Perturbation theory for parabolic differential equations

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Contents

Introduction

The validity of Gaussian bounds for fundamental solutions of second-order parabolic equations in divergence form with non-smooth coefficients goes back to Aronson. In [Aro67], Aronson studied the parabolic equation

$$\partial_t u = \nabla \cdot \big(a(t,x)\nabla u\big) \tag{0.1}$$

with a uniformly parabolic measurable coefficient $a\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^{n\times n}$, i.e., a is bounded and there exists $\varepsilon > 0$ such that $\sum_{j,k=1}^n a_{jk}(t,x)\xi_j\xi_k \geqslant \varepsilon|\xi|^2$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^n$ and all $\xi\in\mathbb{R}^n$. He proved that there are constants $C_1, C_2, c_1, c_2 > 0$ such that the fundamental solution g of (0.1) satisfies the two-sided Gaussian bounds

$$C_1(t-s)^{-n/2}e^{-c_1\frac{|x-y|^2}{t-s}} \leqslant g(t,x;s,y) \leqslant C_2(t-s)^{-n/2}e^{-c_2\frac{|x-y|^2}{t-s}} \tag{0.2}$$

for all $(t,x),(s,y)\in[0,\infty)\times\mathbb{R}^n$ with $t>s$; see [Aro67; Rem. 5] for an explanation why these estimates are *global* in time, i.e., why no restriction $t-s\leqslant T$ is needed.

In [Aro68], Aronson studied a more general parabolic equation that includes lower order terms with measurable coefficients, in particular, a multiplication term (potential) was allowed that was supposed to satisfy a certain integrability property. Starting from the publication of [Aro68] the topic has a rich history, and numerous extensions and generalisations have been obtained. For results in the Euclidean space setting see, e.g., [EiPo84, FaSt86, Dav87a, Dav89, Str92, Sem99, LiSe00, Dan00]; further generalisations will be discussed below.

For the case of Schrödinger operators $-\Delta + V$, in [AiSi82] the Kato class of potentials was shown to be the appropriate class for the L_1-perturbation theory of the corresponding C_0-semigroups, and in [Sim82] it was shown that the fundamental solution of the perturbed heat equation still satisfies upper and lower Gaussian estimates. It was much later, however, that parabolic equations with more general time-dependent potentials were studied and the non-autonomous Kato class as the proper extension of the Kato class was introduced; cf. [Zha96, Zha97, ScVo99, Gul02, Gul04].

Essentially following [Zha96] we say that a potential $V\in L_{1,\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$ is in the *non-autonomous Kato class* **NK** if

$$N^+(V) := \lim_{\alpha\to 0} N_\alpha^+(V) = 0 \quad \text{and} \quad N^-(V) := \lim_{\alpha\to 0} N_\alpha^-(V) = 0,$$

where

$$N_\alpha^\pm(V) = \sup_{x,s}\int_0^\alpha\int_{\mathbb{R}^n} k_t(x-y)|V(s\pm t,y)|\,dy\,dt \qquad (0<\alpha\leqslant\infty),$$

with the free heat kernel $k_t(x) = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}$, and $V(\tau,\cdot) := 0$ for $\tau<0$. We say that V is in the *enlarged* non-autonomous Kato class $\widehat{\mathbf{NK}}$ if $N^+(V)<\infty$ and $N^-(V)<\infty$. (Observe that this differs from [ScVo99], where only $N^+(V)<\infty$ was required.) Note that $N^+(V) = N^-(V)$ in the case of time-independent V. In fact, **NK** just reduces to the Kato class in this case, and $\widehat{\mathbf{NK}}$ to the extended Kato class (cf. [ScVo99]).

In [Zha97], Zhang studied the fundamental solution of the parabolic equation

$$\partial_t u = \nabla \cdot \big(a(t,x)\nabla u\big) - V(t,x)u. \tag{0.3}$$

He proved that if a is uniformly parabolic and $V_\lambda := V(\cdot/\lambda, \cdot) \in \mathbf{NK}$ for a suitable $\lambda > 0$, then the fundamental solution satisfies the two-sided Gaussian bounds (0.2) for $s < t \leqslant s + 1$. (The framework of [Zha97] is actually the more general one of parabolic equations with uniformly subelliptic principal part, see below.) Of course the above bounds can be extended to all $t > s$ by means of the reproducing kernel property. However, this leads to an additional factor $e^{\omega t}$ in the upper bound, and $e^{-\omega t}$ in the lower bound; in general the bounds are not global in time.

