential expression $\mathcal{L}$ is the form method. Since $\mathcal{L}$ is given in divergence form, it corresponds to a sesquilinear form in L_2. If this form is densely defined, sectorial and closed then it is associated with an m-sectorial operator A in L_2, which in turn generates a C_0-semigroup $(e^{-tA}; t \geqslant 0)$ on L_2 ([Kat80; Thms. VI.2.1, IX.1.24]). If $\left\| e^{-tA}\restriction_{L_2\cap L_p} \right\|_{L_p\to L_p} \leqslant Ce^{\omega t}$ for some $p \in [1,\infty)$ then $\left(e^{-tA}\restriction_{L_2\cap L_p}; t \geqslant 0\right)$ extends to a semigroup on L_p. For $p > 1$, this semigroup is always strongly continuous, whereas for $p = 1$ this is the case if, e.g., the semigroup is positive (see [Voi92]).

The above approach was used for constructing semigroups which act in L_p for all $p \in [1,\infty)$ (see [Dav89; Chapter 1] and the references there), or only for p from some subinterval of $[1,\infty)$ containing 2 (see, e.g., [BeSe90], [Lis96]). However, we do not assume that the form corresponding to $\mathcal{L}$ is sectorial, not even that it is bounded from the left, so the traditional approach is not applicable. In the case $b_2 = 0$, $V = 0$, non-sectorial forms were studied in [KoSe90], [Lis96] by approximating the coefficient b_1 in such a way that the approximating forms became sectorial.

Here we develop a new approach to the construction of positive C_0-semigroups associated with sesquilinear forms. It includes cases of forms that can be associated with a C_0-semigroup on L_2 under assumptions when all known representation theorems break down.

Our approach is based upon approximations by sectorial forms, however, not related to approximations of the coefficients of the first order terms. In contrast, we approximate the potential: we introduce a positive potential U which 'absorbs' all the singularities of the lower order terms of $\mathcal{L}$ in the sense that, being added to the corresponding form, it makes the sum sectorial. Under certain conditions, this gives rise to a positive C_0-semigroup on L_p, as described above. Finally, making use of the perturbation theory for positive semigroups developed in [Voi86], [Voi88], we subtract the potential U again. It is crucial for this construction that the resulting semigroup turns out to be independent of the particular choice of U.

In the context of Schrödinger operators with magnetic fields, and dominated semigroups with singular complex potentials, a similar approximation idea was used in [PeSe81] and in [LiMa97]—however, not in order to construct semigroups but to study properties of semigroups constructed in a different way.

The L_p-properties of the semigroups we study here include analyticity with p-independent sector, and p-independence of the spectra of the generators. Assume that the differential expression $\mathcal{L}$ is associated with a C_0-semigroup on L_p, for p from some subinterval of $[1, \infty)$. If one of the semigroups is analytic then by the Stein interpolation theorem one can show that, for p from the interior of this interval, the semigroup on L_p is analytic; the resulting angle of analyticity, however, tends to zero as p approaches the endpoints of the interval.

E. M. Ouhabaz [Ouh95] was the first to establish analyticity of angle $\frac{\pi}{2}$ in $L_p(\mathbb{R}^N)$, $p \in [1, \infty)$, for symmetric semigroups satisfying Gaussian upper bounds. E. B. Davies [Dav95a] extended this result to the more general setting of metric spaces with polynomial volume growth. For symmetric semigroups acting on L_p for p from a subinterval of $[1, \infty)$ only, analyticity of angle $\frac{\pi}{2}$ was first shown in [Sch96], under the assumption of certain weighted estimates. Here we prove an analogous result for general uniformly elliptic second order operators.

Concerning the problem of L_p-spectral independence, let us mention that this is not a general property of second order elliptic operators (see, e.g., [HeVo86]). However, as first dicovered by R. Hempel and J. Voigt, it is generic for Schrödinger type operators. Here we present rather general conditions on the coefficients of second order elliptic operators under which p-independence of the spectrum holds.

In order to treat the three problems described above, namely extension of the semigroup to L_p, analyticity, and L_p-spectral independence, we first provide a general setting in which these problems can be studied, and we formulate proper sufficient conditions in terms of weighted norm estimates. Motivated by the paper [Dav95a] mentioned above, we study semigroups on metric spaces with exponentially bounded volume growth, not just on open subsets of $\mathbb{R}^N$. This enables us to unify and generalise numerous previous results concerning the three problems under consideration—see the discussion in Section 2.2.

As a specific application of the abstract results we obtain that the Schrödinger semigroup on a Riemannian manifold with Ricci curvature bounded below is analytic of angle $\frac{\pi}{2}$ on L_p, for p from a certain subinterval of $[1, \infty)$ ($p \in [1, \infty)$ if the negative part of the potential is in the Kato class). This result seems to be new even for positive potentials—in this case the semigroup operators act as contractions on all L_p-spaces.

Many of the known results are proved under the assumption that the semigroup acts on all L_p-spaces and has an integral kernel satisfying a Gaussian upper bound. Weighted norm estimates, i.e. estimates on the norm of the semigroup operators as operators between weighted L_p-spaces, were first used in [ScVo94] to establish L_p-spectral independence for Schrödinger operators on $\mathbb{R}^N$ with form small negative part of the potential. In this case the semigroup acts on $L_p(\mathbb{R}^N)$ only for p from an interval around $p = 2$.

The proof in [ScVo94] relied on a discrete method where $\mathbb{R}^N$ is subdivided into congruent cubes. It is clear that this method is essentially restricted to the study of semigroups acting on (subsets of) $\mathbb{R}^N$ since in the case of a general metric space there is no natural partition into countably many subsets. Here we present a continuous version of the technique of weighted norm estimates which is suitable for the general context.

The structure of the thesis is as follows. To a large extent, Chapter 1 is a collection of known facts about C_0-semigroups and their generators (Section 1.1), sectorial forms and the associated analytic semigroups (Section 1.2), Dirichlet forms and the associated sub-Markovian semigroups (Section 1.5), and perturbation of positive semigroups by real-valued potentials (Section 1.3). In Section 1.4, the heart of the chapter, we provide the method that, as described above, is needed in Chapter 3 to construct semigroups on L_p associated with (non-sectorial) forms.

Chapter 2 is devoted to the theory of weighted norm estimates for semigroups on metric spaces with exponentially bounded volume growth. In Section 2.1 we present our abstract results on extrapolation, analyticity and L_p-spectral independence. In Section 2.2 we give some account to the history of these three problems, and we relate our results to the existing literature. As an application we study perturbation of sub-Markovian semigroups satisfying Gaussian upper bounds (such as the diffusion semigroup on a Riemannian manifold) by potentials. The proofs of the main theorems are given in Sections 2.3-2.5.

In Chapter 3 we apply the abstract theory of Chapter 2 to our main subject, the L_p-theory of second order elliptic differential operators. The main results concerning the construction of the semigroup on L_p, quasi-contractivity and analyticity of the semigroups, and L_p-spectral independence of the generators are formulated in Section 3.1 and proved in the two subsequent sections. In Section 3.4 we study to what extent the assumptions of our theorems are necessary. The main result in this direction is that, for a wide class of coefficients, we can characterise the set of all p for which the differential expression $\mathcal{L}$ is associated with a quasi-contractive C_0-semigroup on L_p.

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Chapter 1

C_0-semigroups and sesquilinear forms

In this chapter we provide the functional analytic tools needed in the two subsequent chapters. In Sections 1.1 and 1.2 we collect basic facts about C_0-semigroups, analytic semigroups and sectorial forms, mainly in order to fix our notation but also to recall results frequently used in this thesis. In Section 1.3 we recall J. Voigt's perturbation theory for positive C_0-semigroups. This is fundamental for Section 1.4 where we investigate sesquilinear forms τ in $L_2(\mu)$ fulfilling the first Beurling-Deny criterion and show how to associate with τ a positive C_0-semigroup on $L_p(\mu)$. Section 1.5 deals with the theory of (non-symmetric) Dirichlet forms, the main example being the form corresponding to a homogeneous second order elliptic differential operator with real coefficients.

1.1 C_0-semigroups

In this section we recall some basic definitions and results from the theory of C_0-semigroups, the main references being [Dav80] and [Paz83]. Let X be a Banach space over $\mathbb{C}$ (throughout this thesis we assume $\mathbb{C}$ to be the underlying scalar field). By $\mathcal{L}(X)$ we denote the space of all bounded linear operators from X to X.

Definition 1.1. (a) A function $T\colon [0,\infty) \to \mathcal{L}(X)$ is called a *semigroup on X* if $T(0) = I$ and $T(s+t) = T(s)T(t)$ for all $s, t \geqslant 0$. We say that T is *exponentially bounded* if $\|T(t)\| \leqslant Ce^{\omega t}$ for some $C \geqslant 1,\ \omega \in \mathbb{R}$ and all $t \geqslant 0$.

(b) A semigroup T is called *strongly continuous* or a *C_0-semigroup* if $T(t)f \to f$ as $t \to 0$ for all $f \in X$.

(c) The *generator* of a C_0-semigroup T is the operator A in X defined by

$$D(A) := \big\{u \in X;\ Au := \lim_{t\to 0} \tfrac{1}{t}\big(T(t)u - u\big) \text{ exists}\big\}.$$

Every C_0-semigroup is exponentially bounded. An exponentially bounded semigroup T is strongly continuous if and only if $T(t)f \to f$ for all f from some dense subset of X.

The generator of a C_0-semigroup T is a closed densely defined operator which determines the semigroup T uniquely. We can therefore write $e^{tA} := T(t)$ for the semigroup operators. From time to time we will loosely speak of a C_0-semigroup $T(t) = e^{tA}$ in order to express that T is a C_0-semigroup with generator A.

The *type* of a C_0-semigroup T is the infimum of all $\omega \in \mathbb{R}$ for which there exists $C \geqslant 1$ such that $\|T(t)\| \leqslant Ce^{\omega t}$ for all $t \geqslant 0$. If $\omega_0 \in [-\infty, \infty)$ is the type and A the generator of T then

$$\lambda \in \rho(A), \quad (\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt \qquad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda > \omega_0,$$

where the integral is a strong integral.

In the literature, a semigroup T is often denoted by $\big(T(t); t \geqslant 0\big)$, $\big(T(t)\big)_{t\geqslant 0}$ or $T(\cdot)$ in order to indicate that T depends on one parameter. We will mostly use the symbol T only.

Let A be a linear operator in X. The *numerical range* of A is the set

$$\Theta(A) := \big\{x'(Au);\, u \in D(A),\, \|u\| = 1,\, x' \in X',\, \|x'\| = 1,\, x'(u) = 1\big\}.$$

For $X = L_p(\mu)$, where (Ω, μ) is a measure space and $1 < p < \infty$, we obtain

$$\Theta(A) = \big\{\langle Au, |u|^{p-1} \operatorname{sgn} u\rangle;\, u \in D(A),\, \|u\|_p = 1\big\}.$$

Here and in the sequel, $\langle f, g\rangle$ is defined as $\int_\Omega f(x) \cdot \overline{g}(x)\, d\mu(x)$ whenever $f \cdot \overline{g} \in L_1(\mu)$, for $f, g\colon \Omega \to \mathbb{C}$ measurable.

The operator A is called *m-accretive* if $\Theta(A) \subseteq \big\{z \in \mathbb{C};\ \operatorname{Re} z \geqslant 0\big\}$ and $-1 \in \rho(A)$. It is called *quasi-m-accretive* if $\omega + A$ is m-accretive for some $\omega \in \mathbb{R}$. A semigroup T is called *contractive* if $\|T(t)\| \leqslant 1$ for all $t \geqslant 0$, and *quasi-contractive* if $\|T(t)\| \leqslant e^{\omega t}$ for some $\omega \in \mathbb{R}$ and all $t \geqslant 0$. The Lumer-Phillips theorem states that $-A$ generates a contractive C_0-semigroup if and only if A is m-accretive.

Let E, F be Banach spaces, and assume that there exists a Hausdorff topological vector space G such that $E \hookrightarrow G$, $F \hookrightarrow G$ (continuous injections) and $E \cap F$ is dense in both E and F. Let B_E and B_F be bounded operators in E and F, respectively. We say that B_E and B_F are *consistent* if $B_E{\restriction}_{E\cap F} = B_F{\restriction}_{E\cap F}$. Two semigroups T_E, T_F on E, F, respectively, are called *consistent* if the operators $T_E(t)$ and $T_F(t)$ are consistent for all $t \geqslant 0$.

Let T_E be a semigroup on E, D a dense subset of $E \cap F$, and assume that $T_E(t){\restriction}_D$ extends to a bounded operator $T_F(t)$ on F, for all $t \geqslant 0$. Then T_F is a semigroup on F, and T_E and T_F are consistent. In this case we will say that T_E *extrapolates* to the semigroup T_F on F.

Later on, we will make use of the following notion.

Definition 1.2. ([Voi86; Def. 1.5]) Let T_n ($n \in \mathbb{N}$), T be C_0-semigroups on X. We say that T_n *converges strongly* to T, in symbols $T = \text{s-lim}\, T_n$ or simply $T_n \to T$, if $T_n(t)f \to T(t)f$ as $n \to \infty$, uniformly for t in bounded subsets of $[0, \infty)$, for all $f \in X$.

Let A_n, A be the generators of T_n, T, respectively. Then $T_n \to T$ if and only if $A_n \to A$ in the strong resolvent sense, by the Trotter-Kato-Neveu theorem.

1.2 Analytic semigroups and sectorial forms

For the theory of sectorial forms we refer to [Kat80; Chapter VI], for the connection with analytic semigroups to [Kat80; Sec. IX.1]. For symmetric forms see also [Dav80; Sec. 4.2], for analytic semigroups also [Dav80; Sec. 2.5] and [Paz83; Sec. 2.5].

Definition 1.3. (a) For $\theta \in (0, \frac{\pi}{2}]$ let

$$S_\theta := \big\{0 \neq z \in \mathbb{C};\, |\arg z| < \theta\big\}.$$

A function $T\colon S_\theta \to \mathcal{L}(X)$ is called *exponentially bounded* if $\|T(z)\| \leqslant Ce^{\omega \operatorname{Re} z}$ for some $C \geqslant 1$, $\omega \in \mathbb{R}$ and all $z \in S_\theta$.

(b) A C_0-semigroup T on X is called *analytic* if T has an exponentially bounded analytic extension to S_θ, for some $\theta \in (0, \frac{\pi}{2}]$. The supremum θ_0 of such θ is called the *angle of analyticity* of T. The extension of T to S_{θ_0} will again be denoted by T.

Analytic semigroups are usually defined in a different way (see, e.g., [Dav80]), but actually these definitions are equivalent: If a semigroup T has an analytic extension to S_θ for some $\theta \in (0, \frac{\pi}{2}]$ then $T(z + w) = T(z)T(w)$ for all $z, w \in S_\theta$, by unique analytic continuation. If T is strongly continuous, and $\theta < \frac{\pi}{2}$, then the analytic extension to S_θ is exponentially bounded if and only if $\lim_{S_\theta \ni z \to 0} T(z)f = f$ for all $f \in X$. For the "only if" part note that $\big\{T(t)f;\, t > 0,\, f \in X\big\}$ is dense in X and that T is strongly continuous on $t + \overline{S_\theta}$, for all $t > 0$.

A linear operator A in X is called *m-sectorial (of angle θ)* if $\Theta(A) \subseteq \overline{S_\theta} - \omega$ and $-1 - \omega \in \rho(A)$ for some $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$. In particular, if A is m-sectorial then A is quasi-m-accretive. Moreover, $-A$ generates an analytic semigroup of angle at least $\frac{\pi}{2} - \theta$, with $\|e^{-zA}\| \leqslant e^{\omega \operatorname{Re} z}$ for all $z \in S_{\frac{\pi}{2} - \theta}$. However, if e^{-tA} is an analytic semigroup then A need *not* be m-sectorial (the diffusion semigroup on $L_1(\mathbb{R}^N)$ is analytic of angle $\frac{\pi}{2}$ but its generator is not m-sectorial).

Let H be a Hilbert space. A C_0-semigroup T on H is *symmetric*, i.e. all semigroup operators $T(t)$ are selfadjoint, if and only if the generator of T is selfadjoint. In this case, T is in particular quasi-contractive, and analytic of angle $\frac{\pi}{2}$.

A sesquilinear form τ in H is called *symmetric* if $\tau(u, v) = \overline{\tau(v, u)}$ for all $u, v \in D(\tau)$. A symmetric form τ is said to be *bounded below* if $\tau \geqslant -\omega$ for some $\omega \in \mathbb{R}$, i.e.,

$$\tau(u) := \tau(u,u) \geqslant -\omega\|u\|^2 \quad \big(u \in D(\tau)\big).$$

In this case, $\tau(u,v) + (\omega+1)(u,v)$ defines a scalar product on $D(\tau)$, and τ is called *closed* if $D(\tau)$ is a Hilbert space with respect to this scalar product. In the following, if we speak of a closed symmetric form τ then this implicitly means that τ is bounded below.

If τ is a symmetric form which is bounded below then we define $\tau(u) := \infty$ for $u \in H \setminus D(\tau)$. With this definition, τ is closed if and only if $u \mapsto \tau(u)$ is a lower semicontinuous function from H to $(-\infty, \infty]$ (see [Dav80; Thm. 4.12]). Moreover, if we state an inequality of the type $\tau(u) \leqslant c$ then this implicitly expresses that $u \in D(\tau)$.

Let τ be a sesquilinear form in H. We define the form τ^* in H by $\tau^*(u,v) := \overline{\tau(v,u)}$ on $D(\tau^*) := D(\tau)$. Moreover,

$$\operatorname{Re}\tau := \frac{\tau+\tau^*}{2} \quad \text{and} \quad \operatorname{Im}\tau := \frac{\tau-\tau^*}{2i},$$

where the sum of two forms τ_1, τ_2 is defined by $(\tau_1+\tau_2)(u,v) := \tau_1(u,v)+\tau_2(u,v)$ on $D(\tau_1+\tau_2) := D(\tau_1) \cap D(\tau_2)$. Then $\operatorname{Re}\tau$ and $\operatorname{Im}\tau$ are symmetric forms, and $\tau = \operatorname{Re}\tau + i\operatorname{Im}\tau$, $\tau^* = \operatorname{Re}\tau - i\operatorname{Im}\tau$. Note that $(\operatorname{Re}\tau)(u) = \operatorname{Re}(\tau(u))$ for all $u \in D(\tau)$. (But *not* $(\operatorname{Re}\tau)(u,v) = \operatorname{Re}(\tau(u,v))$ for all $u, v \in D(\tau)$!)

The *numerical range* of τ is the set

$$\Theta(\tau) := \big\{\tau(u);\, u \in D(\tau),\, \|u\| = 1\big\}.$$

The form τ is symmetric if and only if $\Theta(\tau) \subseteq \mathbb{R}$ (recall $\mathbb{K} = \mathbb{C}$). Moreover, $\operatorname{Re}\tau \geqslant -\omega$ for some $\omega \in \mathbb{R}$ if and only if $\Theta(\tau) \subseteq \big\{z \in \mathbb{C};\ \operatorname{Re} z \geqslant -\omega\big\}$. In this case, τ is said to be *bounded from the left*.

We say that τ is *sectorial* if $\Theta(\tau+\omega) \subseteq \overline{S_\theta}$ for some $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, or equivalently,

$$|\operatorname{Im}\tau(u)| \leqslant \tan\theta(\operatorname{Re}\tau+\omega)(u) \quad \big(u \in D(\tau)\big).$$

In this case we endow $D(\tau)$ with the *form norm* $\|\cdot\|_\tau$ defined by $\|u\|_\tau^2 := (\operatorname{Re}\tau + \omega + 1)(u)$. It is easy to see that this definition does not depend on the particular choice of ω, up to equivalence of norms.

We say that a sectorial form τ is *closed* if the symmetric form $\operatorname{Re}\tau$ is closed. By Kato's first representation theorem ([Kat80; Thm. II.2.1]), every densely defined closed sectorial form in H is associated with an m-sectorial operator A in H in the sense that $D(A) \subseteq D(\tau)$ and

$$(Au, v) = \tau(u,v) \quad \text{for all } u \in D(A),\ v \in D(\tau).$$

The operator A is selfadjoint if and only if τ is symmetric. We will shortly write $\tau \leftrightarrow T$ to indicate that T is the analytic semigroup generated by $-A$. With this notation, $\tau^* \leftrightarrow T^* := \big(T(t)^*; t \geqslant 0\big)$.

Let τ be a closed sectorial form, A the associated m-sectorial operator in H. By [Kat80; Cor. 2.3] the numerical range $\Theta(A)$ of A is a dense subset of the numerical range $\Theta(\tau)$ of τ. Therefore, $-A$ generates a contractive C_0-semigroup if and only if A is m-accretive, and the latter is true if and only if $\operatorname{Re}\tau \geqslant 0$.

A sectorial form τ is called *closable* if it has a closed extension $\tilde{\tau} \supseteq \tau$. The smallest closed extension of τ is denoted by $\overline{\tau}$. The sum of two closable (closed) sectorial forms is again a closable (closed) sectorial form. A subspace $D \subseteq D(\tau)$ is called a *core for* τ if D is dense in $D(\tau)$ with respect to the form norm. If τ is closable then this is equivalent to $\overline{\tau\restriction_D} = \overline{\tau}$, where we shortly write $\tau\restriction_D$ for the restriction of τ to $D \times D$.

Let τ_1, τ_2 be symmetric forms. We write $\tau_1 \leqslant \tau_2$ if $D(\tau_1) \supseteq D(\tau_2)$ and $\tau_1(u) \leqslant \tau_2(u)$ for all $u \in D(\tau_2)$. (This is consistent with the definition of $\tau \geqslant -\omega$ if $-\omega$ is interpreted as the form defined by $-\omega(u, v)$ with domain H.)

Let $H = L_2(\mu)$ for some measure space (Ω, μ), and $V\colon \Omega \to \mathbb{R}$ measurable. Then $\int V u\overline{v}\, d\mu$ on the domain $Q(V) := \big\{u \in L_2(\mu);\ V|u|^2 \in L_1(\mu)\big\}$ defines a symmetric form which we will also denote by V. The form V satisfies $V \geqslant -\omega$ for some $\omega \in \mathbb{R}$ if and only if $V \geqslant -\omega$ a.e., and V is closed in this case.

We conclude this section by a lemma which is useful for extrapolation of analytic semigroups.

Lemma 1.4. *Let (Ω, μ) be a measure space, D a subspace of $L_1(\mu) \cap L_\infty(\mu)$ which is dense in $L_p(\mu)$ for all $1 \leqslant p < \infty$, and norming for $L_1(\mu)$. Let $1 \leqslant p, q < \infty$, $S \subseteq \mathbb{C}$ open, $F\colon S \to \mathcal{L}(L_q(\mu))$ an analytic function. If $\|F(\cdot)\restriction_D\|_{p\to p}$ is locally bounded then $F(\cdot)\restriction_D$ extends to an analytic function $F_p\colon S \to \mathcal{L}(L_p)$.*

Proof. It is clear that $F(\cdot)\restriction_D$ extends to a locally bounded function $F_p\colon S \to \mathcal{L}(L_p)$. The assumption implies that $\langle F_p(\cdot)f, g\rangle$ is analytic for all $f, g \in D$. A slight modification of [Kat80; Thm. III.3.12] shows that F_p is analytic. (For the case $p = 1$ note that D is not necessarily dense in $L_\infty(\mu)$; it suffices that D is a norming subspace for $L_1(\mu)$.) □

This result will be applied in Chapter 3 in the following situation: (Ω_n) is an increasing sequence of measurable subsets of Ω such that $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup_n \Omega_n$ (in particular, Ω is σ-finite). Then the space of all $f \in L_\infty(\Omega)$ for which there exists $n \in \mathbb{N}$ such that $f = 0$ a.e. on $\Omega \setminus \Omega_n$ is a suitable choice for D.

1.3 Perturbation of positive C_0-semigroups by real-valued potentials

In this section we give a short introduction to J. Voigt's perturbation theory for positive C_0-semigroups developed in [Voi86], [Voi88]. We include the proofs for two reasons: they partly simplify the original proofs, and they demonstrate how the theory works.

Let (Ω, μ) be a measure space, $1 \leqslant p < \infty$. A semigroup T of positive operators on $L_p(\mu)$ is called *positive*, which we denote by $T \geqslant 0$. If T_1, T_2 are two positive semigroups, then $T_1 \leqslant T_2$ means $T_1(t) \leqslant T_2(t)$ as positive operators for all $t \geqslant 0$. If T is a C_0-semigroup on $L_p(\mu)$ with generator $-A$, and $V \in L_\infty(\mu)$, then T_V denotes the C_0-semigroup generated by $-(A+V)$. For the remainder of this section let T, T_1, T_2 be positive C_0-semigroups on $L_p(\mu)$.

The following two inequalities (see [Voi88; Prop. 1.3]) lie at the heart of Voigt's perturbation theory. Most of the subsequent proofs rely on these inequalities only. Let $V, V_1, V_2 \in L_\infty(\mu)$ be real-valued. Then

$$V_1 \leqslant V_2 \implies T_{V_1} \geqslant T_{V_2} \geqslant 0, \tag{1.1}$$

$$V \geqslant 0,\ T_1 \leqslant T_2 \implies 0 \leqslant (T_2)_V - (T_1)_V \leqslant T_2 - T_1. \tag{1.2}$$

The first statement and the first inequality of the second one follow from the Trotter product formula, $T_V(t) = \text{s-lim}_{n\to\infty}\big(T(\frac{t}{n})e^{-tV/n}\big)^n$ for all $t \geqslant 0$ (cf. [EnNa00; Exercise III.5.11]). The second inequality in (1.2) is equivalent to $T_1 - (T_1)_V \leqslant T_2 - (T_2)_V$ which in turn follows from Duhamel's formula, $T(t) - T_V(t) = \int_0^t T(t-s)VT_V(s)\,ds$, since $(T_1)_V \leqslant (T_2)_V$ by the first inequality.

We are going to extend the definition of T_V to unbounded real-valued potentials, approximating V by $V^{(n)} := (V \wedge n) \vee (-n)$ and letting

$$T_V(t) := \operatorname*{s-lim}_{n\to\infty} T_{V^{(n)}}(t) \quad (t \geqslant 0) \tag{1.3}$$

if the limits exist. Obviously, T_V is a semigroup in this case, and inequalities (1.1) and (1.2) carry over to unbounded potentials whenever the corresponding limits exist. Moreover, if $V \geqslant 0$ or $V \leqslant 0$ then $(T_{V^{(n)}})$ is monotone by (1.1). This leads to the following definition.

Definition 1.5. ([Voi86; Def. 2.2], [Voi88; Def. 2.1], [Voi88; Def. 3.1]) Let $V \colon \Omega \to [0, \infty)$ be measurable.

(a) If $V \geqslant 0$ then the limit in (1.3) exists for all $t \geqslant 0$ by dominated convergence. If T_V is strongly continuous, V is called *T-admissible*. In this case, $T_{V^{(n)}} \to T_V$.

(b) If $V \leqslant 0$ then V is called *T-admissible* if the limit in (1.3) exists for all $t \geqslant 0$ and defines a C_0-semigroup. In this case, $T_{V^{(n)}} \to T_V$.

By monotone convergence, the limit exists if and only if $\sup_{n\in\mathbb{N}} \|T_{V^{(n)}}(t)\| < \infty$ for all $t \geqslant 0$. By [Voi88; Prop. 2.2], V is T-admissible if and only if $\sup_{0\leqslant t\leqslant 1,\, n\in\mathbb{N}} \|T_{V^{(n)}}(t)\| < \infty$.

(c) If $V \geqslant 0$ and V is T-admissible then $-V$ is T_V-admissible since $T_V \leqslant (T_V)_{-V\wedge n} \leqslant (T_{V\wedge n})_{-V\wedge n} = T$ for all $n \in \mathbb{N}$ by (1.1) and (1.2). If $T = (T_V)_{-V}$, then V is called *T-regular*.

In the subsequent proofs we will make use of inequalities (1.1) and (1.2) without further notice. The crucial result concerning the notion of admissibility is as follows.

Proposition 1.6. *([Voi88; Thm. 2.6]) Let $U, V \geqslant 0$ be measurable. Assume that $-U$ and V are T-admissible. Then V is T_{-U}-admissible, $-U$ is T_V-admissible, and $(T_V)_{-U} = (T_{-U})_V = \text{s-lim}_{n,m\to\infty} T_{V\wedge n - U\wedge m} = T_{V-U}$.*

Proof. First observe that the semigroup $(T_{-U})_V$ is strongly continuous since $T_V \leqslant (T_{-U})_V \leqslant T_{-U}$, i.e., V is $T_{-U}(\cdot)$-admissible. The fact that $-U$ is $T_V(\cdot)$-admissible will be shown simultaneously with $(T_V)_{-U} = (T_{-U})_V$.

Let $U_n := U \wedge n,\ V_n := V \wedge n$ $(n \in \mathbb{N})$. For $m \in \mathbb{N}$ we have

$$0 \leqslant (T_{V_n})_{m-U_m} - (T_V)_{m-U_m} \leqslant T_{V_n} - T_V \to 0 \quad (n \to \infty)$$

and hence $(T_{-U_m})_{V_n} = (T_{V_n})_{-U_m} \to (T_V)_{-U_m}$ $(n \to \infty)$. Further,

$$0 \leqslant (T_{-U})_{V_n} - (T_{-U_m})_{V_n} \leqslant T_{-U} - T_{-U_m} \quad (n, m \in \mathbb{N}),$$

which implies the second equality. Letting $n \to \infty$ we obtain

$$0 \leqslant (T_{-U})_V - (T_V)_{-U_m} \leqslant T_{-U} - T_{-U_m},$$

so that $(T_V)_{-U_m} \to (T_{-U})_V$ as $m \to \infty$. This shows that $-U$ is $T_V(\cdot)$-admissible and $(T_V)_{-U} = (T_{-U})_V$. To prove the last equality it suffices to note that

$$T_{V_{2n}-U_n} \leqslant T_{(V-U)^{(n)}} \leqslant T_{V_n-U_{2n}} \quad (n \in \mathbb{N})$$

since $V_{2n} - U_n \geqslant (V-U)^{(n)} \geqslant V_n - U_{2n}$. □

Lemma 1.7. *Let $V \geqslant 0$ be measurable, $T_1 \leqslant T_2$.*

(a) *(cf. [LiMa97; Prop. 1.4(a), Prop. 1.5]) V is T_1-admissible if and only if V is T_2-admissible.*

(b) *([LiMa97; Cor. 1.15]) If V is T_2-regular then V is T_1-regular.*

Proof. (a) If V is T_1-admissible then $(T_2)_V$ is strongly continuous since $(T_1)_V \leqslant (T_2)_V \leqslant T_2$. If V is T_2-admissible then $0 \leqslant T_1 - (T_1)_V \leqslant T_2 - (T_2)_V$ implies that $T_1(t) - (T_1)_V(t) \to 0$ strongly as $t \to 0$, hence $(T_1)_V$ is strongly continuous.

(b) V is T_1-admissible by (a). With $V_n := V - V \wedge n$ we have $(T_V)_{-V\wedge n} = T_{V_n}$ by Proposition 1.6. Therefore, $0 \leqslant T_1 - (T_1)_{V_n} \leqslant T_2 - (T_2)_{V_n}$ implies that $\big((T_1)_V\big)_{-V\wedge n} \to T_1$ as $n \to \infty$. □

The converse of (b) is not true in general, but we have the following result.

Lemma 1.8. *Let $U, V \geqslant 0$ be measurable.*

(a) *([Voi88; Prop. 3.4]) If $-U$ is T-admissible and V is T-regular, then V is T_{-U}-regular.*

(b) *(cf. [LiMa97; Cor. 1.16]) Assume that U is T-regular. Then V is T-regular if and only if V is T_U-regular.*

Proof. (a) For $m \in \mathbb{N}$, V is $T_{m-U\wedge m}$-regular by Lemma 1.7(b) and hence $T_{-U\wedge m}$-regular. Further,

$$0 \leqslant T_{-U} - (T_{-U})_{V-V\wedge n} \leqslant \big(T_{-U} - T_{-U\wedge m}\big) + \big(T_{-U\wedge m} - (T_{-U\wedge m})_{V-V\wedge n}\big)$$

for all $n, m \in \mathbb{N}$. The assertion follows by choosing first m and then n large enough.

(b) follows directly from (a) and Lemma 1.7(b) since $(T_U)_{-U} = T$. □

Another important application of Proposition 1.6 is the next result which, roughly speaking, expresses that negative admissible potentials are always regular.

Lemma 1.9. *(cf. [Voi88; Prop. 3.3(b)]) Let $V \geqslant 0$ be measurable. If $-V$ is T-admissible, then $(T_{-V})_V = T$, and V is T-regular.*

Proof. Since $T \leqslant (T_{-V})_V \leqslant T_{-V}$, the semigroup $(T_{-V})_V$ is strongly continuous, i.e., V is T_{-V}-admissible. Lemma 1.7(a) implies that V is T-admissible, hence $(T_V)_{-V} = (T_{-V})_V = T$ by Proposition 1.6. □

In the last two results of this section, let $V \geqslant 0$ be measurable, and T_p, T_q consistent positive C_0-semigroups on $L_p(\mu)$, $L_q(\mu)$, respectively, for some $p, q \in [1, \infty)$.

Lemma 1.10. *([Voi86; Prop. 3.1])* (a) *$(T_p)_V$ and $(T_q)_V$ are consistent, and V is T_p-admissible if and only if V is T_q-admissible.*

(b) *If $-V$ is T_p- and T_q-admissible, then $(T_p)_{-V}$ and $(T_q)_{-V}$ are consistent.*

(c) *V is T_p-regular if and only if V is T_q-regular.*

Proof. (a) First, observe that $(T_p)_{V\wedge n}$, $(T_q)_{V\wedge n}$ are consistent ($n \in \mathbb{N}$) by the Trotter product formula. Therefore, $(T_p)_V$, $(T_q)_V$ are consistent as limits of consistent semigroups. By [Voi92], $(T_p)_V$ is strongly continuous if and only if $(T_q)_V$ is strongly continuous (since $(T_p)_V$ and $(T_q)_V$ are positive semigroups).

(b) is proved in the same way as (a).

(c) follows from (a) and (b) since $-V$ is admissible with respect to $(T_p)_V$ as well as $(T_q)_V$. □

Corollary 1.11. *Assume that $-V$ is T_p-admissible. Then $-V$ is T_q-admissible if and only if $(T_p)_{-V}$ extrapolates to a C_0-semigroup $\hat{T}_q$ on $L_q(\mu)$, and $\hat{T}_q = (T_q)_{-V}$ in this case.*

Proof. If $-V$ is T_q-admissible then the semigroups $(T_p)_{-V}$ and $(T_q)_{-V}$ are consistent, by Lemma 1.10(b). This shows the "only if" part, with $\hat{T}_q = (T_q)_{-V}$. To show the other implication, assume that $(T_p)_{-V}$ extrapolates to a C_0-semigroup $\hat{T}_q$ on $L_q(\mu)$. By Lemma 1.9 and Lemma 1.10(a), $T_p = \big((T_p)_{-V}\big)_V$ and $(\hat{T}_q)_V$ are consistent semigroups, i.e., $(\hat{T}_q)_V = T_q$. This shows the T_q-admissibility of $-V$, and $\hat{T}_q = (T_q)_{-V}$ follows from the consistency of $(T_p)_{-V}$ and $(T_q)_{-V}$. □

1.4 The first Beurling-Deny criterion for sesquilinear forms

In Section 1.2 we recalled the well-known fact that with every densely defined closed sectorial form in a Hilbert space one can associate an analytic semigroup on H (Kato's first representation theorem). In this section we are going to associate a positive C_0-semigroup on $L_p(\mu)$ with a sesquilinear form in $L_2(\mu)$ fulfilling the first Beurling-Deny criterion ((Ω, μ) a measure space), even in cases when the form is not bounded from the left. This is performed in Proposition 1.19 and Definition 1.20 below.

The contents of the present section are partly new. This section, together with the previous one, is fundamental for the understanding of Chapter 3.

Definition 1.12. Let τ be a sesquilinear form in $L_2(\mu)$.

(a) τ is called *real* if $\operatorname{Re} u \in D(\tau)$ for all $u \in D(\tau)$, and $\tau(u, v) \in \mathbb{R}$ for all real-valued $u, v \in D(\tau)$.

(b) τ is said to *fulfil the first Beurling-Deny criterion* if τ is real and $u^+ \in D(\tau)$, $\tau(u^+, u^-) \leqslant 0$ for all real-valued $u \in D(\tau)$.

The following proposition, due to E.-M. Ouhabaz, shows the relevance of these two notions.

Proposition 1.13. *([Ouh92b; Prop. 2.2 and Thm. 2.4]) Let τ be a densely defined closed sectorial form in $L_2(\mu)$, T the associated analytic semigroup on $L_2(\mu)$. Then T is real (i.e., all semigroup operators are real) if and only if τ is real, and T is positive if and only if τ fulfils the first Beurling-Deny criterion.*

The next lemma states that it suffices to verify the conditions of Definition 1.12 on a form core.

Lemma 1.14. *Let τ be a closable sectorial form. If τ fulfils the first Beurling-Deny criterion then so does $\overline{\tau}$.*

Proof. We first show that $\overline{\tau}$ is real. Without restriction $\operatorname{Re} \tau \geqslant 0$. Then

$$\tau(\operatorname{Re} u) \leqslant \tau(\operatorname{Re} u) + \tau(\operatorname{Im} u) = \operatorname{Re} \tau(u) \quad \big(u \in D(\tau)\big)$$

since τ is real. From this we easily deduce: if $u \in D(\overline{\tau})$, $(u_n) \subseteq D(\tau)$ with $u_n \to u$ in $D(\overline{\tau})$, then $\operatorname{Re} u \in D(\overline{\tau})$ and $\operatorname{Re} u_n \to \operatorname{Re} u$ in $D(\overline{\tau})$. By the latter we show that $\overline{\tau}(u, v) \in \mathbb{R}$ for all real-valued $u, v \in D(\overline{\tau})$, i.e., $\overline{\tau}$ is real.

From the above it follows that the set of all real-valued elements of $D(\tau)$ is dense in the set of all real-valued elements of $D(\overline{\tau})$. Now, for real-valued $u \in D(\tau)$ we have $\overline{\tau}(u^+, u - u^+) = -\overline{\tau}(u^+, u^-) \geqslant 0$ and $\overline{\tau}(u - u^+, u^+) = -\overline{\tau}\big((-u)^+, (-u)^-\big) \geqslant 0$. Thus, we can apply [MaRö92; Lemma I.4.9] to conclude that $u^+ \in D(\overline{\tau})$, $\overline{\tau}(u^+, u^-) \leqslant 0$ for all real-valued $u \in D(\overline{\tau})$. ☐

For the remainder of this section let τ be a densely defined sesquilinear form in $L_2(\mu)$ fulfilling the first Beurling-Deny criterion. The next result characterises admissibility of potentials via form conditions, in the case of symmetric forms.

Proposition 1.15. *(cf. [Voi86; Prop. 5.7, Prop. 5.8(a)]) Let τ be symmetric and closed, T the associated positive C_0-semigroup on $L_2(\mu)$, $V\colon \Omega \to [0,\infty)$ measurable.*

(a) *The potential V is T-admissible if and only if $\tau + V$ is densely defined, and $\tau + V \leftrightarrow T_V$ in this case.*

(b) *The potential $-V$ is T-admissible if and only if $V \leqslant \tau+\omega$ for some $\omega \in \mathbb{R}$. In this case, $\tau - V$ is closable and $\overline{\tau - V} \leftrightarrow T_{-V}$.*

Proof. All the assertions of the proposition, except for the closability of $\tau - V$, are shown in [Voi86]. There the proof is given for the case of the diffusion semigroup on $\mathbb{R}^N$ only, but literally the same proof carries over to the general case. The closability of $\tau - V$ is due to A. Manavi ([Man01; Prop. 12.1.7]); we present his argument here.

Note that T_{-V} is a symmetric C_0-semigroup. Let $\tilde\tau$ be the densely defined, closed symmetric form in $L_2(\mu)$ associated with T_{-V}. By part (a) of the proposition we have $\tilde\tau + V \leftrightarrow (T_{-V})_V = T \leftrightarrow \tau$, taking into account Lemma 1.9 and the definition of T. Hence $\tilde\tau + V = \tau$. Since $Q(V) \supseteq D(\tau)$, this implies that $\tilde\tau \supseteq \tau - V$, i.e., $\tau - V$ has a closed extension. ☐

Proposition 1.15(a) is valid even for sectorial forms, see [Man01; Kor. 12.1.4(a)]. Part (b), however, is not valid for sectorial forms τ: the inequality $V \leqslant \operatorname{Re}\tau + \omega$ still implies that $-V$ is T-admissible ([Man01; Prop. 12.1.11]), but the converse is not true as we will see in Example 3.31 in Section 3.4.

It is clear that a sesquilinear form τ fulfils the first Beurling-Deny criterion if and only if the same holds for $\tau + V$, for some measurable function $V\colon \Omega \to \mathbb{R}$ with $Q(V) \supseteq D(\tau)$. Surprisingly, a similar result holds for closability. It is a direct consequence of Proposition 1.15(b).

Corollary 1.16. *(cf. [Man01; Kor. 12.1.14]) Let τ be sectorial. Then τ is closable if and only if $\tau + V$ is closable for some measurable function $V \geqslant 0$ with $Q(V) \supseteq D(\tau)$.*

Proof. Without restriction τ is symmetric. Let $V \geqslant 0$ be measurable with $Q(V) \supseteq D(\tau)$. If τ is closable then it is clear that $\tau + V$ is closable. If $\tau + V$ is closable then $V \leqslant \overline{\tau + V} + \omega$ for some $\omega \in \mathbb{R}$. Proposition 1.15(b) implies that $\overline{\tau + V} - V$ is closable. Thus, τ is closable since $\tau \subseteq \overline{\tau + V} - V$. ☐

Definition 1.17. Let τ be sectorial and closable, $V \geqslant 0$ measurable. We say that V is *τ-regular* if $D(\tau + V)$ is a core for τ, i.e., $D(\tau) \cap Q(V)$ is dense in $D(\tau)$.

Obviously, if V is τ-regular then V is $\overline{\tau}$-regular, but the converse is not true in general ($D(\tau + V)$ may be $\{0\}$ although V is $\overline{\tau}$-regular, see [StVo85]). The following lemma states in particular that form regularity implies semigroup regularity.

Lemma 1.18. *Let τ be sectorial and closable, T the positive C_0-semigroup associated with $\overline{\tau}$, $V \geqslant 0$ τ-regular. Then V is T-regular, and $T_V \leftrightarrow \overline{\tau + V}$.*

Proof. Note that $\overline{\tau + V}$ fulfils the first Beurling-Deny criterion, by Lemma 1.14. Let T_1 be the positive C_0-semigroup associated with $\overline{\tau + V}$.

Since $D(\tau+V)$ is a core for $\overline{\tau}$ and $(\tau+V-V\wedge n)(u) \to \overline{\tau}(u)$ for all $u \in D(\tau+V)$, we can use [Kat80; Thm. VIII.3.6] to obtain $(T_1)_{-V\wedge n} \to T$. Thus, $-V$ is T_1-admissible, and $(T_1)_{-V} = T$. Lemma 1.9 implies that V is T_1-regular and that $T_1 = T_V$. The latter shows the second assertion, and V is regular with respect to $T = (T_1)_{-V}$, by Lemma 1.8(a). □

In [Man01; Kor. 12.1.4(b)] it is shown that form regularity and semigroup regularity are actually equivalent, but we do not need this fact here.

Now we are ready to formulate the main result of this section which is fundamental for Chapter 3.

Proposition 1.19. *Let $U \geqslant 0$ be measurable, $Q(U) \supseteq D(\tau)$, $\tau + U$ sectorial and closable, $\overline{\tau + U} \leftrightarrow T_{U,2}$. Let $V \geqslant 0$ be $(\tau + U)$-regular, $\tau + V$ sectorial and closable, $\overline{\tau + V} \leftrightarrow T_{V,2}$. Let $p \in [1, \infty)$.*

Assume that $T_{U,2}$ extrapolates to a positive C_0-semigroup $T_{U,p}$ on $L_p(\mu)$ and that $-U$ is $T_{U,p}$-admissible. Then the same holds with V in place of U, V is $(T_{U,p})_{-U}$-regular, and $(T_{U,p})_{-U} = (T_{V,p})_{-V}$.

Proof. Let $T_p := (T_{U,p})_{-U}$. It suffices to show that V is $T_{U,p}$-regular and that $T_{V,2}$, $(T_p)_V$ are consistent: then V is T_p-regular by Lemma 1.8(a) and thus $(T_{U,p})_{-U} = \big((T_p)_V\big)_{-V}$.

The potential U is $(\tau + V)$-regular since $Q(U) \supseteq D(\tau + V)$, and V is $(\tau + U)$-regular by the assumptions. Lemma 1.18 implies that $(T_{V,2})_U \leftrightarrow \overline{(\tau + V) + U} = \overline{(\tau + U) + V} \leftrightarrow (T_{U,2})_V$ and that U is $T_{V,2}$-regular. Therefore,

$$T_{V,2} = \big((T_{V,2})_U\big)_{-U} = \big((T_{U,2})_V\big)_{-U}.$$

Moreover, V is $T_{U,2}$-regular and hence $T_{U,p}$-regular by Lemma 1.10(c). Since $-U$ is $T_{U,p}$-admissible we obtain by Proposition 1.6 that

$$(T_p)_V = \big((T_{U,p})_{-U}\big)_V = \big((T_{U,p})_V\big)_{-U}.$$

Now we combine the above two equalities and conclude by Lemma 1.10(a) and (b) that $T_{V,2}$ and $(T_p)_V$ are consistent. □

Proposition 1.19 leads to the following definition. Recall that τ is a densely defined sesquilinear form fulfilling the first Beurling-Deny criterion.

