A sharp condition for the chaotic behaviour of a size structured cell population

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Abstract

We show that the condition $0 \leq \beta \leq \frac{1}{2\ln 2}$ is necessary for the chaoticity of the solution of the cell population model

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = -\frac{\partial(xu(t,x))}{\partial x} + \gamma u(t,x) - \beta u(t,x) + 4\beta u(t,2x)\chi_{(0,\frac{1}{2})}(x), \\ u(0,\cdot) = f \in L^1(0,1). \end{cases} \tag{0.1}$$

(If $\gamma - 3\beta > 0$, then this condition is known to be sufficient.) The analysis depends on solving a forward delay equation.

1 Preliminaries

In [2] we studied the chaotic behaviour of the above simplified model of a size structured cell population. We supposed that $\gamma - 3\beta > 0$ and we proved that the evolution associated with (0.1) is chaotic if $0 \leq \beta \leq \frac{1}{2\ln 2}$, and is subchaotic if $\beta > \frac{1}{2\ln 2}$ (see [2; Theorem 2.3]). In the present note we show that in fact the evolution is not chaotic for $\beta > \frac{1}{2\ln 2}$ (i.e., the space of subchaoticity is a proper subspace).

The following result due to Desch, Schappacher and Webb will be used to rule out chaoticity in our main result.

Theorem 1.1. *([1; Theorem 3.3]) If a C_0-semigroup $T(\cdot)$ generated by A in a Banach space X is hypercyclic, then the adjoint A^* of A and the dual semigroup have the following properties:*

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(i) *If* $\varphi \in X^*$, $\varphi \neq 0$, *then the orbit* $(T^*(t)\varphi : t \geq 0)$ *is unbounded,*

(ii) *the point spectrum of* A^* *is empty.*

Remark 1.2. On a separable Banach space, a chaotic C_0-semigroup is automatically hypercyclic (see [1; Definition 2.1 and Theorem 2.2]; see also [2; Definition 1.2] for "chaotic").

2 Main result

From [2] we know that, by the change of variable $x = e^{-y}$ $(y > 0)$, one transforms equation (0.1) into

$$(2.1)\quad \begin{cases} \frac{\partial v(t,y)}{\partial t} = e^y \frac{\partial (e^{-y} u(t,y))}{\partial y} + (\gamma - \beta) v(t,y) + 4\beta v(t, y - \ln 2)\chi_{(\ln 2, \infty)}(y) =: Av(t,y), \\ v(0,\cdot) = g \in L^1((0,\infty), e^{-y} dy), \end{cases}$$

with $g(y) = f(e^{-y})$ and $D(A) = \{v \in L^1((0,\infty), e^{-y}dy); \frac{\partial v}{\partial y} \in L^1((0,\infty), e^{-y}dy)\}$. By a standard perturbation argument one sees that A generates a C_0-semigroup $T(\cdot)$ on $L^1((0,\infty), e^{-y}dy)$ and hence equation (2.1) (or (0.1)) is well-posed.

The following is the main result of this paper.

Theorem 2.1. *If* $\beta > \frac{1}{2\ln 2}$, *then the* C_0*-semigroup associated with* (2.1) *is not hypercyclic, and therefore not chaotic.*

Remark 2.2. Note that the hypothesis $\gamma - 3\beta > 0$ of [2; Theorem 2.3] is not needed in Theorem 2.1.

Proof of Theorem 2.1. In view of Theorem 1.1 and Remark 1.2 it is sufficient to show that the point spectrum of A^* is not empty.

A standard computation shows that A^* is given by

$$A^*\varphi(y) = -\varphi'(y) + (\gamma - \beta)\varphi(y) + 2\beta\varphi(y + \ln 2) \qquad (y \geq 0),$$

with $D(A^*) = \{\varphi \in W^1_\infty(0,\infty); \varphi(0) = 0\}$.

A function φ is an eigenfunction of A^*, associated with the eigenvalue λ, if and only if φ is a bounded C^1-function satisfying

$$(2.2)\quad \begin{cases} \varphi'(y) = \omega\varphi(y) + 2\beta\varphi(y + \ln 2) \quad (y \geq 0), \\ \varphi(0) = 0, \end{cases}$$

with $\omega := \gamma - \beta - \lambda$.