As is observed in [ScVo99], for the well-posedness of the Cauchy problem for (0.3) in $L_1(\mathbb{R}^n)$ only the condition $N^+(V_\lambda) < \lambda/M$ is needed, where λ, M are such that the fundamental solution g_0 corresponding to $V = 0$ satisfies $g_0(t,x;s,y) \leqslant M k_{\lambda(t-s)}(x-y)$. This is derived by means of the non-autonomous Miyadera perturbation theorem [RRSV00; Thm. 3.4]; in fact, V satisfies the required Miyadera smallness condition under the above assumption. However, a condition controlling $N^+(V_\lambda)$ (a *forward* Kato condition) is not sufficient for obtaining Gaussian bounds: Zhang's result also requires a *backward* Kato condition, i.e., a condition on $N^-(V_\lambda)$. The latter condition is responsible for the well-posedness of the adjoint Cauchy problem

$$-\partial_t u = \nabla \cdot \big(a^\top(t,x)\nabla u\big) - V(t,x)u, \qquad u(T,\cdot) = f$$

in $L_1(\mathbb{R}^n)$, where $T > 0$ (cf. Section 2.3).

It was noted in [Zha97] that the Gaussian bounds (0.2) for the fundamental solution of (0.3) remain valid for $s < t \leqslant s+1$ if $V_\lambda \in \widehat{\mathbf{NK}}$ and the quantities $N^\pm(V_\lambda)$ are sufficiently small, for a suitable $\lambda > 0$. The results of this thesis imply that the upper bound holds if the negative part V^- of V satisfies $N^\pm(V_\lambda^-) < \lambda/M$ (with λ, M as in the previous paragraph); cf. Theorem 3.10. It follows from the general approach of absorption propagators that we use that no restrictions on the positive part V^+ of V are needed. The lower bound in (0.2) holds under the assumption $N^\pm(V_\lambda^+) < \infty$ and the condition that the Cauchy problem for (0.3) is well-posed in $L_1(\mathbb{R}^n)$; cf. Theorem 3.12. In order to obtain global bounds (0.2), we require the same conditions as above but with $N_\infty^\pm$ in place of $N^\pm$. For the upper bound this essentially is proved in [Zha97]. The global lower bound under these conditions is due to [LVV06; Rem. 3.13].

The aim of this thesis is to develop a perturbation method that enables one to derive the above results in the general framework of positive propagators on L_p-spaces, thus allowing for a much wider range of applications. A (linear) propagator, or evolution family, describes the time evolution of a system with time-dependent driving force. The perturbation by a time-dependent potential that we are going to study can be thought of as an operator addition to the local infinitesimal generators of the propagator. Since the latter do not exist, in general, the perturbed propagators are constructed by requiring the validity of a Duhamel formula. We refer to the introduction of [RRSV00] for a more extensive discussion.

This approach using the Duhamel formula and the resulting Dyson-Phillips series works rather directly in the case of bounded perturbations (not necessarily by potentials); see [LVV06; Sec. 1.2]. (In [RäSc99; Sec. 2], bounded perturbations are studied by means of evolution semigroups.) In Chapter 1 we show that the approach can also be implemented in the more general case of Miyadera perturbations; thus we entirely avoid the use of evolution semigroups.

For the special case of positive propagators on L_p-spaces and perturbation by potentials, one can still go further by approximating a general potential by bounded potentials and using monotonicity. This approach, known as the theory of absorption semigroups and absorption propagators, is well-established for C_0-semigroups (cf. [Voi86], [Voi88], [ArBa93], [Man01]). In the non-autonomous case, perturbations by positive potentials have been studied in [RäSc99], and only in [LVV06] the study of sign-changing potentials was initiated.

In Chapter 2 we further develop the theory of absorption propagators. One of the new aspects is that we enlarge the class of admissible perturbations, thus allowing perturbation by highly oscillating potentials. Moreover, we study strongly measurable propagators (as opposed to strongly continuous propagators). This makes the development of the theory more natural since the class of strongly measurable propagators is closed under strong convergence.