Definition 1.20. Let $p \in [1, \infty)$. We say that τ is associated with a positive C_0-semigroup T_p on $L_p(\mu)$, $\tau \leftrightarrow T_p$ on $L_p(\mu)$ for short, if the following holds:

There exists $U \geqslant 0$ such that $Q(U) \supseteq D(\tau)$, $\tau + U$ is sectorial and closable, the positive C_0-semigroup $T_{U,2}$ on $L_2(\mu)$ associated with $\overline{\tau + U}$ extrapolates to a C_0-semigroup $T_{U,p}$ on $L_p(\mu)$, $-U$ is $T_{U,p}$-admissible, and $T_p = (T_{U,p})_{-U}$.

According to Proposition 1.19, the semigroup T_p is uniquely determined by the form τ. If τ itself is sectorial and closable, we can choose $U = 0$. In the context of forms fulfilling the first Beurling-Deny criterion, the above definition is thus an extension of the corresponding definition of the analytic semigroup on $L_2(\mu)$ associated with a closed sectorial form in $L_2(\mu)$.

The following result is a generalisation of Lemma 1.18.

Proposition 1.21. *Let $p \in [1, \infty)$ and assume that τ is associated with a positive C_0-semigroup T_p on $L_p(\mu)$. Let $U \geqslant 0$ be measurable, $Q(U) \supseteq D(\tau)$, and $\tau + U$ is sectorial and closable. If $V \geqslant 0$ is $(\tau + U)$-regular then V is T_p-regular, and $\tau + V \leftrightarrow (T_p)_V$.*

Proof. First assume that $V \geqslant U$. Then $\tau + V$ is a closable sectorial form. Let $T_{V,2}$ be the C_0-semigroup associated with $\overline{\tau + V}$. By Proposition 1.19 we obtain that $T_{V,2}$ extrapolates to a C_0-semigroup $T_{V,p}$ on L_p, $(T_{V,p})_{-V} = T_p$, and V is T_p-regular. Lemma 1.9 implies that $T_{V,p} = (T_p)_V$, i.e., $\tau + V \leftrightarrow (T_p)_V$.

In the general case we apply the above argument to $U + V$ in place of V. We conclude that $(\tau + V) + U \leftrightarrow (T_p)_{U+V}$ and that $U + V$ is T_p-regular. Thus, V is T_p-regular, by [Voi88; Prop. 3.3(a)]. Moreover, $-U$ is $(T_p)_{U+V}(\cdot)$-admissible and $\big((T_p)_{U+V}\big)_{-U} = (T_p)_V$ by [Voi88; Thm. 3.4]. Hence $\tau + V \leftrightarrow (T_p)_V$. □

The next proposition deals with consistent semigroups and the adjoint semigroup. Part (a) is similar to Corollary 1.11.

Proposition 1.22. *Let $p \in [1, \infty)$ and assume that τ is associated with a positive C_0-semigroup T_p on $L_p(\mu)$.*

(a) *Let $q \in [1, \infty)$ and T_q a positive C_0-semigroup on $L_q(\mu)$. Then τ is associated with T_q if and only if T_p, T_q are consistent. In this case, τ is associated with a family of consistent C_0-semigroups T_s on $L_s(\mu)$, $s \in [p \wedge q, p \vee q]$.*

(b) *If $p > 1$ and T_p^* denotes the adjoint semigroup on $L_{p'}(\mu)$ then the form τ^* is associated with T_p^*.*

Note that, since T_p is a real semigroup, it makes no difference whether the adjoint semigroup is taken with respect to the bilinear or with respect to the sesquilinear duality bracket.

Proof of Proposition 1.22. Let $U \geqslant 0$ be such that $Q(U) \supseteq D(\tau)$, $\tau + U$ is sectorial and closable, the positive C_0-semigroup $T_{U,2}$ on $L_2(\mu)$ associated with $\overline{\tau + U}$ extrapolates to a C_0-semigroup $T_{U,p}$ on $L_p(\mu)$, $-U$ is $T_{U,p}$-admissible, and $T_p = (T_{U,p})_{-U}$.

(a) Assume that τ is associated with T_q. Then Proposition 1.19 implies that $T_{U,2}$ extrapolates to a positive C_0-semigroup $T_{U,q}$ on $L_q(\mu)$, that $-U$ is $T_{U,q}$-admissible, and $T_q = (T_{U,q})_{-U}$. The semigroups $T_{U,p}$, $T_{U,q}$ are consistent, so T_p, T_q are consistent by Lemma 1.10(b).

Conversely, assume that T_p, T_q are consistent. Then $(T_p)_U$, $(T_q)_U$ are consistent by Lemma 1.10(a). From Lemma 1.9 we know that

$$(T_p)_U = \big((T_{U,p})_{-U}\big)_U = T_{U,p}.$$

Since $T_{U,2}$, $T_{U,p}$ are consistent, we conclude that $T_{U,2}$ extrapolates to the semigroup $(T_q)_U$ on $L_q(\mu)$. The potential U is T_p-regular by Proposition 1.19 and hence T_q-regular by Lemma 1.10(c). Thus, $-U$ is $(T_q)_U$-admissible, and $\big((T_q)_U\big)_{-U} = T_q$. By Definition 1.20 this shows that τ is associated with T_q.

The last assertion now follows by Riesz-Thorin interpolation.

(b) It is easy to see that $\tau^* + U$ is closable, fulfils the first Beurling-Deny criterion, and that $\overline{\tau^* + U} = \overline{\tau + U}^*$. Thus, $\overline{\tau^* + U}$ is associated with the positive C_0-semigroup $T^*_{U,2}$ which in turn extrapolates to the semigroup $T^*_{U,p}$ on $L_{p'}(\mu)$. Moreover, $\big((T^*_{U,p})_{-U\wedge n}\big)_{n\in\mathbb{N}}$ is an increasing sequence of semigroups, and

$$(T^*_{U,p})_{-U\wedge n} = \big((T_{U,p})_{-U\wedge n}\big)^* \to T^*_p \quad \text{weakly as } n \to \infty$$

since $(T_{U,p})_{-U\wedge n} \to T_p$. We deduce that $(T^*_{U,p})_{-U\wedge n} \to T^*_p$ strongly as $n \to \infty$. Hence, $-U$ is $T^*_{U,p}$-admissible and $(T^*_{U,p})_{-U} = T^*_p$, i.e., τ^* is associated with T^*_p. □

The following corollary shows that, in the case of symmetric forms, Definition 1.20 does not lead to new situations in which τ can be associated with a C_0-semigroup on $L_p(\mu)$.

Corollary 1.23. *Let τ be symmetric.*

(a) *The form τ is associated with a C_0-semigroup on $L_2(\mu)$ if and only if τ is bounded below and closable.*

(b) *If τ is associated with a C_0-semigroup on $L_p(\mu)$, for some $p \in [1,2)$, then τ is associated with a C_0-semigroup on $L_q(\mu)$ for all $q \in [p,p']$, $q \neq \infty$.*

Proof. (a) The "if" part is clear, so we prove the "only if" part. Let $U \geqslant 0$ be such that $Q(U) \supseteq D(\tau)$, $\tau + U$ is sectorial and closable, and $-U$ is admissible with respect to the positive C_0-semigroup $T_{U,2}$ associated with $\overline{\tau + U}$. Then, by Proposition 1.15(b), we have $U \leqslant \overline{\tau + U} + c$ for some $c \in \mathbb{R}$. This implies $\tau \geqslant -c$ since $Q(U) \supseteq D(\tau)$. Since $\tau + U$ is closable we obtain by Corollary 1.16 that τ is closable.

(b) This is a direct consequence of Proposition 1.22. □

For the last result of this section recall that I is an ideal of a lattice X if $u \in I,\ v \in X,\ |v| \leqslant |u|$ implies $v \in I$.

Lemma 1.24. *Let τ be sectorial and closable, $U \geqslant 0$ measurable.*

(a) *Let $D \subseteq D(\tau)$ be a dense ideal. Then $D \cap Q(U)$ is dense in $D(\tau + U)$.*

(b) *If $V \geqslant 0$ is τ-regular then V is $(\tau + U)$-regular. In particular, if U, V are τ-regular then $U + V$ is τ-regular.*

Proof. Without restriction, τ is symmetric and $\tau \geqslant 0$.

(a) Since $\tau + U$ fulfils the first Beurling-Deny criterion it suffices to consider $0 \leqslant u \in D(\tau + U)$. Let $(u_n) \subseteq D$ such that $u_n \to u$ in $D(\tau)$ as $n \to \infty$. Let $v_n := (\operatorname{Re} u_n)^+$. Then $v_n \in D$, and $v_n \to u$ in $L_2(\mu)$. Since τ fulfils the first Beurling-Deny criterion we have $\limsup_{n\to\infty} \tau(v_n) \leqslant \lim_{n\to\infty} \tau(u_n) = \tau(u)$.

From the lower semicontinuity of τ we deduce that $v_n \to u$ in $D(\tau)$ as $n \to \infty$. Moreover, $\tau((u - v_n)^+) \leqslant \tau(u - v_n) \to 0$ and thus $v_n \wedge u = u - (u - v_n)^+ \to u$ in $D(\tau)$ as $n \to \infty$. Finally, $v_n \wedge u \in D \cap Q(U)$, and $v_n \wedge u \to u$ in $Q(U)$ by dominated convergence.

(b) Just apply (a) to $D = D(\tau + V)$. □

1.5 Dirichlet forms

In this section we introduce the notions of sub-Markovian semigroups and (non-symmetric) Dirichlet forms, the main source being [MaRö92]. For this thesis, the main example of a Dirichlet form is the form corresponding to a homogeneous second order elliptic differential operator with real coefficients—the principal part of the type of elliptic operators we investigate in Chapter 3. At the end of the section we present a perturbation result for symmetric sub-Markovian semigroups which is due to V. Liskevich and Yu. Semenov.

Let (Ω, μ) be a measure space. An operator $B \in \mathcal{L}(L_2(\mu))$ is called *sub-Markovian* if B is positive and *L_∞-contractive*, i.e., $\|Bf\|_\infty \leqslant \|f\|_\infty$ for all $f \in L_2(\mu) \cap L_\infty(\mu)$. It is easy to see that B is sub-Markovian if and only if $f \leqslant 1$ a.e. implies that $Bf \leqslant 1$ a.e. for all $f \in L_2(\mu)$: for the "if" part note that $f \geqslant 0$ is equivalent to $-nf \leqslant 1$ for all $n \in \mathbb{N}$.

A C_0-semigroup T on $L_2(\mu)$ is called sub-Markovian if all semigroup operators $T(t)$ are sub-Markovian. The next result gives a characterisation of this property for the case that T is associated with a closed sectorial form $\tau \geqslant 0$. We state the result in a core version which we did not find in the literature, so we include a proof. We write $D(\tau)_r$ for the set of real-valued elements of $D(\tau)$.

Proposition 1.25. *(cf. [MaRö92; Prop. I.4.3, Thm. I.4.4]) Let τ be a densely defined closed sectorial form in $L_2(\mu)$, $\tau \geqslant 0$, and T the associated analytic semigroup on $L_2(\mu)$. Then T is sub-Markovian if and only if τ is real and*

$$u \wedge 1 \in D(\tau),\ \tau(u \wedge 1, (u-1)^+) \geqslant 0 \quad (u \in D) \tag{1.4}$$

for some dense subset D of $D(\tau)_r$.

Proof. If T is sub-Markovian then τ is real by Proposition 1.13, and (1.4) follows from [MaRö92; Prop. I.4.3 (ii) ⇒ (i), Thm. I.4.4 (iv) ⇒ (i)], for $D = D(\tau)$.

Conversely, assume that τ is real and that (1.4) holds for a dense subset D of $D(\tau)_r$. We will show that

$$(u-1)^+ \in D(\tau),\ \tau(u, (u-1)^+) \geqslant 0 \quad (u \in D(\tau)_r);$$

then it follows that T is sub-Markovian, by [MaRö92; Prop. I.4.3 (iii) ⇒ (ii)]. Let first $u \in D$. Note that $u = u \wedge 1 + (u-1)^+$ (this in particular implies that $(u-1)^+ \in D(\tau)$ in (1.4)), hence

$$\tau(u, (u-1)^+) = \tau(u \wedge 1, (u-1)^+) + \tau((u-1)^+) \geqslant \tau((u-1)^+) \geqslant 0.$$

From this we also obtain, by the sectoriality of τ (see [Kat80; Sec. VI.3, (1.31)]), that there exists $c \geqslant 1$ such that

$$\begin{aligned}\|(u-1)^+\|_\tau^2 &= \tau((u-1)^+) + \|(u-1)^+\|_2^2 \\ &\leqslant (\tau+1)(u,(u-1)^+) \leqslant c\|u\|_\tau \|(u-1)^+\|_\tau.\end{aligned}$$

Therefore, $\|(u-1)^+\|_\tau \leqslant c\|u\|_\tau$ for all real-valued $u \in D$.

Let now $u \in D(\tau)_r$. Let $(u_n) \subseteq D$ such that $u_n \to u$ in $D(\tau)$. Then

$$(u_n - 1)^+ \in D(\tau),\ \tau(u_n, (u_n-1)^+) \geqslant 0$$

for all $n \in \mathbb{N}$. Thus, it remains to show $(u-1)^+ \in D(\tau)$ and $(u_n-1)^+ \to (u-1)^+$ weakly in $D(\tau)$. This in turn follows from $(u_n-1)^+ \to (u-1)^+$ in $L_2(\mu)$, $\sup_{n\in\mathbb{N}} \|(u_n-1)^+\|_\tau \leqslant c \sup_{n\in\mathbb{N}} \|u_n\|_\tau < \infty$ and the lower semicontinuity of $\operatorname{Re}\tau$. □

The above proposition leads to the following definition.

Definition 1.26. A sesquilinear form τ in $L_2(\mu)$ is called *Dirichlet form* if τ is densely defined, real, sectorial and closed, $\operatorname{Re}\tau \geqslant 0$ and

$$u \wedge 1 \in D(\tau),\ \tau(u\wedge 1, (u-1)^+) \geqslant 0,\ \tau((u-1)^+, u\wedge 1) \geqslant 0 \quad \big(u \in D(\tau)_r\big),$$

or equivalently, τ and τ^* fulfil condition (1.4) for some dense subset D of $D(\tau)_r$.

Observe that $\operatorname{Re}\tau$ is a Dirichlet form if τ is a Dirichlet form; but it is easy to show that the converse is not true.

By duality and interpolation we immediately obtain the fundamental result about Dirichlet forms.

Theorem 1.27. *Let τ be a densely defined closed sectorial form in $L_2(\mu)$, T the associated analytic semigroup on $L_2(\mu)$. Then τ is a Dirichlet form if and only if T is positive and L_p-contractive for all $1 \leqslant p \leqslant \infty$.*

Now we introduce our main example of a Dirichlet form which will be the starting point in Chapter 3. Let $N \in \mathbb{N}$, $\varnothing \neq \Omega \subseteq \mathbb{R}^N$ an open set and $a\colon \Omega \to \mathbb{R}^N \otimes \mathbb{R}^N$ a measurable matrix-valued function. If $a \in L_{1,loc}$, i.e., $a_{jk} \in L_{1,loc}(\Omega)$ for all $1 \leqslant j,k \leqslant N$, then we can define a sesquilinear form τ in $L_2(\Omega)$ by

$$\tau(u,v) := \langle a\nabla u, \nabla v\rangle, \quad D(\tau) := C_c^\infty(\Omega).$$

Here, $\langle f, g\rangle$ is defined as $\int_\Omega f(x)\cdot\overline{g}(x)\,dx$ whenever $f\cdot\overline{g} \in L_1(\Omega)$, for $f, g\colon \Omega \to \mathbb{C}^N$ measurable.

We say that a is *sectorial (with constant $\alpha \geqslant 0$)* if

$$\big|\operatorname{Im}(a\zeta\cdot\overline{\zeta})\big| \leqslant \alpha \operatorname{Re}(a\zeta\cdot\overline{\zeta}) \quad \text{a.e. for all } \zeta \in \mathbb{C}^N,$$

or equivalently, $a\zeta\cdot\overline{\zeta} \in \overline{S_{\arctan\alpha}}$ a.e. for all $\zeta \in \mathbb{C}^N$ (here, $S_0 := (0,\infty)$). Obviously, τ is sectorial if a is sectorial. By [Vog00] the converse is also true.

In the following we assume that a is sectorial. Define the sectorial form τ_N in $L_2(\Omega)$ by

$$\tau_N(u,v) := \langle a\nabla u, \nabla v\rangle, \quad D(\tau_N) := \big\{u \in W^1_{1,loc}(\Omega) \cap L_2(\Omega);\, a\nabla u\cdot\nabla\overline{u} \in L_1(\Omega)\big\}.$$

The index N indicates that the associated sectorial operator in $L_2(\Omega)$ corresponds to Neumann boundary conditions (in case τ_N is a densely defined closed sectorial form).

In order to see that τ_N is defined on $D(\tau_N)$, i.e., $a\nabla u \cdot \nabla\overline{v} \in L_1(\Omega)$ for all $u, v \in D(\tau_N)$, let $a_s := \frac{a+a^\top}{2}$ denote the *symmetric part* of a. By the sectoriality of a we obtain that

$$D(\tau_N) = \big\{u \in W^1_{1,loc}(\Omega) \cap L_2(\Omega);\, a_s\nabla u \cdot \nabla\overline{u} \in L_1(\Omega)\big\}.$$

Moreover, by [Kat80; Sec. VI.3, (1.31)], there exists $c \geqslant 1$ such that

$$|a\xi\cdot\eta| \leqslant c(a_s\xi\cdot\overline{\xi})^{\frac12}(a_s\eta\cdot\overline{\eta})^{\frac12} \quad \big(\xi,\eta \in \mathbb{C}^N\big). \tag{1.5}$$

Thus,

$$|a\nabla u\cdot\nabla\overline{v}| \leqslant c(a_s\nabla u\cdot\nabla\overline{u})^{\frac12}(a_s\nabla v\cdot\nabla\overline{v})^{\frac12} \in L_1(\Omega) \quad \big(u, v \in D(\tau_N)\big),$$

i.e., τ_N is defined on $D(\tau_N)$.

The matrix function a is called *uniformly elliptic* if $a \in L_\infty$, i.e., $a_{jk} \in L_\infty(\Omega)$ for all $1 \leqslant j, k \leqslant N$, and there exists $\varepsilon > 0$ such that $a \geqslant \varepsilon$ a.e., i.e.,

$$a\xi\cdot\xi = \sum_{j,k=1}^N a_{jk}\xi_j\xi_k \geqslant \varepsilon|\xi|^2 \quad \text{a.e. for all } \xi \in \mathbb{R}^N.$$

It is standard (see, e.g., [Dav89; Thm. 1.3.9]) that τ_N is a Dirichlet form if a is symmetric and uniformly elliptic. But we are going to study much more general cases. First observe that, by the chain rule (cf. [BoMu82; Thm. 4.2]), $\nabla(u \wedge 1) = \chi_{[u<1]}\nabla u$ and $\nabla(u-1)^+ = \chi_{[u>1]}\nabla u$ for all $u \in W^1_{1,loc}(\Omega)$. This implies that $u \wedge 1 \in D(\tau_N)$ and $\tau\big(u\wedge 1, (u-1)^+\big) = \tau\big((u-1)^+, u\wedge 1\big) = 0$ for all real-valued $u \in D(\tau_N)$. Moreover, it is easy to see that τ_N is real, and $\operatorname{Re}\tau_N \geqslant 0$ since a is sectorial. Therefore, τ_N is a Dirichlet form as soon as τ_N is densely defined and closed. More generally, if τ_N is densely defined and closable then $\overline{\tau_N}$ is a Dirichlet form.

If $a \in L_{1,loc}$ then $C_c^\infty(\Omega) \subseteq D(\tau_N)$; in particular, τ_N is densely defined in this case. The following result is an easy criterion guaranteeing that τ_N is closed.

Proposition 1.28. *(cf. [RöWi85; Thm. 3.2]) Assume that a is sectorial, a.e. invertible, and $a^{-1} \in L_{1,loc}$. Then the form τ_N defined above is sectorial and closed. In particular, if τ_N is densely defined then it is a Dirichlet form.*

In the proof of this proposition we will make use of the next lemma which shows that the assumption a invertible, $a^{-1} \in L_{1,loc}$ is equivalent to a_s invertible, $a_s^{-1} \in L_{1,loc}$. For a matrix $a \in \mathbb{R}^N \otimes \mathbb{R}^N$ we write $|a|$ to denote the norm of the operator a in Euclidean space $\mathbb{R}^N$. Observe that $a_s \geqslant 0$ if a is sectorial, so the square root $a_s^{1/2}$ of a_s exists and we obtain

$$\operatorname{Re}(a\zeta \cdot \overline{\zeta}) = a_s\zeta \cdot \overline{\zeta} = |a_s^{1/2}\zeta|^2 \quad (\zeta \in \mathbb{C}^N). \tag{1.6}$$

Lemma 1.29. *Let $a \in \mathbb{R}^N \otimes \mathbb{R}^N$ be sectorial with constant α. Then $|a| \leqslant (\alpha + 1)|a_s|$, and a is invertible if and only if a_s is invertible. If a is invertible then a^{-1} is sectorial with constant α, and*

$$\begin{aligned} a_s &\leqslant a^\top a_s^{-1} a \leqslant (1+\alpha^2) a_s, \\ (a^{-1})_s &\leqslant a_s^{-1} \leqslant (1+\alpha^2)(a^{-1})_s. \end{aligned} \tag{1.7}$$

Proof. Let $\check{a} := a - a_s$ denote the antisymmetric part of a. The sector condition implies that

$$|\check{a}\xi \cdot \eta| \leqslant \alpha |a_s^{1/2}\xi| \cdot |a_s^{1/2}\eta| \quad \big(\xi, \eta \in \mathbb{R}^N\big)$$

(cf. (1.5)). From this we deduce

$$|\check{a}\xi| \leqslant \alpha |a_s^{1/2}||a_s^{1/2}\xi| \quad (\xi \in \mathbb{R}^N).$$

In particular, $|\check{a}| \leqslant \alpha |a_s^{1/2}||a_s^{1/2}| = \alpha |a_s|$ and thus $|a| \leqslant (\alpha+1)|a_s|$.

In order to show the second assertion, assume that a_s is invertible. Then $a\xi \cdot \xi = |a_s^{1/2}\xi|^2 > 0$ and hence $a\xi \neq 0$ for all $0 \neq \xi \in \mathbb{R}^N$, i.e., a is invertible. Conversely, if a_s is not invertible then there exists $0 \neq \xi \in \mathbb{R}^N$ such that $|a_s^{1/2}\xi|^2 = 0$. By the above, $|\check{a}\xi| \leqslant \alpha |a_s^{1/2}||a_s^{1/2}\xi| = 0$ and thus $a\xi = a_s^{1/2}(a_s^{1/2}\xi) + \check{a}\xi = 0$, i.e., a is not invertible.

Let now a be invertible. Then the sector condition implies that

$$a^{-1}\zeta \cdot \overline{\zeta} = a(a^{-1}\overline{\zeta}) \cdot \overline{(a^{-1}\overline{\zeta})} \in \overline{S_{\arctan \alpha}} \quad (\zeta \in \mathbb{C}^N),$$

i.e., a^{-1} is sectorial with constant α. Since $\check{a}$ is antisymmetric, we have

$$a^\top a_s^{-1} a = (a_s + \check{a}^\top) a_s^{-1}(a_s + \check{a}) = a_s + \check{a}^\top + \check{a} + \check{a}^\top a_s^{-1}\check{a} = a_s + \check{a}^\top a_s^{-1}\check{a}.$$

Observe that $\check{a}^\top a_s^{-1}\check{a} = (a_s^{-1/2}\check{a})^\top(a_s^{-1/2}\check{a}) \geqslant 0$. Thus, (1.7) is equivalent to $|a_s^{-1/2}\check{a}\xi| \leqslant \alpha |a_s^{1/2}\xi|$ for all $\xi \in \mathbb{R}^N$. This in turn follows from

$$|a_s^{-1/2}\check{a}\xi \cdot \eta| = |\check{a}\xi \cdot a_s^{-1/2}\eta| \leqslant \alpha |a_s^{1/2}\xi| \cdot |\eta| \quad (\xi, \eta \in \mathbb{R}^N).$$

The last assertion is a direct consequence of (1.7), the identity $a^\top (a^{-1})_s a = a_s$, and the following elementary observation. Let $b, c \in \mathbb{R}^N \otimes \mathbb{R}^N$ be symmetric. Then

$$b \leqslant c \quad \text{if and only if} \quad a^\top b a \leqslant a^\top c a$$

since a is invertible. $\square$

By the functional calculus one can show that for a continuous function $f\colon \mathbb{R} \to \mathbb{C}$ the mapping $B \mapsto f(B)$, in the space of all selfadjoint operators $B \in \mathcal{L}(\mathbb{R}^N)$, is continuous. We infer that the function $x \mapsto a_s(x)^{1/2}$ on Ω is measurable. By (1.6) we obtain that

$$D(\tau_N) = \left\{u \in W^1_{1,loc}(\Omega) \cap L_2(\Omega);\, a_s^{1/2}\nabla u \in L_2(\Omega)^N\right\}$$

and that the norm in $D(\tau_N)$ is given by $\|u\|^2_{\tau_N} = \|u\|_2^2 + \|a_s^{1/2}\nabla u\|_2^2$.

Proof of Proposition 1.28. First we show $a_s^{-1/2} \in L_{2,loc}$. Observe that $a_s^{-1} \in L_{1,loc}$ if and only if $|a_s^{-1}| \in L_{1,loc}$. Since $|a_s^{-1/2}| = |a_s^{-1}|^{1/2}$ we obtain that $a_s^{-1/2} \in L_{2,loc}$ if and only if $a_s^{-1} \in L_{1,loc}$. The latter holds by Lemma 1.29.

We only have to show that $D(\tau_N)$ is complete. Let (u_n) be a Cauchy sequence in $D(\tau_N)$. Then $u_n \to u$ in $L_2(\Omega)$ and $a_s^{1/2}\nabla u_n \to f$ in $L_2(\Omega)^N$ for some $u \in L_2(\Omega)$, $f \in L_2(\Omega)^N$. Since $a_s^{-1/2} \in L_{2,loc}$ we obtain $\nabla u_n \to a_s^{-1/2} f$ in $L_{1,loc}(\Omega)^N$. This implies that $u \in W^1_{1,loc}(\Omega)$, $a_s^{1/2}\nabla u = f \in L_2(\Omega)$. Therefore, $u \in D(\tau_N)$ and $u_n \to u$ in $D(\tau_N)$ as $n \to \infty$. □

Let the assumptions of Proposition 1.28 hold. If $a \in L_{1,loc}$ then we can define the form $\tau_D := \overline{\tau_N\restriction_{C_c^\infty(\Omega)}}$. The index D indicates that the associated sectorial operator in $L_2(\Omega)$ corresponds to Dirichlet boundary conditions.

Proposition 1.30. *Assume that a is sectorial, a.e. invertible, and $a, a^{-1} \in L_{1,loc}$. Then $W^1_{\infty,c}(\Omega) \subseteq D(\tau_D)$, and τ_D is a Dirichlet form.*

Proof. The first assertion follows from a standard convolution argument, using $a \in L_{1,loc}$ and the lower semicontinuity of $\operatorname{Re}\tau_D$ (see [LiVo00; proof of Lemma B4(i)]). By Proposition 1.28, τ_N is a Dirichlet form. Thus, for the second assertion it suffices to show $u \wedge 1 \in D(\tau_D)$ for all $u \in C_c^\infty(\Omega)$. This follows from the first assertion. □

For the remainder of the section, (Ω, μ) will be a measure space. Recall that a function $\varphi\colon \mathbb{C} \to \mathbb{C}$ is called a *normal contraction* if $\varphi(0) = 0$ and $|\varphi(x) - \varphi(y)| \leqslant |x - y|$ for all $x, y \in \mathbb{C}$, i.e., φ is Lipschitz continuous with constant 1. Then we will also say, for all measurable $u\colon \Omega \to \mathbb{C}$, that the function $\varphi \circ u$ is a *normal contraction of u*. The crucial result on normal contractions is as follows (recall that $\operatorname{Re}\tau$ is a symmetric Dirichlet form if τ is a Dirichlet form).

Proposition 1.31. *(cf. [ReSi78; Thm. XIII.51]) Let τ be a Dirichlet form in $L_2(\mu)$. If $u \in D(\tau)$ and v is a normal contraction of u then $v \in D(\tau)$ and* $\operatorname{Re}\tau(v) \leqslant \operatorname{Re}\tau(u)$.

We conclude this section by a result on perturbation of Dirichlet forms by real-valued potentials which is essentially due to V. Liskevich and Yu. Semenov.

Theorem 1.32. *(cf. [LiSe93; Thm. 2], [LiSe96; Thm. 3.2]) Let τ be a symmetric Dirichlet form in $L_2(\mu)$, $V\colon \Omega \to \mathbb{R}$ measurable. Assume that $\tau + V^+$ is densely defined, and $V^- \leqslant \beta\tau + V^+ + c_\beta$ for some $\beta < 1$, $c_\beta \in \mathbb{R}$. Let $p_\pm := \frac{2}{1\mp\sqrt{1-\beta}}$ (the roots of the equation $\frac{4}{pp'} = \beta$).*

(a) *Then $\tau + V$ is a densely defined closable symmetric form, and for all $p \in [p_-, p_+]$ the associated analytic semigroup $T_{V,2}$ on $L_2(\mu)$ extrapolates to a positive C_0-semigroup $T_{V,p}$ on $L_p(\mu)$, with $\|T_{V,p}(t)\| \leqslant e^{c_\beta t}$ $(t \geqslant 0)$. For $p \in (p_-, p_+)$, the semigroup $T_{V,p}$ is analytic.*

(b) *For the generator $-A_{V,p}$ of $T_{V,p}$ we have*

$$\big\langle A_{V,p}u, |u|^{p-1}\operatorname{sgn} u\big\rangle \geqslant \overline{\big(\tfrac{4}{pp'}\tau + V\big)}(|u|^{\frac{p}{2}}\operatorname{sgn} u) \quad \big(u \in D(A_{V,p})\big).$$

In particular, for $p \in (p_-, p_+)$ we obtain $|u|^{\frac{p}{2}}\operatorname{sgn} u \in D(\tau)$ for all $u \in D(A_{V,p})$, and

$$\big\langle A_{V,p}u, |u|^{p-1}\operatorname{sgn} u\big\rangle \geqslant \big(\tfrac{4}{pp'} - \beta\big)\tau(|u|^{\frac{p}{2}}\operatorname{sgn} u) - c_\beta\|u\|_p^p.$$

Remarks 1.33. (a) In [LiSe93], [LiSe96], the above theorem is proved in the more general setting of perturbation by sub-Markovian generators, not only by potentials. But the assumption on the perturbation is slightly more restrictive, namely (expressed for perturbation by a potential) $V_- \leqslant \beta(\tau + V_+) + c_\beta$, with $V_-, V_+\colon \Omega \to [0,\infty)$ such that $V = V_+ - V_-$.

(b) Note that the inequality $V^- \leqslant \beta\tau + V^+ + c_\beta$ obviously implies $\beta\tau + V + c_\beta \geqslant 0$, but the converse is not true: For example, choose $V \leqslant 0$ such that $D(\tau + V) = \{0\}$. Then trivially $\tau + V \geqslant 0$, but $V^- \not\leqslant \tau$. Nevertheless we have the following equivalence: $V^- \leqslant \beta\tau + V^+ + c_\beta$ if and only if $\beta\tau + V + c_\beta \geqslant 0$ and $Q(V^-) \supseteq D(\tau) \cap Q(V^+)$.

(c) For $p \in [1,\infty)$, let T_p be the positive C_0-semigroup on $L_p(\mu)$ associated with τ. By Proposition 1.15 we have $T_{V,2} = \big((T_2)_{V^+}\big)_{-V^-}$. Thus, by Corollary 1.11, part (a) of the theorem can be expressed differently: V^- is $(T_p)_{V^+}$-admissible for all $p \in [p_-, p_+]$. (But V^- is *not* T_p-admissible in general!) By Proposition 1.22(a) we obtain yet another reformulation of part (a): for all $p \in [p_-, p_+]$, the form $\tau + V$ is associated with a quasi-contractive C_0-semigroup on L_p.

(d) The following trivial example shows that the interval $[p_-, p_+]$ obtained in Theorem 1.32 is not always significant. Let $U\colon \Omega \to [0,\infty)$ be measurable and unbounded, $\tau := U$, $V := -\frac{1}{2}U$. Then $V^- \leqslant \beta\tau + c$ holds only if $\beta \geqslant \frac{1}{2}$. But $\tau + V = \frac{1}{2}U$ is associated with a contractive C_0-semigroup on L_p for all $p \in [1,\infty)$.

Nevertheless, we will show in Section 3.4 that the interval $[p_-, p_+]$ is sharp if τ is the form corresponding to a second order elliptic differential operator: let $\Omega \subseteq \mathbb{R}^N$ be open, $a\colon \Omega \to \mathbb{R}^N \otimes \mathbb{R}^N$ measurable and symmetric, $a \geqslant 0$ a.e. in the matrix sense, $a, a^{-1} \in L_{1,loc}$. Let τ_N be the symmetric Dirichlet form given by Proposition 1.28, and $\tau \subseteq \tau_N$ a Dirichlet form. In Theorem 3.22 we will show that $\tau + V$ is associated with a quasi-contractive C_0-semigroup on L_p if and only if $V^- \leqslant \frac{4}{pp'}\tau + V^+ + c$ for some $c \in \mathbb{R}$. For a particular example see Example 3.27.

(e) In Section 2.2 we will show that, under some additional conditions on the measure space and the semigroup, the L_p-scale $[p_-, p_+]$ of existence of the semigroup can be extended, but in general without quasi-contractivity outside $[p_-, p_+]$. The proof of this extension result (Theorem 2.10) will reveal the relevance of Theorem 1.32(b).

The proof of Theorem 1.32 relies on [LiSe96; Thm. 2.1] and the following auxiliary result which is an L_p-version of Proposition 1.15. Here we use the notation $v_p(u) := |u|^{\frac{p}{2}} \operatorname{sgn} u$ and $w_p(u) := |u|^{p-1} \operatorname{sgn} u$ for $u\colon \Omega \to \mathbb{C}$, $1 < p < \infty$. Notice that $u \cdot \overline{w_p(u)} = v_p(u) \cdot \overline{v_p(u)} = |u|^p$.

Lemma 1.34. *Let $1 < p < \infty$, $T(t) = e^{-tA}$ a positive contractive C_0-semigroup on $L_p(\mu)$, and $\mathfrak{h}$ a closed symmetric form in $L_2(\mu)$. Assume that*

$$\big\langle Au, w_p(u)\big\rangle \geqslant \mathfrak{h}(v_p(u)) \quad \big(u \in D(A)\big).$$

Let $V\colon \Omega \to \mathbb{R}$ be measurable, with T-admissible V^+, and $V^- \leqslant \mathfrak{h} + V^+$. Then $-V^-$ is T_{V^+}-admissible, $e^{-tA_V} := (T_{V^+})_{-V^-}(t)$ is contractive, and

$$\big\langle A_V u, w_p(u)\big\rangle \geqslant \overline{\mathfrak{h} + V}(v_p(u)) \quad \big(u \in D(A_V)\big). \tag{1.8}$$

Proof. Notice that $\mathfrak{h} + V$ is closable by Proposition 1.15. It suffices to study the cases $V \geqslant 0$, $V \leqslant 0$. Let first $V \geqslant 0$. Then we only have to show (1.8).

Let $u \in D(A_V)$. Without restriction assume that $0 \in \rho(A + V \wedge n)$ for all $n \in \mathbb{N}$. Then $u_n := (A + V \wedge n)^{-1} A_V u \to u$ in L_p and hence $w_p(u_n) \to w_p(u)$ in $L_{p'}$ as $n \to \infty$. Therefore,

$$(\mathfrak{h} + V \wedge n)(v_p(u_n)) \leqslant \big\langle (A + V \wedge n)u_n, w_p(u_n)\big\rangle \to \big\langle A_V u, w_p(u)\big\rangle \quad (n \to \infty).$$

Without restriction $v_p(u_n) \to v_p(u)$ a.e. and in L_2. By Fatou's lemma and the lower semicontinuity of $\mathfrak{h}$ we infer that $V|v_p(u)|^2 = \lim_{n\to\infty} |v_p(u_n)|^2 \in L_1(\mu)$, $v_p(u) \in D(\mathfrak{h})$ and

$$(\mathfrak{h} + V)(v_p(u)) \leqslant \big\langle A_V u, w_p(u)\big\rangle.$$

Now we study the case $V \leqslant 0$. Recall that $V^{(n)} = V \vee (-n)$. The assumption $V^- \leqslant \mathfrak{h}$ implies that

$$\big\langle (A + V^{(n)})u, w_p(u)\big\rangle \geqslant (\mathfrak{h} + V^{(n)})(v_p(u)) \geqslant 0 \quad \big(u \in D(A),\ n \in \mathbb{N}\big).$$

Thus, by the Lumer-Phillips theorem, $T_{V^{(n)}}$ is a contractive semigroup for all $n \in \mathbb{N}$. By [Voi88; Prop. 2.2] we infer that V is T-admissible. To show (1.8), let $u \in D(A_V)$ and assume without restriction that $0 \in \rho(A + V^{(n)})$ for all $n \in \mathbb{N}$. Then we have $u_n := (A+V^{(n)})^{-1} A_V u \to u$ in L_p as $n \to \infty$. Note that $u_n \in D(A)$ implies that $v_p(u_n) \in D(\mathfrak{h}) \subseteq D(\overline{\mathfrak{h} + V})$. Therefore,

$$\begin{aligned}\overline{\mathfrak{h} + V}(v_p(u_n)) &\leqslant (\mathfrak{h} + V^{(n)})(v_p(u_n)) \\ &\leqslant \big\langle (A + V^{(n)})u_n, w_p(u_n)\big\rangle \to \big\langle A_V u, w_p(u)\big\rangle \quad (n \to \infty),\end{aligned}$$

and the lower semicontinuity of $\overline{\mathfrak{h} + V}$ implies (1.8). ◻

Proof of Theorem 1.32. Without restriction assume that $c_\beta = 0$ (cf. Remark 1.33(b)). By Proposition 1.15, the symmetric form $\frac{4}{pp'}\tau + V$ is closable if $\frac{4}{pp'} \geqslant \beta$, i.e. $p \in [p_-, p_+]$. In particular, $\tau + V$ is closable.

Let $p \in [p_-, p_+]$, $T_p(t) = e^{-tA_p}$ the positive C_0-semigroup on $L_p(\mu)$ associated with τ. By [LiSe96; Thm. 2.1] we have

$$\big\langle A_p u, w_p(u)\big\rangle \geqslant \frac{4}{pp'}\tau(v_p(u)) \quad \big(u \in D(A_p)\big).$$

(In [LiSe96], this inequality was shown for σ-finite measures μ only, for the general case one should argue as in [NaVo96].) Now Lemma 1.34 implies that $-V^-$ is $(T_p)_{V^+}$-admissible, that $T_{V,p}$ is contractive and that (b) holds. The analyticity of $T_{V,p}$ for $p \in (p_-, p_+)$ follows from the analyticity of $T_{V,2}$ and Stein interpolation. $\square$

Chapter 2

Extrapolation, Analyticity, and L_p-spectral independence

Given a measure space (M, μ) and a C_0-semigroup T_q on $L_q(\Omega)$ for some measurable subset $\Omega \subseteq M$, $q \in [1, \infty)$, we are going to investigate the following two problems: under which conditions does the semigroup T_q extrapolate to a consistent family of C_0-semigroups T_p on $L_p(\Omega)$ with p-independent angle of analyticity, for p from some interval in $[1, \infty)$ containing q? Secondly, assuming T_q does extrapolate to consistent C_0-semigroups T_p on $L_p(\Omega)$ for p from some interval, when is the spectrum of the semigroup generators p-independent? The conditions on both the space M and the semigroup T_q will be formulated in terms of a measurable semi-metric on M.

The chapter is organised as follows. In Section 2.1 we introduce the framework and formulate our main results. In Section 2.2 we give some account to the history of the problems of L_p-spectral independence and analyticity of semigroups and relate our results to the existing literature. As an application we continue the study of perturbation of sub-Markovian semigroups by potentials. For our main application, the L_p-theory of second order elliptic differential operators, see Chapter 3.

In Section 2.3 we develop the technique of weighted estimates which constitutes a major tool in the proofs of our results. We reformulate and prove the result on extrapolation and analyticity in a more general form in Section 2.4. In Section 2.5 we prove the theorem on L_p-spectral independence for semigroup generators by reducing it to a theorem on L_p-spectral independence for bounded operators.

2.1 Framework and main results

Throughout this chapter let $1 \leqslant p_0 < q_0 \leqslant \infty$ be fixed, and (M, μ) a σ-finite measure space with $\mu(M) > 0$. Let d a measurable semi-metric on M, i.e., d: $M \times M \to [0, \infty)$ is measurable. Then $d(x, \cdot)$ is measurable for all $x \in M$

since the function $y \mapsto (x,y)$ is measurable. The open ball with respect to d with centre x and radius r will be denoted by $B(x,r)$. We assume $\mu(B(x,r)) < \infty$ for all $x \in M,\ r > 0$. In the case $(p_0, q_0) \neq (1,\infty)$ let $v_r(x) := \mu(B(x,r))$ $(x \in M,$ $r > 0)$, whereas in the case $(p_0, q_0) = (1, \infty)$ we only assume $v_r \colon M \to [0,\infty)$ to be measurable functions satisfying $\mu(B(x,r)) \leqslant v_r(x)$ for all $x \in M,\ r > 0$ and

$$v_r \leqslant v_R \text{ on } M \ (R > r > 0), \quad v_r(x) \leqslant v_{r+d(x,y)}(y) \ (x, y \in M,\ r > 0). \tag{2.1}$$

Note that (2.1) is automatically fulfilled if $v_r(x) = \mu(B(x,r))$ since $B(x,r) \subseteq B(y, r + d(x,y))$.

Fix a measurable subset $\Omega \subseteq M$. We tacitly assume that functions defined on Ω are extended by 0 outside Ω when considered as functions on M. In the following we consider semigroups on $L_p(\Omega),\ 1 \leqslant p < \infty$. The reason for introducing the space M is that the functions $\mu(B(\cdot, r))$ on M can behave much better than the functions $\mu(B(\cdot, r) \cap \Omega)$ on Ω. An important example for this situation is $M = \mathbb{R}^N$ and an open subset $\Omega \subseteq \mathbb{R}^N$.

For the problem of extrapolation and analyticity we will need two volume growth conditions,

$$v_r \leqslant c_0 e^{c_1 r} v_1 \text{ on } M \quad (r > 1), \tag{2.2}$$

$$v_{2r} \leqslant c_0 v_r \text{ on } M \quad (0 < r \leqslant \tfrac{1}{2}), \tag{2.3}$$

for some $c_0 \geqslant 1,\ c_1 > 0$. Condition (2.2) means that the volume of balls grows at most exponentially, condition (2.3) is the doubling property for small balls. The latter is known to be equivalent to

$$v_R \leqslant c_2 \left(\tfrac{R}{r}\right)^N v_r \quad (0 < r < R \leqslant 1) \tag{2.4}$$

for some $N > 0$. (In '(2.3) $\Longrightarrow$ (2.4)' one obtains $c_2 = c_0,\ N = \log_2 c_0$.)

In the case $M = \mathbb{R}^N,\ d$ the supremum metric (this will turn out to be convenient) and μ the Lebesgue measure, conditions (2.2) and (2.3) are trivially fulfilled with $v_r(x) = \mu(B(x,r)) = (2r)^N$. If M is a complete Riemannian manifold with Ricci curvature bounded below, d the Riemannian distance and μ the Riemannian volume, then (2.2) and (2.4) hold for $v_r(x) = \mu(B(x,r))$ and N the dimension of M, by Bishop's comparison principle (see, e.g., [GHL90; Thm. 4.19]). In the latter case, v_r is a function heavily depending on the space variable: in contrast to the flat space case it is not bounded below in general.