By setting $h(y) := e^{-(\omega \ln 2)y}\varphi(y \ln 2)$, $c := 2\beta \ln 2\, e^{\omega \ln 2}$, we transform problem (2.2) into a problem for a "forward delay equation",

$$(2.3)\quad \begin{cases} h'(y) = c\, h(y+1) \quad (y \geq 0), \\ h(0) = 0. \end{cases}$$

The function $h(y) := ye^y$ is a solution of (2.3), for $c = e^{-1}$. For the corresponding solution φ of (2.2) we obtain

$$\varphi(y \ln 2) = e^{(\omega \ln 2)y} h(y) = ye^{(\omega \ln 2 + 1)y}.$$

From $c = e^{-1}$ and $2\beta \ln 2 > 1$ we obtain that $\omega \ln 2 + 1 = \ln \frac{c}{2\beta \ln 2} + 1 = -\ln(2\beta \ln 2) < 0$. We conclude that φ is bounded, and therefore φ is an eigenfunction of A^*. □

Combining Theorem 2.1 and [2; Theorem 2.3], we obtain the following conclusion.

Corollary 2.3. *Assume that* $\gamma - 3\beta > 0$. *Then the* C_0*-semigroup associated with* (2.1) *is chaotic if and only if* $0 \le \beta \le \frac{1}{2 \ln 2}$.

Appendix. On solutions of $h'(y) = c\,h(y+1)$

In this appendix we present further solutions of problem (2.3), and we explain how we found the above solution.

We start with the observation that for any $\lambda \in \mathbb{C}$ the function $g_\lambda(y) := e^{\lambda y}$ satisfies

$$g_\lambda'(y) = \lambda e^{-\lambda} g_\lambda(y+1).$$

If $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \ne \lambda_2$, are such that $\lambda_1 e^{-\lambda_1} = \lambda_2 e^{-\lambda_2}$, then obviously $h := g_{\lambda_1} - g_{\lambda_2}$ is a non-trivial solution of (2.3).

Let $\xi < 1$. We are looking for $\eta > 0$ such that

$$(\xi + i\eta)e^{-(\xi + i\eta)} = (\xi - i\eta)e^{-(\xi - i\eta)},$$

i.e., $(\xi + i\eta)e^{-(\xi+i\eta)}$ is real, $0 = \operatorname{Im}\big((\xi + i\eta)e^{-(\xi+i\eta)}\big) = \eta \cos \eta - \xi \sin \eta$, $\xi = \frac{\eta \cos \eta}{\sin \eta}$. The function $\eta \mapsto \frac{\eta \cos \eta}{\sin \eta}$ maps the interval $[0, \pi)$ continuously and strictly decreasingly onto $(-\infty, 1]$. Therefore there exists a unique η $(=: \eta_\xi) \in (0, \pi)$ such that $\xi = \frac{\eta \cos \eta}{\sin \eta}$. Moreover, η_ξ depends continuously and strictly decreasingly on ξ, and $\eta_\xi \to 0$ as $\xi \to 1$. For a pair (ξ, η) of this kind one concludes that

$$h(y) := \frac{1}{2i}\big(e^{(\xi+i\eta)y} - e^{(\xi - i\eta)y}\big) = e^{\xi y} \sin \eta y$$

is a solution of (2.3), for $c = c_\xi := (\xi + i\eta)e^{-(\xi+i\eta)} = e^{-\xi}(\xi \cos \eta + \eta \sin \eta) = e^{-\xi} \frac{\eta}{\sin \eta}$.

Using the function h above we obtain that

$$h_\xi(y) := e^{\xi y} \frac{\sin \eta_\xi y}{\eta_\xi}$$

is a solution of (2.3), for $c = c_\xi$. Taking the pointwise limit one obtains

$$h(y) := \lim_{\xi \to 1} h_\xi(y) = ye^y,$$

and it is easy to check that h satisfies (2.3) for $c = e^{-1} = \lim_{\xi \to 1} c_\xi$.

References

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