The main ingredient in our proof of stability of Gaussian bounds under perturbation by potentials is an interpolation inequality for absorption propagators (see Section 2.5) that is well-known in the case of Schrödinger semigroups, in which probabilistic tools can be used for the proof. We provide a purely analytical argument that is based on a Trotter product formula for strongly continuous propagators, provided in Section 1.4. Then in Section 3.1 we show how the interpolation inequality can be used to prove both upper and lower Gaussian type bounds, in the general context of positive ultracontractive propagators.

Above we have only discussed parabolic equations on the whole space $\mathbb{R}^n$. Suppose now that Ω is a bounded connected open subset of $\mathbb{R}^n$, let $T \in (0,\infty]$, and consider the parabolic equation with homogeneous Dirichlet boundary condition

$$\partial_t u = \nabla \cdot \big(a(t,x)\nabla u\big), \qquad u = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{0.4}$$

where $a\colon [0,T) \times \Omega \to \mathbb{R}^{n\times n}$ is measurable and uniformly parabolic. It follows from general domination principles that the fundamental solution of (0.4) satisfies the upper bound in (0.2). The corresponding lower bound, however, does not hold, and the upper bound does not reflect the boundary behaviour. Moreover, the long-time behaviour is different since Ω is bounded. The latter is the reason why in the following we assume that $T < \infty$ unless stated otherwise.

The autonomous case was studied in detail by Davies, based on intrinsic ultracontractivity estimates from [DaSi84]. Under the assumption that $n \geqslant 3$ and that a and $\partial\Omega$ are sufficiently smooth, it follows from [Dav87b; Thm. 3 and Thm. 9] that there exist $C, c > 0$ such that the fundamental solution g of (0.4) satisfies the Gaussian type upper bound

$$g(t,x;s,y) \leqslant C(t-s)^{-n/2}\Big(\frac{d(x)d(y)}{t-s} \wedge 1\Big) \cdot \exp\Big(-c\frac{|x-y|^2}{t-s}\Big) \tag{0.5}$$

for all $(t,x),(s,y) \in [0,T) \times \Omega$ with $t > s$, where $d(x)$ denotes the distance from x to the boundary of Ω. An analogous lower bound remained an open question; only in the simpler situation of the long-time behaviour, precise upper and lower bounds were provided.

The missing lower bound was first proved in [Zha02] for the case of the heat equation and $C^{1,1}$-boundary, again under the assumption $n \geqslant 3$. The latter assumption was removed in [Son04a]. It was noted in [Zha02; Rem. 1.3] that the lower bound also holds for space-dependent Hölder continuous coefficients. In [Son04b], global upper and lower bounds of the type (0.5) are proved for the case of the heat equation, assuming that $n \geqslant 3$ and that Ω is the subgraph of a bounded $C^{1,1}$-function $\Gamma\colon \mathbb{R}^{n-1} \to \mathbb{R}$.

The case of time-dependent coefficients was first investigated by Riahi in [Ria01] for the half space and later in [Ria05] for bounded Ω with $C^{1,1}$-boundary. The most general result we know of is due to Cho. In [Cho06] he proves (0.5) and a matching lower bound under the assumption that Ω has $C^{1,\alpha}$-boundary, for some $\alpha \in (0,1)$, and that the coefficients are Dini continuous with respect to the parabolic distance $\left|(t,x)-(s,y)\right| = |t-s|^{1/2} + |x-y|$, i.e., $|a_{jk}(X) - a_{jk}(Y)| \leqslant \omega(|X-Y|)$ for all $X, Y \in [0,T) \times \Omega$ and $j,k = 1,\dots,n$, for some increasing function $\omega\colon [0,\infty) \to [0,\infty)$ satisfying $\int_0^1 \omega(t)/t\,dt < \infty$. (Without some regularity of a and $\partial\Omega$, the kernel estimate (0.5) does not hold in general.)

We point out that already in [Dav87b], the parabolic equation contained a (time-independent) potential V satisfying a suitable condition on infinitesimal form smallness. In [Ria03] it is proved that (0.5) and also the corresponding lower bound are stable under perturbation by potentials from **NK**. This does not cover, however, the boundary singularities of the potentials considered in [Dav87b]. Only in [Ria07] a proper generalisation was found: It follows from [Ria07; Cor. 5.7] that the kernel bound (0.5) and a matching lower bound hold if the parabolic equation (0.4) is perturbed by a potential that is locally in **NK** and that satisfies, for some $\alpha > 0$ and some small enough $c, \varepsilon > 0$, the estimate

$$\sup_{x,s} \int_0^\alpha \int_{\mathbb{R}^n} \frac{d(y)}{d(x)} \gamma_t^c(x,y) |V(s \pm t, y)| \, dy \, dt \leqslant \varepsilon,$$

where $\gamma_t^c(x,y) = t^{-n/2}\bigl(\frac{d(x)d(y)}{t} \wedge 1\bigr) \exp\bigl(-c\frac{|x-y|^2}{t-s}\bigr)$. We point out that Riahi's result deals with the more general case of perturbation by measures, and that an additional drift term is included whose coefficient is a vector-valued measure.