In order to formulate our main results we need the following notation. By means of the semi-metric d, we define weight functions $\rho_{\gamma,y}$ on M,

$$\rho_{\gamma,y}(x) := e^{-\gamma d(x,y)} \quad (x, y \in M,\ \gamma \in \mathbb{R}).$$

Let B be a linear operator in $L_1(M) + L_\infty(M)$, and $1 \leqslant p \leqslant q \leqslant \infty$. We denote the norm of B as an operator from L_p to L_q by

$$\|B\|_{p\to q} := \sup\bigl\{\|Bf\|_q;\ f \in L_p(M) \cap D(B),\ \|f\|_p \leqslant 1\bigr\} \in [0,\infty],$$

and for $\gamma \in \mathbb{R}$ we define the weighted operator norm

$$\begin{aligned}\|B\|_{p\to q,\gamma} &:= \sup_{y\in M} \|\rho_{\gamma,y} B \rho_{\gamma,y}^{-1}\restriction_{\{f\in L_p(M);\, \rho_{\gamma,y}^{-1} f\in D(B)\}}\|_{p\to q} \\ &= \inf\bigl\{c \geqslant 0;\ \forall\, f\in D(B),\ y\in M\colon \|\rho_{\gamma,y} Bf\|_q \leqslant c\|\rho_{\gamma,y} f\|_p\bigr\} \in [0,\infty].\end{aligned}$$

We will call an estimate of the type $\|B\|_{p\to q,\gamma} < \infty$ a *weighted $p{\to}q$-estimate* or *weighted norm estimate.*

Theorem 2.1. *Assume that (M,d) is separable and that* (2.2) *and* (2.3) *hold. Let $p_0 \leqslant s \leqslant q_0$ and T_s a C_0-semigroup on $L_s(\Omega)$. Assume that there exist $m > 1$, $t_0 > 0$ and $\alpha_0, \beta_0 \geqslant 0$ with $\alpha_0 + \beta_0 = p_0^{-1} - q_0^{-1}$ such that*

$$\sup_{0<t\leqslant t_0} \|v_{t^{1/m}}^{\alpha_0} T_s(t) v_{t^{1/m}}^{\beta_0}\|_{p_0\to q_0, t^{-1/m}} < \infty. \tag{2.5}$$

Then T_s extrapolates to a consistent family of C_0-semigroups T_p on $L_p(\Omega)$, $p \in [p_0, q_0] \setminus \{\infty\}$, with angle of analyticity not depending on p.

Moreover, there exist $C > 0$, $\omega \in \mathbb{R}$, $\nu \geqslant 0$ such that

$$\|v_{t^{1/m}}^{\alpha} T_s(t) v_{t^{1/m}}^{\beta}\|_{p\to q,\gamma} \leqslant C e^{\omega t + \nu\gamma^m t} \quad (t>0,\ \gamma \geqslant 0) \tag{2.6}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0$, $\alpha, \beta \geqslant 0$ with $\alpha + \beta = p^{-1} - q^{-1}$.

By the phrase 'angle of analyticity not depending on p' we mean the following. If one of the semigroups T_p is analytic of angle θ, then all of them are analytic of angle θ; if one of the semigroups is *not* analytic, then none of them is analytic. We point out that the above theorem contributes to the solution of *two* problems, extrapolation as well as analyticity of semigroups. So to say, it deals with extension of the L_p-scale as well as extension of the time scale.

For the case of Euclidean space $\mathbb{R}^N$ we immediately obtain, recalling $v_r = (2r)^N$:

Corollary 2.2. *Let $\Omega \subseteq \mathbb{R}^N$ be measurable, $p_0 \leqslant s \leqslant q_0$ and T_s a C_0-semigroup on $L_s(\Omega)$. Assume that there exist $m > 1_1$, $t_0, C > 0$ such that*

$$\|T_s(t)\|_{p_0\to q_0, t^{-1/m}} \leqslant C t^{-\frac{N}{m}(\frac{1}{p_0} - \frac{1}{q_0})} \quad (0 < t \leqslant t_0).$$

Then T_s extrapolates to a consistent family of C_0-semigroups T_p on $L_p(\Omega)$, $p \in [p_0, q_0] \setminus \{\infty\}$, with angle of analyticity not depending on p.

Remarks 2.3. (a) The set of estimates in (2.6) can be considered as a generalised Gaussian upper bound (cf. [Sch96; p. 44]). Notice that (2.6) trivially implies (2.5).

In the case $(p_0, q_0) = (1, \infty)$, estimate (2.6) with $p = 1$, $q = \infty$, $\alpha = \beta = \frac{1}{2}$ is equivalent to the following Gaussian upper bound of order m on the semigroup kernel:

$$|k_t(x,y)| \leqslant C\bigl(v_{t^{1/m}}(x) v_{t^{1/m}}(y)\bigr)^{-\frac{1}{2}} \exp\bigl(\omega t - c(\tfrac{d(x,y)^m}{t})^{\frac{1}{m-1}}\bigr) \quad (t > 0,\ x, y \in \Omega).$$

The equivalence will be shown by Davies' trick, see Proposition 2.7 below. P. Lie and S. T. Yau [LiYa86] proved that a Gaussian upper bound of order $m = 2$ holds for the heat semigroup on Riemannian manifolds with Ricci curvature bounded below.

(b) The measure μ is assumed to be σ-finite since this is a necessary condition for the property that all balls $B(x, r)$ have finite volume: for fixed $x_0 \in M$ we have $M = \bigcup_{n\in\mathbb{N}} B(x_0, n)$.

(c) Observe that (2.2) and (2.3) imply that $v_r > 0$ for all $r > 0$: recall that $\mu(M) > 0$. Thus, for all $x \in M$ there exists $R > 0$ such that $v_R(x) \geqslant \mu(B(x, R)) > 0$. By (2.2) and (2.3) we obtain $v_r(x) > 0$ for all $r > 0$. In the case $(p_0, q_0) \neq (1, \infty)$ we thus have $\mu(B(x, r)) > 0$ for all $x \in M,\ r > 0$.

(d) In Lemma 2.17 below we will show the following. (M, d) is separable as soon as $0 < \mu(B(x, r)) < \infty$ for all $x \in M,\ r > 0$. In particular, (M, d) is separable if (2.2) and (2.3) hold for $v_r(x) = \mu(B(x, r))$. Conversely, if M is separable then $\mu(B(x, r)) > 0$ for all $r > 0$ and almost all $x \in M$. After removing a null set we can (and do) therefore assume $\mu(B(x, r)) > 0$ for *all* $x \in M,\ r > 0$ also in the case $(p_0, q_0) = (1, \infty)$.

For the second result which deals with L_p-spectral independence, the exponential volume growth condition (2.2) is too weak. In fact, it is known that in the case of exponential volume growth the L_p-spectrum of the semigroup generators typically does depend on p (see, e.g., [Stu93; Prop. 2(b)]). Instead, we need the following subexponential volume growth condition

$$\forall \varepsilon > 0 \,\exists\, c_\varepsilon > 0 \,\forall\, r > 1 \colon v_r \leqslant c_\varepsilon e^{\varepsilon r} v_1, \tag{2.7}$$

as in [Stu93; p. 443]. We further assume

$$\mu(B(x, 1)) > 0 \quad (x \in M). \tag{2.8}$$

Observe that, if (2.7) holds then the latter condition is automatically fulfilled in the case $(p_0, q_0) \neq (1, \infty)$ (cf. Remark 2.3(c)). Moreover, if (M, d) is separable then (2.8) holds (without restriction, cf. Remark 2.3(d)). Recall that $v_r(x) = \mu(B(x, r))$ in the case $(p_0, q_0) \neq (1, \infty)$, whereas $\mu(B(x, r)) \leqslant v_r(x)$ in the case $(p_0, q_0) = (1, \infty)$, for all $x \in M,\ r > 0$.

Theorem 2.4. *Assume that* (2.7) *and* (2.8) *hold. Let* $T_p(t) = e^{-tA_p}$ *be consistent* C_0*-semigroups on* $L_p(\Omega)$, $p \in [p_0, q_0]\setminus\{\infty\}$*. Assume that there exist* $C, K, t_0, \gamma_0 > 0$ *and* $\alpha, \beta \geqslant 0$ *with* $\alpha + \beta = p_0^{-1} - q_0^{-1}$ *such that*

$$\| v_1^\alpha T_{p_0}(t) v_1^\beta \|_{p_0 \to q_0, \gamma_0} \leqslant C t^{-K} \quad (0 < t \leqslant t_0). \tag{2.9}$$

Then the spectrum $\sigma(A_p)$ *does not depend on* $p \in [p_0, q_0]\setminus\{\infty\}$*, and the operators* A_p *have consistent resolvents.*

Corollary 2.5. *Let $\Omega \subseteq \mathbb{R}^N$ be measurable, $T_p(t) = e^{-tA_p}$ consistent C_0-semigroups on $L_p(\Omega)$, $p \in [p_0, q_0] \setminus \{\infty\}$. Assume that there exist $C, K, t_0, \gamma_0 > 0$ such that*

$$\|T_{p_0}(t)\|_{p_0 \to q_0, \gamma_0} \leqslant Ct^{-K} \quad (0 < t \leqslant t_0).$$

Then the spectrum $\sigma(A_p)$ does not depend on $p \in [p_0, q_0] \setminus \{\infty\}$, and the operators A_p have consistent resolvents.

Remark 2.6. Condition (2.9) is in particular fulfilled if the doubling property (2.3) holds and there exists $m > 1$ such that

$$\sup_{0<t\leqslant t_0} \|v^{\alpha}_{t^{1/m}} T_{p_0}(t) v^{\beta}_{t^{1/m}}\|_{p_0 \to q_0, \gamma_0} < \infty.$$

In this case, we can choose $K = \frac{N}{m}(\frac{1}{p_0} - \frac{1}{q_0})$ in (2.9), with N from (2.4). Thus, if (2.3) holds then we have the following relation between the assumptions of Theorems 2.1 and 2.4: The volume growth assumption (2.7) for large balls is more restrictive than (2.2) in Theorem 2.1, whereas assumption (2.9) on the semigroup is less restrictive than (2.5) in Theorem 2.1:

In estimate (2.9) the size of the exponent $-K$ of t does not matter. In contrast, it is important that the number N occurring in the corresponding estimate in Corollary 2.2 is the dimension of the underlying space $\mathbb{R}^N$. Moreover, in (2.9) the weighted estimate is only needed for a fixed γ_0 whereas in (2.5) it is crucial that $\gamma(t) = t^{-\frac{1}{m}}$ tends to ∞ in the right way as $t \to 0$.

2.2 Comments and Examples

The problem of L_p-spectral independence for generators of consistent C_0-semigroups has a long history going back to B. Simon [Sim82] where the question was posed for Schrödinger operators. The main breakthrough was made by R. Hempel and J. Voigt [HeVo86] who answered the question in the affirmative for the case that the negative part of the potential is from the Kato class. This result was a starting point for many extensions in different directions.

The three crucial properties of Schrödinger semigroups used in the proof in [HeVo86] are the following. The underlying space is Euclidean space, the semigroup has an integral kernel satisfying a Gaussian upper bound (in particular, the semigroup acts on the whole L_p-scale), and the semigroup on L_2 is symmetric. W. Arendt proved in [Are94] an abstract result saying that these three conditions are already enough to ensure L_p-spectral independence.

In the subsequent investigations, different results were proved assuming only two of the conditions (possibly replacing the third one by another condition). K.-Th. Sturm showed that the method of [HeVo86] can be adapted to the setting of Riemannian manifolds. In [Stu93] he proved L_p-spectral independence for uniformly elliptic second order operators on Riemannian manifolds with Ricci curvature bounded below, assuming a volume growth condition slightly weaker than (2.7). In the setting of metric spaces with polynomially bounded volume, an abstract approach was developed by E. B. Davies in [Dav95a].

G. Schreieck and J. Voigt were the first to investigate the problem for semigroups not acting on the whole L_p-scale. In [ScVo94] they established L_p-spectral independence for Schrödinger operators on $\mathbb{R}^N$ with form small negative part of the potential. In this case we have consistent C_0-semigroups on $L_p(\mathbb{R}^N)$ only for p from an interval around $p = 2$ (see Theorem 1.32). As a result, the semigroup has no integral kernel enjoying a pointwise Gaussian upper bound. The ideas from [ScVo94] were put in a more general context in [Sch96]. A similar method was used in [Dav95b] to show L_p-spectral independence for higher order elliptic operators with bounded measurable coefficients. Further progress was made by Yu. Semenov [Sem97] who studied selfadjoint second order elliptic operators with unbounded coefficients in the principal part, adapting the method from [ScVo94].

In [HiSc99], again in the context of pointwise Gaussian upper bounds, the symmetry assumption on the semigroup was replaced by certain commutator estimates. P. C. Kunstmann showed in [Kun99] that the symmetry assumption can actually be dropped. Further generalisations combining the above ones can be found in [Kun00], [LiVo00] and [KuVo00]. Most of all these extensions of the result in [HeVo86] are unified in Theorem 2.4.

Most of the known results concerning the problem of analyticity are about semigroups acting on the whole L_p-scale. Then the question of analyticity in L_1 is of particular interest since for $1 < p < \infty$, analyticity can be shown by Stein interpolation (but with angle depending on p). In general, L_1-analyticity does not hold, even if the semigroup on L_2 is symmetric and sub-Markovian (see, e.g., [Dav89; Thm. 4.3.6], [Voi96]). Starting from [Ama83], there are several specific results on certain classes of elliptic operators on domains of $\mathbb{R}^N$ stating that the semigroup on L_1 is analytic, but not giving the optimal angle ([Kat86], [CaVe88], [ArBa93], only to mention a few).

E.-M. Ouhabaz was the first to establish analyticity of angle $\frac{\pi}{2}$ in $L_1(\mathbb{R}^N)$. In his thesis ([Ouh92a]) he observed that a Gaussian upper bound on the semigroup kernel for complex times proved in [Dav89; Thm. 3.4.8] can be used to show the following. If T_2 is a symmetric sub-Markovian semigroup on $L_2(\mathbb{R}^N)$ satisfying a Gaussian upper bound then the corresponding consistent C_0-semigroups T_p on $L_p(\mathbb{R}^N)$ are analytic of angle $\frac{\pi}{2}$, for all $p \in [1, \infty)$. See [Ouh95] for a more general version not assuming the semigroup to be sub-Markovian.

Ouhabaz' result was generalised in [Dav95a] from Euclidean space to metric spaces with polynomially bounded volume. Again in the context of Euclidean space, the symmetry assumption was dropped in [Hie96], with a result stating p-independence of the angle of analyticity. For a comprehensive discussion of the case $(p_0, q_0) = (1, \infty)$ see [Are97].

Concerning the case $(p_0, q_0) \neq (1, \infty)$ there are few results so far, and they are restricted to Euclidean space. E. B. Davies proved the following in [Dav95b]. If H is a selfadjoint superelliptic operator on $\mathbb{R}^N$ of order $2m < N$, with bounded measurable coefficients, then e^{-tH} extrapolates to an analytic semigroup on $L_p(\mathbb{R}^N)$ of angle $\frac{\pi}{2}$, for all $p \in [\frac{2N}{N+2m}, \frac{2N}{N-2m}]$. Analyticity of angle $\frac{\pi}{2}$ (but not extrapolation) was also shown in [Sch96; Sec. 3.3] in a more general setting assuming a generalised Gaussian upper bound similar to (2.6). On the other hand, extrapolation was studied in [Sem00] for generalised Schrödinger semigroups with form small negative part of the potential, but without showing analyticity of angle $\frac{\pi}{2}$.

We will show that the results on L_p-spectral independence and analyticity discussed above are covered by Theorems 2.1 and 2.4 (and Theorem 2.26 in Section 2.5 below), except for the following. In [Stu93] a slightly weaker assumption than the subexponential volume growth condition (2.7) was used, but the proof heavily depends on the symmetry of the semigroup in L_2 and existence on the whole L_p-scale. In [Kun00], [KuVo00] there are more sophisticated results concerning L_p-spectral independence in the case $(p_0, q_0) = (1, \infty)$.

For the case $(p_0, q_0) = (1, \infty)$ it is important to observe that the set of estimates in (2.6) in Theorem 2.1 is equivalent to a Gaussian upper bound of order m.

Proposition 2.7. *Assume that (M, d) is separable. Let T_2 be a C_0-semigroup on $L_2(\Omega)$. Then the estimates in (2.6) hold with $\alpha = \beta = \frac{1}{2}$, $p = 1$, $q = \infty$ if and only if the semigroup operators $T_2(t)$ have integral kernels k_t satisfying*

$$|k_t| \leqslant C\big(v_{t^{1/m}} \otimes v_{t^{1/m}}\big)^{-\frac{1}{2}} \exp\big(\omega t - c_m\big(\tfrac{d^m}{\nu t}\big)^{\frac{1}{m-1}}\big) \quad (t > 0), \tag{2.10}$$

with $c_m = (m-1)m^{-\frac{m}{m-1}}$.

Proof. Let $D \subseteq M$ be countable and dense, $t > 0$. By the Dunford-Pettis theorem, (2.6) holds if and only if $T_2(t)$ has an integral kernel k_t satisfying

$$\begin{aligned} |k_t(x,y)| &\leqslant \inf_{w \in D} C\rho_{\gamma,w}(x)^{-1} v_{t^{1/m}}(x)^{-\frac{1}{2}} v_{t^{1/m}}(y)^{-\frac{1}{2}} \rho_{\gamma,w}(y) e^{\omega t + \nu\gamma^m t} \\ &= C\big(v_{t^{1/m}}(x) v_{t^{1/m}}(y)\big)^{-\frac{1}{2}} e^{\omega t + \nu\gamma^m t - \gamma d(x,y)} \end{aligned}$$

for almost all $x, y \in \Omega$ and all rational $\gamma \geqslant 0$. We now optimise with respect to γ (Davies' trick): setting $\gamma = \big(\frac{d(x,y)}{m\gamma t}\big)^{\frac{1}{m-1}}$ yields the desired conclusion. □

Gaussian upper bounds are known to hold for wide classes of uniformly elliptic operators, e.g. for

(a) second order uniformly elliptic operators in divergence form on $\mathbb{R}^N$
with real coefficients [Aro67],
with complex coefficients in dimensions 1 and 2 [AMT98],
with uniformly continuous complex coefficients in higher dimensions [Aus96];

(b) superelliptic operators of order $2m$ in dimensions $N < 2m$ [Dav95b];

(c) second order uniformly elliptic operators in divergence form on Riemannian manifolds with Ricci curvature bounded below [Sal92].

For more detailed discussions of examples for which Gaussian upper bounds are valid, we refer to [HiSc99], [Kun99].

Until recently, all results concerning the case $(p_0, q_0) \neq (1, \infty)$ were restricted to $M = \mathbb{R}^N$, $\mu = \lambda^N$ the Lebesgue measure. The proofs relied on the 'box method' where $\mathbb{R}^N$ is subdivided into congruent cubes Q_j and one works in spaces $l_r(L_p(Q_j))$. In contrast, our proofs of Theorems 2.1 and 2.4 do *not* use the box method but rely on Lemma 2.19 as a substitute. Indeed, working in a general measure space which carries a semi-metric it is not clear what one should use instead of the partition into cubes of equal size.

The weight functions $\rho_{\gamma,y}$ were not used in the context of weighted norm estimates until [KuVo00]. In [ScVo94] the functions ρ_ξ defined by $\rho_\xi(x) := e^{\xi x}$ $(x, \xi \in \mathbb{R}^N)$ were used to prove L_p-spectral independence, in [Sch96] also to prove analyticity of angle $\frac{\pi}{2}$. It was set forth in [Dav95b] that (approximations of) these weight functions ρ_ξ are suitable for studying all three problems of interest in the present chapter: extrapolation, analyticity and L_p-spectral independence. For the technique of weighted estimates, which we develop in the next section, the crucial advantage of the weights $\rho_{\gamma,y} = e^{-\gamma|\cdot - y|_\infty}$ is that they are integrable for $\gamma > 0$ whereas the weights ρ_ξ grow exponentially in direction ξ.

In [Sem97] and later in [LiVo00], the weights had the more general form $\rho_\xi = e^{\xi\psi}$, with an L_1*-regular* function $\psi\colon \mathbb{R}^N \to \mathbb{R}^N$, i.e., ψ is Lipschitz continuous and

$$\sup_{k \in \mathbb{Z}^N} \sum_{j \in \mathbb{Z}^N} e^{-|\psi(k) - \psi(j)|} < \infty. \tag{2.11}$$

With these weight functions at hand, it was possible to study elliptic operators with singular coefficients not only in the lower order terms but also in the principal part. Note that $\psi(x) = x$ is L_1-regular. We will see that the more general weights are suitable for the problem of L_p-spectral independence only.

Before [KuVo00], all results concerning the case $(p_0, q_0) \neq (1, \infty)$ involved weighted operator norms of the type $\sup_{|\xi|=\gamma} \|\rho_\xi B \rho_\xi^{-1}\|_{p\to q}$ instead of the weighted norm $\|B\|_{p\to q,\gamma}$ defined via $\rho_{\gamma,y}$. We point out that in the case of higher order elliptic operators on $\mathbb{R}^N$ (see [Dav95b]), it is hard to estimate $\|B\|_{p\to q,\gamma}$ directly since the functions $\rho_{\gamma,y}$ have only one bounded weak derivative. Nevertheless we have the following result.

Proposition 2.8. *Let $\psi\colon \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz continuous, $\Omega \subseteq \mathbb{R}^N$ open, $B\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ a linear operator, $\gamma > 0$, $1 \leqslant p < q \leqslant \infty$. Define a semi-metric d on $\mathbb{R}^N$ by $d(x,y) := |\psi(x) - \psi(y)|_\infty$, and let $\|\cdot\|_{p\to q,\gamma}$ be the corresponding weighted operator norm. Then*

$$\|B\|_{p\to q,\gamma} \leqslant 2N \sup_{|\xi|=\gamma} \|e^{\xi\psi} B e^{-\xi\psi}\|_{p\to q}.$$

Proof. Let $E := \{\pm\gamma e_j;\, j = 1, \ldots, N\}$ where e_j are the standard unit vectors of $\mathbb{R}^N$. Fix $y \in \mathbb{R}^N$, and for $\xi \in E$ let $\rho_\xi := e^{\xi(\psi - \psi(y))}$. Then

$$\rho_{\gamma,y}^{-1} = e^{\gamma|\psi - \psi(y)|_\infty} = \max_{\xi \in E} \rho_\xi^{-1}.$$

We will make use of the fact that B has a dual operator $B'\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ and that $\|B'\|_{q'\to p'} = \|B\|_{p\to q}$ (see [KuVo00; Lemma 10]). For $f \in L_{\infty,c}(\Omega)$ we obtain

$$\|\rho_{\gamma,y}^{-1} B' \rho_{\gamma,y} f\|_{p'} \leqslant \sum_{\xi\in E} \|\rho_\xi^{-1} B' \rho_\xi \rho_\xi^{-1} \rho_{\gamma,y} f\|_{p'} \leqslant \sum_{\xi\in E} \|\rho_\xi^{-1} B' \rho_\xi\|_{q'\to p'} \|\rho_\xi^{-1}\rho_{\gamma,y} f\|_{q'}.$$

Note that $\rho_\xi^{-1}\rho_{\gamma,y} \leqslant 1$. By duality and the definition of ρ_ξ we conclude that

$$\|\rho_{\gamma,y} B \rho_{\gamma,y}^{-1}\|_{p\to q} = \|\rho_{\gamma,y}^{-1} B' \rho_{\gamma,y}\|_{q'\to p'} \leqslant \sum_{\xi\in E} \|\rho_\xi^{-1} B' \rho_\xi\|_{q'\to p'} \leqslant 2N \max_{\xi\in E} \|e^{\xi\psi} B e^{-\xi\psi}\|_{p\to q}$$

which completes the proof. □

Let us now first consider the case $\psi(x) = x$. Then the semi-metric d defined above is just the supremum metric, and we have $\lambda^N(B(x,r)) = (2r)^N$ for all $x \in \mathbb{R}^N$, $r > 0$. Hence, the volume growth conditions of both Corollaries 2.2 and 2.5 are satisfied. Moreover, we can use Proposition 2.8 to estimate $\|T(t)\|_{p\to q,\gamma}$.

A typical example for an L_1-regular function on $\mathbb{R}^1$ is as follows. Define ψ on $\mathbb{R}$ by $\psi(2n + x) = 2n + 2x^+$ for all $n \in \mathbb{Z}$, $|x| \leqslant 1$. Then $\psi\restriction_{[2n-1,2n]} = 2n$ for all $n \in \mathbb{Z}$ and hence $\lambda^N(B(x,r)) \geqslant 1$ if $(x - \frac{r}{2}, x + \frac{r}{2}) \cap [2n-1, 2n] \neq \varnothing$ for some $n \in \mathbb{Z}$. From this we easily see that the doubling property (2.3) does not hold, i.e., the conditions of Theorem 2.1 are not fulfilled. But we show that the conditions of Theorem 2.4 do hold for L_1-regular ψ:

First observe the following. If ψ is Lipschitz continuous then there exists $L > 0$ such that $d(x,y) \leqslant L|x-y|_\infty$ for all $x, y \in \mathbb{R}^N$. This implies that $B(x,1) \supseteq x + [-\frac{1}{L}, \frac{1}{L}]^N$. Hence $\lambda^N(B(x,1)) \geqslant \left(\frac{2}{L}\right)^N$ for all $x \in \mathbb{R}^N$, i.e., (2.8) holds. Now, (i) $\Longrightarrow$ (ii) of the following lemma shows that (2.7) holds for L_1-regular ψ and, by Proposition 2.8, that

$$\|v_1^\alpha B v_1^\beta\|_{p\to q,\gamma} \leqslant \left(c\cdot 2^N\right)^{\alpha+\beta} \cdot 2N \sup_{|\xi|=\gamma} \|e^{\xi\psi} B e^{-\xi\psi}\|_{p\to q}.$$

Lemma 2.9. *(cf. [KuVo00; Lemma 6]) Let $\psi\colon \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz continuous. Define the semi-metric d on $\mathbb{R}^N$ as in Proposition 2.8. Then the following are equivalent:*

(i) *ψ is L_1-regular,*

(ii) *there exists $c > 0$ such that $\lambda^N(B(x,r)) \leqslant c(1+r)^N$ for all $r > 0$,*

(iii) *there exists $c > 0$ such that $\lambda^N(B(x,r)) \leqslant ce^{r/2}$ for all $r > 0$.*

Proof. As above, let $L > 0$ such that $d(x,y) \leqslant L|x-y|_\infty$ for all $x, y \in \mathbb{R}^N$. For $j \in \mathbb{Z}^N$ let $Q_j := j + [-\frac{1}{2}, \frac{1}{2}]^N$.

(i)⇒(ii) (cf. [LiVo00; Appendix A]). Let $r > 0$ and let $n \in \mathbb{N}$ with $n - 1 < r \leqslant n$. Then

$$\begin{aligned} B(x,r) &= \left\{y \in \mathbb{R}^N;\ \psi(y) \in \psi(x) + (-r,r)^N\right\} \\ &\subseteq \bigcup \left\{\psi^{-1}(\psi(x) + j + [0,1]^N);\ j \in \{-n,\ldots,n-1\}^N\right\}. \end{aligned}$$

It follows that $\lambda^N(B(x,r)) \leqslant (2n)^N \sup_{z\in\mathbb{R}^N} \lambda^N\big(\psi^{-1}(z+[0,1]^N)\big)$. Since $(2n)^N \leqslant 2^N(1+r)^N$ it remains to show that the supremum is finite.

To this end, let $z \in \mathbb{R}^N$ and $Q := z+[0,1]^N$. Let $x_0, y \in \psi^{-1}(Q)$ and choose $k, j \in \mathbb{Z}^N$ with $x_0 \in Q_k$ and $y \in Q_j$. Then $d(x_0,y) \leqslant 1$ and $d(x_0,k), d(y,j) \leqslant L/2$, hence $d(j,k) \leqslant L+1$. Therefore $\psi^{-1}(Q) \subseteq \bigcup\big\{Q_j;\, j \in \mathbb{Z}^N, d(j,k) \leqslant L+1\big\}$ and

$$\lambda^N\big(\psi^{-1}(Q)\big) \leqslant \#\big\{j \in \mathbb{Z}^N;\, d(j,k) \leqslant L+1\big\} \leqslant \sum_{j\in\mathbb{Z}^N} e^{L+1-d(j,k)}.$$

By the L_1-regularity of ψ this shows that $\lambda^N\big(\psi^{-1}(Q)\big)$ can be estimated from above independently of the cube Q.

The implication (ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i). For all $k \in \mathbb{Z}^N$ we have

$$\sum_{j\in\mathbb{Z}^N} e^{-|\psi(k)-\psi(j)|} \leqslant \sum_{j\in\mathbb{Z}^N} e^{L/2} \int_{Q_j} e^{-|\psi(k)-\psi(y)|}\, dy = e^{L/2} \int_{\mathbb{R}^N} e^{-d(k,y)}\, dy.$$

This shows (i) since

$$\int_{\mathbb{R}^N} e^{-d(k,y)}\, dy \leqslant \sum_{n=1}^{\infty} e^{-(n-1)} \lambda^N\big(B(k,n)\big) \leqslant \sum_{n=1}^{\infty} c e^{-(n/2-1)} < \infty. \qquad \square$$

To conclude this section we want to show how our theorems can be applied in the context of perturbation of Dirichlet forms by potentials. Let M, Ω, μ, d be as in the introduction of Section 2.1. Based on Theorem 1.32 we are going to prove the following result.

Theorem 2.10. *Assume that* (2.2) *and* (2.4) *hold for* $v_r(x) = \mu(B(x,r))$. *Let* τ *be a symmetric Dirichlet form in* $L_2(\Omega)$, *and assume that the associated symmetric sub-Markovian semigroup* T *on* $L_2(\Omega)$ *satisfies the Gaussian upper bound* (2.10) *with* $m = 2$. *Let* $V\colon \Omega \to \mathbb{R}$ *be measurable such that* $\tau + V^+$ *is densely defined and* $V^- \leqslant \beta\tau + V^+ + c_\beta$ *for some* $\beta < 1$, $c_\beta \in \mathbb{R}$. *Assume* $N > 2$ *in* (2.4) *and let* $p_+ := \frac{2}{1-\sqrt{1-\beta}}$, $p_{\max} := \frac{N}{N-2}p_+$, $p_{\min} := p_{\max}'$.

(a) *Then* T_V, *the analytic semigroup on* $L_2(\Omega)$ *associated with* $\overline{\tau + V}$, *extrapolates to an analytic semigroup* $T_{V,p}$ *on* $L_p(\Omega)$ *of angle* $\frac{\pi}{2}$, *for all* $p \in (p_{\min}, p_{\max})$.

(b) *If the subexponential volume growth condition* (2.7) *holds instead of* (2.2), *then the spectrum of the generators of the semigroups* $T_{V,p}$ *is independent of* $p \in (p_{\min}, p_{\max})$.

Remark 2.11. (a) It was first observed by Yu. Semenov that the L_p-scale $[p_-, p_+]$ given in Theorem 1.32 can be extended: in [Sem00] he studied the form τ corresponding to a selfadjoint second order uniformly elliptic operator on $\mathbb{R}^N$. He showed that T_V extrapolates to an analytic semigroup on $L_p(\mathbb{R}^N)$, for all $p \in (p_{\min}, p_{\max})$, but he did not obtain the (optimal) angle $\frac{\pi}{2}$.

More generally, the above theorem can be applied to the following situation. Let M be a complete Riemannian manifold with Ricci curvature bounded below,

d the Riemannian distance and μ the Riemannian volume. Then (2.2) and (2.4) hold for $v_r(x) = \mu(B(x,r))$ and N the dimension of M, by Bishop's comparison principle. Let $\Omega \subseteq M$ be open, τ the form corresponding to a selfadjoint second order uniformly elliptic operator on Ω subject to Dirichlet boundary conditions. Then the associated semigroup on $L_2(\Omega)$ satisfies a Gaussian upper bound of order $m = 2$ (see [Sal92; Thm. 6.3]).

(b) For $p \in [1,\infty)$, let T_p be the positive C_0-semigroup on $L_p(\Omega)$ associated with τ. By Proposition 1.15, the above theorem can be reformulated as follows.

Assume that (2.7) and (2.3) hold for $v_r(x) = \mu(B(x,r))$ and that T_2 satisfies (2.10) with $m = 2$. Let $V\colon \Omega \to \mathbb{R}$ be measurable such that $\frac{1}{\beta}V^+$ is T-admissible and $-\frac{1}{\beta}V^-$ is $T_{\frac{1}{\beta}V^+}$-admissible, for some $\beta < 1$. Assume $N > 2$ in (2.4) and define $p_{\min}$, $p_{\max}$ as above.

Then, for all $p \in (p_{\min}, p_{\max})$, $-V^-$ is $(T_p)_{V^+}$-admissible, $\big((T_p)_{V^+}\big)_{-V^-}(t) = e^{-tA_{V,p}}$ is analytic of angle $\frac{\pi}{2}$, and the spectrum $\sigma(A_{V,p})$ is independent of $p \in (p_{\min}, p_{\max})$.

Assume, more restrictively, that there exists $\alpha > 1$ such that αV^+ is T_1-admissible and $-\alpha V^-$ is $(T_1)_{\alpha V^+}$-admissible. Then one can show, with a similar proof, that the assertions of Theorem 2.10 hold for all $p \in [1,\infty)$. In fact, only the second part of the proof given below is needed, with slight changes and additions.

(c) An interesting point about Theorem 2.10 is the following. If $V^- \leqslant \beta\tau + V^+ + c$ for some $\beta < 1$, $c \in \mathbb{R}$ then $\tau + V$ is associated with a C_0-semigroup on $L_p(\Omega)$ for all p in $[\frac{2N}{N+2}, \frac{2N}{N-2}]$, an interval not depending on β. If one only knows $V^- \leqslant \tau + V^+ + c$ for some $c \in \mathbb{R}$, but not $V^- \leqslant \beta\tau + V^+ + c$ for any $\beta < 1$, $c \in \mathbb{R}$, then $\tau + V$ is associated with a C_0-semigroup on $L_2(\Omega)$, by Proposition 1.15(b). In this situation it is not known whether $\tau + V$ is associated with a C_0-semigroup on $L_p(\Omega)$ for some $p \neq 2$.

In the proof of Theorem 2.10 we will make use of the following immediate consequence of the Stein interpolation theorem (which is in fact a 'pre-version' of the Stein interpolation theorem). We fix an increasing sequence (Ω_n) of measurable subsets of Ω that have finite d-diameter (and hence finite μ-volume) such that $\Omega = \bigcup_n \Omega_n$. By $L_{\infty,c}$ we denote the space of all $f \in L_\infty(\Omega)$ for which there exists $n \in \mathbb{N}$ such that $f = 0$ a.e. on $\Omega \setminus \Omega_n$.

Lemma 2.12. *Let $S := \{z \in \mathbb{C};\ 0 \leqslant \operatorname{Re} z \leqslant 1\}$, and $F\colon S \to L_1(\mu) + L_\infty(\mu)$. Assume that $\langle F(\cdot), f\rangle$ is continuous and bounded, and analytic in the interior of S, for all $f \in L_{\infty,c}$. Let $p_0, p_1 \in [1,\infty]$. If there exist $C_0, C_1 > 0$ such that*

$$\|F(j+it)\|_{p_j} \leqslant C_j \quad (j = 0, 1,\ t \in \mathbb{R})$$

then $\|F(\theta)\|_{p_\theta} \leqslant C_0^{1-\theta} C_1^\theta$ for all $\theta \in (0,1)$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof of Theorem 2.10. By symmetry, we only need to study the case $p \geqslant 2$. From Theorem 1.32 we already know that T_V extrapolates to a positive C_0-semigroup $T_{V,p}$ on $L_p(\Omega)$, for all $p \in [2, p_+]$. The idea is to apply Theorems 2.1 and 2.4 with $p_0 = p \in [2, p_+)$ and $q_0 = rp$, for all $1 < r < \frac{N}{N-2}$. For that purpose we need a weighted $p \to rp$-estimate for $v_{\sqrt{t}}^{\frac{1}{p}-\frac{1}{rp}} T_V(t)$. By means of Theorem 1.32 we will show an unweighted estimate for $v_{\sqrt{t}}^{\frac{1}{p}-\frac{1}{rp}} T_{\alpha V}(t)$ for some $\alpha > 1$ and then use Lemma 2.12 to derive the desired weighted estimate.

Since (2.10) holds with $m = 2$ we obtain by Theorem 2.1 that

$$\|v_{\sqrt{t}}^{\frac{1}{p}-\frac{1}{q}} T(t)\|_{p\to q,\gamma} \leqslant Ce^{\omega t+\nu\gamma^2 t} \quad (t > 0,\ \gamma \geqslant 0,\ 1 \leqslant p \leqslant q \leqslant \infty) \tag{2.12}$$

for some $C \geqslant 1,\ \omega \in \mathbb{R},\ \nu \geqslant 0$. Without restriction assume that $c_\beta = \omega = 0$. Our first aim is to show the following Sobolev type inequality

$$\|v_{\sqrt{t}}^{\frac{1}{2r'}} u\|_{2r} \leqslant C_r \big(\|u\|_2^2 + t\tau(u)\big)^{\frac{1}{2}} \quad (u \in D(\tau),\ 0 < t \leqslant 1) \tag{2.13}$$

for all $1 < r < \frac{N}{N-2}$. Observe that $\|u\|_2^2 + t\tau(u) = \|(1+tA)^{\frac{1}{2}} u\|_2^2$, where A is the selfadjoint operator in $L_2(\Omega)$ associated with τ. We thus have to show $\|v_{\sqrt{t}}^{\frac{1}{2r'}} (1+tA)^{-\frac{1}{2}}\|_{2\to 2r} \leqslant C_r$. By the spectral theorem,

$$(1+tA)^{-\frac{1}{2}} = \Gamma(\tfrac{1}{2})^{-1} \int_0^\infty s^{-\frac{1}{2}} e^{-s(1+tA)} ds.$$

Therefore,

$$\|v_{\sqrt{t}}^{\frac{1}{2r'}} (1+tA)^{-\frac{1}{2}}\|_{2\to 2r} \leqslant \Gamma(\tfrac{1}{2})^{-1} \int_0^\infty s^{-\frac{1}{2}} e^{-s} \|v_{\sqrt{t}}^{\frac{1}{2r'}} T(st)\|_{2\to 2r}\, ds.$$

Note that $\frac{1}{2r'} = \frac{1}{2} - \frac{1}{2r}$. For $s \geqslant 1$ we have $v_{\sqrt{t}} \leqslant v_{\sqrt{st}}$, so (2.12) yields

$$\|v_{\sqrt{t}}^{\frac{1}{2r'}} T(st)\|_{2\to 2r} \leqslant C \quad (s \geqslant 1).$$

For $s \leqslant 1$ we estimate $v_{\sqrt{t}} \leqslant c_2 \big(\frac{\sqrt{t}}{\sqrt{st}}\big)^N v_{\sqrt{st}}$ by (2.4), since $\sqrt{t} \leqslant 1$. By (2.12) we infer that

$$\|v_{\sqrt{t}}^{\frac{1}{2r'}} T(st)\|_{2\to 2r} \leqslant \big(c_2 s^{-\frac{N}{2}}\big)^{\frac{1}{2r'}} C \quad (s \leqslant 1).$$

Moreover, $\frac{N}{4r'} < \frac{1}{2}$ by the choice of r and hence

$$\|v_{\sqrt{t}}^{\frac{1}{2r'}} (1+tA)^{-\frac{1}{2}}\|_{2\to 2r} \leqslant \Gamma(\tfrac{1}{2})^{-1} C \left(c_2 \int_0^1 s^{-\frac{1}{2}-\frac{N}{4r'}} e^{-s} ds + \int_1^\infty s^{-\frac{1}{2}} e^{-s} ds \right) =: C_r.$$

This proves (2.13).

Let now $2 \leqslant p < p_+,\ 1 < r < \frac{N}{N-2}$. By Theorem 1.32 we know that T_V extrapolates to a contractive analytic semigroup $T_{V,p}$ on $L_p(\Omega)$, and for the generator $-A_{V,p}$ of $T_{V,p}$ we have

$$\tau(u^{\frac{p}{2}}) \leqslant \big(\tfrac{4}{pp'} - \beta\big)^{-1} \langle A_{V,p} u, u^{p-1}\rangle \quad \big(0 \leqslant u \in D(A_{V,p})\big).$$

Let $0 \leqslant f \in L_p(\Omega)$, $u_t := T_{V,p}(t)f \bigl(\in D(A_{V,p})\bigr)$ $(t > 0)$. By (2.13) we obtain

$$\|v_{\sqrt{t}}^{\frac{1}{pr'}} u_t\|_{rp}^p = \|v_{\sqrt{t}}^{\frac{1}{2r'}} u_t^{\frac{p}{2}}\|_{2r}^2 \leqslant C_r^2\bigl(\|u_t\|_p^p + t(\tfrac{4}{pp'} - \beta)^{-1}\langle A_{V,p}u_t, u_t^{p-1}\rangle\bigr) \quad (0 < t \leqslant 1).$$

Since $T_{V,p}$ is a contractive analytic semigroup, there exists $c > 0$ such that $\|A_{V,p}T_{V,p}(t)\| \leqslant \frac{c}{t}$ $(t > 0)$. We conclude that

$$\|v_{\sqrt{t}}^{\frac{1}{pr'}} T_{V,p}(t)f\|_{rp}^p \leqslant C_r^2\bigl(\|f\|_p^p + (\tfrac{4}{pp'} - \beta)^{-1}c\|f\|_p^p\bigr) \quad (0 < t \leqslant 1),$$

which amounts to an unweighted $p \to rp$-estimate.

In order to derive the weighted estimate by Lemma 2.12, observe that $\frac{4}{pp'} > \frac{4}{p_+p_-} = \beta$ since $p \in [2, p_+)$. Choose $\alpha > 1$ such that $\alpha\beta < \frac{4}{pp'}$. Then $(\alpha V)^- \leqslant \alpha\beta\tau + (\alpha V)^+$, and the above implies that there exists $K \geqslant 1$ such that

$$\|v_{\sqrt{t}}^{\frac{1}{pr'}} T_{\alpha V,p}(t)f\|_{rp} \leqslant K\|f\|_p \quad \bigl(0 < t \leqslant 1,\ 0 \leqslant f \in L_p(\Omega)\bigr).$$

Let $0 < t \leqslant 1$, $0 \leqslant f \in L_{\infty,c}$. Note that

$$T_{\alpha(V^+\wedge m - V^-\wedge n)}(t)f \downarrow T_{\alpha(V^+ - V^-\wedge n)}(t)f \ (\leqslant T_{\alpha V}(t)f) \quad \text{as } m \to \infty.$$

Hence there exists $m_n \in \mathbb{N}$ (depending on f!) such that, with $V_n := V^+ \wedge m_n - V^- \wedge n$,

$$\|v_{\sqrt{t}}^{\frac{1}{pr'}} T_{\alpha V_n}(t)f\|_{rp} \leqslant (K+1)\|f\|_p \quad (n \in \mathbb{N}).$$

By (2.12) we also have, noting $\frac{1}{p} - \frac{1}{rp} = \frac{1}{pr'}$, that

$$\|\rho_{\gamma,y} v_{\sqrt{t}}^{\frac{1}{pr'}} T(t)\rho_{\gamma,y}^{-1} f\|_{rp} \leqslant Ce^{\nu\gamma^2 t}\|f\|_p \quad (y \in M,\ \gamma \geqslant 0).$$

Lemma 2.12 yields

$$\|\rho_{\frac{\alpha-1}{\alpha}\gamma,y} v_{\sqrt{t}}^{\frac{1}{pr'}} T_{V_n}(t)\rho_{\frac{\alpha-1}{\alpha}\gamma,y}^{-1} f\|_{rp} \leqslant \bigl(Ce^{\nu\gamma^2 t}\bigr)^{\frac{\alpha-1}{\alpha}}(K+1)^{\frac{1}{\alpha}}\|f\|_p.$$

Finally, we use $T_{V_n}(t)g \geqslant T_{V^+ - V^-\wedge n}(t)g \uparrow T_V(t)g$ $(0 \leqslant g \in L_2(\Omega),\ n \to \infty)$ and the positivity of T_V to obtain

$$\|v_{\sqrt{t}}^{\frac{1}{pr'}} T_V(t)\|_{p\to rp,\gamma} \leqslant C^{1-\frac{1}{\alpha}}(K+1)^{\frac{1}{\alpha}} e^{\frac{\alpha}{\alpha-1}\nu\gamma^2 t} \quad (0 < t \leqslant 1,\ \gamma \geqslant 0).$$

Now we are in a position to apply Theorem 2.1 with $p_0 := p$ and $q_0 := rp$. We obtain that $T_{V,p}$ extrapolates to a semigroup $T_{V,q}$ on $L_q(\Omega)$ which is analytic of the same angle as $T_{V,p}$, for all $q \in (p, \frac{N}{N-2}p)$. This holds for all $p \in [2, p_+)$, so the proof of (a) is complete. In the same way we can apply Theorem 2.4 to prove part (b) (observe that estimate (2.9) is fulfilled by Remark 2.6). $\square$

2.3 Technique of weighted estimates

In this section we provide some technical tools needed in the proofs of Theorems 2.1 and 2.4. The main goal is to show Proposition 2.16 below. Let the notation and assumptions be as in the introduction of Section 2.1. Our first result deals with norm estimates for integral operators on M.

Lemma 2.13. *Let $1 \leqslant p \leqslant q \leqslant \infty$. Let $k\colon M \times M \to \mathbb{C}$ be measurable, and for $s \in [1,\infty]$ define*

$$n_s(k) := \max\bigl(\operatorname*{ess\,sup}_{x\in M} \|k(x,\cdot)\|_s, \operatorname*{ess\,sup}_{y\in M} \|k(\cdot,y)\|_s\bigr) \in [0,\infty].$$

(a) *Let $s \in [1,\infty]$ with $s^{-1}+p^{-1} = 1+q^{-1}$. If $n_s(k)$ is finite then k defines a bounded integral operator $I_k\colon L_p(M) \to L_q(M)$, and $\|I_k\|_{p\to q} \leqslant n_s(k)$.*

(b) *Let $r,s \in [p,q]$ with $p^{-1}+q^{-1} = r^{-1}+s^{-1}$, and $f \in L_r(\Omega)$. Then*

$$\bigl\|y \mapsto \|k(y,\cdot)f\|_{L_p(\Omega)}\bigr\|_{L_q(M)} \leqslant n_s(k)\|f\|_r.$$

Proof. (a) is well-known and can be proved by an application of Fubini's theorem and Riesz-Thorin interpolation.