Owing to the method of [Ria07], the constants c and ε in the above condition on V are not sharp. In Section 3.2 we prove stability of kernel estimates of the type (0.5) in a more general setting, under a condition on V that is sharp up to a factor of 2.

In Section 3.3, the concluding section of the thesis, we show that our methods can also be applied to stability of kernel estimates on complete Riemannian manifolds with Ricci curvature bounded below. The main breakthrough in this context was accomplished by Li and Yau, who in [LiYa86] proved global upper and lower bounds for the heat kernel in the case of non-negative Ricci curvature. It is shown in [Sal92] that these estimates remain valid if one replaces the heat equation with a uniformly parabolic equation with time-dependent measurable coefficients. For further developments see [DaPa89, Dav93, Stu95, Gri99, Gri06]. In particular, in [Gri06; Sec. 10], Grigor'yan has obtained sharp estimates for the long time behaviour in the time-independent symmetric case.

Estimates analogous to the Li-Yau estimate also hold for uniformly subelliptic operators of Hörmander type; see [Sán84], [JeSá86] for compact manifolds and [KuSt87], [KuSt88] for the setting of $\mathbb{R}^n$. In fact, it is the latter setting in which the stability results of [Zha97] are proved, not only the uniformly parabolic setting described in the beginning. In Section 3.3 we prove stability of kernel estimates in a more general framework that includes the above applications, under a miminal assumption on the perturbation.

The outline of the thesis is as follows. In Chapter 1 we investigate the general perturbation theory for strongly measurable propagators. The notion of strongly measurable propagators is introduced in Section 1.1, and some basic properties are proved. In Section 1.2 we prove the Miyadera perturbation theorem for strongly measurable propagators and related results; the special case of closed Miyadera perturbations is studied in Section 1.3 in more detail. In Section 1.4 we present a convergence result for bounded perturbations and derive the Trotter product formula for strongly continuous propagators.

Chapter 2 is devoted to the theory of absorption propagators, i.e., to the particular case of perturbation of positive propagators on L_p-spaces by real-valued potentials. The general theory is developed in Section 2.1, and in Section 2.2 the relation to the Miyadera perturbation theorem is established. In Section 2.3 we introduce a backward Miyadera condition that is responsible for local L_∞-boundedness of perturbed propagators. The interpolation inequalities for absorption propagators that are mentioned above are studied in Sections 2.5 and 2.6; the proofs are based on the notion of logarithmically convex operator-valued functions that is presented in Section 2.4.

In Chapter 3 we apply the abstract theory of Chapter 2 to our main subject, the stability of kernel estimates for strongly continuous propagators under perturbation by potentials. Section 3.1 deals with general kernel estimates in the setting of ultracontractive propagators. In Section 3.2 we prove refined results for more special Gaussian type estimates involving boundary terms. Finally, in Section 3.3 we show that our method can also be adapted to cover the case of heat kernel estimates on Riemannian manifolds.

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Chapter 1

Perturbation theory for strongly measurable propagators

Strongly continuous propagators are a standard device in the study of non-autonomous evolution equations. Here, more generally, we investigate strongly measurable propagators since they arise naturally as absorption propagators; cf. Section 2.1. Only at the end of the chapter, strong continuity will play a crucial role. This will have the consequence that in the kernel estimates in Chapter 3 we will have to start from an unperturbed propagator that is strongly continuous.

In Section 1.1 we give a short introduction to strongly measurable and strongly continuous propagators. In Sections 1.2 and 1.3 we prove the Miyadera perturbation theorem for strongly measurable propagators and related results concerning Duhamel formulas and Dyson-Phillips series. Section 1.4 deals with the special case of bounded perturbations; there we prove a Trotter product formula for strongly continuous propagators.

1.1 Strongly measurable propagators

Let $J \subseteq \mathbb{R}$ be an interval, and let $D_J := \big\{(t,s) \in J \times J;\, t \geqslant s\big\}$. For $t \in \mathbb{R}$ we use the notation $J_{\geqslant t} := \{s \in J;\, s \geqslant t\}$, and analogously $J_{>t}$, $J_{\leqslant t}$, $J_{<t}$.

Let X be a (real or complex) Banach space. A *propagator* on X (with *parameter interval* J) is a function $U\colon D_J \to \mathcal{L}(X)$ satisfying

$$U(t,t) = I, \quad U(t,r) = U(t,s)\,U(s,r) \qquad (t \geqslant s \geqslant r \text{ in } J).$$

With a slight abuse of language, we say that U is a *strongly measurable propagator* if additionally U is *separately* strongly measurable, i.e., $U(\cdot,s)$ is strongly measurable on $J_{\geqslant s}$, for all $s \in J$, and $U(t,\cdot)$ is strongly measurable on $J_{\leqslant t}$, for all $t \in J$. This definition is explained by the fact that separate strong measurability of propagators implies joint strong measurability, but not vice versa if X is not separable, as shown in the following lemma and in Example 1.3 below.

1.1 Lemma. (a) *Let U be a strongly measurable propagator on X. Then U is jointly strongly measurable.*

(b) *Let U be a propagator on X, and assume that U is jointly strongly measurable. Then $U(t,\cdot)$ is strongly measurable on $J_{\leqslant t}$, for all $t \in J$. If, in addition, X is separable* *then $U(\cdot, s)$ is strongly measurable on $J_{\geqslant s}$, for all $s \in J$, so U is a strongly measurable propagator.*

Proof. (a) Let $\tau \in J$ and $R_\tau := J_{\geqslant\tau} \times J_{\leqslant\tau}$. Then $(t,s) \mapsto U(t,s) = U(t,\tau)U(\tau,s)$ is strongly measurable on R_τ, by Lemma 1.2 below. The assertion follows since $\bigcup_{\tau\in J\cap\mathbb{Q}} R_\tau$ has full measure in D_J.

(b) Let $t \in J$ and $x \in X$; we prove that $U(t,\cdot)x$ is measurable on $J_{\leqslant t}$. In the case $t = \inf J$ there is nothing to show. If $t > \inf J$ then $U(\tau,\cdot)x$ is measurable on $J_{\leqslant\tau}$ for a.e. $\tau \in J_{<t}$, so $U(t,\cdot)x = U(t,\tau)U(\tau,\cdot)x$ is measurable on $J_{\leqslant\tau}$ for a.e. $\tau \in J_{<t}$ and hence measurable on $J_{\leqslant t}$.

Assume now that X is separable, and let X_0 be a countable dense subset of X. Then there exists a null set $N \subseteq J$ such that $U(\cdot,s)x$ is measurable on $J_{\geqslant s}$ for all $s \in J \setminus N$ and all $x \in X_0$. It follows that $U(\cdot,s)$ is strongly measurable on $J_{\geqslant s}$ for all $s \in J \setminus N$, and with an argument analogous to the above we obtain the same for all $s \in J$, which completes the proof. □

1.2 Lemma. *Let $f\colon J \to X$ be measurable, and let $B\colon J \to \mathcal{L}(X)$ be strongly measurable. Then $t \mapsto B(t)f(t)$ is strongly measurable. If, in addition, $A\colon J \to \mathcal{L}(X)$ is strongly measurable then $t \mapsto B(t)A(t)$ is strongly measurable.*

Proof. Let $f_n\colon J \to X$ be step functions such that $f_n \to f$ a.e. as $n \to \infty$. Then $t \mapsto B(t)f_n(t)$ is measurable for each $n \in \mathbb{N}$ and $B(\cdot)f_n(\cdot) \to B(\cdot)f(\cdot)$ a.e. as $n \to \infty$, which implies the first assertion. The second assertion is immediate from the first one. □

The following example shows that one cannot dispense with the separability assumption in Lemma 1.1(b).

1.3 Example. Let $J := \Omega := [0,1]$, and let μ be the counting measure on Ω. We define $U\colon D_J \to \mathcal{L}(L_1(\mu))$ by $U(t,t) := I$ for all $t \in J$ and

$$U(t,s)f := f(s)\delta_t \qquad \big(0 \leqslant s < t \leqslant 1,\ f \in L_1(\mu)\big),$$

where $\delta_t = \mathbb{1}_{\{t\}}$ is the indicator function of $\{t\}$. One easily checks that U is a propagator on $L_1(\mu)$, and U is jointly strongly measurable since $U(t,s)f = 0$ for a.e. $s \in J$, for every $f \in L_1(\mu)$. However, the function $t \mapsto U(t,0)\delta_0 = \delta_t$ is not measurable, so U is not a strongly measurable propagator.