(b) In the case $p=\infty$ there is nothing to show. For $p<\infty$, (b) is equivalent to

$$\bigl\|y \mapsto \int_\Omega |k|^p(y,\cdot)|f|^p d\mu\bigr\|_{L_{q/p}(M)} \leqslant n_s(k)^p\|f\|_r^p = n_{s/p}(|k|^p)\||f|^p\|_{r/p}$$

which in turn follows from (a) since $(\frac{s}{p})^{-1}+(\frac{r}{p})^{-1} = 1+(\frac{q}{p})^{-1}$. □

We are going to apply Lemma 2.13 to integral kernels of the type $k(x,y) = v_r(x)^{-\alpha}v_r(y)^{-\beta}e^{-\gamma d(x,y)}$. The next result gives an estimate for $n_s(k)$.

Lemma 2.14. *Let $r > 0$ and assume $v_R \leqslant c_0e^{c_rR}v_r$ for all $R > r$, for some $c_0 \geqslant 1$, $c_r > 0$. Then*

$$n_s\bigl((x,y) \mapsto v_r(x)^{-\alpha}v_r(y)^{-\beta}e^{-\gamma d(x,y)}\bigr) \leqslant c_0^2(1-e^{-\gamma r/3})^{-1}e^{2c_rr}$$

for all $1 \leqslant s \leqslant \infty$, $\gamma \geqslant 3c_r$ and $a,\beta \leqslant 1$ with $\alpha+\beta = s^{-1}$.

Proof. First note that assumption (2.1) implies

$$v_r(x) \leqslant v_{r+d(x,y)}(y) \leqslant c_0e^{c_r(r+d(x,y))}v_r(y) \quad (x,y \in M),$$

and hence

$$v_r^\delta(x)/v_r^\delta(y) \leqslant c_0e^{c_rr+c_rd(x,y)} \quad (x,y \in M,\ |\delta| \leqslant 1). \tag{2.14}$$

If $s=\infty$ then $\alpha=-\beta$. By (2.14) we infer, since $\gamma \geqslant c_r$, that

$$v_r(x)^{-\alpha}v_r(y)^{-\beta}e^{-\gamma d(x,y)} \leqslant c_0e^{c_rr+(c_r-\gamma)d(x,y)} \leqslant c_0e^{c_rr} \quad (x,y \in M).$$

This proves the lemma in the case $s = \infty$.

Secondly, we study the case $s = 1$. To this end, we first estimate the integral $\int_M e^{-\gamma d(x,y)} d\mu(y)$ for all $x \in M$ and $\gamma > c_r$. For $n \in \mathbb{N}$ let $K_n := B(x, nr) \setminus B(x, (n-1)r)$. Due to the assumption we have $\mu(K_n) \leqslant v_{nr}(x) \leqslant c_0 e^{c_r nr} v_r(x)$, and therefore

$$\begin{aligned}\int_M e^{-\gamma d(x,y)} d\mu(y) &= \sum_{n=1}^{\infty} \int_{K_n} e^{-\gamma d(x,y)} d\mu(y) \leqslant \sum_{n=1}^{\infty} c_0 e^{c_r nr} v_r(x) e^{-\gamma(n-1)r} \\ &= c_0 e^{\gamma r} \sum_{n=1}^{\infty} e^{(c_r - \gamma)nr} v_r(x) = \frac{c_0 e^{c_r r}}{1 - e^{(c_r-\gamma)r}} v_r(x).\end{aligned}$$

By (2.14) we have $v_r(x)^{-\alpha} v_r(y)^{-\beta} \leqslant c_0 e^{c_r r + c_r d(x,y)} v_r(x)^{-1}$ since $\alpha + \beta = s^{-1} = 1$. For $\gamma \geqslant 3c_r$ we conclude that

$$\int_M v_r(x)^{-\alpha} v_r(y)^{-\beta} e^{-\gamma d(x,y)} d\mu(y) \leqslant c_0 e^{c_r r} \cdot \frac{c_0 e^{c_r r}}{1 - e^{(c_r - (\gamma - c_r))r}} = \frac{c_0^2 e^{2c_r r}}{1 - e^{(2c_r - \gamma)r}}.$$

The same holds with α, β interchanged, or equivalently, with x, y interchanged. This completes the proof in the case $s = 1$ since $2c_r \leqslant \frac{2}{3}\gamma$.

Finally, the case $1 < s < \infty$ will be reduced to the cases $s = 1, \infty$ by Lemma 2.12. (Alternatively, it could be reduced to the case $s = 1$ by direct computation.) Without restriction assume $\alpha \geqslant \beta$. Then $0 \leqslant \alpha \leqslant 1$. For $0 \leqslant \operatorname{Re} z \leqslant 1$ let $k_z(x,y) := v_r(x)^{-\alpha} v_r(y)^{\alpha - z} e^{-\gamma d(x,y)}$. Then $|k_z| = k_{\operatorname{Re} z}$. Above we obtained estimates for $n_1(k_{1+it}) = n_1(k_1)$ and $n_\infty(k_{it}) = n_\infty(k_0)$ $(t \in \mathbb{R})$. Making use of Lemma 2.12 we deduce the desired estimate for $n_s(k_{1/s})$. □

Let us remark that a volume growth assumption of the type $v_R \leqslant c e^{cR} v_r$ is necessary in the above lemma: assume that the assertion of the lemma holds for $s = 1,\ \alpha = 1,\ \beta = 0,\ \gamma = 3c_r$. Then we have, for almost all $x \in M$:

$$\begin{aligned}\mu(B(x,R)) &\leqslant \int_M e^{\gamma R - \gamma d(x,y)} d\mu(y) \leqslant n_1\big((x,y) \mapsto v_r(x)^{-1} e^{-\gamma d(x,y)}\big) e^{\gamma R} v_r(x) \\ &\leqslant c_0^2 (1 - e^{-c_r r})^{-1} e^{2c_r r} e^{3c_r R} v_r(x).\end{aligned}$$

For $\gamma \geqslant 0$ and $C \geqslant 1$ we introduce a class of weight functions on M,

$$P(\gamma, C) := \big\{\rho\colon M \to (0,\infty) \text{ measurable; } \rho(x)/\rho(y) \leqslant C e^{\gamma d(x,y)} \text{ for all } x, y \in M\big\}.$$

For $\rho \in P(\gamma, C)$ and $y \in M$ we have, by the definition of $\rho_{\gamma,y}$,

$$\rho \leqslant C \rho_{-\gamma,y} \rho(y) \quad \text{and} \quad \rho(y) \leqslant C \rho_{-\gamma,y} \rho. \tag{2.15}$$

Notice that, by the triangle inequality, $\rho_{\gamma,y} \in P(|\gamma|, 1)$ for all $\gamma \in \mathbb{R},\ y \in M$. Moreover, if $v_R \leqslant c_0 e^{c_r R} v_r$ for some $r > 0$ and all $R > r$ then $v_r^\delta \in P(c_0 e^{c_r r}, c_r)$ for all $|\delta| \leqslant 1$, by (2.14).

We fix an increasing sequence (Ω_n) of measurable subsets of Ω that have finite d-diameter (and hence finite μ-volume) such that $\Omega = \bigcup_n \Omega_n$. By $L_{1,loc}$ we denote the set of (equivalence classes of) all measurable functions f on Ω with $\|\chi_{\Omega_n} f\|_1 < \infty$ for all $n \in \mathbb{N}$, and by $L_{\infty,c}$ the space of all $f \in L_\infty(\Omega)$ for which there exists $n \in \mathbb{N}$ such that $f = 0$ a.e. on $\Omega \setminus \Omega_n$. Note that the elements of $P(\gamma, C)$ are multiplication operators on $L_{1,loc}$ and $L_{\infty,c}$.

Remark 2.15. Let $p, q \in [1, \infty]$, B a bounded operator on $L_p(\Omega)$ and $\rho_1, \rho_2 \in L_{\infty,loc}$, i.e., $\|\chi_{\Omega_n}\rho_j\|_\infty < \infty$ for all $n \in \mathbb{N}$, $j = 1, 2$. Let $D(\rho_2, L_p)$ denote the domain of the multiplication operator ρ_2 on $L_p(\Omega)$. Then

$$\|\rho_1 B \rho_2\restriction_{D(\rho_2,L_p)}\|_{p\to q} = \|\rho_1 B \rho_2\restriction_{L_{\infty,c}}\|_{p\to q}.$$

This follows from an application of Fatou's lemma.

The following result is one of the crucial tools in the proofs of Theorems 2.1 and 2.4. The archetype of this result is due to G. Schreieck and J. Voigt ([ScVo94; Prop. 3.2]; see [Sem97; Lemma 5.2] and [KuVo00; Prop. 13] for improved versions). In the present form the result is new.

Proposition 2.16. *Let $r > 0$, and assume $v_R \leqslant c_0 e^{c_r R} v_r$ for all $R > r$, for some $c_0 \geqslant 1$, $c_r > 0$. Further assume that $\mu(B(x, r)) > 0$ for all $x \in M$. Let $\gamma_0 \geqslant 8c_r$ and $\alpha_0, \beta_0 \geqslant 0$ with $\alpha_0 + \beta_0 = p_0^{-1} - q_0^{-1}$. Then, for any linear operator $B\colon L_{\infty,c} \to L_{1,loc}$ satisfying $\|v_r^{\alpha_0} B v_r^{\beta_0}\|_{p_0\to q_0,\gamma_0} \leqslant 1$ we have*

$$\|\rho v_r^{\alpha} B v_r^{\beta} \rho^{-1}\|_{p\to q} \leqslant C^2 c_0^6 (1 - e^{-\gamma_0 r/8})^{-2} e^{5\gamma_0 r}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0$, $\alpha, \beta \geqslant 0$ with $\alpha + \beta = p^{-1} - q^{-1}$, $C \geqslant 1$, $\rho \in P(\gamma_0/2, C)$. In particular,

$$\|v_r^{\alpha} B v_r^{\beta}\|_{p\to q,\gamma} \leqslant c_0^6 (1 - e^{-\gamma_0 r/8})^{-2} e^{5\gamma_0 r} \quad (|\gamma| \leqslant \gamma_0/2).$$

Observe that the assumption $\mu(B(x, r)) > 0$ is automatically fulfilled in the case $(p_0, q_0) \neq (1, \infty)$ (cf. Remark 2.3(c)). For the proof of Proposition 2.16 we need some preparatory lemmas. The first one will be needed in the case $(p_0, q_0) = (1, \infty)$. At the same time it proves Remark 2.3(d).

Lemma 2.17. *Assume that there exists $r > 0$ such that $\mu(B(x, r)) > 0$ for all $x \in M$. Then there exists a sequence $(x_n) \subseteq M$ such that $M = \bigcup_{n\in\mathbb{N}} B(x_n, 2r)$. In particular, if $\mu(B(x, r)) > 0$ for all $x \in M$, $r > 0$ then M is separable. Conversely, if M is separable then $\mu(B(x, r)) > 0$ for all $r > 0$ and almost all $x \in M$.*

Proof. Fix $x_0 \in M$ and let $k \in \mathbb{N}$. By Zorn's lemma we can choose $M_k \subseteq B(x_0, k)$ such that $\big(B(x, r)\big)_{x\in M_k}$ is a maximal family of pairwise disjoint balls with radius r and centre in $B(x_0, k)$. Then M_k is countable since $\mu(B(x_0, k + r)) < \infty$ and $\mu(B(x, r)) > 0$ for all $x \in M_k$. By the maximality we have $B(x_0, k) \subseteq \bigcup_{x\in M_k} B(x, 2r)$. This proves the first assertion since $M = \bigcup_{k\in\mathbb{N}} B(x_0, k)$.

Assume now that M is separable. Let $r > 0$ and choose $(x_n) \subseteq M$ such that $M = \bigcup_{n\in\mathbb{N}} B(x_n, \frac{r}{2})$. Then for all $n \in \mathbb{N}$ with $\mu(B(x_n, \frac{r}{2})) > 0$ we obtain $\mu(B(x,r)) > 0$ for all $x \in B(x_n, \frac{r}{2})$. This implies the second assertion. □

Corollary 2.18. *Let $r > 0$ and assume that $\mu(B(x,r)) > 0$ for all $x \in M$. Let $\gamma > 0$, $B\colon L_{\infty,c} \to L_{1,loc}$ a linear operator. If $\|B\|_{1\to\infty,\gamma} \leqslant 1$ then B has an integral kernel k satisfying $|k(x,y)| \leqslant e^{4\gamma r - \gamma d(x,y)}$ for almost all $x, y \in \Omega$.*

Proof. Let $w \in M$. By the Dunford-Pettis theorem, the assumption implies that the operator $B_w := \rho_{\gamma,w} B \rho_{\gamma,w}^{-1}$ has an integral kernel k_w satisfying $|k_w| \leqslant 1$ a.e. Therefore, B has an integral kernel k satisfying

$$k(x,y) = \rho_{\gamma,w}(x)^{-1} k_w(x,y) \rho_{\gamma,w}(y) \quad \text{(almost all } x, y \in \Omega).$$

According to Lemma 2.17 choose a sequence $(w_n) \subseteq M$ such that $M = \bigcup_n B(w_n, 2r)$. For all $n \in \mathbb{N}$ and almost all $x \in B(w_n, 2r)$, $y \in \Omega$ we obtain

$$|k(x,y)| = e^{\gamma d(x,w_n) - \gamma d(y,w_n)} \leqslant e^{4\gamma r - \gamma d(x,y)},$$

which concludes the proof. □

The next lemma will be used in the case $(p_0, q_0) \neq (1, \infty)$. Here is the place where the assumption $v_r(x) = \mu(B(x,r))$ enters. Though being elementary, this lemma constitutes the main trick in the proof of Proposition 2.16.

Lemma 2.19. *Let $r > 0$ and assume $v_r(x) = \mu(B(x,r))$ ($x \in M$). For $\gamma > 0$, $1 \leqslant q \leqslant \infty$ and $f \in L_q(\Omega)$ we then have*

$$\|f\|_q \leqslant e^{\gamma r} \big\| y \mapsto \|\rho_{\gamma,y} v_r^{-1/q} f\|_{L_q(\Omega)} \big\|_{L_q(M)}.$$

Proof. By Fubini's theorem we have, since $\rho_{\gamma,y}(x) = \rho_{\gamma,x}(y)$ for all $x, y \in M$,

$$\big\| y \mapsto \|\rho_{\gamma,y} v_r^{-1/q} f\|_{L_q(\Omega)} \big\|_{L_q(M)} = \big\| x \mapsto \|\rho_{\gamma,x}\|_{L_q(M)} v_r(x)^{-1/q} f(x) \big\|_{L_q(\Omega)}.$$

This implies the assertion since $e^{\gamma r}\|\rho_{\gamma,x}\|_{L_q(M)} \geqslant \|\chi_{B(x,r)}\|_{L_q(M)} = \mu(B(x,r))^{1/q}$ for all $x \in \Omega$. □

In order to present the idea of the proof of Proposition 2.16 in the case $(p_0, q_0) \neq (1, \infty)$, let us first show a simple variant of Proposition 2.16: let $r > 0$ and assume that $v_R \leqslant c_0 e^{c_r R} v_r$ for all $R > r$, with $v_r(x) = \mu(B(x,r))$. Let $1 \leqslant p \leqslant q \leqslant \infty$, $\gamma \geqslant 3c_r$ and $B\colon L_{\infty,c} \to L_{1,loc}$ a linear operator satisfying $\|v_r^{-1/q} B v_r^{1/p}\|_{p\to q,\gamma} \leqslant 1$. Then $\|\rho_{\gamma,y} v_r^{-1/q} Bf\|_q \leqslant \|\rho_{\gamma,y} v_r^{-1/p} f\|_p$ for all $f \in L_{\infty,c}$, $y \in M$. By Lemma 2.19 and Lemma 2.13(b) we obtain

$$\begin{aligned}\|Bf\|_q &\leqslant e^{\gamma r} \big\| y \mapsto \|\rho_{\gamma,y} v_r^{-1/q} Bf\|_q \big\|_{L_q(M)} \\ &\leqslant e^{\gamma r} \big\| y \mapsto \|\rho_{\gamma,y} v_r^{-1/p} f\|_p \big\|_{L_q(M)} \leqslant e^{\gamma r} n_p\big((x,y) \mapsto \rho_{\gamma,y}(x) v_r(x)^{-1/p}\big) \|f\|_q.\end{aligned}$$

Thus, Lemma 2.14 yields $\|B\|_{q\to q} \leqslant c_0^2 (1 - e^{-\gamma r/3})^{-1} e^{2c_r r + \gamma r}$.

Proof of Proposition 2.16. First case: $(p_0, q_0) = (1, \infty)$. By Corollary 2.18, the assumption implies that the operator $v_r^{\alpha_0} B v_r^{\beta_0}$ has an integral kernel k satisfying

$$|k(x,y)| \leqslant e^{4\gamma_0 r - \gamma_0 d(x,y)} \quad \text{(almost all } x, y \in \Omega).$$

Therefore, the kernel h of $\rho v_r^{\alpha} B v_r^{\beta} \rho^{-1}$ satisfies

$$|h(x,y)| \leqslant v_r(x)^{\alpha-\alpha_0} v_r(y)^{\beta-\beta_0} \rho(x)\rho(y)^{-1} e^{4\gamma_0 r - \gamma_0 d(x,y)} \quad \text{(almost all } x, y \in \Omega).$$

Note that $\alpha_0 - \alpha, \beta_0 - \beta \leqslant 1$, $(\alpha_0 - \alpha) + (\beta_0 - \beta) = 1 - p^{-1} + q^{-1} =: s^{-1}$, and $\rho(x)\rho(y)^{-1} \leqslant C e^{\frac{\gamma_0}{2} d(x,y)}$. Thus, Lemma 2.14 implies that

$$n_s(h) \leqslant C e^{4\gamma_0 r} \cdot c_0^2 (1 - e^{-\gamma_0 r/6})^{-1} e^{2c_r r}.$$

By Lemma 2.13(a) we obtain the assertion since $2c_r \leqslant \gamma_0$.

Second case: $(p_0, q_0) \neq (1, \infty)$. Then $v_r(x) = \mu(B(x,r))$ $(x \in M)$. Let $\rho \in P(\gamma_0/2, C)$, $f \in L_{\infty,c}$. We have to show that $\|\rho v_r^{\alpha} B f\|_q \leqslant C^2 c_0^6 (1 - e^{-\gamma_0 r/8})^{-2} \cdot e^{5\gamma_0 r} \|\rho v_r^{-\beta} f\|_p$.

By Lemma 2.19 we have

$$\|\rho v_r^{\alpha} B f\|_q \leqslant e^{2\gamma_0 r} \big\| y \mapsto \|\rho_{2\gamma_0, y} v_r^{-1/q} \cdot \rho v_r^{\alpha} B f\|_q \big\|_{L_q(M)}. \tag{2.16}$$

Let $s^{-1} := q^{-1} - q_0^{-1}$ and $\rho_1 := \rho v_r^{\alpha - q_0^{-1} - \alpha_0}$. Then $v_r^{-1/q} \cdot \rho v_r^{\alpha} = v_r^{-1/s} \rho_1 v_r^{\alpha_0}$, and by Hölder's inequality we estimate

$$\|\rho_{2\gamma_0,y} v_r^{-1/q} \rho v_r^{\alpha} B f\|_q \leqslant \|\rho_{3\gamma_0/8,y} v_r^{-1/s}\|_s \|\rho_{5\gamma_0/8,y} \rho_1 \cdot \rho_{\gamma_0,y} v_r^{\alpha_0} B f\|_{q_0} \tag{2.17}$$

for all $y \in M$. Since $3\gamma_0/8 \geqslant 3c_r$ we can apply Lemma 2.14 to the first factor on the right hand side of (2.17) and obtain

$$\|\rho_{3\gamma_0/8,y} v_r^{-1/s}\|_s \leqslant c_0^2 (1 - e^{-\gamma_0 r/8})^{-1} e^{2c_r r} \quad \text{(almost all } y \in M).$$

To estimate the second factor, note that (2.14) implies that $v_r^{\delta} \in P(c_r, c_0 e^{c_r r})$ for all $|\delta| \leqslant 1$. Since $c_r \leqslant \gamma_0/8$, this yields $\rho_1 \in P(\frac{5}{8}\gamma_0, C c_0 e^{c_r r})$. Using (2.15) and the assumption $\|\rho_{\gamma_0,y} v_r^{\alpha_0} B f\|_{q_0} \leqslant \|\rho_{\gamma_0,y} v_r^{-\beta_0} f\|_{p_0}$, we thus obtain

$$\begin{aligned}
\|\rho_{5\gamma_0/8,y} \rho_1 \cdot \rho_{\gamma_0,y} v_r^{\alpha_0} B f\|_{q_0} &\leqslant \|C c_0 e^{c_r r} \rho_1(y) \cdot \rho_{\gamma_0,y} v_r^{\alpha_0} B f\|_{q_0} \\
&\leqslant C c_0 e^{c_r r} \|\rho_1(y) \cdot \rho_{\gamma_0,y} v_r^{-\beta_0} f\|_{p_0} \\
&\leqslant C^2 c_0^2 e^{2c_r r} \|\rho_{-5\gamma_0/8,y} \rho_1 \cdot \rho_{\gamma_0,y} v_r^{-\beta_0} f\|_{p_0} \\
&\leqslant C^2 c_0^2 e^{2c_r r} \|\rho_{3\gamma_0/8,y} \rho v_r^{a - q_0^{-1} - a_0 - \beta_0} f\|_{p_0}.
\end{aligned}$$

Due to the assumption, $\alpha - q_0^{-1} - \alpha_0 - \beta_0 = p^{-1} - q^{-1} - p_0^{-1} - \beta =: -\beta - s_1^{-1}$. By (2.17) we therefore conclude that

$$\|\rho_{2\gamma_0,y} v_r^{-1/q} \rho v_r^{\alpha} B f\|_q \leqslant C^2 c_0^4 (1 - e^{-\gamma_0 r/8})^{-1} e^{4c_r r} \|\rho_{3\gamma_0/8,y} v_r^{-1/s_1} \cdot \rho v_r^{-\beta} f\|_{p_0}.$$

Inserting this into (2.16) yields, by Lemma 2.13(b) and Lemma 2.14,

$$\begin{aligned}\|\rho v_r^\alpha Bf\|_q &\leqslant e^{2\gamma_0 r}C^2c_0^4(1-e^{-\gamma_0 r/8})^{-1}e^{4c_r r}n_{s_1}\big(\rho_{3\gamma_0/8,y}(x)v_r(x)^{-1/s_1}\big)\cdot\|\rho v_r^{-\beta}f\|_p\\ &\leqslant C^2c_0^6(1-e^{-\gamma_0 r/8})^{-2}e^{6c_r r+2\gamma_0 r}\|\rho v_r^{-\beta}f\|_p.\end{aligned}$$

To complete the proof, note that $6c_r \leqslant \gamma_0$. □

We conclude this section by a lemma which will be used to show strong continuity of consistent semigroups.

Lemma 2.20. *Let $1 \leqslant p < \infty$ and (B_α) a bounded net in $\mathcal{L}(L_p(\Omega))$. Assume that $C := \sup_\alpha \|B_\alpha\|_{p\to p,\gamma} < \infty$ for some $\gamma < 0$, and that there exists $q > p$ such that $B_\alpha f \to f$ in $L_q(\Omega)$ for all $f \in L_{\infty,c}$. Then $B_\alpha \to I$ strongly in $\mathcal{L}(L_p(\Omega))$.*

Proof. Since (B_α) is bounded, it suffices to show $B_\alpha f \to f$ in $L_p(\Omega)$ for all $f \in L_{\infty,c}$. Given f, let $x_0 \in \Omega$, $r_0 > 0$ such that $\operatorname{supp} f \subseteq B(x_0, r_0)$. For $r > r_0$ we obtain (note that $\gamma < 0$!)

$$\begin{aligned}\|B_\alpha f - f\|_p &\leqslant \|(B_\alpha f)\restriction_{B(x_0,r)} - f\|_p + \|(B_\alpha f)\restriction_{B(x_0,r)^c}\|_p\\ &\leqslant \mu(B(x_0,r))^{p^{-1}-q^{-1}}\|B_\alpha f - f\|_q + \|e^{\gamma r}\rho_{\gamma,x_0}B_\alpha f\|_p.\end{aligned}$$

By the assumption we have $\|\rho_{\gamma,x_0}B_\alpha f\|_p \leqslant C\|\rho_{\gamma,x_0}f\|_p$. Since $\gamma < 0$ this implies that $\|e^{\gamma r}\rho_{\gamma,x_0}B_\alpha f\|_p \leqslant e^{\gamma r}C\|\rho_{\gamma,x_0}f\|_p \to 0$ as $r \to \infty$, and we conclude that $B_\alpha f \to f$ in $L_p(\Omega)$. □

2.4 Extrapolation and analyticity

In order to apply Proposition 2.16 in the proof of Theorem 2.1, we need a reformulation of our volume growth conditions: assume that (2.2) and (2.3) hold. We claim that then for all $\varepsilon > 0$ there exists $c_0(\varepsilon) \geqslant 1$ such that

$$v_R \leqslant c_0(\varepsilon)e^{(c_1\vee\frac{\varepsilon}{r})R}v_r \quad (R > r > 0). \tag{2.18}$$

This is shown by a distinguishing three cases. If $R > r \geqslant 1$ then $v_R \leqslant c_0 e^{c_1 R}v_1$ by (2.2), and we are done since $v_1 \leqslant v_r$. If $r < R \leqslant 1$ then $v_R \leqslant c_2\big(\frac{R}{r}\big)^N v_r$ by (2.4), and there exists $c_0(\varepsilon) > 1$ such that $c_2\big(\frac{R}{r}\big)^N \leqslant c_0(\varepsilon)e^{\frac{\varepsilon}{r}R}$ for all $R > r > 0$.

Let now $0 < r \leqslant 1 \leqslant R$. Then $v_R/v_r = v_R/v_1 \cdot v_1/v_r \leqslant c_0e^{c_1R}\cdot c_2\big(\frac{1}{r}\big)^N$. If $r \geqslant \frac{\varepsilon}{2c_1}$ then we conclude that $v_R/v_r \leqslant c_0c_2\big(\frac{2c_1}{\varepsilon}\big)^N e^{c_1R}$; if $r < \frac{\varepsilon}{2c_1}$ then $c_1 < \frac{\varepsilon}{2r}$ and hence $v_R/v_r \leqslant c_0c_2e^{\frac{\varepsilon}{2r}R}\big(\frac{R}{r}\big)^N \leqslant c_0(\varepsilon)e^{\frac{\varepsilon}{r}R}$ for some $c_0(\varepsilon) \geqslant 1$ not depending on r, R. This shows (2.18).

Conversely, assume that (2.18) holds for some $\varepsilon > 0$, $c_0(\varepsilon) \geqslant 1$. Then we easily show estimates of the type (2.2) and (2.3).

Remark 2.21. As a direct consequence of Proposition 2.16 we obtain the following. Assume that (2.18) holds and that (Ω, d) is separable. Let $B\colon L_{\infty,c} \to L_{1,loc}$ be a linear operator, $\varepsilon > 0$. Then, with $K_\varepsilon := \big(c_0(\frac{\varepsilon}{8})\big)^6 (1 - e^{-\varepsilon/8})^{-2}$,

$$\|v_r^\alpha B v_r^\beta\|_{p\to q,\gamma} \leqslant K_\varepsilon e^{5\gamma_0 r} \|v_r^{\alpha_0} B v_r^{\beta_0}\|_{p_0\to q_0,\gamma_0}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0,\ r > 0,\ \gamma_0 \geqslant \frac{\varepsilon}{r} \vee (8c_1),\ |\gamma| \leqslant \gamma_0/2$, and $\alpha_0, \beta_0, \alpha, \beta \geqslant 0$ with $\alpha_0 + \beta_0 = p_0^{-1} - q_0^{-1},\ \alpha + \beta = p^{-1} - q^{-1}$.

For the proof recall from Remark 2.3(d) that the separability of (Ω, d) is essentially equivalent to $\mu(B(x,r)) > 0$ for all $x \in M,\ r > 0$. Then apply Proposition 2.16 with $c_0 := c_0(\frac{\varepsilon}{8})$ and $c_r := c_1 \vee \frac{\varepsilon}{8r}$, and observe that $(1 - e^{-\gamma_0 r/8})^{-2} \leqslant (1 - e^{-\varepsilon/8})^{-2}$ for all $\gamma_0 \geqslant 8c_r = \frac{\varepsilon}{r} \vee (8c_1)$.

In the proof of Theorem 2.1 we will use the following result to pass from small times to large times.

Lemma 2.22. *Let T be a semigroup on $L_p(\Omega)$. Assume that there exist $\varepsilon, t_0 > 0$, $C \geqslant 1$ and $m > 1$ such that $\|T(t)\|_{p\to p,\gamma} \leqslant C$ for all $t \leqslant t_0,\ 0 \leqslant \gamma \leqslant \varepsilon t^{-1/m}$. Then there exist $\omega, \nu \geqslant 0$ such that*

$$\|T(t)\|_{p\to p,\gamma} \leqslant C e^{\omega t + \nu\gamma^m t} \quad (t, \gamma \geqslant 0).$$

Proof. For $\gamma = 0$ the assertion is well-known. In case $\gamma > 0$ let $t_\gamma := (\gamma/\varepsilon)^{-m} \wedge t_0$. For $t \geqslant 0$ choose $n \in \mathbb{N}$ with $(n-1)t_\gamma \leqslant t < nt_\gamma$. Then $n - 1 \leqslant t/t_\gamma \leqslant \big((\gamma/\varepsilon)^m + t_0^{-1}\big)t$ and $\frac{t}{n} \leqslant (\gamma/\varepsilon)^{-m} \wedge t_0$, in particular $\gamma \leqslant \varepsilon\big(\frac{t}{n}\big)^{-1/m}$. By the assumption it follows that

$$\|T(t)\|_{p\to p,\gamma} \leqslant \|T(\tfrac{t}{n})\|_{p\to p,\gamma}^n \leqslant C \cdot C^{n-1} \leqslant C e^{\ln C\cdot\left((\gamma/\varepsilon)^m + t_0^{-1}\right)t} = C e^{\omega t + \nu\gamma^m t}$$

with $\omega = t_0^{-1} \ln C$ and $\nu = \varepsilon^{-m} \ln C$. ∎

We further need to extend the weighted estimate in Lemma 2.22 from real to complex times. The next proposition serves this purpose. Comparable results are shown in [Dav89], [Sch96] and [Hie96] by means of the Phragmen-Lindelöf theorem. But it seems to be more natural to use Stein interpolation, similar to the proof of [Dav95b; Lemma 9] by means of the three lines theorem.

Proposition 2.23. *Let $\rho\colon \Omega \to (0,\infty)$ with $\rho, \rho^{-1} \in L_{\infty,loc}$, and $\theta \in (0, \frac{\pi}{2}]$. Let $F\colon S_\theta \to \mathcal{L}(L_p)$ be a bounded continuous function, analytic in the interior of S_θ, satisfying the inequality*

$$\|\rho^\gamma F(t) \rho^{-\gamma}\| \leqslant C e^{\nu\gamma^m t} \quad (t, \gamma \geqslant 0)$$

for some $C \geqslant 1,\ \nu > 0,\ m > 1$. Then for all $\varphi \in (0, \theta)$ there exists $\nu_\varphi > 0$ such that

$$\|\rho^\gamma F(z) \rho^{-\gamma}\| \leqslant C_1 e^{\nu_\varphi \gamma^m \operatorname{Re} z} \quad (z \in S_\varphi,\ \gamma \geqslant 0),$$

with $C_1 = \max\{\|F\|_\infty, C\}$. If $\theta = \frac{\pi}{2}$ then one can choose $\nu_\varphi = (1 - \frac{2}{\pi}\varphi)^{-m}\nu$.

Proof. Fix $\gamma \geqslant 0$ and let $\varphi(z) := \exp\big(-\frac{\nu\gamma^m}{\sin\theta} e^{i(\frac{\pi}{2}-\theta z)}\big)$ for $0 \leqslant \operatorname{Re} z \leqslant 1$. Then $|\varphi(z)| = \exp\big(-\nu\gamma^m \frac{\sin\theta x}{\sin\theta} e^{\theta y}\big)$, where $z = x + iy$. We apply the Stein interpolation theorem to the function G defined by

$$G(z) := \varphi(z)\rho^{z\gamma} F(e^{i\theta(1-z)})\rho^{-z\gamma} \quad (0 \leqslant \operatorname{Re} z \leqslant 1).$$

For $\operatorname{Re} z = 0$, the function $z \mapsto e^{i\theta(1-z)}$ describes the upper ray of the boundary of S_θ, for $\operatorname{Re} z = 1$ it describes the positive real semi-axis. For $f, g \in L_{\infty,c}$, the function $z \mapsto \langle G(z)f, g\rangle$ is analytic, and we have

$$|\langle G(z)f, g\rangle| \leqslant |\varphi(z)| \|F(e^{i\theta(1-z)})\| \cdot \|\rho^{-z\gamma} f\|_p \|\rho^{z\gamma} g\|_{p'} \leqslant \|F\|_\infty \cdot c\|f\|_p \|g\|_{p'} < \infty,$$

where c depends on γ and on the supports of f and g, but not on z.

The function φ is chosen in such a way that $\|G(z)\| \leqslant C_1 = \max\{\|F\|_\infty, C\}$ for $\operatorname{Re} z = 0, 1$. We infer that $\|G(z)\| \leqslant C_1$ for all $0 \leqslant \operatorname{Re} z \leqslant 1$, so

$$\|\rho^{x\gamma} F(e^{i\theta(1-x)} e^{\theta y})\rho^{-x\gamma}\| \leqslant C_1/|\varphi(x+iy)| = C_1 \exp\big(\nu\gamma^m \tfrac{\sin\theta x}{\sin\theta} e^{\theta y}\big).$$

Choose now $x = 1 - \frac{\varphi}{\theta}$ and let $z := e^{i\theta(1-x)} e^{\theta y} = e^{i\varphi} e^{\theta y}$. Then

$$\|\rho^{x\gamma} F(z)\rho^{-x\gamma}\| \leqslant C_1 \exp\big(\nu\gamma^m \tfrac{\sin(\theta-\varphi)}{\sin\theta} \tfrac{\operatorname{Re} z}{\cos\varphi}\big).$$

Writing $\frac{\gamma}{x} = \frac{\theta}{\theta-\varphi}\gamma$ instead of γ we obtain the assertion with $\nu_\varphi = \nu(\frac{\theta}{\theta-\varphi})^m \frac{\sin(\theta-\varphi)}{\sin\theta\cos\varphi}$. □

Now we are in a position to show the following improved version of Theorem 2.1.

Theorem 2.24. *Assume that* (2.18) *holds and that* (Ω, d) *is separable. Let* $p_0 \leqslant s \leqslant q_0$ *and* T *a semigroup on* $L_s(\Omega)$. *Assume that there exist* $\varepsilon, t_0 > 0$, $m > 1$ *and* $\alpha_0, \beta_0 \geqslant 0$ *with* $\alpha_0 + \beta_0 = p_0^{-1} - q_0^{-1}$ *such that*

$$C_0 := \sup_{0<t\leqslant t_0} \|v_{t^{1/m}}^{\alpha_0} T(t) v_{t^{1/m}}^{\beta_0}\|_{p_0\to q_0, \varepsilon t^{-1/m}} < \infty.$$

(a) *Then* T *extrapolates to an exponentially bounded semigroup* T_p *on* $L_p(\Omega)$, *for all* $p \in [p_0, q_0] \setminus \{\infty\}$.

(b) *If one of the semigroups* T_p *is strongly continuous then so are all of them.*

(c) *Assume that* T_{p_1} *has an exponentially bounded analytic extension to* S_θ *for some* $\theta \in (0, \frac{\pi}{2}]$, $p_1 \in [p_0, q_0] \setminus \{\infty\}$. *Then* T_p *has an exponentially bounded analytic extension to* S_φ *for all* $\varphi \in (0, \theta)$, $p \in [p_0, q_0] \setminus \{\infty\}$.

(d) *In case the assumption of* (c) *holds let* $I := [0, \theta)$, *otherwise* $I := \{0\}$, $S_0 := (0, \infty)$. *Then for all* $\varphi \in I$ *there exist* $C_\varphi \geqslant 1$, $\omega_\varphi \in \mathbb{R}$ *and* $\nu_\varphi \geqslant 0$ *such that*

$$\|v_{(\operatorname{Re} z)^{1/m}}^{\alpha} T(z) v_{(\operatorname{Re} z)^{1/m}}^{\beta}\|_{p\to q,\gamma} \leqslant C_\varphi e^{(\omega_\varphi + \nu_\varphi|\gamma|^m)\operatorname{Re} z} \tag{2.19}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0$, $z \in S_\varphi$, $\gamma \in \mathbb{R}$, $\alpha, \beta \geqslant 0$ *with* $\alpha + \beta = p^{-1} - q^{-1}$. *If* $\theta = \frac{\pi}{2}$ *the one can choose* $\omega_\varphi = \tilde{\omega} + (8\pi c_1)^m (\frac{\pi}{2} - \varphi)^{-m}$ *for some* $\tilde{\omega} \in \mathbb{R}$.

Observe that, by our definition of an analytic semigroup, the above theorem implies Theorem 2.1.

Proof of Theorem 2.24. It suffices to show (d): taking $\alpha = \beta = \gamma = 0$, $z \in (0,\infty)$, $p = q$, the weighted norm estimate (2.19) implies (a). If T_p is strongly continuous for some $p \in [p_0, q_0] \setminus \{\infty\}$ then, for $q \in [p_0, q_0] \setminus \{1,\infty\}$, the strong continuity of T_q follows from [Voi92]; in the case $q = p_0 = 1$ it follows from Lemma 2.20. Finally, part (c) follows from Lemma 1.4.

We first show (2.19) for $p = q$ and $\varphi = 0$, i.e., $z \in (0,\infty)$. Without restriction assume $\varepsilon t_0^{-1/m} \geqslant 8c_1$. Let $t \leqslant t_0$ and $\gamma_0 := \varepsilon t^{-1/m}$. Note that $\gamma_0 \geqslant \varepsilon t_0^{-1/m} \geqslant 8c_1$ and $\gamma_0 t^{1/m} = \varepsilon$. By Remark 2.21 we obtain

$$\begin{aligned} \|v^{\alpha}_{t^{1/m}} T(t) v^{\beta}_{t^{1/m}}\|_{p\to q,\gamma} &\leqslant K_\varepsilon e^{5\varepsilon} \|v^{\alpha_0}_{t^{1/m}} T(t) v^{\beta_0}_{t^{1/m}}\|_{p_0\to q_0,\varepsilon t^{-1/m}} \\ &\leqslant K_\varepsilon e^{5\varepsilon} C_0 := C_1 \quad (t \leqslant t_0,\ \gamma \leqslant \varepsilon t^{-1/m}/2) \end{aligned} \tag{2.20}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = p^{-1} - q^{-1}$. By Lemma 2.22 we infer that there exist $\omega_1, \nu_1 \geqslant 0$ such that

$$\|T(t)\|_{p\to p,\gamma} \leqslant C_1 e^{\omega_1 t + \nu_1 \gamma^m t} \quad (p \in [p_0, q_0],\ t, \gamma \geqslant 0).$$

In particular, T extrapolates to an exponentially bounded semigroup on $L_p(\Omega)$ for all $p \in [p_0, q_0] \setminus \{\infty\}$.

Assume that T_{p_1} has an exponentially bounded extension to S_θ for some $\theta \in (0, \frac{\pi}{2}]$, $p_1 \in [p_0, q_0] \setminus \{\infty\}$. Let $C_2 \geqslant C_1$, $\omega_2 \geqslant \omega_1$ such that $\|T_{p_1}(z)\| \leqslant C_2 e^{\omega_2 \operatorname{Re} z}$ for all $z \in S_\theta$. Then Proposition 2.23 implies that for all $\varphi \in (0, \theta)$ there exists $\nu_2 = \nu_2(\varphi) \geqslant 0$ such that

$$\|T_{p_1}(z)\|_{p_1\to p_1,\gamma} \leqslant C_2 e^{(\omega_2 + \nu_2\gamma^m)\operatorname{Re} z} \quad (z \in S_{\frac{\varphi+\theta}{2}},\ \gamma \geqslant 0). \tag{2.21}$$

If $\theta = \frac{\pi}{2}$ then we can choose $\nu_2(\varphi) = (\frac{1}{2} - \frac{\varphi}{\pi})^{-m}\nu_1$. If there is no exponentially bounded extension then $I = \{0\}$, and (2.21) still holds for $\varphi = 0$, with $\theta := 0$.

Now we show (2.19) for $p = p_0$, $q = q_0$, $\alpha = \alpha_1 := p_1^{-1} - q_0^{-1}$, $\beta = \beta_1 := p_0^{-1} - p_1^{-1}$ and $\gamma > 0$. Let $\varphi \in I$ and choose $\delta > 0$ in such a way that $z - 2\delta \operatorname{Re} z \in S_{\frac{\varphi+\theta}{2}}$ for all $z \in S_\varphi$. Let $z \in S_\varphi$. Then $t_z := (\delta \operatorname{Re} z) \wedge (2\gamma/\varepsilon)^{-m} \wedge t_0 \leqslant t_0$, $\gamma \leqslant \varepsilon t_z^{-1/m}/2$, and $z - 2t_z \in S_{\frac{\varphi+\theta}{2}}$. From (2.20) and (2.21) we therefore obtain, taking into account Remark 2.15,

$$\begin{aligned} &\|v^{\alpha_1}_{t_z^{1/m}} T_{p_1}(z) v^{\beta_1}_{t_z^{1/m}}\|_{p_0\to q_0,\gamma} \\ &\qquad \leqslant \|T_{p_0}(t_z) v^{\beta_1}_{t_z^{1/m}}\|_{p_0\to p_1,\gamma} \|T_{p_1}(z - 2t_z)\|_{p_1\to p_1,\gamma} \|v^{\alpha_1}_{t_z^{1/m}} T_{p_1}(t_z)\|_{p_1\to q_0,\gamma} \\ &\qquad \leqslant C_1 \cdot C_2 e^{(\omega_2 + \nu_2\gamma^m)\operatorname{Re}(z - 2t_z)} \cdot C_1. \end{aligned} \tag{2.22}$$

To obtain the desired estimate (2.19) for $v^{\alpha_1}_{r_z} T_{p_1}(z) v^{\beta_1}_{r_z}$, with $r_z := (\operatorname{Re} z)^{1/m}$, we have to estimate $(v_{r_z}/v_{t_z^{1/m}})^{\alpha_1+\beta_1}$. By (2.18), $v_R/v_r \leqslant c_0(1)e^{R/r}$ for all $r \leqslant$

c_1^{-1}, $R > r$. Without restriction assume $t_0^{1/m} \leqslant c_1^{-1}$ so that $t_z^{1/m} \leqslant c_1^{-1}$. We have $t_z^{-1/m} = (\delta \operatorname{Re} z)^{-1/m} \vee \frac{2\gamma}{\varepsilon} \vee t_0^{-1/m}$. Since $m > 1$ there exists $c \geqslant 0$ such that

$$r_z/t_z^{1/m} \leqslant \delta^{-1/m} + \big(\tfrac{2\gamma}{\varepsilon} \vee t_0^{-1/m}\big)(\operatorname{Re} z)^{1/m} \leqslant c + (1+\gamma^m)\operatorname{Re} z \quad (\operatorname{Re} z > 0). \tag{2.23}$$

Therefore, $v_{r_z}/v_{t_z^{1/m}} \leqslant c_0(1)e^{c+(1+\gamma^m)\operatorname{Re} z}$. Since $\alpha_1+\beta_1 \leqslant 1$, estimate (2.22) yields

$$\|v_{r_z}^{\alpha_1} T_{p_1}(z) v_{r_z}^{\beta_1}\|_{p_0\to q_0,\gamma} \leqslant C_3 e^{(\omega_3+\nu_3\gamma^m)\operatorname{Re} z} \quad (z \in S_\varphi),$$

with $C_3 = C_1^2 C_2 c_0(1) e^c$, $\omega_3 = \omega_2 + 1$, $\nu_3 = \nu_3(\varphi) = \nu_2(\varphi) + 1$.

Finally, let $p_0 \leqslant p \leqslant q \leqslant q_0$, $z \in S_\varphi$, $\gamma \in \mathbb{R}$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = p^{-1} - q^{-1}$. For $\gamma_0 := (2|\gamma|) \vee r_z^{-1} \vee (8c_1)$ we obtain, noting $r_z^{-m} = (\operatorname{Re} z)^{-1}$, that

$$\begin{aligned}\|v_{r_z}^{\alpha_1} T_{p_1}(z) v_{r_z}^{\beta_1}\|_{p_0\to q_0,\gamma_0} &\leqslant C_3 e^{\omega_3 \operatorname{Re} z + \nu_3\left((2|\gamma|)^m + r_z^{-m} + (8c_1)^m\right)\operatorname{Re} z} \\ &= C_4 e^{(\omega_4+\nu_4|\gamma|^m)\operatorname{Re} z},\end{aligned}$$

with $C_4 = C_4(\varphi) = C_3 e^{\nu_3(\varphi)}$, $\omega_4 = \omega_4(\varphi) = \omega_3 + (8c_1)^m \nu_3(\varphi)$, $\nu_4 = \nu_4(\varphi) = 2^m \nu_3(\varphi)$. Note that $\gamma_0 \geqslant \frac{1}{r_z} \vee (8c_1)$ and $|\gamma| \leqslant \gamma_0/2$. Thus, Remark 2.21 yields

$$\|v_{r_z}^{\alpha} T_{p_1}(z) v_{r_z}^{\beta}\|_{p\to q,\gamma} \leqslant K_1 e^{5\gamma_0 r_z} \cdot C_4 e^{(\omega_4+\nu_4|\gamma|^m)\operatorname{Re} z}.$$

As in (2.23), there exists $c \geqslant 1$ such that $5\gamma_0 r_z \leqslant c + (1 + |\gamma|^m)\operatorname{Re} z$, and we obtain the desired estimate (2.19).

For the case $\theta = \frac{\pi}{2}$ we compute

$$\omega_\varphi := \omega_4(\varphi) + 1 = \omega_2 + 2 + (8c_1)^m \big((\tfrac{1}{2} - \tfrac{\varphi}{\pi})^{-m}\nu_1 + 1\big).$$

This yields the last assertion. □

Remark 2.25. Let $T(t) = e^{-tA}$ be a semigroup on a Banach space X which is analytic of angle $\frac{\pi}{2}$. Assume that there exist $C \geqslant 1$, $\omega \in \mathbb{R}$, $\nu > 0$, $m > 1$ such that

$$\|T(z)\| \leqslant C e^{\left(\omega + \nu(\frac{\pi}{2} - \arg z)^{-m}\right)\operatorname{Re} z} \quad (\operatorname{Re} z > 0).$$

Then there exist $\omega_1 \in \mathbb{R}$, $\nu_1 > 0$ such that the spectrum $\sigma(A)$ lies in the 'filled generalised parabola'

$$\big\{x + iy \in \mathbb{C};\, x \geqslant \nu_1 |y|^{\frac{m}{m-1}} - \omega_1\big\}.$$

This easily follows by optimisation from the following fact. If $\varphi \in [0, \frac{\pi}{2})$ and $\|T(z)\| \leqslant C e^{\omega \operatorname{Re} z}$ for all $z \in S_\varphi$ then

$$\sigma(A) \subseteq \big\{x + iy \in \mathbb{C};\, x \geqslant |y|\tan\varphi - \omega\big\}.$$

2.5 L_p-spectral independence

Throughout this section we assume that conditions (2.7) and (2.8) hold. We are going to prove Theorem 2.4 which deals with L_p-spectral independence for generators $-A_p$ of consistent C_0-semigroups T_p. By the formula

$$(\lambda + A_p)^{-n} = \int_0^\infty \tfrac{t^{n-1}}{(n-1)!} e^{-\lambda t} T_p(t)\, dt, \tag{2.24}$$

which holds for λ greater than the type of T_p, the theorem will be reduced to the following result on L_p-spectral independence for consistent bounded operators.

Theorem 2.26. *(cf. [KuVo00; Thms. 1 and 2]) Let B: $L_{\infty,c} \to L_{1,loc}$ be a linear operator satisfying*

$$\|v_1^\alpha B v_1^\beta\|_{p_0 \to q_0, \gamma_0} < \infty$$

for some $\gamma_0 > 0$, $\alpha, \beta \geqslant 0$ with $\alpha + \beta = p_0^{-1} - q_0^{-1}$. Then B extends to a consistent family of bounded operators B_p on $L_p(\Omega)$, $p \in [p_0, q_0] \setminus \{\infty\}$, the spectrum $\sigma(B_p)$ is independent of $p \in [p_0, q_0] \setminus \{\infty\}$, and the operators B_p have consistent resolvents.

In [KuVo00] this theorem was proved under the additional assumption that $\varepsilon \leqslant v_1 \leqslant \varepsilon^{-1}$ for some $\varepsilon > 0$. The general theorem can be reduced to this case by resorting to the weighted measure space $(M, v_1^{-1}\mu)$ and using [KuVo00; Thms. 23 and 26] (see [Vog01]). As we are going to present a selfcontained proof here, we will use a direct approach instead.

Using the methods introduced in this section, one can show the following for the case $q_0 = \infty$: under the assumptions of Theorem 2.26, B extends to a weak*-continuous operator B_∞ on $L_\infty(\Omega)$, and $\sigma(B_\infty) = \sigma(B_{p_0})$. Similarly, the other results of this section hold without the restriction $p < \infty$, after suitable reformulation for the case $p = \infty$. We don't pursue this for the sake of simplicity.

In the case $(p_0, q_0) = (1, \infty)$, Theorem 2.26 is not optimal in two respects: firstly, a weighted $1{\to}\infty$-estimate is a strong assumption on the integral kernel of B (cf. Corollary 2.18). It is possible to replace this assumption by an appropriate integrability assumption. Secondly, if we know more than the subexponential volume growth in (2.7), e.g. if the volume growth is polynomial, then the weight functions $\rho_{\gamma,y}$ can be adjusted to this particular volume growth. See [KuVo00; Thm. 2].

For the proof of Theorem 2.26 we need several preparatory results. The proof of Theorem 2.4 will be given at the end of the section.

In order to prove the inclusion $\rho(B_p) \subseteq \rho(B_q)$ for $p, q \in [p_0, q_0] \setminus \{\infty\}$ one has to show that for $\lambda \in \rho(B_p)$ the operator $(\lambda - B_p)^{-1}\restriction_{L_{\infty,c}}$ extends to a bounded operator on $L_q(\Omega)$. This is expressed in the following elementary lemma which is stated in the general context of topological spaces (cf. [LiVo00; Prop. 4], [Are94; Prop. 2.3]).

Let E, F, G be Hausdorff spaces with $E, F \hookrightarrow G$ (continuous injections) such that $E \cap F$ is dense in both E and F. Let $D \subseteq E \cap F$ be a subset which is dense with respect to the initial topology induced by the imbeddings $E \cap F \hookrightarrow E$ and $E \cap F \hookrightarrow F$.

Lemma 2.27. *([KuVo00; Lemma 9]) Let B_E: $E \to E$ and B_F: $F \to F$ be continuous mappings satisfying $B_E \restriction_D = B_F \restriction_D$. Assume that B_E is continuously invertible and that $B_E^{-1} \restriction_D$ extends to a continuous mapping R: $F \to F$. Then B_F is continuously invertible, and $B_F^{-1} = R$.*

Proof. Since D is dense in $E \cap F$ and $E, F \hookrightarrow G$, we have $B_E \restriction_{E\cap F} = B_F \restriction_{E\cap F}$ and $B_E^{-1} \restriction_{E\cap F} = R \restriction_{E\cap F}$. Hence $RB_F = B_F R = I$ on $E \cap F$. This yields the claim since $E \cap F$ is dense in F. □

We will apply this lemma in the situation $E = L_p(\Omega)$, $F = L_q(\Omega)$, $G = L_{1,loc}$ and $D = L_{\infty,c}$.

In the proof of Theorem 2.26, we will make use of Proposition 2.16 in the following form.

Remark 2.28. Let B: $L_{\infty,c} \to L_{1,loc}$ be a linear operator, $\gamma_0 > 0$. Then

$$\|v_1^{\alpha} B v_1^{\beta}\|_{p\to q,\gamma} \leqslant K_{\gamma_0} \|v_1^{\alpha_0} B v_1^{\beta_0}\|_{p_0 \to q_0,\gamma_0}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0$, $|\gamma| \leqslant \gamma_0/2$, and $\alpha_0, \beta_0, \alpha, \beta \geqslant 0$ with $\alpha_0 + \beta_0 = p_0^{-1} - q_0^{-1}$, $\alpha + \beta = p^{-1} - q^{-1}$. Here we can choose $K_{\gamma_0} := c_{\gamma_0/8}^6 (1 - e^{-\gamma_0/8})^{-2} e^{5\gamma_0/8}$, with $c_{\gamma_0/8}$ from (2.7).

For the proof, apply Proposition 2.16 with $c_r := \frac{\gamma_0}{8}$ and $c_0 := c_{\gamma_0/8}$.

The crucial part in the proof of Theorem 2.26 is the following estimate which implies convergence of weighted operators (cf. [Sch96; Lemma 3.2.3], [LiVo00; Prop. 5(iii)]).

Proposition 2.29. *(cf. [KuVo00; Prop. 15]) Let B: $L_{\infty,c} \to L_{1,loc}$ be a linear operator, $1 \leqslant p < \infty$, $\gamma_0 > 0$. There exist $\delta_{\gamma_0,\gamma} > 0$ with $\delta_{\gamma_0,\gamma} \to 0$ as $\gamma \to 0$ such that*

$$\|\rho B \rho^{-1} - B\|_{p\to p} \leqslant \delta_{\gamma_0,\gamma} \|B\|_{p \to p,\gamma_0}$$

for all $0 < \gamma < \gamma_0$ and $\rho \in P(\gamma, 1)$. In particular, for all $|\gamma| < \gamma_0$, $\rho_{\gamma,y} B \rho_{\gamma,y}^{-1}$ extends to a bounded operator $B_{\gamma,y}$ on $L_p(\Omega)$, and $B_{\gamma,y} \to B_{0,y}$ in the norm as $\gamma \to 0$, uniformly in $y \in M$.

Proof. Let $0 < \gamma < \gamma_0$ and $\rho \in P(\gamma, 1)$. By (2.7) there exists $c_0 > 0$ such that $v_r \leqslant c_0 e^{(\gamma_0 - \gamma)r} v_1$ for all $r > 1$. By (2.14) this implies $v := v_1^{-1/p} \in P(\gamma_0 - \gamma, C)$, with $C := c_0 e^{\gamma_0 - \gamma}$.

Let $f \in L_{\infty,c}$. By Lemma 2.19 we have

$$\|(\rho B \rho^{-1} - B) f\|_p \leqslant e^{2\gamma_0} \big\| y \mapsto \|\rho_{2\gamma_0,y} v (\rho B \rho^{-1} - B) f\|_p \big\|_{L_p(M)}. \tag{2.25}$$

We now write

$$\rho B \rho^{-1} - B = \rho\rho(y)^{-1} B(\rho(y)\rho^{-1} - 1) + (\rho\rho(y)^{-1} - 1)B =: B_1 + B_2,$$

insert this into (2.25), use the triangle inequality and estimate the two resulting terms separately. For the second term we have, using Lemma 2.13(b) and (2.15),

$$\big\|y \mapsto \|\rho_{2\gamma_0,y} v B_2 f\|_p\big\|_{L_p(M)} \leqslant n_p\big(v(x)\rho_{2\gamma_0,y}(x)(\rho_{-\gamma,y}(x)-1)\big)\|B\|_{p\to p}\|f\|_p.$$

Using (2.15) and Lemma 2.13(b) again, we can estimate the first term by

$$\begin{aligned}\big\|y \mapsto \|\rho_{2\gamma_0,y} v B_1 f\|_p\big\|_{L_p(M)} &\leqslant C\big\|y \mapsto \|\rho_{\gamma_0,y} v(y) B(\rho(y)\rho^{-1}-1) f\|_p\big\|_{L_p(M)} \\ &\leqslant C\|B\|_{p\to p,\gamma_0}\big\|y \mapsto \|\rho_{\gamma_0,y} v(y)(\rho_{-\gamma,y}-1) f\|_p\big\|_{L_p(M)} \\ &\leqslant C\|B\|_{p\to p,\gamma_0} n_p\big(v(y)\rho_{\gamma_0,y}(x)(\rho_{-\gamma,y}(x)-1)\big)\|f\|_p.\end{aligned}$$

To complete the proof, note that $\|B\|_{p\to p} \leqslant K_{\gamma_0}\|B\|_{p\to p,\gamma_0}$ by Remark 2.28 and that

$$n_p\big(v(x)\rho_{\gamma_0,y}(x)(\rho_{-\gamma,y}(x)-1)\big) \leqslant n_p\big(v(x)e^{-\gamma_0 d(x,y)/2}\big)\sup_{r\geqslant 0} e^{-\gamma_0 r/2}(e^{\gamma r}-1) \to 0$$

as $\gamma \to 0$, where we used $\rho_{\gamma_0,y} = (\rho_{\gamma_0/2,y})^2$. □

The following consequence of Proposition 2.29 will be used in the proof of Theorem 2.26.

Corollary 2.30. *(cf. [KuVo00; Cor. 16]) Let $1 \leqslant p < \infty$ and B a bounded operator on $L_p(\Omega)$ satisfying $\|B\|_{p\to p,\gamma_0} < \infty$ for some $\gamma_0 > 0$. Let $\lambda \in \rho(B)$. Then there exists $\gamma_1 > 0$ such that $\|(\lambda - B)^{-1}\|_{p\to p,\gamma} < \infty$ for all $|\gamma| \leqslant \gamma_1$.*

Proof. Let $\lambda \in \rho(B)$. By the assumption we have $\|\lambda - B\|_{p\to p,\gamma_0} < \infty$. Recall that inversion is continuous in the open set of invertible elements in $\mathcal{L}(L_p(\Omega))$. Thus, by Proposition 2.29 there exists $\gamma_1 > 0$ such that $\lambda - B_{\gamma,y}$ is invertible for all $y \in M$, $|\gamma| < \gamma_1$, and

$$\sup_{y\in M}\|(\lambda - B_{\gamma,y})^{-1}\|_{p\to p} < \infty.$$

In order to prove $\|(\lambda - B)^{-1}\|_{p\to p,\gamma} < \infty$ it remains to show $(\lambda - B_{\gamma,y})^{-1}\restriction_{D(\rho_{\gamma,y}^{-1},L_p)} = \rho_{\gamma,y}(\lambda - B)^{-1}\rho_{\gamma,y}^{-1}$ for all $y \in M$. The latter is a consequence of the following two facts: $(\lambda - B_{\gamma,y})\restriction_{D(\rho_{\gamma,y}^{-1},L_p)} = \rho_{\gamma,y}(\lambda - B)\rho_{\gamma,y}^{-1}$ for all $y \in M$, and the operator $\rho_{\gamma,y}^{-1}\colon D(\rho_{\gamma,y}^{-1}, L_p) \to L_p(\Omega)$ is bijective since $\rho_{\gamma,y}$ is bounded. □

Proof of Theorem 2.26. By Remark 2.28 it is clear that B extends to a bounded operator B_p on $L_p(\Omega)$, for all $p \in [p_0, q_0]\setminus\{\infty\}$. Let $p, q \in [p_0, q_0]\setminus\{\infty\}$. The operators B_p and B_q are consistent since $L_{\infty,c}$ is dense in $L_p(\Omega) \cap L_q(\Omega)$. So we have to prove the inclusion $\rho(B_p) \subseteq \rho(B_q)$ and the consistency of the resolvents. Let $\lambda \in \rho(B_p)$.

First we study the case $\lambda \neq 0$. Then we can rewrite the resolvent $R(\lambda)$ of B_p as

$$R(\lambda) = \lambda^{-1}I + \lambda^{-2}B_p + \lambda^{-2}B_p R(\lambda) B_p.$$

We have to show that $\lambda \in \rho(B_q)$ and that $(\lambda - B_p)^{-1}$, $(\lambda - B_q)^{-1}$ are consistent, which by Lemma 2.27 amounts to showing L_q-boundedness of $R(\lambda)\restriction_{L_{\infty,c}}$.

It is clear that $\lambda^{-1}I + \lambda^{-2}B$ is L_q-bounded; we will show that $B_pR(\lambda)B$ is L_q-bounded. According to Corollary 2.30 we have $\|R(\lambda)\|_{p\to p,\gamma} < \infty$ for some $0 < \gamma < \frac{\gamma_0}{2}$. Let $\alpha := p_0^{-1} - p^{-1}$, $\beta := p^{-1} - q_0^{-1}$. Then $\|Bv_1^\alpha\|_{p_0\to p,\gamma} < \infty$ and $\|v_1^\beta B\|_{p\to q_0,\gamma} < \infty$ by Remark 2.28. Remark 2.15 implies that $\|v_1^\beta B_pR(\lambda)Bv_1^\alpha\|_{p_0\to q_0,\gamma} < \infty$. Another application of Remark 2.28 yields L_q-boundedness of $B_pR(\lambda)B$.

In the case $\lambda = 0$ we simply write $R(\lambda) = B_pR(\lambda)^3B_p$. By the above we have $\|v_1^\beta R(\lambda)v_1^\alpha\|_{p_0\to q_0,\gamma} < \infty$. Again $\|R(\lambda)\|_{q\to q} < \infty$ by Remark 2.28. □

Remark 2.31. Let $1 \leqslant p < q \leqslant \infty$ and $\alpha := p^{-1} - q^{-1}$. Let B_p be a bounded operator on $L_p(\Omega)$ with $0 \in \rho(B_p)$ and $\|v_1^\alpha B_p\|_{p\to q} < \infty$. Then $\|v_1^\alpha\colon L_p(\Omega) \to L_q(\Omega)\| \leqslant \|B_p^{-1}\|_{p\to p}\|v_1^\alpha B_p\|_{p\to q} < \infty$. Therefore (Ω,μ) cannot contain a sequence (M_n) of bounded subsets satisfying $M_n \supseteq M_{n+1}$ $(n \in \mathbb{N})$ and $0 < \mu(M_n) \to 0$ $(n \to \infty)$.

Then $\Omega = \bigcup_{n=0}^\infty M_n$ where $\mu(M_0) = 0$, and M_n are pairwise disjoint atoms of (Ω,μ) $(n \geqslant 1)$. Therefore, for all $s < \infty$, the space $L_s(\Omega)$ is isometrically isomorphic to the weighted space of sequences $\big\{(x_n);\ \sum_n |x_n|^s\mu(M_n) < \infty\big\}$, and $L_\infty(\Omega)$ is isometrically isomorphic to l_∞. In this case we have $\|B_p\restriction_{L_1(\Omega)\cap L_p(\Omega)}\|_{1\to\infty} < \infty$.

In order to derive Theorem 2.4 from Theorem 2.26 we need some more preparation. Part (b) of the following lemma is inspired by [Kar00; Lemma 6.3].

Lemma 2.32. *Let A be a closed operator in a Banach space X, and $n \in \mathbb{N}$.*

(a) *Let $\lambda_0 \in \rho(A)$, $\lambda \in \mathbb{C}$ such that $(\lambda_0 - \lambda)^{-n} \in \rho\big((\lambda_0 - A)^{-n}\big)$. Then $\lambda \in \rho(A)$, and*

$$(\lambda - A)^{-1} = \sum_{k=1}^n (\lambda_0 - \lambda)^{k-n-1}(\lambda_0 - A)^{-k}\big((\lambda_0 - \lambda)^{-n} - (\lambda_0 - A)^{-n}\big)^{-1}.$$

(b) *Assume that $\sigma(A) \subseteq \big\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \leqslant \omega\big\}$ for some $\omega \in \mathbb{R}$. Then for all $\lambda \in \rho(A)$ there exists $\lambda_0 > \omega$ such that $(\lambda_0 - \lambda)^{-n} \in \rho\big((\lambda_0 - A)^{-n}\big)$.*

Proof. (a) Let $B := (\lambda_0 - A)^{-1}$, $\alpha := (\lambda_0 - \lambda)^{-1}$, $S := \sum_1^n (\lambda_0 - \lambda)^{k-n-1}(\lambda_0 - A)^{-k}$. Then

$$(\lambda - A)S = (B^{-1} - \alpha^{-1})\sum_{k=1}^n \alpha^{n+1-k}B^k = (\alpha - B)\sum_{k=1}^n \alpha^{n-k}B^{k-1} = \alpha^n - B^n,$$

and hence $(\lambda - A)S(\alpha^n - B^n)^{-1} = \mathrm{id}$. In the same way, $S(\alpha^n - B^n)^{-1}(\lambda - A) = \mathrm{id}_{D(A)}$ (note that S and $(\alpha^n - B^n)^{-1}$ commute). This proves (a).

(b) Let $\lambda \in \rho(A)$. By the spectral mapping theorem for bounded operators we have to show that there exists $\lambda_0 > \omega$ such that

$$(\lambda_0 - \lambda)^{-1}e^{2\pi i\frac{k}{n}} \in \rho\big((\lambda_0 - A)^{-1}\big) \quad (k = 0, \dots, n-1).$$

By the spectral mapping theorem for the resolvent, the latter is equivalent to $\mu_k := \lambda_0 - (\lambda_0 - \lambda)e^{2\pi i\frac{k}{n}} \in \rho(A)$ $(k = 0, \dots, n-1)$. For $k = 0$ this is true since

$\lambda \in \rho(A)$. For $k = 1, \ldots, n-1$ we have $\operatorname{Re} \mu_k \geqslant \lambda_0(1 - \cos \frac{2\pi k}{n}) - |\lambda|$, so by the assumption on $\sigma(A)$ we obtain that $\mu_k \in \rho(A)$ if λ_0 is sufficiently large. □

For the next result let E, F be Banach spaces, and assume that there exists a Hausdorff topological vector space G such that $E \hookrightarrow G$, $F \hookrightarrow G$ (continuous injections) and $E \cap F$ is dense in both E and F.

Proposition 2.33. *Let A_E, A_F be closed operators in E, F, respectively, with $\sigma(A_E), \sigma(A_F) \subseteq \{\lambda \in \mathbb{C};\ \operatorname{Re} \lambda \leqslant \omega\}$ for some $\omega \in \mathbb{R}$. Let $n \in \mathbb{N}$. Assume that, for all $\lambda_0 > \omega$, the resolvents $(\lambda_0 - A_E)^{-1}$, $(\lambda_0 - A_F)^{-1}$ are consistent, and $(\lambda_0 - A_E)^{-n}$, $(\lambda_0 - A_F)^{-n}$ have equal spectra and consistent resolvents. Then A_E, A_F have equal spectra and consistent resolvents.*

Proof. We have to show $\rho(A_E) \subseteq \rho(A_F)$ and the consistency of the resolvents. Let $\lambda \in \rho(A_E)$. By Lemma 2.32(b) there exists $\lambda_0 > \omega$ such that $(\lambda_0 - \lambda)^{-n} \in \rho\big((\lambda_0 - A_E)^{-n}\big)$, so $(\lambda_0 - \lambda)^{-n} \in \rho\big((\lambda_0 - A_F)^{-n}\big)$ by the assumption. Since $\lambda_0 \in \rho(A_E) \cap \rho(A_F)$, the assertions of Lemma 2.32(a) are fulfilled for $A = A_E$ as well as $A = A_F$. Thus, $\lambda \in \rho(A_F)$, and $(\lambda - A_E)^{-1}$, $(\lambda - A_F)^{-1}$ are consistent since $(\lambda_0 - A_E)^{-1}$, $(\lambda_0 - A_F)^{-1}$ are consistent and $(\lambda_0 - A_E)^{-n}$, $(\lambda_0 - A_F)^{-n}$ have consistent resolvents. □

Proof of Theorem 2.4. We are going to show a weighted norm estimate for $(\lambda + A_p)^{-n}$ in order to apply Theorem 2.26 to this operator.

By Remark 2.28 it follows from assumption (2.9) that

$$\|v_1^{p_0^{-1} - p^{-1}} T_{p_0}(t)\|_{p_0 \to p, \gamma_0/2} \leqslant K_{\gamma_0} \cdot C\big(\tfrac{t_0}{2}\big)^{-K} =: C_0 \quad \big(\tfrac{t_0}{2} \leqslant t \leqslant t_0,\ p_0 \leqslant p \leqslant q_0\big).$$

With $\alpha := p_0^{-1} - q_0^{-1}$ and $\omega := t_0^{-1} \ln C_0$ we deduce that

$$\|v_1^{\alpha} T_{p_0}(t)\|_{p_0 \to q_0, \gamma_0/2} \leqslant C_0 e^{\omega t} \quad (t \geqslant t_0).$$

For small times, Remark 2.28 applied to assumption (2.9) yields

$$\|v_1^{\alpha} T_{p_0}(t)\|_{p_0 \to q_0, \gamma_0/2} \leqslant K_{\gamma_0} \cdot C t^{-K} \quad (0 < t \leqslant t_0).$$

Let $n \in \mathbb{N}$ with $n > K$. For $\lambda_0 > \omega$ we obtain, using the representation of $(\lambda_0 + A_{p_0})^{-n}$ given in (2.24),

$$\|v_1^{\alpha} (\lambda_0 + A_{p_0})^{-n}\|_{p_0 \to q_0, \gamma_0/2} < \infty.$$

Note that the operators $(\lambda_0 + A_p)^{-n}$ are consistent, by (2.24). Therefore, by Theorem 2.26, they have spectrum independent of $p \in [p_0, q_0] \setminus \{\infty\}$ and consistent resolvents. We conclude the proof by an application of Proposition 2.33. □

In the case $(p_0, q_0) \neq (1, \infty)$ we could avoid using (2.24), and we would need Proposition 2.33 only for the simple case $n = 1$: By the assumption of Theorem 2.4, for all $r \in [p_0, q_0] \setminus \{\infty\}$ there exists $C_r \geqslant C$ such that $\|T_r(t)\| \leqslant C_r$ $(t \leqslant t_0)$. Let $p, q \in [p_0, q_0] \setminus \{\infty\}$ with $p < q$, and let $\theta := \theta_{p,q} \in (0, 1]$ such that

$p^{-1}-q^{-1}=\theta(p_0^{-1}-q_0^{-1})$. Then Stein interpolation between the bounds (2.9) and $\|T_r(t)\| \leqslant C_0$ for suitable $r \in [p,q]$ yields

$$\|v_1^{\theta\alpha} T_p(t) v_1^{\theta\beta}\|_{p\to q,\theta\gamma_0} \leqslant C_r t^{-\theta K} \quad (0 < t \leqslant t_0).$$

Now assume that $\theta_{p,q} < \frac{1}{K}$, i.e., $p^{-1}-q^{-1} < \frac{1}{K}(p_0^{-1}-q_0^{-1})$. Then we obtain, as in the above proof,

$$\|v_1^{\theta\alpha}(\lambda_0 + A_p)^{-1} v_1^{\theta\beta}\|_{p\to q,\theta\gamma_0} < \infty. \tag{2.26}$$

It remains to apply Theorem 2.26 and the spectral mapping theorem for the resolvent to obtain that $\sigma(A_r)$ is independent of $r \in [p,q]$, whenever $p^{-1}-q^{-1} < \frac{1}{K}(p_0^{-1}-q_0^{-1})$. The consistency of the resolvents of the operators A_r is straightforward from the consistency of the resolvents of the operators $(\lambda_0 + A_r)^{-1}$. This proves Theorem 2.4.

Observe that the above idea of proof is not applicable in the case $(p_0, q_0) = (1,\infty)$: then we only have the assumption $v_r(x) \leqslant \mu(B(x,r))$. Hence the weighted $p{\to}q$-estimate (2.26) is of no use — Theorem 2.26 requires a weighted $1{\to}\infty$-estimate.

Chapter 3

L_p-properties of elliptic differential operators

This chapter is devoted to the L_p-theory of second order elliptic differential operators on an open set $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, corresponding to the formal differential expression

$$\mathcal{L} := -\nabla \cdot (a\nabla) + b_1 \cdot \nabla + \nabla \cdot b_2 + V$$

with singular measurable coefficients $a\colon \Omega \to \mathbb{R}^N \otimes \mathbb{R}^N$, $b_1, b_2\colon \Omega \to \mathbb{R}^N$, $V\colon \Omega \to \mathbb{R}$. We are going to construct a positive C_0-semigroup on $L_p := L_p(\Omega)$, whose generator is associated with $\mathcal{L}$ in a natural way which will be made precise below. As it is well-known, this implies well-posedness of the corresponding Cauchy problem.

There is vast literature concerning the case that one can associate a consistent family of C_0-semigroups on all L_p-spaces with the differential expression $\mathcal{L}$. This, however, is not the case of major interest here. In general, $\mathcal{L}$ will be associated with a C_0-semigroup on L_p for p from a proper subinterval of $[1, \infty)$.

The chapter is organised as follows. In Section 3.1 we motivate and formulate our main results concerning the construction of a family of positive C_0-semigroups on L_p associated with $\mathcal{L}$. Moreover, we investigate the problems of extrapolation, analyticity and L_p-spectral independence for these semigroups. The proofs of the main theorems are given in the two subsequent sections. Finally, in Section 3.4 we discuss to what extent our results are sharp.

The contents of this chapter are partly contained in [SoVo00], [LSV00].

3.1 Construction of the semigroup on L_p and main properties

Elliptic operators in divergence form with measurable coefficients are usually defined by means of the form method. The form associated with the above differential expression $\mathcal{L}$ is defined by

$$\tau(u,v) := \langle a\nabla u, \nabla v\rangle + \langle \nabla u, b_1 v\rangle - \langle b_2 u, \nabla v\rangle + \langle Vu, v\rangle \tag{3.1}$$

on a suitable domain $D(\tau)$ corresponding to the boundary conditions. Here and in the following, $\langle f, g\rangle$ is defined as $\int_\Omega f(x)\cdot \overline{g}(x)\,dx$ whenever $f\cdot\overline{g} \in L_1$, for $f, g\colon \Omega \to \mathbb{C}$ or $f, g\colon \Omega \to \mathbb{C}^N$ measurable. We will consider the lower order terms of τ as (not necessarily form bounded) perturbations of the second order term.

Assume that τ is densely defined and fulfils the first Beurling-Deny criterion. (The latter holds if and only if $(\operatorname{Re} u)^+ \in D(\tau)$ for all $u \in D(\tau)$; in this case we have $\tau(u,v) \in \mathbb{R}$ and $\tau(u^+, u^-) = 0$ for all real-valued $u, v \in D(\tau)$—see Definition 1.12.) Then the precise formulation of the problem is as follows: Given $p \in [1,\infty)$, under which conditions on the coefficients a, b_1, b_2, V and the domain $D(\tau)$ is τ associated with a positive C_0-semigroup T_p on L_p, in the sense of Definition 1.20? If τ is associated with $T_p(t) = e^{-tA_p}$, we can regard A_p as the L_p-realisation of $\mathcal{L}$ with boundary conditions prescribed by $D(\tau)$.

The present section is organised as follows. First we formulate conditions on the form τ ensuring that τ is densely defined and fulfils the first Beurling-Deny criterion. We then investigate for which $p \in [1,\infty)$ one can expect τ to be associated with a *quasi-contractive* C_0-semigroup on L_p. A comprehensive answer is given in Theorem 3.2; see also Corollary 3.5.

In general, the set of *all* $p \in [1,\infty)$ such that τ is associated with a positive C_0-semigroup T_p on L_p is strictly larger than the set I determined in Theorem 3.2. In Theorem 3.8 we will show, under some additional restrictions, that I can be extended to the left and to the right if a is uniformly elliptic. Moreover, we show that under these restrictions we have p-independence of the angle of analyticity and of the L_p-spectrum. Our last result, Theorem 3.10, shows that the conditions needed to obtain L_p-spectral independence, for $p \in I$ only, are considerably weaker.

We make the following qualitative assumptions on the coefficients of $\mathcal{L}$.

(a) a is a.e. invertible, $a, a^{-1} \in L_{1,loc}$, and a is sectorial, i.e.,

$$\left|\operatorname{Im}(a\zeta\cdot\overline{\zeta})\right| \leqslant \alpha \operatorname{Re}(a\zeta\cdot\overline{\zeta}) \quad \text{a.e. for all } \zeta \in \mathbb{C}^N$$

for some $\alpha \geqslant 0$. Recall from Proposition 1.28 that

$$\tau_N(u,v) := \langle a\nabla u, \nabla v\rangle,\ D(\tau_N) := \left\{u \in W^1_{1,loc} \cap L_2;\, a\nabla u\cdot\nabla\overline{u} \in L_1\right\}$$

defines a (non-symmetric) Dirichlet form in L_2. Let $\tau_a \subseteq \tau_N$ be a Dirichlet form.

(bV) The potentials $W_j := b_j^\top a_s^{-1} b_j$ $(j = 1,2)$ and V^+ are τ_a-regular, and $Q(V^-) \supseteq D(\tau_a) \cap Q(W_1 + W_2 + V^+)$ (recall that $a_s = \frac{a+a^\top}{2}$).

We define the form τ on $D(\tau) := D(\tau_a) \cap Q(W_1 + W_2 + V^+)$ by (3.1). This is possible since for all $u, v \in D(\tau)$ and $j = 1, 2$ we have, by the Cauchy-Schwarz inequality,

$$|\nabla u \cdot b_j \overline{v}| = \left|a_s^{1/2} \nabla u \cdot a_s^{-1/2} b_j \overline{v}\right| \leqslant \left(a_s \nabla u \cdot \nabla \overline{u}\right)^{1/2} \left(W_j |v|^2\right)^{1/2} \in L_1. \tag{3.2}$$

Since τ_a is a Dirichlet form we have $(\operatorname{Re} u)^+ \in D(\tau)$ for all $u \in D(\tau)$. Therefore, τ fulfils the first Beurling-Deny criterion. Of course, we can define τ in the same way as above without assuming **(bV)**. The reason for assuming **(bV)** is that then $D(\tau)$ is dense in $D(\tau_a)$, by Lemma 1.24(b); in particular, τ is densely defined.

Our first aim is to determine the interval of those $p \in [1, \infty)$ for which τ is associated with a *quasi-contractive* C_0-semigroup on L_p. The only quantitative condition we need is seen from the Lumer-Phillips theorem by a formal computation. Suppose τ is associated with a positive quasi-contractive C_0-semigroup $T_p(t) = e^{-tA_p}$ on L_p, for some $1 < p < \infty$. Then A_p is quasi-accretive which, by the positivity of T_p, is equivalent to $\langle A_p u, u^{p-1} \rangle \geqslant -\omega_p \|u\|_p^p$ for all $0 \leqslant u \in D(A_p)$, for some $\omega_p \in \mathbb{R}$.

Formally, $A_p u = \mathcal{L} u$, $\nabla u^{p-1} = (p-1) u^{p-2} \nabla u = \frac{2}{p'} u^{\frac{p}{2}-1} \nabla u^{\frac{p}{2}}$, and similarly $\nabla u = \frac{2}{p} u^{1-\frac{p}{2}} \nabla u^{\frac{p}{2}}$. Thus,

$$\begin{aligned} \langle A_p u, u^{p-1} \rangle &= \left\langle -\nabla \cdot (a \nabla u) + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + V u, u^{p-1} \right\rangle \\ &= \tfrac{4}{pp'} \langle a \nabla u^{\frac{p}{2}}, \nabla u^{\frac{p}{2}} \rangle + \langle (\tfrac{2}{p} b_1 - \tfrac{2}{p'} b_2) u^{\frac{p}{2}}, \nabla u^{\frac{p}{2}} \rangle + \langle V u^p \rangle \end{aligned}$$

(here and in the following, $\langle f \rangle := \int_\Omega f(x)\,dx$ for all $f \in L_1$). We define symmetric forms τ_p on $D(\tau_p) := D(\tau)$ $(1 < p < \infty)$, as real parts of sesquilinear forms, by

$$\begin{aligned} \tau_p(u) &:= \operatorname{Re}\left(\tfrac{4}{pp'} \langle a \nabla u, \nabla u \rangle + \tfrac{2}{p} \langle \nabla u, b_1 u \rangle - \tfrac{2}{p'} \langle b_2 u, \nabla u \rangle + \langle V u, u \rangle \right) \\ &= \tfrac{4}{pp'} \operatorname{Re} \tau_a(u) + \tfrac{2}{p} \langle \nabla |u|, b_1 |u| \rangle - \tfrac{2}{p'} \langle b_2 |u|, \nabla |u| \rangle + \langle V |u|^2 \rangle \end{aligned}$$

(note that $\operatorname{Re}(\overline{u} \nabla u) = |u| \nabla |u|$ for all $u \in W^1_{1,loc}$ [LeSi81; Appendix, Cor. 1]). Then the natural condition for L_p-accretivity is

$$\tau_p(u) \geqslant -\omega_p \|u\|_2^2 \quad \text{for all } 0 \leqslant u \in D(\tau).$$

This is equivalent to $\tau_p \geqslant -\omega_p$ since τ_p fulfils the first Beurling-Deny criterion. Note that $\tau_2 = \operatorname{Re} \tau$ (as to be expected). Further, we define the symmetric forms τ_1, τ_∞ by setting $p = 1, \infty$ in the above definition:

$$\begin{aligned} \tau_1(u) &:= 2 \langle \nabla |u|, b_1 |u| \rangle + \langle V |u|^2 \rangle, \\ \tau_\infty(u) &:= -2 \langle b_2 |u|, \nabla |u| \rangle + \langle V |u|^2 \rangle \end{aligned}$$

on $D(\tau_1) := D(\tau_\infty) := D(\tau)$.

Now we can explain why we do not just assume V^- to be τ_a-regular in **(bV)**. Suppose that $\tau_p \geqslant -\omega_p$ for some $p \in [1, \infty]$, $\omega_p \in \mathbb{R}$. Then

$$\langle V^- u^2 \rangle \leqslant \tfrac{4}{pp'} \operatorname{Re} \tau_a(u) + \langle (\tfrac{2}{p} b_1 - \tfrac{2}{p'} b_2) u, \nabla u \rangle + \langle (V^+ + \omega_p) u^2 \rangle \quad \left(0 \leqslant u \in D(\tau)\right).$$

If V^- is τ_a-regular then it is $(\tau_a + W_1 + W_2 + V^+)$-regular, by Lemma 1.24(b). Thus, the above inequality is valid for all $0 \leqslant u \in D(\tau_a) \cap Q(W_1 + W_2 + V^+)$, so **(bV)** holds. In other words, if the assumption on V^- in **(bV)** does not hold then none of the forms τ_p is bounded below.

The forms τ_p play a crucial role in all our results on elliptic operators. In the following proposition we collect several simple properties of the forms τ and τ_p which are important for the understanding of the subsequent theorems.

Proposition 3.1. *Assume that* **(a)** *and* **(bV)** *hold. Let I be the set of all $p \in [1,\infty)$ such that $\omega_p := \inf\{\omega \in \mathbb{R};\, \tau_p \geqslant -\omega\} < \infty$ (then $\tau_p \geqslant -\omega_p$ for all $p \in I$).*

(a) *For all potentials $U \geqslant W_1 + W_2 + V^-$, the form $\tau + U$ is sectorial and closed. For all $1 < p < \infty$ and $U \geqslant p'W_1 + pW_2 + V^-$, the symmetric form $\tau_p + U$ is closed. In particular, τ_p is closable for all $p \in I \setminus \{1\}$.*

(b) *The set I is an interval and, for all $p \in \mathring{I}$, there exist $\varepsilon_p > 0,\ c_p \in \mathbb{R}$ such that $\tau_p \geqslant \varepsilon_p \operatorname{Re}\tau_a - c_p$. If $\tau_{p_j} \geqslant -\omega_{p_j}$ $(j = 0, 1)$ for some $1 \leqslant p_0 < p < p_1 \leqslant \infty$ then we can choose $\varepsilon_p = 4(\frac{1}{p_0} - \frac{1}{p})(\frac{1}{p} - \frac{1}{p_1}),\ c_p = \omega_{p_0} \vee \omega_{p_1}$.*

(c) *For all $p, q \in \mathring{I}$, the norms $\|\cdot\|_{\tau_p}$ and $\|\cdot\|_{\tau_q}$ are equivalent.*

Note that, in case $\tau_1, \tau_\infty \geqslant 0$, part (b) of the proposition reads $\tau_p \geqslant \frac{4}{pp'} \operatorname{Re}\tau_a$.

Proof of Proposition 3.1. (a) From (3.2) we deduce by Euclid's inequality ($|ab| \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ for all $a, b \in \mathbb{R},\ \varepsilon > 0$) that the sum of the first order terms of τ is form small with respect to $\tau_a + W_1 + W_2$. Thus, $\tau + U$ is a closed sectorial form for any potential $U \geqslant W_1 + W_2 + V^-$. The analogous argument works for τ_p if $1 < p < \infty$. By Corollary 1.16 we obtain that τ_p is closable if it is bounded below, i.e., $p \in I$.

The proof of (b) and (c) relies on the following identity which results directly from the definition of the forms τ_p: for all $p_0, p_1 \in I,\ \theta \in (0,1)$ and p_θ defined by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$\tau_{p_\theta} = (1-\theta)\tau_{p_0} + \theta\tau_{p_1} + 4\left(\frac{1}{p_\theta p_\theta'} - \frac{1-\theta}{p_0 p_0'} - \frac{\theta}{p_1 p_1'}\right)\operatorname{Re}\tau_a. \tag{3.3}$$

In order to prove (b), it now suffices to show that

$$\frac{1}{p_\theta p_\theta'} - \frac{1-\theta}{p_0 p_0'} - \frac{\theta}{p_1 p_1'} = \left(\frac{1}{p_\theta'} - \frac{1}{p_0'}\right)\left(\frac{1}{p_\theta} - \frac{1}{p_1}\right) \left(= \left(\frac{1}{p_0} - \frac{1}{p_\theta}\right)\left(\frac{1}{p_\theta} - \frac{1}{p_1}\right)\right),$$

which in turn follows from the equality

$$\frac{1}{p_\theta p_0'} + \frac{1}{p_\theta' p_1} = \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)\frac{1}{p_0'} + \left(\frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}\right)\frac{1}{p_1} = \frac{1-\theta}{p_0 p_0'} + \frac{\theta}{p_1 p_1'} + \frac{1}{p_0' p_1}.$$

(c) By (3.3) we have $\tau_{p_\theta} \geqslant (1-\theta)\tau_{p_0} + \theta\tau_{p_1}$. We deduce that, for all $p, q \in \mathring{I}$, there exist $\varepsilon > 0,\ \omega \in \mathbb{R}$ such that $\tau_p \geqslant \varepsilon\tau_q - \omega$ and $\tau_q \geqslant \varepsilon\tau_p - \omega$. □

The form τ itself need not be sectorial. In fact, Theorem 3.2 below includes cases where τ is not even bounded from the left. However, the form $\tau + W_1 + W_2 + V^-$ is sectorial and closed by Proposition 3.1(a). This enables us to make use of Definition 1.20 in the first main theorem of this chapter which reads as follows.

Theorem 3.2. *Assume that* **(a)** *and* **(bV)** *hold. Let I be the interval of all $p \in [1, \infty)$ such that $\omega_p = \inf\{\omega \in \mathbb{R};\ \tau_p \geqslant -\omega\} < \infty$. Then the following assertions hold.*

(a) *The form τ is associated with a consistent family of positive C_0-semigroups $T_p(t) = e^{-tA_p}$ on L_p, $p \in I$, with $\|T_p(t)\| \leqslant e^{\omega_p t}$ for all $p \in I$, $t \geqslant 0$.*

(b) *For all $p \in I \setminus \{1\}$ and $u \in D(A_p)$ we have $|u|^{\frac{p}{2}} \operatorname{sgn} u \in D(\overline{\tau_p})$ and*

$$\operatorname{Re}\langle A_p u, u|u|^{p-2}\rangle \geqslant \overline{\tau_p}(|u|^{\frac{p}{2}} \operatorname{sgn} u). \tag{3.4}$$

(c) *If, in addition,*

$$\big|\operatorname{Im}\langle (b_1 + b_2)u, \nabla u\rangle\big| \leqslant c_1 \tau_p(u) + c_2 \|u\|_2^2 \quad \big(u \in D(\tau)\big) \tag{3.5}$$

for some $p \in \mathring{I}$, $c_1 \geqslant 0$, $c_2 \in \mathbb{R}$ then A_p is an m-sectorial operator for all $p \in \mathring{I}$, in particular, T_p extends to an analytic semigroup on L_p.

Remarks 3.3. (a) We point out that the semigroups T_p are associated with the form τ, not with the forms τ_p. These forms, however, determine important properties of the semigroups T_p.

The method used in the theorem to construct the semigroups T_p is much more natural than it may seem at first sight. The construction amounts to an approximation of the lower order perturbations: not of the first order terms as one might expect but of the potential—recall that we apply Definition 1.20. See also Corollary 3.4(b) below.

(b) The domain of τ_a determines the 'boundary conditions' under consideration (the standard examples are the case of Neumann boundary conditions $\tau_a = \tau_N$ and of Dirichlet boundary conditions $\tau_a = \tau_D := \overline{\tau_N \restriction_{C_c^\infty(\Omega)}}$). Assumption **(bV)** expresses that the lower order perturbations must not disturb the boundary conditions prescribed by $D(\tau_a)$. In the case of Dirichlet boundary conditions, assumption **(bV)** is fulfilled in particular if $W_1, W_2, V \in L_{1,loc}$.

Suppose that assumption **(bV)** is not fulfilled, but $D(\tau)$ is dense in L_2. Let $\widetilde{\tau}_a := \overline{\tau_N \restriction_{D(\tau)}}$. Then assumptions **(a)** and **(bV)** are fulfilled with $\widetilde{\tau}_a$ in place of τ_a, so Theorem 3.2 is still applicable to the form τ. (Note that $\widetilde{\tau}_a$ is a Dirichlet form since condition (1.4) is fulfilled for $D = D(\tau)$—see Definition 1.26.)

(c) If the form τ itself is sectorial then it is closable by Proposition 3.1(a) and Corollary 1.16. In this case we have $2 \in I$, and the operator A_2 constructed in Theorem 3.2 is just the m-sectorial operator associated with $\overline{\tau}$ (cf. the paragraph following Definition 1.20).

(d) We point out that the case $I = \{1\}$ is quite possible. By definition, $1 \in I$ if $\tau_1 \geqslant -\omega$ for some $\omega \in \mathbb{R}$. Note that the coefficient b_2 is not involved in this condition. In particular, if **(a)** holds, $b_1 = 0$ and $V \geqslant 0$ then τ is associated with a positive contractive C_0-semigroup on L_1, whenever $b_2{}^\top a_s^{-1} b_2$ is τ_a-regular.