Unlike strong measurability, strong continuity of propagators is defined in the obvious way. A propagator U on X with parameter interval J is called a *strongly continuous propagator* if $U\colon D_J \to \mathcal{L}(X)$ is strongly continuous. Clearly, a strongly continuous propagator is separately strongly continuous, but it follows from Example 1.5 below that the converse is not true. The next lemma shows that the converse is true for locally bounded propagators. (The remainder of the section is taken from [LVV06; Sec. 1].)

1.4 Lemma. *([Gul04; Thm. 2.2]) Let U be a propagator on X. Suppose that*

(i) *U is locally bounded;*

(ii) *for any $s \in J$ the mapping $J_{\geqslant s} \ni t \mapsto U(t,s)$ is strongly continuous;*

(iii) *for any $t \in J$ the mapping $J_{\leqslant t} \ni s \mapsto U(t,s)$ is strongly continuous at $s = t$.*

Then U is strongly continuous.

Proof. Fix $(t_0, s_0) \in D_J$, and let $J_0 := J_{<s_0}$ ($J_0 := \{s_0\}$ in case $s_0 = \inf J$). Due to (iii) the set $\bigcup_{r\in J_0} U(s_0, r)X$ is dense in X. Thus, due to (i), it suffices to show that the mapping $D_J \ni (t, s) \mapsto U(t, s)x_0$ is continuous at (t_0, s_0), for $x_0 := U(s_0, r)x$ with $r \in J_0$ and $x \in X$. Let $t_n \to t_0$, $s_n \to s_0$, without loss of generality $s_n \geqslant r$ for all $n \in \mathbb{N}$. Denote $x_n := U(s_n, r)x$. Condition (ii) implies that $x_n \to x_0$. Using (i) and (ii) again we conclude that

$$U(t_n, s_n)x_0 = U(t_n, r)x + U(t_n, s_n)(x_0 - x_n) \to U(t_0, r)x = U(t_0, s_0)x_0. \qquad \square$$

The uniform boundedness theorem implies that strongly continuous propagators are locally bounded. The following example shows that separate strong continuity of a propagator is not sufficient to obtain local boundedness (as was claimed in [CuPr78; Def. 2.32]). In particular it shows that condition (i) of Lemma 1.4 is needed for the conclusion.

1.5 Example. We indicate an example on the Hilbert space ℓ_2.

Let $\big((a_n, b_n)\big)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint subintervals of $[0, 1]$, and for each $n \in \mathbb{N}$ let $a_n < s_n < t_n < b_n$. For $n \in \mathbb{N}$ let $\varphi_n\colon [0, 1] \to [\frac{1}{n}, 1]$ be a continuous function, $\varphi_n = 1$ on $[0, a_n]$, $\varphi_n(s_n) = \frac{1}{n}$, $\varphi_n(t_n) = 1$ and $\varphi_n = \frac{1}{n}$ on $[b_n, 1]$. We define $U\colon D_{[0,1]} \to \mathcal{L}(\ell_2)$ by

$$U(t, s)\big((x_n)_n\big) := \left(\frac{\varphi_n(t)}{\varphi_n(s)} x_n\right)_n,$$

for $0 \leqslant s \leqslant t \leqslant 1$ and $(x_n)_n \in \ell_2$.

Then U is a propagator on ℓ_2 with parameter interval $[0, 1]$. For $(t, s) \in D_{[0,1]}$ and $n \in \mathbb{N}$ observe that $\frac{\varphi_n(t)}{\varphi_n(s)} \leqslant 1$ except when $a_n < s < t < b_n$. From the pairwise disjointness of the intervals (a_n, b_n) it thus follows that $[0, t] \ni s \mapsto U(t, s)$ is strongly continuous for all $t \in [0, 1]$, and that $[s, 1] \ni t \mapsto U(t, s)$ is strongly continuous for all $s \in [0, 1]$. However, $\|U(t_n, s_n)\| = n$ for all $n \in \mathbb{N}$, so U is not bounded.