(e) For the case $p = \infty$ we obtain the following by considering the adjoint picture in L_1. If $\tau_\infty \geqslant -\omega_\infty$ for some $\omega_\infty \in \mathbb{R}$ then we can associate a weak*-continuous semigroup T_∞ on L_∞ with the form τ, which satisfies $\|T_\infty(t)\| \leqslant e^{\omega_\infty t}$ for all $t \geqslant 0$. Observe that the condition on τ_∞ imposes no restriction on b_1.

(f) Estimate (3.4) is analogous to the corresponding estimate in Theorem 1.32(b). In former results on second order elliptic operators with singular first order terms ([Lis96], [LiSe96]), an inequality similar to (3.4) was proved only for $|u|^{\frac{p}{2}}$ in place of $|u|^{\frac{p}{2}} \operatorname{sgn} u$. (Note that $\overline{\tau_p}(v) \geqslant \overline{\tau_p}(|v|)$ for all $v \in D(\overline{\tau_p})$ since $\overline{\tau_p}$ is a closed symmetric form fulfilling the first Beurling-Deny criterion.)

Corollary 3.4. *Let the assumptions and notation be as in Theorem 3.2, and $p \in I$.*

(a) Let $U \geqslant 0$ be τ_a-regular. Then U is T_p-regular, and $\tau + U \leftrightarrow (T_p)_U$.

(b) Let $(U_n)_{n\in\mathbb{N}_0}$ be a sequence of positive potentials such that U_0 is τ_a-regular, $U_n \leqslant U_0$, $\tau + U_n$ is sectorial ($n \in \mathbb{N}_0$) and $U_n \to 0$ a.e. as $n \to \infty$. Then $\tau + U_n$ is closable, the analytic semigroup $T_{U_n,2}$ associated with $\overline{\tau + U_n}$ extrapolates to the C_0-semigroup $(T_p)_{U_n}$ on L_p, and U_n is T_p-regular. In particular, $(T_p)_{U_n} \to T_p$ as $n \to \infty$.

Proof. (a) Let $W := W_1 + W_2 + V^-$. Then $\tau + W$ is a closed sectorial form, by Proposition 3.1(a). By Lemma 1.24(b), U is $(\tau_a + W)$-regular and hence $(\tau + W)$-regular. By Proposition 1.21 we obtain the assertions of (a).

(b) Let $n \in \mathbb{N}_0$. By Proposition 3.1(a), $\tau + U_n + W$ is closed, so $\tau + U_n$ is closable by Corollary 1.16. By (a), U_n is T_p-regular, and $\tau + U_n \leftrightarrow (T_p)_{U_n}$, i.e., $T_{U_n,2}$ and $(T_p)_{U_n}$ are consistent. Finally, by [Voi88; Cor. 3.6], $(T_p)_{U_n} \to T_p$ as $n \to \infty$ since U_0 is T_p-regular. ☐

As a direct consequence of Theorem 3.2 we obtain a more explicit version of that theorem.

Corollary 3.5. *Let $V_+, V_- \geqslant 0$ be τ_a-regular with $V_+ - V_- = V$, and $\tau_+ := \operatorname{Re}\tau_a + V_+$. Assume that* **(a)** *and* **(bV)** *hold and that*

$$(-1)^j \langle b_j u, \nabla u\rangle \leqslant \beta_j \tau_+(u) + B_j\|u\|_2^2, \quad \langle V_- u^2\rangle \leqslant \gamma\tau_+(u) + G\|u\|_2^2$$

($0 \leqslant u \in D(\tau)\cap Q(V_+)$, $j = 1,2$) for some constants $\beta_1, \beta_2, \gamma \geqslant 0$, $B_1, B_2, G \in \mathbb{R}$. Let $I_0 := \big\{p \in [1,\infty);\ \frac{4}{pp'} - \frac{2}{p}\beta_1 - \frac{2}{p'}\beta_2 - \gamma \geqslant 0\big\}$. Then, with the notation of Theorem 3.2, $I \supseteq I_0$, and $\omega_p \leqslant \frac{2}{p}B_1 + \frac{2}{p'}B_2 + G$ for all $p \in I_0$. Moreover, for all $p \in \mathring{I}_0$ and $u \in D(A_p)$ we have $v_p := |u|^{\frac{p}{2}} \operatorname{sgn} u \in D(\tau_+)$ and

$$\operatorname{Re}\langle A_p u, u|u|^{p-2}\rangle \geqslant \big(\tfrac{4}{pp'} - \tfrac{2}{p}\beta_1 - \tfrac{2}{p'}\beta_2 - \gamma\big)\tau_+(v_p) - \big(\tfrac{2}{p}B_1 + \tfrac{2}{p'}B_2 + G\big)\|u\|_p^p.$$

If, in addition,

$$\big|\operatorname{Im}\langle (b_1 + b_2)u, \nabla u\rangle\big| \leqslant c_1\tau_+(u) + c_2\|u\|_2^2 \quad \big(u \in D(\tau) \cap Q(V_+)\big)$$

for some $c_1 \geqslant 0,\ c_2 \in \mathbb{R}$ *then* T_p *extends to an analytic semigroup on* L_p *for all* $p \in \mathring{I}$.

Proof. Since $\tau_+(|u|) \leqslant \tau_+(u)$ for all $u \in D(\tau_+)$, and $1 \geqslant \frac{4}{pp'}$, the assumptions imply that

$$\begin{aligned}\tau_p(u) &= \tfrac{4}{pp'} \operatorname{Re} \tau_a(u) + \langle V_+|u|^2\rangle - \big(-\tfrac{2}{p}\langle b_1|u|, \nabla|u|\rangle\big) - \tfrac{2}{p'}\langle b_2|u|, \nabla|u|\rangle - \langle V_-|u|^2\rangle \\ &\geqslant \big(\tfrac{4}{pp'} - \tfrac{2}{p}\beta_1 - \tfrac{2}{p'}\beta_2 - \gamma\big)\tau_+(u) - \big(\tfrac{2}{p}B_1 + \tfrac{2}{p'}B_2 + G\big)\|u\|_2^2\end{aligned}$$

for all $p \in [1,\infty),\ u \in D(\tau) \cap Q(V_+)$. Let $W := W_1 + W_2 + |V|$. Then τ_p is a bounded form on $D(\tau_a + W)$. Since V_+ is $(\tau_a + W)$-regular by Lemma 1.24(b), we deduce that $\tau_p \geqslant -\big(\frac{2}{p}B_1 + \frac{2}{p'}B_2 + G\big)$ for all $p \in I_0$. Thus, Theorem 3.2(a) implies the first two assertions. In order to obtain the remaining assertions, note that the above also implies that

$$\tau_p \geqslant \big(\tfrac{4}{pp'} - \tfrac{2}{p}\beta_1 - \tfrac{2}{p'}\beta_2 - \gamma\big)\tau_+ - \big(\tfrac{2}{p}B_1 + \tfrac{2}{p'}B_2 + G\big)$$

for all $p \in \mathring{I}_0$. □

Remarks 3.6. (a) The interval I_0 defined in Corollary 3.5 can be non-empty only in the cases $\gamma < 1$, and $\gamma = 1,\ \beta_1 = \beta_2 = 0$ (then $I_0 = \{2\}$).

(b) The assumptions of Corollary 3.5 are in particular fulfilled if $W_j \leqslant \beta_j^2\tau_+ + 2\beta_j B_j$ for $j = 1,2$ and $V_- \leqslant \gamma\tau_+ + G$: then we have, by (3.2) and Euclid's inequality,

$$\begin{aligned}|\langle b_j u, \nabla u\rangle| &\leqslant \frac{\beta_j}{2}\langle a_s \nabla u, \nabla u\rangle + \frac{1}{2\beta_j}\langle W_j|u|^2\rangle \\ &\leqslant \frac{\beta_j}{2}\operatorname{Re}\tau_a(u) + \frac{1}{2\beta_j}\big(\beta_j^2\tau_+(u) + 2\beta_j B_j\|u\|_2^2\big) \leqslant \beta_j\tau_+(u) + B_j\|u\|_2^2\end{aligned}$$

and thus also

$$\big|\operatorname{Im}\langle(b_1+b_2)u, \nabla u\rangle\big| \leqslant (\beta_1+\beta_2)\tau_+(u) + (B_1+B_2)\|u\|_2^2$$

for all $u \in D(\tau) \cap Q(V_+)$. In this way, we reobtain [Lis96; Thms. 1-5] as special cases of Corollary 3.5.

We point out that, besides some additional restrictions, b_2 was assumed to be 0 in [Lis96]. In this case, the semigroup associated with $\tau + V^-$ is L_∞-contractive (see Remark 3.3(e)). This leads to considerable simplifications in the proofs. However, if $b_2 \neq 0$ then it is not clear whether there exists a τ_a-regular potential U such that the semigroup associated with $\tau + U$ is L_∞-bounded.

One of the major disadvantages of the assumption $W_j \leqslant \beta_j^2\tau_+ + 2\beta_j B_j$ (in comparison with the corresponding assumption in Corollary 3.5) is that it does not respect the sign of the drift: if the assumption is fulfilled for b_j then also for $-b_j$.

(c) The example of the Ornstein-Uhlenbeck operator $\mathcal{L} = -\Delta + Bx\cdot\nabla$ shows that the conditions posed on b_1, b_2 in Corollary 3.5 are much less restrictive than the conditions studied in (b). Let $\Omega = \mathbb{R}^N$, $a = \mathrm{id}$, $D(\tau_a) = W_2^1(\mathbb{R}^N)$. Let $V = 0$, $b_2 = 0$, and define b_1 by $b_1(x) = Bx$ for some $B \in \mathbb{R}^N \otimes \mathbb{R}^N$. Then $W_1 = |b_1|^2$ is τ_a-regular. Moreover,

$$-\langle b_1 \nabla u, u\rangle = -\frac{1}{2}\langle b_1, \nabla u^2\rangle = \frac{1}{2}\operatorname{tr} B\|u\|_2^2 \quad \big(0 \leqslant u \in C_c^1(\mathbb{R}^N)\big),$$

so we obtain that the assumptions of Corollary 3.5 hold with $V_+ = V_- = 0$, $\beta_1 = \beta_2 = B_2 = \gamma = G = 0$, $B_1 = \frac{1}{2}\operatorname{tr} B$. Hence, τ is associated with a consistent family of positive quasi-contractive C_0-semigroups T_p on L_p, $p \geqslant 1$, with $\|T_p(t)\| \leqslant e^{\frac{t}{p}\operatorname{tr} B}$ for all $t \geqslant 0$, $p \geqslant 1$.

Now we show that the conditions posed in (b) are not fulfilled with $\gamma < 1$ (cf. (a)) unless $B = 0$. Assume that there exist $\beta_1 \geqslant 0$, $\gamma \in [0,1)$, $B_1, G \in \mathbb{R}$, $V_+ = V_-$ measurable (so that $V_+ - V_- = 0$) such that $W_1 \leqslant \beta_1^2(\tau_a + V_+) + 2\beta_1 B_1$ and $V_- \leqslant \gamma(\tau_a + V_+) + G$. Then $(1-\gamma)V_+ \leqslant \gamma\tau_a + G$ and hence $W_1 \leqslant c(\tau_a + 1)$ for some $c > 0$. If $B \neq 0$ then $W_1 = |b_1|^2$ increases at infinity, in some direction x. More precisely, there exists $x \in \mathbb{R}^N$ such that $W_1 \geqslant \lambda^2$ on $B(\lambda x, 1)$, for all $\lambda \geqslant 1$. It is easy so see that this contradicts $W_1 \leqslant c(\tau_a + 1)$.

For further examples, in particular where τ is associated with a semigroup on L_p only for p from some subinterval of $[1, \infty)$, see Section 3.4.

In the remainder of the section we make the following assumption (cf. [Sem97; Def. 3.1]).

(BC) For all $0 \leqslant \varphi \in W_\infty^1$ (i.e., φ bounded and Lipschitz continuous) that satisfy

$$a_s \nabla\varphi \cdot \nabla\varphi \leqslant c(\operatorname{Re}\tau_a + W_1 + W_2 + V^+ + 1) \quad \text{for some } c > 0,$$

$u \in D(\tau)$ implies $\varphi u \in D(\tau)$.

The above assumption is a restriction on the type of boundary conditions. Below we show that it holds in the case of Neumann and of Dirichlet boundary conditions. However, **(BC)** does not hold for periodic type boundary conditions. In the case $a \in L_\infty$, **(BC)** simply reads $\varphi u \in D(\tau)$ for all $0 \leqslant \varphi \in W_\infty^1$, $u \in D(\tau)$. Thus, if $a \in L_\infty$ then we have the following. Assume that **(BC)** holds, $\tilde{\tau}_a \subseteq \tau_a$ is a Dirichlet form, and that $D(\tilde{\tau}_a)$ is an ideal of $D(\tau_a)$ ($u \in D(\tilde{\tau}_a)$, $v \in D(\tau_a)$ and $|v| \leqslant |u|$ imply $v \in D(\tilde{\tau}_a)$). Then **(BC)** holds with $\tilde{\tau}_a$ in place of τ_a.

Proposition 3.7. *([LiVo00; Prop. 9]) Let* **(a)** *and* **(bV)** *hold. If $\tau_a = \tau_N$ or $\tau_a = \tau_D$ (see Propositions 1.28 and 1.30) then assumption* **(BC)** *is fulfilled.*

Proof. Let $W := W_1 + W_2 + V^+ + 1$. Recall that $D(\tau) = D(\tau_a + W)$.

Let first $\tau_a = \tau_N$. Let $0 \leqslant \varphi \in W_\infty^1$ with $a_s\nabla\varphi\cdot\nabla\varphi \leqslant c(\operatorname{Re}\tau_N + W)$ for some $c > 0$, and $0 \leqslant u \in D(\tau_N + W)$. Then, by Euclid's inequality,

$$\begin{aligned} a_s\nabla(\varphi u)\cdot\nabla(\varphi u) &= \varphi^2 a_s \nabla u\cdot\nabla u + u^2 a_s\nabla\varphi\cdot\nabla\varphi + 2\varphi u a_s \nabla\varphi\cdot\nabla u \\ &\leqslant 2\|\varphi\|_\infty^2 a_s\nabla u\cdot\nabla u + 2u^2 a_s \nabla\varphi\cdot\nabla\varphi \ \in L_1. \end{aligned} \tag{3.6}$$

Hence $\varphi u \in D(\tau_N)$, and it is clear that $\varphi u \in Q(W)$. This proves the assertion for $\tau_a = \tau_N$ since $\tau_N + W$ fulfils the first Beurling-Deny criterion.

Now let $\tau_a = \tau_D$. Let $0 \leqslant \varphi \in W^1_\infty$ with $a_s \nabla\varphi \cdot \nabla\varphi \leqslant c(\operatorname{Re}\tau_D + W)$ for some $c > 0$ (this does not imply $a_s\nabla\varphi\cdot\nabla\varphi \leqslant c(\operatorname{Re}\tau_N + W)$!). Obviously, $D(\tau_D) \cap L_{\infty,c}$ is a dense ideal of $D(\tau_D)$. Thus, by Lemma 1.24(a), $D(\tau_D + W) \cap L_{\infty,c} =: D_{\infty,c}$ is dense in $D(\tau_D + W)$. Let $0 \leqslant u \in D_{\infty,c}$. Below we show that $\varphi u \in D(\tau_D)$. Then we obtain, by (3.6) and the assumption on φ,

$$\begin{aligned}(\tau_D + W)(\varphi u) &\leqslant 2\|\varphi\|_\infty^2 \tau_D(u) + 2\langle a_s\nabla\varphi\cdot\nabla\varphi, u^2\rangle + \|\varphi\|_\infty^2\langle Wu^2\rangle \\ &\leqslant 2(\|\varphi\|_\infty^2 + c)(\tau_D + W)(u).\end{aligned}$$

Since $D_{\infty,c}$ is a lattice and $\tau_D + W$ is a Dirichlet form, we conclude that φ acts as a bounded multiplication operator from $D_{\infty,c} \subseteq D(\tau_D + W)$ to $D(\tau_D + W)$. This yields the assertion since $D_{\infty,c}$ is dense in $D(\tau_D + W)$.

So, let $0 \leqslant u \in D_{\infty,c}$. We have to show $\varphi u \in D(\tau_D)$. Let $(v_n) \subseteq C_c^\infty$ such that $v_n \to u$ in $D(\tau_D)$ as $n \to \infty$. Let $u_n := |v_n| \wedge \|u\|_\infty$ $(\in W^1_{\infty,c})$. Then $\sup_{n\in\mathbb{N}} \tau_D(u_n) \leqslant \sup_{n\in\mathbb{N}} \operatorname{Re}\tau_D(v_n) < \infty$ since τ_D is a Dirichlet form. Let $\psi \in C_c^\infty$ such that $\psi u = u$. Let $w_n := \varphi\psi u_n$. Then $\varphi\psi, w_n \in W^1_{\infty,c} \subseteq D(\tau_D)$ by Proposition 1.30. By (3.6) we obtain that

$$\tau_D(w_n) \leqslant 2\|\varphi\psi\|_\infty^2 \tau_D(u_n) + 2\|u_n\|_\infty^2 \tau_D(\varphi\psi),$$

which implies that $\sup_{n\in\mathbb{N}} \tau_D(w_n) < \infty$ (recall that $\|u_n\|_\infty \leqslant \|u\|_\infty$). Further, $w_n \to \varphi\psi u = \varphi u$ in L_2 as $n \to \infty$. Hence $\varphi u \in D(\tau_D)$, by the lower semicontinuity of τ_D. □

The following is our main result on extrapolation for second order elliptic differential operators.

Theorem 3.8. *Let* **(a)**, **(bV)** *and* **(BC)** *hold, and* $N \geqslant 3$. *Let* I *and* T_p $(p \in I)$ *be defined as in Theorem 3.2. Assume that*

(i) $a \in L_\infty$, *and there exist* $p \in \mathring{I}$, $\varepsilon_p > 0$, $c_p \in \mathbb{R}$ *such that the form* τ_p *admits the Sobolev imbedding*

$$\tau_p(u) \geqslant \varepsilon_p \|u\|^2_{\frac{2N}{N-2}} - c_p\|u\|_2^2 \quad \big(u \in D(\tau)\big);$$

(ii) *there exist* $p \in \mathring{I}$, $C_p \geqslant 0$ *such that* T_p *is analytic, and*

$$\big|\langle (b_1 + b_2)u^2\rangle\big| \leqslant C_p \|u\|_{\tau_p} \|u\|_2 \quad \big(0 \leqslant u \in D(\tau)\big).$$

Let $p_+ := \sup I$, $p_- := \inf I$, $p_{\max} := \frac{N}{N-2}p_+$, $p_{\min} := \big(\frac{N}{N-2}p_-'\big)'$. *In case* $1 \in I$ *let* $I_{\max} := [1, p_{\max})$, *otherwise* $I_{\max} := (p_{\min}, p_{\max})$. *Then* τ *is associated with an analytic semigroup* $T_p(t) = e^{-tA_p}$ *on* L_p, *for all* $p \in I_{\max}$. *The angle of analyticity of* T_p *and the spectrum* $\sigma(A_p)$ *are independent of* $p \in I_{\max}$.

Remarks 3.9. (a) Note that, by Proposition 3.1(c) and Stein interpolation, both assumptions (i) and (ii) of the above theorem are fulfilled for *some* $p \in \mathring{I}$ if and only if they are fulfilled for *all* $p \in \mathring{I}$.

(b) Concerning assumption (i), the reader should think of a uniformly elliptic matrix function a. Then, by Proposition 3.1(b), (i) is fulfilled if either of the following three conditions holds: $D(\tau_a) \subseteq W^1_{2,0}$, Ω has the cone property, or Ω has the extension property (see [Ada75]).

(c) Concerning assumption (ii), note that it is much less restrictive to pose a condition on $\left|\langle (b_1+b_2)u^2\rangle\right|$ than on $\langle |b_1+b_2|u^2\rangle$.

Assumption (ii) is in particular fulfilled if a is uniformly elliptic and

$$|b_1+b_2|^2 \leqslant K_p\tau_p + \tilde{\omega}_p$$

for some $K_p > 0$, $\tilde{\omega}_p \in \mathbb{R}$. To see this, first observe that the latter condition is equivalent to $\|(b_1+b_2)u\|_2 \leqslant k_p\|u\|_{\tau_p}$ for some $k_p > 0$ and all $u \in D(\tau_p)$.

In order to show that T_p is analytic, we check condition (3.5) of Theorem 3.2. For all $u \in D(\tau)$ we have

$$\left|\langle (b_1+b_2)u, \nabla u\rangle\right| \leqslant \|(b_1+b_2)u\|_2\|\nabla u\|_2 \leqslant k_p\|u\|_{\tau_p}\|\nabla u\|_2.$$

By Proposition 3.1(b) and the uniform ellipticity of a, there exists $c > 0$ such that $\|\nabla u\|_2 \leqslant c\|u\|_{\tau_p}$ for all $u \in D(\tau_p)$. Thus,

$$\left|\langle (b_1+b_2)u, \nabla u\rangle\right| \leqslant k_pc\|u\|^2_{\tau_p} = k_pc\big(\tau_p(u) + (\omega_p+1)\|u\|_2^2\big) \quad \big(u \in D(\tau)\big).$$

By Theorem 3.2(c) we infer that T_p is analytic.

Moreover, for all $0 \leqslant u \in D(\tau)$ we have

$$\left|\langle (b_1+b_2)u^2\rangle\right| \leqslant \|(b_1+b_2)u\|_2\|u\|_2 \leqslant k_p\|u\|_{\tau_p}\|u\|_2,$$

i.e., assumption (ii) of Theorem 3.8 is fulfilled.

(d) In the cases $N = 1, 2$, it should be possible to prove an analogue of Theorem 3.8 with $I_{\max} = (1, \infty)$ or even $I_{\max} = [1, \infty)$. Of course assumption (i) has to be reformulated for $N = 1, 2$.

We conclude the section by a result on L_p-spectral independence which is a generalisation of [LiVo00; Thm. 2]. Recall the notion of L_1-regularity from Section 2.2, (2.11).

Theorem 3.10. *Let* **(a)**, **(bV)** *and* **(BC)** *hold. Let* I *and* $T_p(t) = e^{-tA_p}$ $(p \in I)$ *be defined as in Theorem 3.2. Assume that there exist* $p \in \mathring{I}$, $r > 1$, $\varepsilon_p > 0$, $c_p \in \mathbb{R}$ *such that the form* τ_p *admits the Sobolev imbedding*

$$\tau_p(u) \geqslant \varepsilon_p\|u\|^2_{2r} - c_p\|u\|_2^2 \quad \big(u \in D(\tau)\big).$$

(a) *Assume that there exists an* L_1*-regular function* $\psi = (\psi_1, \ldots, \psi_N)\colon \mathbb{R}^N \to \mathbb{R}^N$ *fulfilling the following two inequalities for some* $p \in \mathring{I}$, $c_0 \geqslant 0$, $c_1 \in \mathbb{R}$, *for* *all* $j = 1, \ldots, N$*:*

$$a\nabla\psi_j \cdot \nabla\psi_j \leqslant c_0\tau_p + c_1,$$
$$\left|\langle (b_1 + b_2) \cdot \nabla\psi_j, u^2\rangle\right| \leqslant c_0\tau_p(u) + c_1\|u\|_2^2 \quad \big(0 \leqslant u \in D(\tau)\big).$$

Then the spectrum $\sigma(A_p)$ *is independent of* $p \in I \setminus \{1\}$*, and the operators* A_p *have consistent resolvents.*

(b) *If, more restrictively than in* (a)*, there exist* $c > 0$*,* $\varepsilon \in (0, 2]$*,* $p \in \mathring{I}$ *such that*

$$\langle a\nabla\psi_j \cdot \nabla\psi_j, u^2\rangle + \left|\langle (b_1 + b_2) \cdot \nabla\psi_j, u^2\rangle\right| \leqslant c\|u\|_{\tau_p}^{2-\varepsilon}\|u\|_2^{\varepsilon}$$

for all $0 \leqslant u \in D(\tau)$*,* $j = 1, \ldots, N$ *then* $\sigma(A_p)$ *is independent of* $p \in I$*, and the operators* A_p *have consistent resolvents.*

Remarks 3.11. (a) Note that the function $\psi(x) = x$ is L_1-regular, and $\nabla\psi_j = e_j$ for all $j = 1, \ldots, N$, where e_j are the standard unit vectors of $\mathbb{R}^N$. Thus, the assumptions of Theorem 3.10 are much weaker than the assumptions of Theorem 3.8. In particular, we do not need to assume the matrix function a to be bounded—if $\psi(x) = x$ then the first condition in Theorem 3.10(a) is just form boundedness of a_{jj} with respect to τ_p, for all $j = 1, \ldots, N$. The attentive reader will note that the latter is a self-referential condition on a. It allows a to have strong local singularities.

(b) The following example reveals the relevance of the notion of L_1-regularity: Let $K_n \neq \varnothing$ ($n \in \mathbb{N}$) be compact subsets of $\mathbb{R}^N$ such that

$$\sup_{n\in\mathbb{N}} \operatorname{diam}(K_n) < \infty, \quad \inf_{n\neq m} \operatorname{dist}(K_n, K_m) > 0.$$

For $n \in \mathbb{N}$ fix some $x_n \in K_n$. Define the function ψ_0 on $\bigcup_{n\in\mathbb{N}} K_n$ by $\psi_0(x) := x_n$ for all $n \in \mathbb{N}$, $x \in K_n$. Then ψ_0 is Lipschitz continuous.

By Kirszbraun's theorem (see, e.g., [Fed96; 2.10.43.]), ψ_0 has a Lipschitz continuous extension ψ to $\mathbb{R}^N$. Without restriction assume that for all $j \in \mathbb{Z}^N$ there exists $n \in \mathbb{N}$ such that $\operatorname{dist}(j, K_n) \leqslant 1$ (otherwise add $\{j\}$ to the collection of K_n). Then it is easy to see that $\sup_{j\in\mathbb{Z}^N} |\psi(j) - j| < \infty$. From this we deduce that ψ is L_1-regular. By the construction, $\nabla\psi_j\restriction_{K_n} = 0$ for all $n \in \mathbb{N}$, $j = 1, \ldots, N$. Thus, by the assumptions on $a\nabla\psi_j \cdot \nabla\psi_j$ and $(b_1 + b_2) \cdot \nabla\psi_j$ in Theorem 3.10 we pose no restrictions on the coefficients a, b_1, b_2 on $\bigcup_{n\in\mathbb{N}} K_n$.

3.2 Quasi-contractive C_0-semigroups

In this section we prove Theorem 3.2. We separate the core of the proof into a lemma. Fix $p \in (1, \infty)$. For $u \in L_{1,loc}$, $n \in \mathbb{N}$ let $u_{n,p} := \big(|u|^{\frac{p}{2}-1}\big) \wedge n$, $v_{n,p} := uu_{n,p}$, $w_{n,p} := uu_{n,p}^2$, and $v_p(u) := u|u|^{\frac{p}{2}-1}$, $w_p(u) := u|u|^{p-2}$ as in Lemma 1.34.

Lemma 3.12. *Let τ be a densely defined sesquilinear form in L_2 fulfilling the first Beurling-Deny criterion. Let $\mathfrak{h}$ be a closed symmetric form in L_2, $\mathfrak{h} \geqslant -\omega$ for some $\omega \in \mathbb{R}$. Assume that there exists a sequence $(U_n)_{n\in\mathbb{N}_0}$ of positive potentials such that $D(U_0) \supseteq D(\tau)$, $\tau + U_0$ is sectorial and closed, $U_n \downarrow 0$ as $n \to \infty$, and*

$$w_{n,p} \in D(\tau),\ v_{n,p} \in D(\mathfrak{h}),\ \operatorname{Re}\tau(u, w_{n,p}) \geqslant \mathfrak{h}(v_{n,p}) - \langle U_n |v_{n,p}|^2\rangle \tag{3.7}$$

for all $u \in D(\tau)$, $n \in \mathbb{N}$.

(a) *Then τ is associated with a positive C_0-semigroup $T_p(t) = e^{-tA_p}$ on L_p with $\|T_p(t)\| \leqslant e^{\omega t}$ $(t \geqslant 0)$, and for all $u \in D(A_p)$ we have*

$$\operatorname{Re}\langle A_p u, w_p(u)\rangle \geqslant \mathfrak{h}(v_p(u)). \tag{3.8}$$

(b) *If, in addition,*

$$|\operatorname{Im}\tau(u, w_{n,p})| \leqslant M(\operatorname{Re}\tau + U_n + \tilde{\omega})(u, w_{n,p}) \quad (u \in D(\tau),\ n \in \mathbb{N})$$

for some $M \geqslant 0$, $\tilde{\omega} \in \mathbb{R}$, then A_p is m-sectorial of angle $\arctan M$. *In particular, T_p is an analytic semigroup.*

Proof. (a) Without restriction assume $\omega = 0$. Let A_0 be the m-sectorial operator in L_2 associated with $\tau + U_0$. In a first step we show that e^{-tA_0} extrapolates to a contractive C_0-semigroup $T_{0,p}(t) = e^{-tA_{0,p}}$ on L_p. Then we make use of Lemma 1.34 to prove the assertions of (a).

(i) By the exponential formula, it suffices to show that, given $f \in L_2 \cap L_p$ and $0 < \lambda \in \rho(-A_0)$, one has $\|(\lambda + A_0)^{-1} f\|_p \leqslant \frac{1}{\lambda}\|f\|_p$. Let $u := (\lambda + A_0)^{-1} f$. Then $u \in D(\tau + U_0) = D(\tau)$. This implies that $v_{n,p} \in Q(U_0)$. By assumption (3.7) and the equality $u\overline{w}_{n,p} = |v_{n,p}|^2$ we have, for all $n \in \mathbb{N}$,

$$\begin{aligned} \lambda\|v_{n,p}\|_2^2 + (\mathfrak{h} + U_0 - U_n)(v_{n,p}) &\leqslant \lambda\langle u, w_{n,p}\rangle + \operatorname{Re}(\tau + U_0)(u, w_{n,p}) \\ &= \operatorname{Re}\big\langle(\lambda + A_0)u, w_{n,p}\big\rangle \leqslant \|f\|_p\|w_{n,p}\|_{p'}. \end{aligned} \tag{3.9}$$

Observe that $|w_{n,p}|^{p'} = |u|^{p'} u_{n,p}^{2p'} \leqslant |u|^2 u_{n,p}^2 = |v_{n,p}|^2$. Hence $\|w_{n,p}\|_{p'} \leqslant \|v_{n,p}\|_2^{\frac{2}{p'}}$, and from (3.9) we obtain, noting $\mathfrak{h} + U_0 - U_n \geqslant \mathfrak{h} \geqslant 0$,

$$\|v_{n,p}\|_2^{\frac{2}{p}} \leqslant \tfrac{1}{\lambda}\|f\|_p \quad (n \in \mathbb{N}).$$

Since $|v_{n,p}| \uparrow |v_p(u)|$ we conclude by monotone convergence that $v_p(u) \in L_2$, and

$$\|(\lambda + A_0)^{-1} f\|_p = \|u\|_p = \|v_p(u)\|_2^{\frac{2}{p}} \leqslant \tfrac{1}{\lambda}\|f\|_p.$$

(ii) With the quantities introduced in (i) we proceed as follows. By dominated convergence, $v_{n,p} \to v_p(u)$ in L_2 and $w_{n,p} \to w_p(u)$ in $L_{p'}$ as $n \to \infty$. Further, $A_0 u = f - \lambda u \in L_p$. From estimate (3.9) we obtain

$$\liminf_{n\to\infty}\big(\mathfrak{h}(v_{n,p}) + \langle(U_0 - U_n)|v_{n,p}|^2\rangle\big) \leqslant \lim_{n\to\infty}\operatorname{Re}\langle A_0 u, w_{n,p}\rangle = \operatorname{Re}\langle A_0 u, w_p(u)\rangle.$$

By monotone convergence, $(U_0 - U_n)|v_{n,p}|^2 \uparrow U_0|v_p(u)|^2$ in L_1. Hence, the left hand side of the previous inequality equals $\liminf_{n\to\infty} \mathfrak{h}(v_{n,p}) + \langle U_0|v_p(u)|^2\rangle$. The lower semicontinuity of $\mathfrak{h}$ implies that

$$v_p(u) \in D(\mathfrak{h}), \quad (\mathfrak{h} + U_0)(v_p(u)) \leqslant \operatorname{Re}\langle A_0 u, w_p(u)\rangle. \tag{3.10}$$

So far we have proved inequality (3.10) for all u from the core $D := (\lambda + A_0)^{-1}(L_2 \cap L_p)$ of $A_{0,p}$, where $\lambda > 0$ is some element of $\rho(-A_0)$.

Let now $u \in D(A_{0,p})$. Choose $(u^{(m)}) \subseteq D$ such that $u^{(m)} \to u$ in $D(A_{0,p})$. Then $v_p(u^{(m)}) \to v_p(u)$ in L_2 and $w_p(u^{(m)}) \to w_p(u)$ in $L_{p'}$. By (3.10) we obtain

$$\liminf_{m\to\infty} (\mathfrak{h} + U_0)(v_p(u^{(m)})) \leqslant \lim_{m\to\infty} \operatorname{Re}\langle A_{0,p} u^{(m)}, w_p(u^{(m)})\rangle = \operatorname{Re}\langle A_{0,p} u, w_p(u)\rangle,$$

so the lower semicontinuity of $\mathfrak{h} + U_0$ implies that (3.10) holds for all $u \in D(A_{0,p})$.

Now we are in a position to apply Lemma 1.34, with $\mathfrak{h} + U_0$ in place of $\mathfrak{h}$, $A = A_{0,p}$, and $V = -U_0$. By (3.10) we infer that $-U_0$ is $T_{0,p}$-admissible, that $T_p := (T_{0,p})_{-U_0}$ is a contractive C_0-semigroup, and that (3.8) holds, with $-A_p$ the generator of T_p.

(b) Recall that $\tau + U_0 \leftrightarrow e^{-tA_{0,p}}$. For $m \in \mathbb{N}$ let $A_m := A_{0,p} - U_0 \wedge m$. Then $\tau + (U_0 - m)^+ \leftrightarrow e^{-tA_m}$. Let $u \in D(A_0) \cap D(A_{0,p})$. Then, since $u\overline{w}_{n,p} = |v_{n,p}|^2$ is real and $A_{0,p} u = A_0 u$,

$$\operatorname{Im}\langle A_m u, w_{n,p}\rangle = \operatorname{Im}\langle (A_0 - U_0 \wedge m)u, w_{n,p}\rangle = \operatorname{Im}\tau(u, w_{n,p}) \quad (m, n \in \mathbb{N}). \tag{3.11}$$

By (3.10) we know that $U_n|u\overline{w}_{n,p}| \leqslant U_0|v_p(u)|^2 \in L_1$. Thus, $\langle U_n u, w_{n,p}\rangle \to 0$ by dominated convergence. By (3.11) and the assumption of (b) we conclude that

$$\begin{aligned}
|\operatorname{Im}\langle A_m u, w_p(u)\rangle| &= \lim_{n\to\infty} |\operatorname{Im}\tau(u, w_{n,p})| \\
&\leqslant \lim_{n\to\infty} M(\operatorname{Re}\tau + (U_0 - m)^+ + U_n + \tilde{\omega})(u, w_{n,p}) = M \operatorname{Re}\langle (A_m + \tilde{\omega})u, w_p(u)\rangle.
\end{aligned}$$

This estimate carries over to all $u \in D(A_m)$ since $D(A_0) \cap D(A_{0,p})$ is a core for A_m.

Let now $u \in D(A_p)$. We have $A_m \to A_p$ in the strong resolvent sense. Thus, $u^{(m)} := (1 + A_m)^{-1}(1 + A_p)u \to u$ in L_p and $w_p(u^{(m)}) \to w_p(u)$ in $L_{p'}$ as $m \to \infty$. Since

$$u^{(m)} + A_m u^{(m)} = u + A_p u,$$

we also have $A_m u^{(m)} \to A_p u$ in L_p as $m \to \infty$. Hence,

$$\begin{aligned}
|\operatorname{Im}\langle A_p u, w_p(u)\rangle| &= \lim_{m\to\infty} |\operatorname{Im}\langle A_m u^{(m)}, w_p(u^{(m)})\rangle| \\
&\leqslant \lim_{m\to\infty} M \operatorname{Re}\langle (A_m + \tilde{\omega})u^{(m)}, w_p(u^{(m)})\rangle = M \operatorname{Re}\langle (A_p + \tilde{\omega})u, w_p(u)\rangle,
\end{aligned}$$

which shows the m-sectoriality of A_p, with angle $\arctan M$. □

From the proof we easily see: In order to show $\tau \leftrightarrow T_p$ on L_p with $\|T_p(t)\| \leqslant e^{\omega t}$ it suffices to require $\tau(u, w_{n,p}) + \langle U_n v_{n,p}^2 \rangle \geqslant -\omega$ for all $0 \leqslant u \in D(\tau),\ n \in \mathbb{N}$.

In order to use Lemma 3.12 in the proof of Theorem 3.2, we need to know that $v_{n,p}$ and $w_{n,p}$ are multiples of normal contractions of u. This is a consequence of the following lemma.

Lemma 3.13. *Let* $\varphi\colon [0,\infty) \to \mathbb{R}$ *such that* $\varphi(0) = 0$ *and* $|\varphi(s) - \varphi(t)| \leqslant |s-t|$ *for all* $s,t \geqslant 0$. *Then* $\hat\varphi(z) := \varphi(|z|) \operatorname{sgn} z$ *defines a normal contraction* $\hat\varphi$ *on* $\mathbb{C}$.

Proof. Let $s,t \geqslant 0,\ \alpha,\beta \in [0,2\pi)$. Then we have

$$\begin{aligned}\big|\hat\varphi(se^{i\alpha}) - \hat\varphi(te^{i\beta})\big|^2 &= |\varphi(s)e^{i\alpha} - \varphi(t)e^{i\beta}|^2 = \varphi(s)^2 + \varphi(t)^2 - 2\varphi(s)\varphi(t)\operatorname{Re} e^{i\alpha - i\beta}\\ &= \big(\varphi(s) - \varphi(t)\big)^2 + 2\varphi(s)\varphi(t)\big(1 - \operatorname{Re} e^{i\alpha - i\beta}\big)\\ &\leqslant (s-t)^2 + 2st\big(1 - \operatorname{Re} e^{i\alpha-i\beta}\big) = \big|se^{i\alpha} - te^{i\beta}\big|^2. \qquad \square\end{aligned}$$

In the application to the functions $v_{n,p}$, $w_{n,p}$, the function φ is of the type

$$\varphi_{\alpha,r}(x) = x(x^\alpha \wedge r) = \begin{cases} x^{\alpha+1} & \text{if } x^\alpha < r,\\ rx & \text{if } x^\alpha \geqslant r,\end{cases}$$

with $\alpha \in \mathbb{R},\ r > 0$. Here, $x^0 := 1$ for all $x \geqslant 0$. In the next lemma we compute the gradient of $\hat\varphi_{\alpha,r} \circ u$ for $u \in W^1_{1,loc}$.

Lemma 3.14. *Let* $\alpha \in \mathbb{R},\ r > 0,\ u \in W^1_{1,loc},\ u_{\alpha,r} := |u|^\alpha \wedge r$. *Then* $uu_{\alpha,r} = \hat\varphi_{\alpha,r} \circ u$ *is a multiple of a normal contraction of* u, $uu_{\alpha,r} \in W^1_{1,loc}$, *and*

$$\nabla(uu_{\alpha,r}) = u_{\alpha,r}\big(\nabla u + \alpha \operatorname{sgn} u \cdot \chi_{[|u|^\alpha < r]} \nabla |u|\big).$$

Proof. The first assertion follows from Lemma 3.13 and the Lipschitz continuity of the function $\varphi_{\alpha,r}$ defined above. In the case $\alpha \notin (0,1)$, the function $[0,\infty) \ni x \mapsto x^\alpha \wedge r$ is Lipschitz continuous, hence $u_{\alpha,r} \in W^1_{1,loc}$, so the remaining assertions follow by an application of the product rule and the chain rule (for the latter see, e.g., [BoMu82; appendix]—the proof given there for $u \in W^1_{2,0}$ works also for $u \in W^1_{1,loc}$).

In the case $\alpha \in (0,1)$, we approximate $|u|$ by $u_\delta := |u| + \delta$ ($\delta > 0$). Note that $(u_\delta^\alpha \wedge r)\nabla u \in L_{1,loc}$ and

$$u\nabla(u_\delta^\alpha \wedge r) = \alpha u u_\delta^{\alpha-1} \chi_{[u_\delta^\alpha < r]} \nabla u_\delta = (u_\delta^\alpha \wedge r)\cdot \alpha \frac{u}{u_\delta} \chi_{[u_\delta^\alpha < r]} \nabla |u| \in L_{1,loc}.$$

By the product rule we obtain that

$$\nabla\big(u(u_\delta^\alpha \wedge r)\big) = (u_\delta^\alpha \wedge r)\big(\nabla u + \alpha \frac{u}{u_\delta} \chi_{[u_\delta^\alpha < r]} \nabla |u|\big).$$

Finally, $u(u_\delta^\alpha \wedge r) \to uu_{\alpha,r}$ and

$$\nabla\big(u(u_\delta^\alpha \wedge r)\big) \to u_{\alpha,r}\big(\nabla u + \alpha \operatorname{sgn} u \cdot \chi_{[|u|^\alpha < r]} \nabla |u|\big)$$

in $L_{1,loc}$ as $\delta \to 0$. This implies the assertion. $\square$

Proof of Theorem 3.2. Let $p \in I$, i.e., $\tau_p \geqslant -\omega_p$. Let $U_0 := W_1 + W_2 + V^-$. By Proposition 3.1(a), $\tau + U_0$ is a closed sectorial form.

First we study the case $p > 1$. Let $u \in D(\tau)$. Then $v_{n,p}, w_{n,p} \in D(\tau)$ as multiples of normal contractions of u. At the end of the proof we will show that

$$\operatorname{Re} \tau(u, w_{n,p}) \geqslant \tau_p(v_{n,p}) - \tfrac{1}{2}\langle \chi_n (W_1 + W_2)|v_{n,p}|^2\rangle, \tag{3.12}$$

where χ_n is the indicator of the set $[|u|^{\frac{p-2}{2}} \geqslant n]$. Applying Lemma 3.12(a) with $\mathfrak{h} = \overline{\tau_p}$ and $U_n = \frac{1}{2}\chi_n(W_1 + W_2)$ ($n \in \mathbb{N}$), we obtain all the assertions of Theorem 3.2(a) and (b).

Let now assumption (3.5) hold for some $p \in \mathring{I}$. Then it holds for all $p \in \mathring{I}$, by Proposition 3.1(b). To prove the analyticity of T_p we need the inequality

$$|\operatorname{Im} \tau(u, w_{n,p})| \leqslant |\operatorname{Im} \tau_a(v_{n,p})| + |\tfrac{1}{p} - \tfrac{1}{p'}| \operatorname{Re} \tau_a(v_{n,p}) + |\operatorname{Im}\langle (b_1 + b_2)v_{n,p}, \nabla v_{n,p}\rangle|, \tag{3.13}$$

which is also shown at the end of the proof. The first term on the right hand side of (3.13) can be estimated by $\alpha \operatorname{Re} \tau_a(v_{n,p})$, due to assumption **(a)**. Thus, by (3.5) we obtain that

$$|\operatorname{Im} \tau(u, w_{n,p})| \leqslant \bigl(\alpha + |\tfrac{1}{p} - \tfrac{1}{p'}|\bigr) \operatorname{Re} \tau_a(v_{n,p}) + c_1 \tau_p(v_{n,p}) + c_2 \|v_{n,p}\|^2.$$

By Proposition 3.1(b) and estimate (3.12) we have

$$\operatorname{Re} \tau_a(v_{n,p}) \leqslant C(\tau_p + \tilde{\omega}_1)(v_{n,p}) \leqslant C(\operatorname{Re} \tau + U_n + \tilde{\omega}_1)(u, w_{n,p})$$

for some $\tilde{\omega}_1 \in \mathbb{R}$, $C > 0$ depending on p. We conclude that

$$|\operatorname{Im} \tau(u, w_{n,p})| \leqslant \bigl[C\bigl(\alpha + |\tfrac{1}{p} - \tfrac{1}{p'}|\bigr) + c_1\bigr](\operatorname{Re} \tau + U_n + \tilde{\omega}_2)(u, w_{n,p})$$

for some $\tilde{\omega}_2 \in \mathbb{R}$, so Lemma 3.12(b) implies that A_p is an m-sectorial operator.

The proof for the case $p = 1$ is based on the assertions of the theorem in the case $p > 1$, applied to the form $\tilde{\tau} := \tau + U_0$, with U_0 as above. Recall that $\tilde{\tau}$ is a closed sectorial form in L_2. Let T_0 be the associated analytic semigroup on L_2. Let $1 < p < \infty$ and $\tilde{\tau}_p := \tau_p + U_0$. For all $0 \leqslant u \in D(\tilde{\tau}) = D(\tau)$ we have

$$\tilde{\tau}_p(u) = \tfrac{4}{pp'}\tau_a(u) - \tfrac{2}{p'}\langle b_2 u, \nabla u\rangle + \tfrac{1}{p}\bigl(2\langle \nabla u, b_1 u\rangle + \langle V u^2\rangle\bigr) + \langle (\tfrac{1}{p'}V + U_0)u^2\rangle.$$

We apply Euclid's inequality to the second term, and the estimate

$$\tau_1(u) = 2\langle \nabla u, b_1 u\rangle + \langle V u^2\rangle \geqslant -\omega_1 \|u\|_2^2$$

to the third term on the right hand side, to obtain

$$\begin{aligned}\tilde{\tau}_p(u) &\geqslant \tfrac{4}{pp'}\tau_a(u) - \tfrac{2}{p'}\bigl(\tfrac{1}{2}\tau_a(u) + \tfrac{1}{2}\langle W_2 u^2\rangle\bigr) - \tfrac{\omega_1}{p}\|u\|_2^2 + \langle (U_0 - \tfrac{1}{p'}V^-)u^2\rangle \\ &= \tfrac{1}{p'}(\tfrac{4}{p} - 1)\tau_a(u) - \tfrac{\omega_1}{p}\|u\|_2^2 + \langle (U_0 - \tfrac{1}{p'}(V^- + W_2))u^2\rangle.\end{aligned}$$

For $1 < p \leqslant 4$, Theorem 3.2(a) and (b) applied to $\tilde{\tau}$ imply: T_0 extrapolates to a C_0-semigroup $T_{0,p}$ on L_p, and for the generator $-A_{0,p}$ of $T_{0,p}$ we have

$$\langle A_{0,p}u, u^{p-1}\rangle \geqslant \langle (U_0 - \tfrac{1}{p'}(V^- + W_2))u^p\rangle - \tfrac{\omega_1}{p}\|u\|_p^p \quad \big(0 \leqslant u \in D(A_{0,p})\big). \tag{3.14}$$

In particular, $\|T_{0,p}(t)\|_{p\to p} \leqslant e^{\frac{\omega_1}{p}t}$ for all $t \geqslant 0$, $1 < p \leqslant 4$ (recall $U_0 = W_1 + W_2 + V^-$). Since T_0 is a positive C_0-semigroup, [Voi92] implies that T_0 extrapolates to a C_0-semigroup $T_{0,1}$ on L_1.

Let now $U_{n,m} := \big(U_0 - \frac{1}{m}(V^- + W_2)\big) \wedge n$ for $n, m \in \mathbb{N}$. It follows from (3.14) that

$$\|(T_{0,p})_{-U_{n,m}}(t)\|_{p\to p} \leqslant e^{\frac{\omega_1}{p}t} \quad (t \geqslant 0)$$

for all $n \in \mathbb{N}$, $m \geqslant 2$ and $1 < p \leqslant \frac{m}{m-1}$ (i.e., $\frac{1}{p'} \leqslant \frac{1}{m}$). Since $(T_{0,p})_{-U_{n,m}}$ and $(T_{0,1})_{-U_{n,m}}$ are consistent by Lemma 1.10(b), we obtain $\|(T_{0,1})_{-U_{n,m}}(t)\|_{1\to 1} \leqslant e^{\omega_1 t}$ for all $t \geqslant 0$, $n \in \mathbb{N}$, $m \geqslant 2$. Since $U_{n,m} \uparrow U_0 \wedge n$ as $m \to \infty$, we have $(T_{0,1})_{-U_{n,m}} \to (T_{0,1})_{-U_0\wedge n}$ for all $n \in \mathbb{N}$, by [Voi86; Prop. A.2]. Hence

$$\sup_{n\in\mathbb{N}} \|(T_{0,1})_{-U_0\wedge n}(t)\|_{1\to 1} \leqslant e^{\omega_1 t} \quad (t \geqslant 0).$$

Finally, [Voi88; Prop. 2.2] implies that $-U_0$ is $T_{0,1}$-admissible, and we obtain $\tau \leftrightarrow (T_{0,1})_{-U_0} =: T_1$, with $\|T_1(t)\|_{1\to 1} \leqslant e^{\omega_1 t}$ for all $t \geqslant 0$.

To complete the proof it remains to show inequalities (3.12) and (3.13). Let $\chi_n^c := 1 - \chi_n$, i.e., the indicator of the set $[|u|^{\frac{p-2}{2}} < n]$. We write $u_n = u_{n,p}$, $v_n = v_{n,p}$ $\big(= u(|u|^{\frac{p-2}{2}} \wedge n)\big)$ and $w_n = w_{n,p}$ $\big(= u(|u|^{p-2} \wedge n^2)\big)$ for short. Lemma 3.14 implies that

$$\nabla v_n = u_n\big(\nabla u + \tfrac{p-2}{2}\chi_n^c \operatorname{sgn} u \nabla|u|\big) = \operatorname{sgn} u\big(u_n \operatorname{sgn} \overline{u} \nabla u + \tfrac{p-2}{2}\chi_n^c u_n \nabla|u|\big).$$

Let $\varphi_n := u_n \operatorname{Re}(\operatorname{sgn} \overline{u}\nabla u) = u_n \nabla|u|$ and $\psi_n := u_n \operatorname{Im}(\operatorname{sgn} \overline{u} \nabla u)$. Then we have

$$\operatorname{sgn} \overline{u} \nabla v_n = (\varphi_n + i\psi_n) + \tfrac{p-2}{2}\chi_n^c \varphi_n = (\tfrac{p}{2}\chi_n^c + \chi_n)\varphi_n + i\psi_n.$$

In the same way, with $\rho_n := (p-1)\chi_n^c + \chi_n$, we have

$$\nabla \overline{w}_n = u_n^2\big(\nabla \overline{u} + (p-2)\chi_n^c \operatorname{sgn} \overline{u} \nabla |u|\big) = u_n \operatorname{sgn} \overline{u}(\rho_n \varphi_n - i\psi_n).$$

Now we compute the different terms occurring in $\tau(u, w_n)$ and $\tau_p(v_n)$ separately.

$$a\nabla u \cdot \nabla \overline{w}_n = a(u_n \operatorname{sgn} \overline{u} \nabla u) \cdot (\rho_n\varphi_n - i\psi_n) = a(\varphi_n + i\psi_n)(\rho_n\varphi_n - i\psi_n), \tag{3.15}$$

$$\begin{aligned} a\nabla v_n \cdot \nabla \overline{v}_n &= a(\operatorname{sgn} \overline{u} \nabla v_n) \cdot (\operatorname{sgn} u \nabla \overline{v}_n) \\ &= (\tfrac{p^2}{4}\chi_n^c + \chi_n) a_s \varphi_n \cdot \varphi_n + a_s\psi_n \cdot \psi_n + i(a - a_s)\psi_n \cdot (p\chi_n^c + 2\chi_n)\varphi_n. \end{aligned} \tag{3.16}$$

Therefore $\operatorname{Re} a\nabla u \cdot \nabla \overline{w}_n = \big((p-1)\chi_n^c + \chi_n\big) a_s\varphi_n \cdot \varphi_n + a_s\psi_n \cdot \psi_n$. Noting $\frac{4}{pp'}\frac{p^2}{4} = p-1$ we obtain

$$\operatorname{Re} \tau_a(u, w_n) = \tfrac{4}{pp'} \operatorname{Re} \tau_a(v_n) + (1 - \tfrac{4}{pp'})\langle \chi_n a_s \varphi_n \cdot \varphi_n + a_s\psi_n \cdot \psi_n\rangle.$$

For the first order terms we compute

$$\begin{aligned} \overline{v}_n \nabla v_n &= |v_n|\big((\tfrac{p}{2}\chi_n^c + \chi_n)\varphi_n + i\psi_n\big), \\ \overline{w}_n \nabla u &= |v_n| u_n \operatorname{sgn} \overline{u} \nabla u = |v_n|(\varphi_n + i\psi_n), \\ u \nabla \overline{w}_n &= |v_n|(\rho_n \varphi_n - i\psi_n). \end{aligned} \tag{3.17}$$

Thus, $\operatorname{Re} \overline{v}_n \nabla v_n = \operatorname{Re} v_n \nabla \overline{v}_n = (\frac{p}{2}\chi_n^c + \chi_n)|v_n|\varphi_n$. We obtain that

$$\operatorname{Re} \overline{w}_n \nabla u = (\chi_n^c + \chi_n)|v_n|\varphi_n = \tfrac{2}{p} \operatorname{Re}(\overline{v}_n \nabla v_n) + (1 - \tfrac{2}{p})\chi_n |v_n| \varphi_n$$

and, since $\frac{2}{p'}\frac{p}{2} = p - 1$,

$$\operatorname{Re} u \nabla \overline{w}_n = \big((p-1)\chi_n^c + \chi_n\big)|v_n|\varphi_n = \tfrac{2}{p'} \operatorname{Re}(v_n \nabla \overline{v}_n) + (1 - \tfrac{2}{p'})\chi_n |v_n| \varphi_n.$$

Let now $\varepsilon_p := \frac{1}{p'} - \frac{1}{p} = 1 - \frac{2}{p} = -(1 - \frac{2}{p'})$. Then $\varepsilon_p^2 = 1 - \frac{4}{pp'}$. We get

$$\begin{aligned} \operatorname{Re} \tau(u, w_n) &= \operatorname{Re} \tau_a(u, w_n) + \operatorname{Re}\langle \nabla u, b_1 w_n\rangle - \operatorname{Re}\langle b_2 u, \nabla w_n\rangle + \langle V u, w_n\rangle \\ &= \tau_p(v_n) + \varepsilon_p^2 \langle \chi_n a_s \varphi_n \cdot \varphi_n + a_s \psi_n \cdot \psi_n\rangle + \varepsilon_p \langle \chi_n (b_1 + b_2)|v_n| \cdot \varphi_n\rangle. \end{aligned}$$

This implies (3.12) since $\varepsilon_p \chi_n |(b_1+b_2)v_n \cdot \varphi_n| \leqslant \varepsilon_p^2 \chi_n a_s \varphi_n \cdot \varphi_n + \frac{1}{2}\chi_n(W_1+W_2)|v_n|^2$, by Euclid's inequality.

To prove (3.13), we first compute $\operatorname{Im} \tau_a(u, w_n)$. By (3.15),

$$\begin{aligned} \operatorname{Im}(a \nabla u \cdot \nabla \overline{w}_n) &= ((p-1)\chi_n^c + \chi_n) a \psi_n \cdot \varphi_n - a\varphi_n \cdot \psi_n \\ &= (p-2)\chi_n^c a_s \psi_n \cdot \varphi_n + (p\chi_n^c + 2\chi_n)(a - a_s)\psi_n \cdot \varphi_n. \end{aligned}$$

The second term on the right hand side equals $\operatorname{Im}(a \nabla v_n \cdot \nabla \overline{v}_n)$, by (3.16). The first term we estimate, using Euclid's inequality and (3.16), as follows:

$$\begin{aligned} |(p-2)\chi_n^c a_s \psi_n \cdot \varphi_n| &\leqslant |p-2| \chi_n^c \big(\tfrac{p}{4} a_s \varphi_n \cdot \varphi_n + \tfrac{1}{p} a_s \psi_n \cdot \psi_n\big) \\ &= |1 - \tfrac{2}{p}| \chi_n^c \big(\tfrac{p^2}{4} a_s \varphi_n \cdot \varphi_n + a_s \psi_n \cdot \psi_n\big) \leqslant |\tfrac{1}{p} - \tfrac{1}{p'}| \operatorname{Re}(a \nabla v_n \cdot \nabla \overline{v}_n). \end{aligned}$$

For the first order terms we have, by (3.17),

$$\operatorname{Im}\big(\langle \nabla u, b_1 w_n\rangle - \langle b_2 u, \nabla w_n\rangle\big) = \langle (b_1 + b_2)|v_n|, \psi_n\rangle = -\operatorname{Im}\langle (b_1 + b_2) v_n, \nabla v_n\rangle.$$

Thus, inequality (3.13) follows. □

3.3 Weighted estimates for second order elliptic differential operators

In this section we prove Theorems 3.8 and 3.10. In order to apply the abstract results of Section 2.1, we need to show appropriate weighted estimates for the semigroups T_p constructed in Theorem 3.2. Recall that the semigroups T_p are associated with the form τ which is defined in (3.1). We will establish estimates on the 'twisted semigroups' $\rho T_p \rho^{-1}$, where ρ is a weight function, by studying the 'twisted form' τ_ρ which is formally defined by $\tau_\rho(u,v) = \tau(\rho^{-1}u, \rho v)$. We point out that it is a nontrivial technical problem to establish relationship between τ_ρ and $\rho T_p \rho^{-1}$ (see, e.g., [Sem97; Prop. 3.4]). A comprehensive solution is given in Theorem 3.15 below.

Throughout this section we assume that **(a)** and **(bV)** are fulfilled. Let τ_a, τ, τ_p $(1 \leqslant p < \infty)$ be the forms defined in Section 3.1. Recall that

$$I = \big\{p \in [1,\infty);\, \omega_p = \inf\{\omega \in \mathbb{R};\, \tau_p \geqslant -\omega\} < \infty\big\}.$$

Let $\rho\colon \Omega \to (0,\infty)$ be locally Lipschitz continuous, $\Phi_\rho := \rho^{-1}\nabla\rho = \nabla \ln \rho$, $p \in \mathring{I}$. Assume that

$$a_s\Phi_\rho \cdot \Phi_\rho = \rho^{-2} a_s \nabla\rho \cdot \nabla\rho \leqslant \delta\tau_p + c_\rho \tag{3.18}$$

for some $\delta \geqslant 0$, $c_\rho \in \mathbb{R}$. Then we can define a form τ_ρ by

$$\tau_\rho(u,v) := \tau(u,v) + \langle \nabla u, a^\top \Phi_\rho v\rangle - \langle a\Phi_\rho u, \nabla v\rangle - \big\langle \big[a_s\Phi_\rho\cdot\Phi_\rho + (b_1+b_2)\cdot\Phi_\rho\big]u, v\big\rangle$$

on $D(\tau_\rho) := D(\tau)$, due to the following observations. Firstly, by Euclid's inequality, $|b_j \cdot \Phi_\rho| \leqslant \frac14 a_s\Phi_\rho\cdot\Phi_\rho + W_j$ for $j = 1,2$. Secondly, by **(a)** and by (1.7) from Lemma 1.29,

$$(a\Phi_\rho)^\top a_s^{-1}(a\Phi_\rho) = {\Phi_\rho}^\top a^\top a_s^{-1} a\Phi_\rho \leqslant (1+\alpha^2) a_s\Phi_\rho\cdot\Phi_\rho, \tag{3.19}$$

and in the same way $(a^\top\Phi_\rho)^\top a_s^{-1}(a^\top\Phi_\rho) \leqslant (1+\alpha^2)a_s\Phi_\rho\cdot\Phi_\rho$. In particular, the form τ_ρ is of the same type as the form τ, with new lower order coefficients

$$\tilde b_1 = b_1 + a^\top\Phi_\rho, \quad \tilde b_2 = b_2 + a\Phi_\rho, \quad \tilde V = V - a_s\Phi_\rho\cdot\Phi_\rho - (b_1+b_2)\cdot\Phi_\rho$$

satisfying assumption **(bV)**.

By a straightforward computation we obtain, using the product rule,

$$\tau_\rho(u,v) = \tau(\rho^{-1}u, \rho v) \quad \big(u, v \in D(\tau_\rho) \text{ such that } \rho^{-1}u, \rho v \in D(\tau)\big).$$

Theorem 3.15. *Assume that* **(a)** *and* **(bV)** *hold. Let $p \in \mathring{I}$, $\varepsilon \in (0,1)$. Then there exist $\delta > 0$, $k \geqslant 1$, $\tilde\omega \in \mathbb{R}$ such that, for all locally Lipschitz continuous weights $\rho\colon \Omega \to (0,\infty)$ satisfying*

$$\begin{aligned} a_s\Phi_\rho\cdot\Phi_\rho &\leqslant \delta\tau_p + c_\rho, \\ \langle (b_1+b_2)\cdot\Phi_\rho, u^2\rangle &\leqslant \frac{\varepsilon}{2}\tau_p(u) + c_\rho\|u\|_2^2 \quad \big(0 \leqslant u \in D(\tau)\big) \end{aligned}$$

for some $c_\rho \in \mathbb{R}$, the following assertions hold:

(a) *The form τ_ρ is associated with a positive C_0-semigroup $T_{\rho,p}(t) = e^{-tA_{\rho,p}}$ on L_p, and*

$$\langle A_{\rho,p}u, u|u|^{p-2}\rangle \geqslant (1-\varepsilon)\overline{\tau_p}\big(|u|^{\frac p2}\operatorname{sgn} u\big) - (\tilde\omega + kc_\rho)\|u\|_p^p \quad \big(u \in D(A_{\rho,p})\big).$$

(b) *If $\rho \wedge n$ is a multiplication operator on $D(\tau)$ for all $n \in \mathbb{N}$ then*

$$T_{\rho,p}(t)f = \rho T_p(t)\rho^{-1}f \quad (f \in L_{\infty,c},\ t \geqslant 0),$$

where T_p is the positive C_0-semigroup on L_p associated with τ.

Remark 3.16. If $\rho(x) = \rho_\xi(x) = e^{\xi x}$ for some $\xi \in \mathbb{R}^N$ then $\Phi_{\rho_\xi} = \xi$. Thus, if $a \in L_\infty$ then (3.18) is fulfilled with $\delta = 0,\ c_\rho = \|a_s\xi \cdot \xi\|_\infty \leqslant \|a_s\|_\infty |\xi|^2$. If condition (ii) of Theorem 3.8 holds then by Proposition 3.1(c) we obtain: for all $p \in \mathring{I}$ there exists $C_p \geqslant 0$ such that $\big|\langle (b_1 + b_2)u^2\rangle\big| \leqslant C_p\|u\|_{\tau_p}\|u\|_2$ for all $0 \leqslant u \in D(\tau)$. Therefore,

$$\begin{aligned}\big|\langle (b_1 + b_2)\cdot\xi, u^2\rangle\big| &\leqslant |\xi| \cdot \big|\langle (b_1+b_2)u^2\rangle\big| \leqslant \|u\|_{\tau_p}\cdot C_p|\xi|\|u\|_2 \\ &\leqslant \frac{1}{4}\|u\|_{\tau_p}^2 + C_p^2|\xi|^2\|u\|_2^2 = \frac{1}{4}\tau_p(u) + \big(\frac{1}{4}(\omega_p+1) + C_p^2|\xi|^2\big)\|u\|_2^2.\end{aligned}$$

Let $n \in \mathbb{N},\ \varphi_n := \rho_\xi \wedge n$. Then $\varphi_n \in W_\infty^1$ and $a_s\nabla\varphi_n \cdot \nabla\varphi_n \leqslant \|a_s\|_\infty \|\nabla\varphi_n\|_\infty^2 < \infty$. Thus, if **(BC)** holds then φ_n is a multiplication operator on $D(\tau)$, and by Theorem 3.15 we obtain the following.

Let $p \in \mathring{I}$. Then there exist $\nu_p > 0,\ \tilde\omega_p \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}^N$, the form τ_{ρ_ξ} is associated with a positive C_0-semigroup $T_{\xi,p}$ on L_p satisfying

$$\|T_{\xi,p}(t)\| \leqslant e^{\tilde\omega_p t + \nu_p|\xi|^2 t}, \quad T_{\xi,p}(t) = e^{\xi x}T_p(t)e^{-\xi x} \text{ on } L_{\infty,c} \quad (t \geqslant 0).$$

Proof of Theorem 3.15. (a) In order to apply Theorem 3.2 we need to introduce the symmetric form $\tau_{\rho,p}$ defined by

$$\begin{aligned}\tau_{\rho,p}(u) &:= \operatorname{Re}\tau_a(u) + \tfrac{2}{p}\langle \nabla|u|, \tilde b_1|u|\rangle - \tfrac{2}{p'}\langle \tilde b_2|u|, \nabla|u|\rangle + \langle \tilde V|u|^2\rangle \\ &= \tau_p(u) + \big\langle (\tfrac{2}{p}a^\top - \tfrac{2}{p'}a)\Phi_\rho|u|, \nabla|u|\big\rangle - \big\langle \big[a_s\Phi_\rho\cdot\Phi_\rho + (b_1+b_2)\cdot\Phi_\rho\big]|u|^2\big\rangle\end{aligned}$$

on $D(\tau_{\rho,p}) := D(\tau_\rho)$. By Euclid's inequality we obtain, using (3.19),

$$\tau_{\rho,p}(u) \geqslant \tau_p(u) - \lambda(\alpha^2+1)\tau_a(|u|) - \big\langle \big[(1+\tfrac{1}{\lambda})a_s\Phi_\rho\cdot\Phi_\rho + (b_1+b_2)\cdot\Phi_\rho\big]|u|^2\big\rangle$$

for all $\lambda > 0,\ u \in D(\tau)$. By Proposition 3.1(b) we can choose $\lambda > 0,\ \tilde\omega \in \mathbb{R}$ such that

$$\lambda(\alpha^2+1)\operatorname{Re}\tau_a \leqslant \frac{\varepsilon}{4}\tau_p + \tilde\omega.$$

Using the assumption on $(b_1+b_2)\cdot\Phi_\rho$, we thus obtain

$$\tau_{\rho,p} \geqslant \big(1 - \tfrac{\varepsilon}{4} - \tfrac{\varepsilon}{2}\big)\tau_p - \tilde\omega - c_\rho - (1+\tfrac{1}{\lambda})a_s\Phi_\rho\cdot\Phi_\rho.$$

Now choose $\delta = \frac{\varepsilon}{4}(1+\frac{1}{\lambda})^{-1}$ and $k = 2 + \frac{1}{\lambda}$. Then $\tau_{\rho,p} \geqslant (1-\varepsilon)\tau_p - (\tilde\omega + kc_\rho)$ $\big(\geqslant -(1-\varepsilon)\omega_p - \tilde\omega - kc_\rho\big)$ by the assumption on $a_s\Phi_\rho\cdot\Phi_\rho$. An application of Theorem 3.2 completes the proof of (a).

(b) Let $U_\rho := (\alpha^2+1)a_s\Phi_\rho\cdot\Phi_\rho + W_1 + W_2 + |V|$. Then $U := 5U_\rho$ is τ_a-regular by **(bV)**, the assumption on $a_s\Phi_\rho\cdot\Phi_\rho$, and Lemma 1.24(b). Making use of (3.19), (3.2) and Euclid's inequality, we obtain that

$$\operatorname{Re}\tau_\rho \geqslant \frac{1}{4}\operatorname{Re}\tau_a - 4U_\rho \tag{3.20}$$

and that $\tau + U$, $\tau_\rho + U$ are densely defined closed sectorial forms, with domains $D(\tau_a + U_\rho)$. For $m \in \mathbb{N}$ let $U_m := (U-m)^+$, and A_m, $A_{\rho,m}$ the m-sectorial operators associated with $\tau + U_m$, $\tau_\rho + U_m$, respectively. Due to Corollary 3.4(b), part (b) will follow by passing to the limit in

$$e^{-tA_{\rho,m}}f = \rho e^{-tA_m}\rho^{-1}f \quad (f \in L_{\infty,c},\ t \geqslant 0).$$

The latter formula in turn is equivalent to

$$(\lambda + A_{\rho,m})^{-1}f = \rho(\lambda + A_m)^{-1}\rho^{-1}f \quad (m \in \mathbb{N},\ \lambda > m,\ f \in L_{\infty,c}). \tag{3.21}$$

Let m, λ, f be given. First we show that (3.21) holds if ρ satisfies the condition

$$\rho v \in Q := D(\tau_a + U_\rho) \quad \text{for all } v \in D := (\lambda + A_m)^{-1}L_{\infty,c}. \tag{3.22}$$

Then $u := \rho(\lambda + A_m)^{-1}\rho^{-1}f \in Q$ since $\rho^{-1}f \in L_{\infty,c}$. Moreover, $\rho^{-1}u \in D(A_m) \subseteq Q$. For all $v \in D$ we have $\rho v \in Q$ and hence

$$(\tau_\rho + U_m)(u,v) = (\tau + U_m)(\rho^{-1}u, \rho v) = \langle A_m\rho^{-1}u, \rho v\rangle.$$

Observe that D is dense in $D(A_m)$ and hence dense in $Q = D(\tau_\rho + U_m)$. Thus we obtain that $u \in D(A_{\rho,m})$ and $A_{\rho,m}u = \rho A_m\rho^{-1}u$. Therefore $(\lambda + A_{\rho,m})u = f$, i.e., (3.21) holds.

It remains to show that the assumption of (b) implies (3.22). Let $g \in L_{\infty,c}$, $v := (\lambda + A_m)^{-1}g\,(\in Q)$, $\rho_n := \rho \wedge n$ $(n \in \mathbb{N})$. Then $\rho_n v \in Q$ for all $n \in \mathbb{N}$, by the assumption on ρ. In particular, ρ_n satisfies condition (3.22). Thus, (3.21) holds with ρ_n in place of ρ, and we obtain

$$(\lambda + A_{\rho_n,m})^{-1}(\rho g) = \rho_n(\lambda + A_m)^{-1}\rho_n^{-1}(\rho g) = \rho_n v$$

for $n \in \mathbb{N}$ so large that $\rho_n^{-1}\rho g = g$. Moreover, by (3.20) we can estimate

$$\operatorname{Re}\tau_{\rho_n} + U_m \geqslant \frac{1}{4}\operatorname{Re}\tau_a - 4U_{\rho_n} + U_m \geqslant \frac{1}{4}\operatorname{Re}\tau_a + U_\rho - m\,\big(\geqslant -m\big).$$

This implies $\|(\lambda + A_{\rho_n,m})^{-1}\| \leqslant \frac{1}{\lambda - m}$ and (since $\lambda > m$)

$$\big(\tfrac{1}{4}\operatorname{Re}\tau_a + U_\rho\big)(\rho_n v) \leqslant \operatorname{Re}(\tau_{\rho_n} + U_m + \lambda)(\rho_n v) = \operatorname{Re}\langle \rho g, \rho_n v\rangle \leqslant \frac{1}{\lambda - m}\|\rho g\|_2^2.$$

Therefore, $(\rho_n v)$ is a bounded sequence in Q. Moreover, $(|\rho_n v|)$ is pointwise increasing, and $\rho_n v \to \rho v$ a.e. as $n \to \infty$. Hence $\rho v \in L_2$ by monotone convergence, and $\rho_n v \to \rho v$ in L_2 by dominated convergence. We conclude that $\rho v \in Q$, i.e., ρ satisfies (3.22). □

The proof of Theorem 3.8 is based on Theorem 3.15 and the following consequence of Corollaries 2.2 and 2.5.

Proposition 3.17. *Let $1 \leqslant p < \infty$, T a C_0-semigroup on L_p. Assume that there exist $q > p$, $C \geqslant 1$, $\omega \in \mathbb{R}$, $\nu \geqslant 0$, $m > 1$ such that*

$$\|e^{\xi x}T(t)e^{-\xi x}\|_{p\to p} \leqslant Ce^{\omega t+\nu|\xi|^m t},$$
$$\|T(t)\|_{p\to q} \leqslant Ct^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q})}e^{\omega t},$$

for all $t > 0$, $\xi \in \mathbb{R}^N$. Then T extrapolates to a family of consistent C_0-semigroups $T_r(t) = e^{-tA_r}$ on L_r, $r \in [p,q)$, and the angle of analyticity of T_r and the spectrum $\sigma(A_r)$ do not depend on $r \in [p,q)$.

If, in addition, T is L_1-contractive then the same holds with $[1,q)$ in place of $[p,q)$.

Proof. Without restriction $\omega = 0$. Let $\theta \in (0,1)$ and define q_θ by $\frac{1}{q_\theta} = \frac{1-\theta}{q} + \frac{\theta}{p}$. Then $(1-\theta)(\frac{1}{p}-\frac{1}{q}) = \frac{1}{p}-\frac{1}{q_\theta}$. By Stein interpolation we obtain from the assumptions that

$$\|e^{\theta\xi x}T(t)e^{-\theta\xi x}\|_{p\to q_\theta} \leqslant Ct^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q_\theta})}e^{\theta\nu|\xi|^m t} \quad (t > 0,\ \xi \in \mathbb{R}^N).$$

Let d be the supremum metric on $\mathbb{R}^N$, $\rho_{\gamma,y} := e^{-\gamma d(\cdot,y)}$ for all $\gamma \in \mathbb{R}$, $y \in \mathbb{R}^N$, and $\|\cdot\|_{p\to q,\gamma}$ the corresponding weighted operator norm. Replace ξ by $\frac{\xi}{\theta}$ in the above estimate. Then Proposition 2.8, applied with $\psi(x) = x$, yields

$$\|T(t)\|_{p\to q_\theta,\gamma} \leqslant 2NCt^{-\frac{N}{m}(\frac{1}{p}-\frac{1}{q_\theta})}e^{\theta^{1-m}\nu\gamma^m t} \quad (t,\gamma > 0).$$

Now we are in a position to apply Corollaries 2.2 and 2.5, and we obtain the first set of assertions.

The last assertion follows from the estimate

$$\|T(t)\|_{1\to q_\theta,\gamma} \leqslant C_1t^{-\frac{N}{m}(1-\frac{1}{q_\theta})}e^{\nu_1\gamma^m t} \quad (t,\gamma > 0),$$

for some $C_1, \nu_1 > 0$, which in turn is a consequence of the next proposition. ☐

Proposition 3.18. *Let T be a contractive C_0-semigroup on L_1 and $\rho\colon \Omega \to (0,\infty)$ a weight function. Assume that*

$$\|\rho^\gamma T(t)\rho^{-\gamma}\|_{p\to q} \leqslant Ct^{-\alpha(\frac{1}{p}-\frac{1}{q})}e^{\nu\gamma^m t} \quad (t,\gamma > 0)$$

for some $1 < p < q$, $C, \alpha, \nu > 0$, $m > 1$. Then there exist $C_1, \nu_1 > 0$ (not depending on T, ρ) such that

$$\|\rho^\gamma T(t)\rho^{-\gamma}\|_{1\to q} \leqslant C_1t^{-\alpha(1-\frac{1}{q})}e^{\nu_1\gamma^m t} \quad (t,\gamma > 0).$$

Proof. For $0 < \theta \leqslant 1$ let $p_\theta := \big(\frac{\theta}{p} + \frac{1-\theta}{1}\big)^{-1}$, $q_\theta := \big(\frac{\theta}{q} + \frac{1-\theta}{1}\big)^{-1}$. By Stein interpolation, the assumptions imply that

$$\|\rho^{\theta\gamma} T(t)\rho^{-\theta\gamma}\|_{p_\theta\to q_\theta} \leqslant C^\theta t^{-\theta\alpha(\frac{1}{p}-\frac{1}{q})} e^{\theta\nu\gamma^m t} \quad (t,\gamma > 0). \tag{3.23}$$

Let $t, \gamma > 0$, determine $\theta \in (0,1)$ such that $q_\theta = p$, and let $\theta_k := \theta^k$, $t_k := \theta_k^m t$ $(k \in \mathbb{N}_0)$ and $\beta := \alpha(\frac{1}{p} - \frac{1}{q})$. Then $p_{\theta_k} = q_{\theta_{k+1}}$ $(k \in \mathbb{N}_0)$, and (3.23) yields

$$\|\rho^\gamma T(t_k)\rho^{-\gamma}\|_{q_{\theta_{k+1}}\to q_{\theta_k}} \leqslant C^{\theta_k} t_k^{-\theta_k\beta} e^{\theta_k\nu(\gamma/\theta_k)^m t_k} = C^{\theta_k}(\theta^{mk}t)^{-\theta_k\beta} e^{\theta_k\nu\gamma^m t}$$

for all $k \in \mathbb{N}_0$. We use this as a starting point for a Moser type iteration: let $f \in L_{\infty,c}$. Since $q = q_{\theta_0}$ we obtain by Fatou's lemma that

$$\begin{aligned}\|\rho^\gamma T(\tfrac{t}{1-\theta^m})\rho^{-\gamma} f\|_q &\leqslant \liminf_{n\to\infty} \Big\|\rho^\gamma T\big(\sum_{k=0}^n t_k\big)\rho^{-\gamma} f\Big\|_{q_{\theta_0}} \\ &\leqslant \liminf_{n\to\infty} \prod_{k=0}^n \big(C^{\theta_k}\theta^{-m\beta k\theta_k} t^{-\theta_k\beta} e^{\theta_k\nu\gamma^m t}\big) \cdot \|f\|_{q_{\theta_{n+1}}}.\end{aligned}$$

Let $r := \sum_{k=0}^\infty \theta_k = \frac{1}{1-\theta}$ and $s := \sum_{k=0}^\infty k\theta_k (= \frac{\theta}{(1-\theta)^2})$. By the choice of θ we have $\sum_{k=0}^\infty \theta_k\beta = \frac{\alpha}{1-\theta}(\frac{1}{q_\theta} - \frac{1}{q}) = \alpha(1 - \frac{1}{q})$. We conclude that

$$\|\rho^\gamma T(\tfrac{t}{1-\theta^m})\rho^{-\gamma} f\|_q \leqslant C^r\theta^{-m\beta s} t^{-\alpha(1-\frac{1}{q})} e^{r\nu\gamma^m t}\|f\|_1.$$

This yields the assertion with $C_1 = C^r\theta^{-m\beta s}(1-\theta^m)^{-\alpha(1-\frac{1}{q})}$ and $\nu_1 = (1-\theta^m)r\nu$. □

Proof of Theorem 3.8. Recall that assumptions (i) and (ii) of the theorem are fulfilled for all $p \in \mathring{I}$ (see Remark 3.9(a)). Let $p \in \mathring{I} = (p_-, p_+)$, $T_p(t) = e^{-tA_p}$ the positive C_0-semigroup on L_p associated with the form τ. We are going to apply Proposition 3.17 with $q = \frac{N}{N-2}p$ and $m = 2$. By Remark 3.16 we have

$$\|e^{\xi x} T_p(t) e^{-\xi x}\|_{p\to p} \leqslant e^{\tilde{\omega}_p t + \nu_p|\xi|^2 t} \quad (t > 0,\ \xi \in \mathbb{R}^N),$$

i.e., the first estimate assumed in Proposition 3.17 holds.

In order to show the second estimate, let $0 \leqslant f \in L_p$, $t > 0$, $u := T_p(t)f$. Then $u \in D(A_p)$ since T_p is analytic, and $u \geqslant 0$. By Theorem 3.2(b) and assumption (i) of the theorem we have

$$\langle A_p u, u^{p-1}\rangle \geqslant \overline{\tau_p}(u^{\frac{p}{2}}) \geqslant \varepsilon_p\|u^{\frac{p}{2}}\|^2_{\frac{2N}{N-2}} - c_p\|u^{\frac{p}{2}}\|_2^2.$$

Without restriction assume $c_p = 0$. Then T_p is contractive. Since $(p-1)p' = p$ we obtain, using Hölder's inequality and the analyticity of T_p,

$$\langle A_p u, u^{p-1}\rangle \leqslant \|A_p T_p(t) f\|_p \|(T_p(t)f)^{p-1}\|_{p'} \leqslant \frac{c}{t}\|f\|_p\|T_p(t)f\|_p^{p-1} \leqslant \frac{c}{t}\|f\|_p^p$$

for some $c > 0$ (not depending on t, f). Combining the above two estimates, we arrive at

$$\varepsilon_p \|T_p(t)f\|^p_{\frac{N}{N-2}p} \leqslant \frac{c}{t}\|f\|_p^p,$$

so that, by the positivity of T_p,

$$\|T_p(t)\|_{p\to\frac{N}{N-2}p} \leqslant Ct^{-\frac{1}{p}} = Ct^{-\frac{N}{2}(p^{-1}-(\frac{N}{N-2}p)^{-1})} \quad (t > 0). \tag{3.24}$$

By Proposition 3.17 we conclude that T_p extrapolates to an analytic semigroup $T_r(t) = e^{-tA_r}$ on L_r, for all $r \in [p, \frac{N}{N-2}p)$, and that the angle of analyticity of T_r and the spectrum $\sigma(A_r)$ are r-independent. In case $1 \in I$, the same holds with $[1, \frac{N}{N-2}p)$ in place of $[p, \frac{N}{N-2}p)$. Moreover, the semigroups T_r are associated with τ, by Proposition 1.22(a). Thus, in case $1 \in I$ the proof is complete while otherwise we obtain the assertions of the theorem only with $(p_-, p_{\max})$ in place of $(p_{\min}, p_{\max})$.

To complete the proof in the case $1 \notin I$, we apply the above to the form τ^* in place of τ. The form τ^* is of the same type as the form τ, with coefficients $\tilde{a} = a^\top$, $\tilde{b}_1 = -b_2$, $\tilde{b}_2 = -b_1$, $\tilde{V} = V$. It is easy to see that $(\tau^*)_p = \tau_{p'}$ for all $p \in (1, \infty)$. Therefore, τ^* is associated with an analytic semigroup $\hat{T}_r(t) = e^{-t\hat{A}_r}$ on L_r, for all $r \in (p_+', p_{\min}')$, and the angle of analyticity of $\hat{T}_r$ and the spectrum $\sigma(\hat{A}_r)$ are r-independent. By Proposition 1.22(b) we conclude that τ is associated with the positive C_0-semigroup $\hat{T}^*_{r'}(t) = e^{-tA^*_{r'}}$ on L_r, for all $r \in (p_{\min}, p_+)$. It now remains to note the following: $\hat{T}^*_{r'}$ is analytic of angle $\theta \in (0, \frac{\pi}{2}]$ if and only if $\hat{T}_{r'}$ is analytic of angle θ, and $\sigma(\hat{A}^*_{r'}) = \overline{\sigma(\hat{A}_{r'})}$ $(= \sigma(\hat{A}_{r'})$ since $\hat{T}_{r'}$ is a real semigroup). ☐

In the proof of Theorem 3.10, it will be a bit more difficult to prove $p{\to}q$-smoothing for the semigroup since the assumptions do not ensure analyticity. In fact, it is easier to show $p{\to}q$-smoothing for the *resolvent* and use Theorem 2.26 instead of Theorem 2.4 to obtain L_p-spectral independence. We will use the latter approach for the case $p > 1$. As above, the case $p = 1$ requires additional expense. In order to show a weighted $1{\to}q$-estimate, we are going to use Proposition 3.18. For this reason we need to show $p{\to}q$-smoothing for the *semigroup*. Alternatively, we could prove an analogue of Proposition 3.18 for resolvents, cf. [Sem97; proof of Prop. 4.2].

Both $p{\to}q$-smoothing for the semigroup and for the resolvent are consequences of an improved accretivity estimate. We present this well-known technique in a separate lemma. As a preparation we need the following fact.

Lemma 3.19. *Let (Ω, μ) be a measure space, $p \in (1, \infty)$, $\varnothing \neq J \subseteq \mathbb{R}$ an interval. Let $u\colon J \to L_p(\mu)$ be differentiable, $u(t) \geqslant 0$ for all $t \in J$. Then the function $u^p\colon J \to L_1(\mu)$ is differentiable, with*

$$(u^p)' = pu^{p-1}u'.$$

Proof. We first show that for $x, y \geqslant 0$ we have

$$\begin{aligned} &0 \leqslant x^p - y^p - py^{p-1}(x-y) \leqslant |x-y|^p \quad (1 < p \leqslant 2), \\ &0 \leqslant x^p - y^p - py^{p-1}(x-y) \leqslant \tfrac{1}{2}p(p-1)(x \vee y)^{p-2}|x-y|^2 \quad (p \geqslant 2). \end{aligned} \tag{3.25}$$

For the proof let $f \colon [0,\infty) \to [0,\infty)$, $f(x) := x^p$. Then f is convex, and $x \mapsto y^p + py^{p-1}(x-y)$ is the tangent to f in (y, y^p), for all $y \geqslant 0$. This shows the left hand sides of the inequalities. For the proof of the right hand side for $p \geqslant 2$, just apply Taylor's theorem with Lagrangian remainder to f. It remains to show that $g_y(x) := x^p - y^p - py^{p-1}(x-y) - |x-y|^p \leqslant 0$ in case $1 < p \leqslant 2$, for all $x, y \geqslant 0$. We have $g_y(y) = 0$, and $\frac{1}{p}g_y'(x) = x^{p-1} - y^{p-1} - |x-y|^{p-1}\operatorname{sgn}(x-y)$. The subadditivity of the function $x \mapsto x^{p-1}$ shows that $g_y'(x) \geqslant 0$ for $x \leqslant y$ and $g_y'(x) \leqslant 0$ for $x \geqslant y$. This completes the proof of (3.25).

Let now $s, t \in J$, $s \neq t$, and assume that $p \leqslant 2$. Applying (3.25) to $x = u(s)$ and $y = u(t)$ we obtain, after division by $|s-t|$,

$$\left| \frac{u(s)^p - u(t)^p}{s-t} - pu(t)^{p-1}\frac{u(s)-u(t)}{s-t} \right| \leqslant |s-t|^{p-1} \left| \frac{u(s)-u(t)}{s-t} \right|^p.$$

Letting $s \to t$ yields the assertion for $1 < p \leqslant 2$. The case $p > 2$ is proved similarly (use the fact that $\|v^{p-2}w^2\|_1 \leqslant \|v\|_p^{p-2}\|w\|_p^2$ for all $v, w \in L_p(\mu)$). ☐

If we allow u to be complex-valued in the above lemma then we obtain $\big(|u|^p\big)' = p|u|^{p-1}\operatorname{Re}(u' \operatorname{sgn} u)$. This follows from $|u|'(t) = \operatorname{Re}\big(u'(t)\operatorname{sgn} u(t)\big) + |u'(t)|\chi_{[u(t)=0]}$ (cf. [Nag86; Prop. B-II.2.3, Example C-II.2.3]) and the chain rule.

Lemma 3.20. *Let (Ω, μ) be a measure space, $p \in (1,\infty)$. Let $T(t) = e^{-tA}$ be a positive C_0-semigroup on $L_p(\mu)$ satisfying*

$$\langle Au, u^{p-1}\rangle \geqslant \varepsilon\|u\|_{rp}^p \quad \big(0 \leqslant u \in D(A)\big)$$

for some $r > 1$, $\varepsilon > 0$.

(a) *Then A is m-accretive, and $\|(\lambda + A)^{-1}\|_{(rp')' \to rp} \leqslant \frac{1}{\varepsilon}$ for all $\lambda > 0$.*

(b) *Assume that $\|T(t)\|_{rp\to rp} \leqslant C$ $(t \geqslant 0)$ for some $C \geqslant 1$. Then $\|T(t)\|_{p\to rp} \leqslant C(\varepsilon pt)^{-\frac{1}{p}}$ for all $t > 0$.*

Proof. (a) The semigroup T is contractive since it is positive and $\langle Au, u^{p-1}\rangle \geqslant 0$ for all $0 \leqslant u \in D(A)$. Thus, A is m-accretive. Let now $\lambda > 0$, $0 \leqslant f \in L_p(\mu) \cap L_{(rp')'}(\mu)$, and $u := (\lambda + A)^{-1}f$. Then $u \geqslant 0$ and hence

$$\langle f, u^{p-1}\rangle = \langle (\lambda + A)u, u^{p-1}\rangle \geqslant \varepsilon\|u\|_{rp}^p.$$

We estimate the left hand side, using Hölder's inequality and noting $(p-1)p' = p$,

$$\langle f, u^{p-1}\rangle \leqslant \|f\|_{(rp')'}\|u^{p-1}\|_{rp'} = \|f\|_{(rp')'}\|u\|_{rp}^{p-1}.$$

Combining the above two estimates, we obtain $\varepsilon\|u\|_{rp} \leqslant \|f\|_{(rp')'}$. By the positivity of T, this proves (a).

(b) Let $0 \leqslant f \in D(A)$, and $u_t := T(t)f$ for all $t \geqslant 0$. Then $-\frac{d}{ds}u_s = Au_s$ for all $s \geqslant 0$ and hence, by Lemma 3.19,

$$-\frac{d}{ds}\|u_s\|_p^p = -\frac{d}{ds}\langle u_s^p\rangle = p\langle Au_s, u_s^{p-1}\rangle \geqslant \varepsilon p\|u_s\|_{rp}^p.$$

By the assumption of (b), $\|u_t\|_{rp} \leqslant C\|u_s\|_{rp}$ for all $0 \leqslant s \leqslant t$, so we obtain

$$\varepsilon pt\|u_t\|_{rp}^p \leqslant \int_0^t \varepsilon pC^p\|u_s\|_{rp}^p\,ds \leqslant C^p\int_0^t \big(-\frac{d}{ds}\|u_s\|_p^p\big)\,ds \leqslant C^p\|f\|_p^p \quad (t > 0),$$

noting $u_0 = f$ and $\|u_t\|_p^p \geqslant 0$. Thus, $\|T(t)f\|_{rp} \leqslant C(\varepsilon pt)^{-\frac{1}{p}}\|f\|_p$ for all $t > 0$, $0 \leqslant f \in D(A)$. The set of those f is dense in $\big\{f \in L_p(\mu);\ f \geqslant 0\big\}$, so the proof is complete by the positivity of T. □

If we do not assume the semigroup T to be positive then the assertions of the above lemma still hold if we replace the assumption by

$$\mathrm{Re}\langle Au, u|u|^{p-2}\rangle \geqslant \varepsilon\|u\|_{rp}^p \quad \big(u \in D(A)\big).$$

We will apply Theorem 2.26 in the proof of Theorem 3.10 via the following result.