1.6 Remark. Note that, in the previous example, the unboundedness of U occurs close to the diagonal. In fact, as in the proof of Lemma 1.1(a) one obtains that a separately strongly continuous propagator U with parameter interval J is strongly continuous on $J_{\geqslant\tau} \times J_{\leqslant\tau}$, for all $\tau \in J$, and hence strongly continuous on $\big\{(t, s) \in D_J;\, t - s \geqslant \delta\big\}$, for all $\delta > 0$.

1.2 Miyadera perturbations of strongly measurable propagators

In this and the next section we investigate Miyadera perturbations of strongly measurable propagators. In [RRSV00] (see also [LVV06; Sec. 1.3]), the method of evolution semigroups is used to prove the Miyadera perturbation theorem for strongly continuous propagators. We do not assume strong continuity of the propagator, so the corresponding evolution semigroup need not be strongly continuous either. That is why in the following it is not suitable to make use of evolution semigroups.

Let X be a Banach space, and let U be a locally bounded strongly measurable propagator on X with parameter interval $J \subseteq \mathbb{R}$. A *Miyadera perturbation* of U (*with constants* (α, γ)) is a family $(B(t))_{t\in J}$ of linear operators in X satisfying the following condition.

(M) For all $s \in J$, the set

$$X_s := \{x \in X;\ B(\cdot)U(\cdot, s)x \text{ is a.e. defined and measurable on } J_{\geqslant s}\}$$

is a dense subspace of X, and there exist $\alpha \in (0, \infty]$ and $\gamma \geqslant 0$ such that

$$\int_s^t \|B(\tau)U(\tau, s)x\|\, d\tau \leqslant \gamma\|x\| \qquad \big((t,s) \in D_J,\ t - s \leqslant \alpha,\ x \in X_s\big).$$

We say that $(B(t))_{t\in J}$ is a *small* Miyadera perturbation of U if α, γ can be chosen such that $\gamma < 1$. The case $\alpha = \infty$ is included since it is important for the global behaviour of perturbed propagators; see, e.g., Remark 3.11(c) and Remark 3.24(a).

1.7 Remarks. (a) We say that U is *exponentially bounded* if there exist $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t, s)\| \leqslant Me^{\omega(t-s)} \qquad \big((t, s) \in D_J\big),$$

or equivalently, if $\sup\big\{\|U(t, s)\|;\, (t, s) \in D_J,\ t-s \leqslant 1\big\} < \infty$. If U is exponentially bounded and $B\colon J \to \mathcal{L}(X)$ is bounded and strongly measurable, then by Lemma 1.2 one easily sees that $(B(t))_{t\in J}$ is an *infinitesimally small* Miyadera perturbation of U, i.e., the infimum of all possible γ in condition (M) is zero.

(b) Let $(B(t))_{t\in J}$ be a Miyadera perturbation of U. Since U is a propagator, we obtain from the definition of the spaces X_s that $U(t, s)X_s \subseteq X_t$ for all $(t, s) \in D_J$. Further we show that, for all $(t, s) \in D_J$, there exists $\gamma_{t,s} \geqslant 0$ such that

$$\int_s^t \|B(\tau)U(\tau, s)x\|\, d\tau \leqslant \gamma_{t,s}\|x\| \qquad (x \in X_s). \tag{1.1}$$

This is trivial in the case $\alpha = \infty$, so assume that $\alpha < \infty$. Choose $n \in \mathbb{N}$ such that $(n-1)\alpha \leqslant t - s < n\alpha$, and let $t_j := s + j\alpha$ for $j = 0, \dots, n-1$ and $t_n := t$. Then

$$\int_s^t \|B(\tau)U(\tau, s)x\|\, d\tau = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|B(\tau)U(\tau, t_j)U(t_j, s)x\|\, d\tau \leqslant \sum_{j=0}^{n-1} \gamma\|U(t_j, s)x\|$$

for all $x \in X_s$, so (1.1) holds with $\gamma_{t,s} = \gamma\sum_{j=0}^{n-1}\|U(t_j, s)\|$.