Proposition 3.21. *([LiVo00; Thm. 1]) Given $1 \leqslant p < q < \infty$, let A_p, A_q be closed operators in L_p, L_q, respectively. Assume that there exist $p_0 \leqslant p$, $q_0 \geqslant q$, $\gamma > 0$, $C < \infty$, $\lambda_0 \in \rho(A_p) \cap \rho(A_q)$, and an L^1-regular function $\psi\colon \mathbb{R}^N \to \mathbb{R}^N$ such that $(\lambda_0 - A_p)^{-1}$, $(\lambda_0 - A_q)^{-1}$ are consistent and*

$$\|e^{\xi\psi}(\lambda_0 - A_p)^{-1}e^{-\xi\psi}\|_{p_0\to q_0} \leqslant C \quad \text{for all } \xi \in \mathbb{R}^N,\ |\xi| \leqslant \gamma.$$

Then $\sigma(A_p) = \sigma(A_q)$, and $(\lambda - A_p)^{-1}$, $(\lambda - A_q)^{-1}$ are consistent for all $\lambda \in \rho(A_p) = \rho(A_q)$.

Proof. Let μ be the Lebesgue measure on $\mathbb{R}^N$ and d the semi-metric on $\mathbb{R}^N$ defined by $d(x,y) := |\psi(x) - \psi(y)|_\infty$. For $x \in \mathbb{R}^N =: M$, $r > 0$ let $v_r(x) := \mu(B(x,r))$. By the paragraph preceding Lemma 2.9 we know the following. Conditions (2.7) and (2.8) are fulfilled, and there exists $C_1 > 0$ such that, with the weighted operator norm corresponding to the semi-metric d, we have

$$\|v_1^\alpha(\lambda_0 - A_p)^{-1}\|_{p_0\to q_0,\gamma} \leqslant C_1 \quad (0 \leqslant \alpha \leqslant 1).$$

By Remark 2.28 we conclude that

$$\|v_1^{\frac{1}{p}-\frac{1}{q}}(\lambda_0 - A_p)^{-1}\|_{p\to q,\frac{\gamma}{2}} \leqslant K_\gamma C_1.$$

Now Theorem 2.26 implies that $(\lambda_0 - A_p)^{-1}, (\lambda_0 - A_q)^{-1}$ have equal spectra and consistent resolvents. The spectral mapping theorem for the resolvent yields $\sigma(A_p) = \sigma(A_q)$, and the consistency of the resolvents follows from the identity $(\lambda - A)^{-1} = (\lambda_0-\lambda)^{-1}(\lambda_0-A)^{-1}\big((\lambda_0-\lambda)^{-1}-(\lambda_0-A)^{-1}\big)^{-1}$ (cf. Proposition 2.33). □

Proof of Theorem 3.10. Recall from Proposition 3.1(c) that the first assumption of the theorem implies that for all $p \in \mathring{I}$ there exist $\varepsilon_p > 0,\ c_p \in \mathbb{R}$ such that

$$\tau_p(u) \geqslant \varepsilon_p \|u\|_{2r}^2 - c_p \|u\|_2^2 \quad \big(u \in D(\tau)\big). \tag{3.26}$$

(a) For $\xi \in \mathbb{R}^N$ let $\rho_\xi := e^{\xi\psi}$, and τ_{ρ_ξ} the corresponding form defined in the beginning of the section. Then $\Phi_{\rho_\xi} = \nabla \ln \rho_\xi = \nabla(\xi\psi)$. By Euclid's inequality it follows that

$$a_s \Phi_{\rho_\xi} \cdot \Phi_{\rho_\xi} = \sum_{j,k=1}^N \xi_j \xi_k a_s \nabla\psi_j \cdot \nabla\psi_k \leqslant c|\xi|^2 \sum_{j=1}^N a_s \nabla\psi_j \cdot \nabla\psi_j, \tag{3.27}$$

with a constant $c \geqslant 1$ depending only on the dimension N. Similarly,

$$\big|\langle (b_1 + b_2) \cdot \Phi_{\rho_\xi}, u^2 \rangle\big| \leqslant |\xi| \sum_{j=1}^N \big|\langle (b_1 + b_2) \cdot \nabla\psi_j, u^2\rangle\big| \quad \big(0 \leqslant u \in D(\tau)\big).$$

Let $p \in \mathring{I},\ T_p$ the C_0-semigroup associated with τ. Let $\varepsilon = \frac{1}{2}$ and choose $\delta,\ k,\ \tilde\omega$ as in Theorem 3.15. By Proposition 3.1(c) and the assumptions of (a), the above two estimates imply

$$\begin{aligned} a_s \Phi_{\rho_\xi} \cdot \Phi_{\rho_\xi} &\leqslant |\xi|^2 \big(c_{0,p}\tau_p + c_{1,p}\big), \\ \langle (b_1+b_2)\cdot\Phi_{\rho_\xi}, u^2\rangle &\leqslant |\xi| \big(c_{0,p}\tau_p(u) + c_{1,p}\|u\|_2^2\big) \quad \big(0 \leqslant u \in D(\tau)\big), \end{aligned} \tag{3.28}$$

for some $c_{0,p}, c_{1,p} > 0$ and all $\xi \in \mathbb{R}^N$. Thus the assumptions of Theorem 3.15 are fulfilled for $\rho = \rho_\xi$, with $c_\rho = 1$, if $|\xi|$ is sufficiently small.

We conclude that there exists $\gamma > 0$ such that, for all $|\xi| \leqslant \gamma$, the form τ_{ρ_ξ} is associated with a positive C_0-semigroup $T_{\xi,p}(t) = e^{-tA_{\xi,p}}$ on L_p, and

$$\langle A_{\xi,p}u, u^{p-1}\rangle \geqslant \frac{1}{2}\overline{\tau_p}\big(u^{\frac{p}{2}}\big) - (\tilde\omega + k)\|u\|_p^p \quad \big(0 \leqslant u \in D(A_{\xi,p})\big).$$

By (3.26) we obtain that there exists $C_p \in \mathbb{R}$ such that

$$\langle A_{\xi,p}u, u^{p-1}\rangle \geqslant \frac{1}{2}\varepsilon_p \|u^{\frac{p}{2}}\|_{2r}^2 - C_p\|u^{\frac{p}{2}}\|_2^2 = \frac{\varepsilon_p}{2}\|u\|_{rp}^p - C_p\|u\|_p^p \quad \big(0 \leqslant u \in D(A_{\xi,p})\big).$$

Let $n \in \mathbb{N},\ \varphi_n := e^{\xi\psi} \wedge n$. Then $\varphi_n \in W_\infty^1,\ \nabla\varphi_n = e^{\xi\psi}\chi_{[e^{\xi\psi}<n]}\nabla(\xi\psi) = \varphi_n \chi_{[\varphi_n<n]}\Phi_{\rho_\xi}$. Hence, by (3.28),

$$a_s \nabla\varphi_n \cdot \nabla\varphi_n \leqslant n^2 a_s \Phi_{\rho_\xi}\cdot\Phi_{\rho_\xi} \leqslant c_\xi(\operatorname{Re}\tau_a + W_1 + W_2 + V^+ + 1)$$

for some $c_\xi > 0$. Since **(BC)** holds, φ_n is a multiplication operator on $D(\tau)$ for all $n \in \mathbb{N}$, so Theorem 3.15(b) yields $T_{\xi,p}(t)f = e^{\xi\psi}T_p(t)e^{-\xi\psi}f$ for all $f \in L_{\infty,c}$, $t \geqslant 0$. Thus, $(\lambda + A_{\xi,p})^{-1}f = e^{\xi\psi}(\lambda + A_p)^{-1}e^{-\xi\psi}f$ for all $f \in L_{\infty,c},\ \lambda > C_p$.

Now, since $\langle (C_p + A_{\xi,p})u, u^{p-1}\rangle \geqslant \frac{\varepsilon_p}{2}\|u\|_{rp}^p$ $\big(0 \leqslant u \in D(A_{\xi,p})\big)$, Lemma 3.20(a) implies that $\|e^{\xi\psi}(\lambda + A_p)^{-1}e^{-\xi\psi}\|_{(rp')'\to rp} \leqslant \frac{2}{\varepsilon_p}$ for all $\lambda > C_p$, $|\xi| \leqslant \gamma$. By Proposition 3.21 we infer that $\sigma(A_q)$ is independent of $q \in [(rp')', rp] \cap I$, for all $p \in \mathring{I}$, and that the operators A_q have consistent resolvents. This proves (a).

(b) Assume that $1 \in I$; otherwise there is nothing to show. Let $p \in \mathring{I}$ such that $p \leqslant r$ and $p^2 \in I$. In a first step, we use Theorem 3.15 to obtain a weighted $p\to p$-estimate for the semigroup T_p. Then we use Proposition 3.18 to derive a weighted $1\to q$-estimate, for some $q \in (1, p)$.

(i) For all $\delta > 0$, $u \in D(\tau)$ we have

$$\|u\|_{\tau_p}^{2-\varepsilon}\|u\|_2^{\varepsilon} = \|\delta u\|_{\tau_p}^{2-\varepsilon}\|\delta^{1-\frac{2}{\varepsilon}}u\|_2^{\varepsilon} \leqslant \|\delta u\|_{\tau_p}^2 + \|\delta^{1-\frac{2}{\varepsilon}}u\|_2^2.$$

We deduce that, for all $\delta > 0$, $u \in D(\tau)$, $\xi \in \mathbb{R}^N$,

$$|\xi|^2\|u\|_{\tau_p}^{2-\varepsilon}\|u\|_2^{\varepsilon} \leqslant |\xi|^2\big(\tfrac{\sqrt{\delta}}{|\xi|}\big)^2\|u\|_{\tau_p}^2 + |\xi|^2\big(\tfrac{\sqrt{\delta}}{|\xi|}\big)^{2-\frac{4}{\varepsilon}}\|u\|_2^2 = \delta\|u\|_{\tau_p}^2 + \delta^{1-\frac{2}{\varepsilon}}|\xi|^{\frac{4}{\varepsilon}}\|u\|_2^2.$$

Let $\varepsilon = \frac{1}{2}$ and choose δ, k, $\tilde{\omega}$ as in Theorem 3.15. By (3.27) and the assumption of (b), the above implies that

$$a_s\Phi_{\rho_\xi}\cdot\Phi_{\rho_\xi} \leqslant c|\xi|^2\sum_{j=1}^N a_s\nabla\psi_j\cdot\nabla\psi_j \leqslant \delta\tau_p + C(1 + |\xi|^{\frac{4}{\varepsilon}}) \quad (\xi \in \mathbb{R}^N),$$

for some $C \geqslant 1$. In the same way,

$$\big|\langle (b_1 + b_2)\cdot\Phi_{\rho_\xi}, u^2\rangle\big| \leqslant \frac{1}{4}\tau_p(u) + C(1 + |\xi|^{\frac{2}{\varepsilon}})\|u\|_2^2 \leqslant \frac{1}{4}\tau_p(u) + C\big(2 + |\xi|^{\frac{4}{\varepsilon}}\big)\|u\|_2^2$$

for all $u \in D(\tau)$, $\xi \in \mathbb{R}^N$. Thus, for all $\xi \in \mathbb{R}^N$ the assumptions of Theorem 3.15 are fulfilled for $\rho = \rho_\xi$, with $c_\rho = C(2 + |\xi|^{\frac{4}{\varepsilon}})$. Since assumption **(BC)** holds we can apply Theorem 3.15, as in the proof of (a), to conclude that

$$\|e^{\xi\psi}T_p(t)e^{-\xi\psi}\|_{p\to p} \leqslant e^{\left(\tilde{\omega} + kC(2+|\xi|^{\frac{4}{\varepsilon}})\right)t} \quad (t \geqslant 0,\ \xi \in \mathbb{R}^N).$$

(ii) Recall that $p \leqslant r$. Thus, $\|u\|_{2r}^2 \geqslant \|u\|_{2p}^2 - \|u\|_2^2$ for all $u \in L_2 \cap L_r$. By Theorem 3.2(b) and (3.26) we obtain that

$$\langle A_p u, u^{p-1}\rangle \geqslant \overline{\tau_p}(u^{\frac{p}{2}}) \geqslant \varepsilon_p\big(\|u^{\frac{p}{2}}\|_{2p}^2 - \|u^{\frac{p}{2}}\|_2^2\big) - c_p\|u^{\frac{p}{2}}\|_2^2 \quad \big(0 \leqslant u \in D(A_p)\big).$$

Let $\omega := (c_p + \varepsilon_p) \vee \omega_{p^2}$. Then $\langle (\omega + A_p)u, u^{p-1}\rangle \geqslant \varepsilon_p\|u\|_{p^2}^p$ for all $0 \leqslant u \in D(A_p)$. Moreover, $e^{-\omega t}T_{p^2}(t)$ is a contractive semigroup, so by Lemma 3.20(b) we infer that

$$\|T_p(t)\|_{p\to p^2} \leqslant \big(\varepsilon_p pt\big)^{-\frac{1}{p}}e^{\omega t} = \big(\varepsilon_p pt\big)^{-p'(\frac{1}{p}-\frac{1}{p^2})}e^{\omega t} \quad (t > 0).$$

Let now $q \in (1, p)$. Then Stein interpolation between the above $p\to p^2$-estimate and the weighted $p\to p$-estimate obtained in step (i) yields

$$\|e^{\xi\psi}T_p(t)e^{-\xi\psi}\|_{p\to pq} \leqslant C_1 t^{-p'(\frac{1}{p}-\frac{1}{pq})}e^{\nu_1(1+|\xi|^{\frac{4}{\varepsilon}})t} \quad (t > 0,\ \xi \in \mathbb{R}^N)$$

for some $C_1, \nu_1 > 0$. Finally, we apply Proposition 3.18, and once again Stein interpolation, to obtain

$$\|e^{\xi\psi}T_p(t)e^{-\xi\psi}\|_{1\to q} \leqslant C_2 t^{-p'(1-\frac{1}{q})} e^{\nu_2(1+|\xi|^{\frac{4}{\varepsilon}})t} \quad (t > 0,\ \xi \in \mathbb{R}^N)$$

for some $C_2, \nu_2 > 0$. Note that the t-exponent is greater that -1 since $q < p$. From this we deduce that the assumptions of Proposition 3.21 are fulfilled, with $p_0 = p = 1$ and $q_0 = q$. Thus, A_1 and A_q have equal spectra and consistent resolvents. □

3.4 Sharpness of the results

In this section we show that, under some conditions additional to **(a)** and **(bV)**, the interval I of quasi-contractivity obtained in Theorem 3.2 cannot be enlarged (up to possibly adding $p = 1$). Expressed differently, if τ is associated with a quasi-contractive C_0-semigroup on L_p, for some $p \in (1, \infty)$, then τ_p is bounded below.

Later on, we give an example of coefficients b_1, b_2, V where the interval of existence of the semigroup obtained in Theorem 3.8 cannot be further extended. In this sense, the interval extension given in Theorem 3.8 is optimal. The contents of this section are partly due to Z. Sobol.

The following theorem is the main part of our sharpness result. It is valid under an assumption slightly weaker than **(bV)**, namely

(bV') the potentials W_1, W_2, $|V|$ are τ_a-regular.

Under this assumption, the forms τ and τ_p can be defined in the same way as in Section 3.1, on $D(\tau) = D(\tau_p) = D(\tau_a + W_1 + W_2 + |V|)$.

Theorem 3.22. *Let* **(a)** *and* **(bV')** *hold, and* $p \in (1, \infty)$. *Assume that* $\tau \leftrightarrow T_p$ *on* L_p, *with* $\|T_p(t)\| \leqslant e^{\omega_p t}$ $(t \geqslant 0)$ *for some* $\omega_p \in \mathbb{R}$. *In the case* $p > 2$ $(p < 2)$ *additionally assume that* $\sup_{t\geqslant 0} \|(T_p)_U(t)\|_{\infty\to\infty} < \infty$ $(\sup_{t\geqslant 0} \|(T_p)_U(t)\|_{1\to 1} < \infty)$ *for some* τ_a*-regular potential* $U \geqslant 0$. *Then* $\tau_p \geqslant -\omega_p$.

The additional assumption in the case $p > 2$ is in particular fulfilled in the following situation. Suppose that $\langle \nabla u, b_2 u\rangle \leqslant \omega \|u\|_2^2$ for all $0 \leqslant u \in D(\tau)$ (e.g. $b_2 = 0$). Let $U := V^- + \omega$. Then $\tau + U \leftrightarrow (T_p)_U$ by Corollary 3.4(a). Note that $(\tau_\infty + U)(u) \geqslant 0$ for all $0 \leqslant u \in D(\tau)$. Thus, $\|(T_p)_U(t)\|_{\infty\to\infty} \leqslant 1$ for all $t \geqslant 0$, by Remark 3.3(e).

The proof of Theorem 3.22 is based on the following lemma.

Lemma 3.23. *Let* (M, μ) *be a measure space,* $\mathfrak{h}$ *a Dirichlet form in* $L_2(\mu)$, $D(\mathfrak{h})_+$ *the set of positive elements of* $D(\mathfrak{h})$, *and* $r \geqslant 1$.

(a) *Then* $D_1 := \{u \in D(\mathfrak{h})_+ \cap L_\infty(\mu);\ u^{1/r} \in D(\mathfrak{h})\}$ *is dense in* $D(\mathfrak{h})_+$.

(b) *Let* $\mathfrak{h}_1$ *be a densely defined closed sectorial form in* $L_2(\mu)$ *fulfilling the first Beurling-Deny criterion,* A *the m-sectorial operator associated with* $\mathfrak{h}_1$. *Assume* *that* $D(\mathfrak{h}_1) = D(\mathfrak{h})$, *and* $\|e^{-tA}\|_{\infty\to\infty} \leqslant C$ $(0 \leqslant t \leqslant 1)$ *for some* $C > 0$. *Then* $D_2 := \{u^r;\, 0 \leqslant u \in D(A) \cap L_\infty(\mu),\, Au \in L_\infty(\mu)\}$ *is dense in* $D(\mathfrak{h})_+$.

Proof. (a) For $n \in \mathbb{N}$ define $\varphi_n\colon [0,\infty) \to [0,n]$ by $\varphi_n(s) := s \wedge (ns^r) \wedge n$. It is easy to show that φ_n is Lipschitz continuous with constant r, $\varphi_n^{1/r}$ is Lipschitz continuous, and $\varphi_n(s) \to s$ as $n \to \infty$ $(s \geqslant 0)$. For $u \in D(\mathfrak{h})_+$ we conclude that $\varphi_n(u) \in D_1$, and from [Anc76; Prop. 11] we deduce that $\varphi_n(u) \to u$ in $D(\mathfrak{h})$ as $n \to \infty$.

(b) By (a), it remains to show that D_2 is dense in D_1 with respect to $\|\cdot\|_{\mathfrak{h}}$. Let $u \in D_1$ and $v := u^{1/r}$. Then $v \in D(\mathfrak{h}) \cap L_\infty(\mu)$. By [MaRö92; Thm. I.2.13(ii)] we have $v_\lambda := \lambda(\lambda + A)^{-1}v \to v$ in $D(\mathfrak{h}_1)$ and thus in $D(\mathfrak{h})$ as $\lambda \to \infty$. The assumptions on $\mathfrak{h}_1$ and A imply that $0 \leqslant v_\lambda \in D(A) \cap L_\infty$ and $\|v_\lambda\|_\infty \leqslant 2C\|v\|_\infty$ for sufficiently large λ. Moreover, we have $Av_\lambda = \lambda(v - v_\lambda) \in L_\infty$. Therefore, $v_\lambda^r \in D_2$. Note that the function $s \mapsto s^r$ in Lipschitz continuous on $[0, 2C\|v\|_\infty]$. Hence, by [Anc76; Théorème 10], $v_\lambda^r \to v^r = u$ in $D(\mathfrak{h})$ as $\lambda \to \infty$. □

We further need the following trivial but nevertheless important fact. Let E, F be Banach spaces, and assume that there exists a vector space G such that $E \hookrightarrow G$, $F \hookrightarrow G$.

Lemma 3.24. *Let* A_E, A_F *be closed operators in* E, F, *respectively. Assume that there exists* $\lambda \in \rho(A_E) \cap \rho(A_F)$ *such that* $(\lambda - A_E)^{-1}$, $(\lambda - A_F)^{-1}$ *are consistent. Let* $u \in D(A_E) \cap F$ *such that* $A_E u \in F$. *Then* $u \in D(A_F)$, $A_F u = A_E u$.

Proof. We have $(\lambda - A_E)u \in E \cap F$. Hence

$$u = (\lambda - A_E)^{-1}(\lambda - A_E)u = (\lambda - A_F)^{-1}(\lambda - A_E)u.$$

This implies $u \in D(A_F)$, $(\lambda - A_F)u = (\lambda - A_E)u$. □

Proof of Theorem 3.22. It suffices to study the case $p \geqslant 2$. Then the assertion for the case $p < 2$ follows by an application of Proposition 1.22(b) (recall from the proof of Theorem 3.8 that $\tau_p = (\tau^*)_{p'}$). In the case $p > 2$ assume without restriction that $U \geqslant U_0 := W_1 + W_2 + 2|V|$ (see Lemma 1.24(b)). In the case $p = 2$ let $U := U_0$. Then $\tau + U$ is a closed sectorial form in L_2 (cf. Proposition 3.1(a); the factor 2 is needed since we do not assume $Q(V^-) \supseteq D(\tau_a) \cap Q(W_1 + W_2 + V^+)$).

Let A_p, $A_{p,U}$ be the generators of T_p, $(T_p)_U$, respectively. By the Lumer-Phillips theorem and the assumption on T_p we have $\langle A_p u, u|u|^{p-2}\rangle \geqslant -\omega_p\|u\|_p^p$ for all $u \in D(A_p)$. By Lemma 1.34 we infer that

$$\langle A_{p,U}u, u|u|^{p-2}\rangle \geqslant \langle (U - \omega_p)|u|^p\rangle \quad \big(u \in D(A_{p,U})\big). \tag{3.29}$$

We have to prove $\tau_p \geqslant -\omega_p$ on $D(\tau_p) = D(\tau_a + U_0)$. Notice that τ_p is a bounded form on $D(\tau_a + U_0)$. Since U is $(\tau_a + U_0)$-regular, by Lemma 1.24(b), it therefore suffices to show $\tau_p(u) \geqslant -\omega_p\|u\|_2^2$ for all $u \in D(\tau_a + U)$. Let first $p = 2$. Then $A_{2,U}$ is the m-sectorial operator associated with $\tau + U$, so

$$(\tau_2 + U)(u) = \mathrm{Re}(\tau + U)(u) = \mathrm{Re}\langle A_{2,U}u, u\rangle \geqslant \langle (U - \omega_2)u^2\rangle \quad \big(u \in D(A_{2,U})\big),$$

by (3.29). This shows the assertion for $p = 2$ since $D(A_{2,U})$ is dense in $D(\tau+U) = D(\tau_a + U)$.

Let now $p > 2$. Let A_U be the m-sectorial operator in L_2 associated with $\tau + U$. Below we show that

$$\tau_p(u^{\frac{p}{2}}) \geqslant -\omega_p \|u^{\frac{p}{2}}\|_2^2 \tag{3.30}$$

for all $0 \leqslant u \in D(A_U) \cap L_\infty$ such that $A_U u \in L_\infty$. Then, an application of Lemma 3.23(b) with $\mathfrak{h} = \tau_a + U$, $\mathfrak{h}_1 = \tau + U$, and $A = A_U$ shows that $\tau_p(u) \geqslant -\omega_p \|u\|_2^2$ for all $0 \leqslant u \in D(\tau_a + U)$. This completes the proof since τ_p fulfils the first Beurling-Deny criterion.

So, let $0 \leqslant u \in D(A_U) \cap L_\infty$, $A_U u \in L_\infty$. Then $u \in D(\tau_a + U) \cap L_\infty$ and hence $u^r \in D(\tau_a + U) \cap L_\infty$, $\nabla u^r = r u^{r-1} \nabla u$ for all $r \geqslant 1$. From this we easily obtain $\tau(u, u^{p-1}) = \tau_p(u^{\frac{p}{2}})$ (cf. the computation on page 59) and thus, by the definition of A_U, $\langle A_U u, u^{p-1} \rangle = (\tau_p + U)(u^{\frac{p}{2}})$. By Corollary 3.4(a) we have $\tau + U \leftrightarrow e^{-tA_{p,U}}$. Since $u, A_U u \in L_2 \cap L_\infty \subseteq L_p$, Lemma 3.24 implies that $u \in D(A_{p,U})$ and $A_{p,U} u = A_U u$. By (3.29) we obtain

$$(\tau_p + U)(u^{\frac{p}{2}}) = \langle A_{p,U} u, u^{p-1} \rangle \geqslant \langle (U - \omega_p) u^p \rangle,$$

i.e., (3.30) holds. □

Remark 3.25. Theorem 3.22 is in particular applicable in the case of weakly differentiable b_1 and b_2. For $j = 1, 2$, we assume that b_j is of *τ_a-regular divergence*, i.e., there exists a measurable function div b_j such that $|\text{div } b_j|$ is τ_a-regular and

$$2\langle b_j u, \nabla u \rangle = -\langle (\text{div } b_j) u^2 \rangle \quad \big(0 \leqslant u \in D(\tau) \cap Q(|\text{div } b_j|)\big).$$

Let $U := V^- + |\text{div } b_1| + |\text{div } b_2|$. Then

$$(\tau_1 + U)(u) = \big\langle (-\text{div } b_1 + V + U) u^2 \big\rangle \geqslant 0,$$
$$(\tau_\infty + U)(u) = \big\langle (\text{div } b_2 + V + U) u^2 \big\rangle \geqslant 0$$

for all $0 \leqslant u \in D(\tau + U)$, so $(T_p)_U$ is L_1- and L_∞-contractive.

As an example, we are going to study the formal differential expression

$$-\Delta + c_1 |x|^\alpha x \cdot \nabla + c_2 |x|^\alpha \quad \text{on } \Omega := \mathbb{R}^N,$$

with $\alpha, c_1, c_2 \in \mathbb{R}$. In Remark 3.6(c) we already studied the case $\alpha = 0$ in slightly greater generality, so we will assume $\alpha \neq 0$ here. The case $\alpha = -2$ will be of particular interest (see also Example 3.31 below). Before we proceed to the example we collect some facts about form bounds of the potential r^α with respect to $-\Delta$. Here and in the following, $r \colon \mathbb{R}^N \to \mathbb{R}$, $r(x) := |x|$.

Remark 3.26. Let a be the identity matrix, $D(\tau_a) = W_2^1(\mathbb{R}^N)$. Then τ_a is the standard Dirichlet form, $\tau_a(u) = \|\nabla u\|_2^2$. We will make use of the following Hardy inequality (see [KaWa72; Lemma 1]). If $N \geqslant 3$ then

$$\Big\|\frac{u}{r}\Big\|_2 \leqslant \frac{2}{N-2}\|\nabla u\|_2 \quad \big(u \in W_2^1(\mathbb{R}^N)\big).$$

This inequality is sharp in the sense that there exist no $\beta < \frac{2}{N-2}$, $c \in \mathbb{R}$ with $\|\frac{u}{r}\|_2 \leqslant \beta\|\nabla u\|_2 + c\|u\|_2$ for all $u \in W_2^1(\mathbb{R}^N)$. Moreover, for $N = 1, 2$ there exist no β, c at all for which the inequality is valid. Expressed in the form language, Hardy's inequality states that r^{-2} is form bounded with respect to τ_a, $\frac{(N-2)^2}{4r^2} \leqslant \tau_a$.

Let now $\alpha < -2$. Then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $r^{-2} \leqslant \varepsilon r^\alpha + c_\varepsilon$. Since Hardy's inequality is sharp this implies that r^α is not form bounded with respect to τ_a, i.e., $r^\alpha \not\leqslant c(\tau_a + 1)$ for all $c > 0$. The same holds for $\alpha > 0$ since then $|x|^\alpha \to \infty$ as $|x| \to \infty$.

Finally, let $\alpha \in (-2, 0)$. Then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $r^\alpha \leqslant \varepsilon r^{-2} + c_\varepsilon$. If $N \geqslant 3$ then Hardy's inequality implies that r^α has zero form bound with respect to τ_a. The same holds for $N = 2$. This is an easy consequence of the Sobolev imbedding $\|u\|_q \leqslant c_q\|u\|_{2,1}$ for all $q \in [2, \infty)$, $u \in W_2^1(\mathbb{R}^2)$:

$$\langle r^\alpha |u|^2\rangle \leqslant n\|u\|_2^2 + \|(r^\alpha - n)^+\|_{(\frac{q}{2})'}\||u|^2\|_{\frac{q}{2}} \leqslant n\|u\|_2^2 + \|(r^\alpha - n)^+\|_{(\frac{q}{2})'} c_q\|u\|_{2,1}^2.$$

It remains to note that $\|(r^\alpha - n)^+\|_{(\frac{q}{2})'} \to 0$ as $n \to \infty$ if q is sufficiently large.

Example 3.27. Let $N \geqslant 2$ and, as above, τ_a the standard Dirichlet form on $\mathbb{R}^N$. Let $b_2 = 0$ and define b_1, V by

$$b_1(x) := b(x) := c_1|x|^\alpha x, \quad V(x) := c_2|x|^\alpha,$$

for some $\alpha, c_1, c_2 \in \mathbb{R}$, $\alpha \neq 0$. Then $W_1 = |b|^2$ and $|V|$ are τ_a-regular, i.e., **(bV')** is fulfilled. Let

$$\tau(u, v) := \langle\nabla u, \nabla v\rangle + \langle\nabla u, bv\rangle + \langle Vu, v\rangle$$

on $D(\tau) := D(\tau_a + |b|^2 + |V|)$. If $c_1 = 0$ then $D(\tau) = D(\tau_a + r^\alpha)$ (unless $c_2 = 0$). If $c_2 \neq 0$ then $D(\tau) = D(\tau_a + r^{2(\alpha+1)})$ since $Q(r^\alpha) \supseteq D(\tau_a + r^{2(\alpha+1)})$ (for $\alpha > 0$ and for $\alpha \leqslant -2$ this is trivial; for $\alpha \in (-2, 0)$ it follows from Remark 3.26).

Observe that b is of τ_a-regular divergence,

$$2\langle bu, \nabla u\rangle = -c_1(N + \alpha)\langle r^\alpha u^2\rangle \quad \big(0 \leqslant u \in D(\tau)\big).$$

(For $0 \leqslant u \in C_c^1(\mathbb{R}^N \setminus \{0\})$ apply partial integration; for general u the claim follows by density.) For $p \in [1, \infty]$ we thus obtain

$$\tau_p(u) = \tfrac{4}{pp'}\|\nabla u\|_2^2 + \big(c_2 - \tfrac{1}{p}c_1(N + \alpha)\big)\langle r^\alpha u^2\rangle \quad \big(0 \leqslant u \in D(\tau)\big). \tag{3.31}$$

Now let us investigate for which values of α, c_1, c_2 and for which $p \in [1, \infty)$ the form τ is associated with a positive quasi-contractive C_0-semigroup on $L_p(\mathbb{R}^N)$.

First we consider the three cases $\alpha < -2$, $\alpha > 0$, and $N = 2 = -\alpha$. Then, by Remark 3.26, the potential r^α in not form bounded with respect to τ_a. Thus, the above implies that τ_p is bounded below if and only if $c_2 - \frac{1}{p}c_1(N+\alpha) \geqslant 0$, and $\tau_p \geqslant 0$ in this case. By Theorems 3.2(a) and 3.22 we obtain, for $p \in (1,\infty)$, that τ is associated with a quasi-contractive C_0-semigroup on L_p if and only if $c_2 - \frac{1}{p}c_1(N+\alpha) \geqslant 0$. Let I be the set of all $p \in (1,\infty)$ fulfilling this inequality.

The set I is of one of the types $(1,\infty)$, $(1,p_0]$, $[p_0,\infty)$, $\varnothing$. If $c_2 \geqslant \big(c_1(N+\alpha)\big)^+$ then $I = (1,\infty)$; if $c_2 \leqslant \big(c_1(N+\alpha)\big) \wedge 0$ then $I = \varnothing$. If $c_1(N+\alpha) < c_2 < 0$ then $I = (1, \frac{c_1}{c_2}(N+\alpha)]$; if $0 < c_2 < c_1(N+\alpha)$ then $I = [\frac{c_1}{c_2}(N+\alpha), \infty)$. In particular we obtain: if $\alpha = -N$, $c_2 \geqslant 0$ then, for arbitrary $c_1 \in \mathbb{R}$, the form τ is associated with a consistent family of contractive C_0-semigroups on L_p, $p \geqslant 1$.

Now we consider the case $\alpha \in [-2,0)$, $(N,\alpha) \neq (2,-2)$. Of course, τ is associated with a contractive C_0-semigroup on L_p in the cases discussed above, but we obtain more. If $\alpha \in (-2,0)$ then, by Remark 3.26, for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $\tau_p \geqslant \big(\frac{4}{pp'} - \varepsilon\big)\tau_a - c_\varepsilon$. Thus, τ is associated with a consistent family of quasi-contractive C_0-semigroups on L_p, $p > 1$.

Now assume $\alpha = -2$ (and $N \geqslant 3$). Then, by (3.31) and Hardy's inequality,

$$\tau_p \geqslant \left(\frac{4}{pp'}\frac{(N-2)^2}{4} - \frac{c_1}{p}(N-2) + c_2\right) r^{-2}.$$

Since Hardy's inequality is sharp we obtain that τ_p is bounded below if and only if $\frac{(N-2)^2}{pp'} - c_1\frac{N-2}{p} + c_2 \geqslant 0$, and $\tau_p \geqslant 0$ in this case. Thus, by Theorems 3.2(a) and 3.22, τ is associated with a quasi-contractive C_0-semigroup on L_p, for $p \in (1,\infty)$, if and only if $\frac{(N-2)^2}{pp'} - c_1\frac{N-2}{p} + c_2 \geqslant 0$. For $c_2 = 0$ this condition simplifies to $\frac{N-2}{p'} - c_1 \geqslant 0$. Thus, if $c_1 \geqslant N-2$ then there is no quasi-contractive semigroup on any L_p associated with τ. If $c_1 \in (0, N-2)$ (or $c_1 \leqslant 0$) then τ is associated with a contractive C_0-semigroup on L_p for all $p \geqslant \frac{N-2}{N-2-c_1}$ (or $p \geqslant 1$).

Finally we show that, for $\alpha = -2$, $N \geqslant 3$, the form τ studied above constitutes an example in which the interval in the L_p-scale obtained in Theorem 3.8 cannot be extended. So let, from now on, $N \geqslant 3$, $b(x) := c_1|x|^{-2}x$, $V(x) := c_2|x|^{-2}$. As above let $r(x) = |x|$, and

$$\tau(u,v) = \langle \nabla u, \nabla v\rangle + \langle \nabla u, bv\rangle + \langle Vu, v\rangle$$

on $D(\tau) = W_2^1(\mathbb{R}^N)$.

Proposition 3.28. *Assume that τ is associated with a C_0-semigroup e^{-tA_p} on L_p, for some $p \geqslant 1$. Then*

$$D(A_p) \supseteq D_p := \big\{u \in L_p;\ \Delta u, \tfrac{1}{r}|\nabla u|, \tfrac{1}{r^2}u \in L_p\big\},$$

and $A_p u = (-\Delta + b\cdot\nabla + V)u$ for all $u \in D_p$.

Proof. Define the operator $\mathcal{L}$ in L_p by $\mathcal{L}u := (-\Delta + b\cdot\nabla + V)u,\ D(\mathcal{L}) := D_p$. Let $U := |b|^2 + |V|$. Then $\tau + U$ is a closed sectorial form in L_2. Let A_U be the associated m-sectorial operator in L_2. By Proposition 1.19, e^{-tA_U} extrapolates to a C_0-semigroup $T_{U,p}(t) = e^{-tA_{U,p}}$ on L_p, and $T_p = (T_{U,p})_{-U}$.

Let first $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Then $u \in D(\tau + U)$ and

$$(\tau + U)(u, v) = \langle(\mathcal{L} + U)u, v\rangle \quad \big(v \in D(\tau + U)\big).$$

Hence $u \in D(A_U)$ and $A_U u = (\mathcal{L} + U)u$. Since $u, A_U u \in L_p$, Lemma 3.24 implies that $u \in D(A_{U,p}),\ A_{U,p}u = (\mathcal{L} + U)u$. Thus, $A_{U,p}$ and $\mathcal{L} + U$ coincide on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$. By [Voi86; Cor. 2.7] we have $A_p \supseteq A_{U,p} - U$, so A_p and $\mathcal{L}$ coincide on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$.

Let now $u \in D_p \cap L_{p,c}(\mathbb{R}^N \setminus \{0\}),\ \rho_n$ the standard mollifier. Then $\rho_n * u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ for large n, and it is easy to see that $\rho_n * u \to u,\ \mathcal{L}(\rho_n * u) \to \mathcal{L}u$ in L_p. This implies $A_p u = \mathcal{L}u$ since A_p is a closed operator.

Finally, let $u \in D_p$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leqslant \varphi \leqslant 1,\ \varphi\restriction_{B(0,1)} = 1$, $\operatorname{supp}\varphi \subseteq B(0,2)$. Let $\varphi_n(x) := \varphi(\frac{x}{n}) \cdot (1 - \varphi)(2nx)$ for $x \in \mathbb{R}^N$. Then $\varphi_n \in C_c^\infty(\mathbb{R}^N \setminus \{0\}),\ 0 \leqslant \varphi_n \leqslant 1$, and $\varphi_n \to 1$ a.e. as $n \to \infty$. Let

$$B_n := \big(B(0, 2n) \setminus B(0, n)\big) \cup \big(B(0, \tfrac{1}{n}) \setminus B(0, \tfrac{1}{2n})\big).$$

It is straightforward that

$$|\nabla\varphi_n| \leqslant \tfrac{2}{r}\|\nabla\varphi\|_\infty \chi_{B_n}, \quad |\Delta\varphi_n| \leqslant \tfrac{4}{r^2}\|\Delta\varphi\|_\infty \chi_{B_n} \quad (n \in \mathbb{N}).$$

Thus, by dominated convergence we obtain that

$$\mathcal{L}(\varphi_n u) = \varphi_n \mathcal{L}u - 2\nabla\varphi_n \cdot \nabla u + (b \cdot \nabla\varphi_n - \Delta\varphi_n)u \to \mathcal{L}u \quad \text{in } L_p$$

as $n \to \infty$. So $u \in D(A_p)$ and $A_p u = \mathcal{L}u$ since $\varphi_n u \to u$ in L_p. □

Corollary 3.29. *Assume that τ is associated with a C_0-semigroup e^{-tA_p} on L_p, for some $p \in [1, \infty)$. Let $\sigma < \frac{N}{p} - 2,\ u := r^{-\sigma}e^{-\frac{r^2}{2}}$. Then $u \in D(A_p)$ and*

$$A_p u = \big(-\tfrac{1}{r^2}(\sigma^2 - (N - 2 - c_1)\sigma - c_2) + N - c_1 - 2\sigma - r^2\big)u.$$

Proof. By Proposition 3.28 we have $u \in D(A_p)$ and $A_p u = (-\Delta + b \cdot \nabla + V)u$ since $u \in D_p$. The second assertion now results from a direct computation. □

The next extrapolation lemma is a modification of the result from [Cou91] with literally the same proof.

Lemma 3.30. *Let $p_0 \leqslant p < q \leqslant p_1$. Let T be a semigroup on L_p satisfying $\|T(t)\|_{p_0\to p_0} \leqslant C,\ \|T(t)\|_{p_1\to p_1} \leqslant C$, and*

$$\|T(t)\|_{p\to q} \leqslant Ct^{-\alpha(\frac{1}{p}-\frac{1}{q})} \quad (t > 0),$$

for some $C \geqslant 1,\ \alpha > 0$. Then

$$\|T(t)\|_{p_0\to p_1} \leqslant Ct^{-\alpha(\frac{1}{p_1}-\frac{1}{p_0})} \quad (t > 0).$$

Example 3.31. Let $c_1 := \beta\frac{N-2}{2}$ and $c_2 := -\gamma\frac{(N-2)^2}{4}$ with $0 \leqslant \beta < 2,\ 0 < \gamma < (1-\frac{\beta}{2})^2$. By (3.31) we have

$$\tau_p(u) = \tfrac{4}{pp'}\|\nabla u\|_2^2 - \tfrac{(N-2)^2}{4}\big(\tfrac{2}{p}\beta + \gamma\big)\|\tfrac{u}{r}\|_2^2 \quad \big(u \in D(\tau)\big).$$

Hence, by Hardy's inequality,

$$\tau_p(u) \geqslant \tfrac{(N-2)^2}{4}\big(4\tfrac{1}{p}(1-\tfrac{1}{p}) - 2\beta\tfrac{1}{p} - \gamma\big)\|\tfrac{u}{r}\|_2^2 \quad \big(u \in D(\tau)\big).$$

Let $1 < p_- < p_+ < \infty$ such that $\frac{1}{p_\pm}$ are the roots of the equation

$$4x(1-x) - 2\beta x - \gamma = -4\big(x^2 - (1-\tfrac{\beta}{2})x + \tfrac{\gamma}{4}\big) = 0. \tag{3.32}$$

Then τ_p is bounded below if and only if $p \in [p_-, p_+]$. Hence, by Theorem 3.8, τ is associated with a consistent family of C_0-semigroups e^{-tA_p} on L_p,

$$p_{\min} := \big(\tfrac{N}{N-2}p'_-\big)' < p < \tfrac{N}{N-2}p_+ =: p_{\max}.$$

By Theorems 3.2(a) and 3.22, e^{-tA_p} is quasi-contractive if and only if $p \in [p_-, p_+]$.

We are going to show that, for $q \notin (p_{\min}, p_{\max})$, the form τ is *not* associated with a C_0-semigroup on L_q. Let

$$\sigma := \frac{N}{p_{\max}} = \frac{N-2}{p_+}, \quad p_0 := \frac{N}{\sigma+2} = \big(\tfrac{N}{N-2}p'_+\big)'.$$

Then $\frac{\sigma}{N-2}$ is a root of equation (3.32). Hence, $\sigma^2 - (N-2-\beta\frac{N-2}{2})\sigma + \gamma\frac{(N-2)^2}{4} = 0$. Observe that $p_0 \in (p_{\min}, p_+)$. Let $u := r^{-\sigma}e^{-\frac{r^2}{2}}$, $c := N - \beta\frac{N-2}{2} - 2\sigma$, $p \in (p_{\min}, p_0)$. Then $\sigma < \frac{N}{p} - 2$. By Corollary 3.29, $A_p u = (c - r^2)u \leqslant cu$ and hence $u \leqslant e^{t(c-A_p)}u$ for all $t \geqslant 0$.

Now assume that τ is associated with a C_0-semigroup T_q on L_q, for some $q \geqslant p_{\max}$. Then T_p, T_q are consistent by Proposition 1.22(a). By (3.24) and Lemma 3.30, $e^{-tA_p}\colon L_p \to L_q$ for all $t > 0$. In particular, $e^{-tA_p}u \in L_q$. Since $e^{t(c-A_p)}u \geqslant u$, this contradicts the fact that $u \notin L_q$ (recall $\sigma = \frac{N}{p_{\max}} \geqslant \frac{N}{q}$). Considering the adjoint semigroup we show that e^{-tA_p} does not extrapolate to a semigroup on L_q, for any $q \leqslant p_{\min}$.

In the case of Schrödinger semigroups, i.e. $\beta = 0$, this example was first given in [KPS81]. More precisely, for a certain class of potentials it was shown that the Schrödinger semigroup acts on $L_p(\mathbb{R}^N)$ for $p \in (p_{\min}, p_{\max})$, and it was claimed that for potentials of the type $\frac{c}{r^2}$ with $c < 0$ the interval is maximal. A strict proof of this claim was given by Yu. Semenov (private communication).

Finally, let us return to the remark in the paragraph following the proof of Proposition 1.15. There the following was claimed. Given a densely defined closed sectorial form τ fulfilling the first Beurling-Deny criterion, T the associated C_0-semigroup, and $V \geqslant 0$ measurable, the T-admissibility of $-V$ does not imply

$V \leqslant \operatorname{Re}\tau + \omega$ for some $\omega \in \mathbb{R}$. We now show that the above constitutes an example.

Let $0 < \beta < 1$. Then there exists $0 < \gamma < (1 - \frac{\beta}{2})^2$ such that $p_{\min} < 2 < p_-$, where $p_{\min}, p_-$ are defined as above. We have shown that τ is associated with a positive C_0-semigroup T on L_2 which is not quasi-contractive. Define the form τ_0 in L_2 by

$$\tau_0(u, v) := \langle \nabla u, \nabla v \rangle + \langle \nabla u, bv \rangle$$

on $D(\tau_0) = W_2^1(\mathbb{R}^N)$. Then τ_0 is a densely defined closed sectorial form fulfilling the first Beurling-Deny criterion. Let T_0 be the associated (contractive) C_0-semigroup on L_2. Let $U := -V = \gamma\frac{(N-2)^2}{4r^2}$. Then $\tau_0 = \tau + U$. Since τ is associated with the C_0-semigroup T on L_2, Proposition 1.19 implies that $-U$ is T_0-admissible. But we do not have $U \leqslant \operatorname{Re}\tau_0 + \omega$ for some $\omega \in \mathbb{R}$: this would imply $\tau_2 = \operatorname{Re}\tau = \operatorname{Re}\tau_0 - U \geqslant -\omega$, contradicting $2 \notin [p_-, p_+]$.

We point out that, in the above situation, τ is a form that is not sectorial—it is not even bounded from the left. Nevertheless, by means of Definition 1.20, τ is associated with a C_0-semigroup on L_2.

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken over directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this or any other country.

The present thesis was written at Dresden University of Technology, Institute of Analysis, under the supervision of Prof. Dr. Jürgen Voigt.

Versicherung

Hiermit versichere ich, daß ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

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