(c) Miyadera perturbations of propagators have first been studied in [RRSV00] (see also [RRS96] for non-autonomous Miyadera perturbations of C_0-semigroups). We point out that in [RRSV00], strong continuity of U is assumed, and the authors work with a slightly different condition that at the first sight seems more general than condition (M) above. Namely, they assume the estimate in (M) only for $x \in Y_s$, where $(Y_s)_{s\in J}$ is a given family of dense subspaces of X satisfying $Y_s \subseteq X_s$ and $U(t, s)Y_s \subseteq Y_t$ for all $(t, s) \in D_J$ ([RRSV00; p. 350], case $p = 1$). Obviously, this is an assumption on the parts $\widetilde{B}(t) := B(t)|_{D(B(t))\cap Y_t}$ of $B(t)$ only.

We now show that in this situation $(\widetilde{B}(t))_{t\in J}$ satisfies condition (M) with the same constants α, γ as in the above condition on $(B(t))_{t\in J}$. First observe that for the spaces $\widetilde{X}_s$

corresponding to $(\widetilde{B}(t))_{t\in J}$ we have $Y_s \subseteq \widetilde{X}_s$, so that $\widetilde{X}_s$ is dense in X. Now let $(t,s) \in D_J$ with $t-s \leqslant \alpha$, and let $x \in \widetilde{X}_s$. Then $U(\sigma,s)x \in Y_\sigma$ for a.e. $\sigma \in [s,t]$, by the definition of $\widetilde{X}_s$. For every such σ we have

$$\int_\sigma^t \|\widetilde{B}(\tau)U(\tau,\sigma)U(\sigma,s)x\|\,d\tau \leqslant \gamma\|U(\sigma,s)x\|.$$

It follows that $\int_s^t \|\widetilde{B}(\tau)U(\tau,s)x\|\,d\tau \leqslant \gamma \limsup_{\sigma\to s}\|U(\sigma,s)\|\cdot\|x\|$, so

$$A_{t,s}x := \widetilde{B}(\cdot)U(\cdot,s)x \in L_1(s,t;X)$$

defines a bounded operator from $\widetilde{X}_s$ to $L_1(s,t;X)$. Since Y_s is dense in X, we conclude that the estimate $\|A_{t,s}x\|_1 = \int_s^t \|\widetilde{B}(\tau)U(\tau,s)x\|\,d\tau \leqslant \gamma\|x\|$, which was assumed for all $x \in Y_s$, extends to all $x \in \widetilde{X}_s$. (Note that the last part of the argument is not needed if U is strongly continuous, as assumed in [RRSV00].)

The following Miyadera perturbation theorem is the main result of this section. It generalises [LVV06; Thm. 1.16(a)], where the result is proved in the case of strongly continuous propagators. Also in that case, a slightly weaker version has already been shown in [RRSV00; Thm. 3.4(a), Cor. 3.5]. In these papers, the approach of evolution semigroups is used to derive the result from the Miyadera perturbation theorem for C_0-semigroups [Voi77; Thm. 1]; see also [Miy66; Thm. 2] for the orginal version of the theorem.

1.8 Theorem. *Let $(B(t))_{t\in J}$ be a small Miyadera perturbation of U. Then there exists a unique locally bounded strongly measurable propagator U_B on X satisfying the Duhamel formula*

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau \qquad \big((t,s)\in D_J,\ x\in X_s\big), \tag{1.2}$$

with X_s as defined in condition (M).

If U is strongly continuous then U_B is strongly continuous, too.

For the proof we will use the following lemma, which brings a new perspective on condition (M). For the convenience of a simplified notation, we extend U to a strongly measurable function from $J\times J$ to $\mathcal{L}(X)$ by setting $U(t,s) := 0$ for $t < s$.

1.9 Lemma. *Let $(B(t))_{t\in J}$ be a Miyadera perturbation of U, and assume that J is compact. Then*

$$A(s)x := B(\cdot)U(\cdot,s)x \qquad (x\in X_s) \tag{1.3}$$

extends to a bounded operator $A(s)$ from X to $L_1(J;X)$, for each $s\in J$, and the function $A\colon J\to\mathcal{L}\big(X;L_1(J;X)\big)$ thus defined is bounded and strongly measurable.

Proof. Note that U is bounded since J is compact. We thus obtain from Remark 1.7(b) that there exists $c \geqslant 0$ such that

$$\|A(s)x\|_1 = \int_s^{\sup J} \|B(\tau)U(\tau,s)x\|\,d\tau \leqslant c\|x\|$$

for all $s \in J$ and all $x \in X_s$. Therefore, (1.3) determines a bounded operator $A(s)\colon X \to L_1(J;X)$, for each $s \in J$, and the resulting function A is bounded.

Let now $(t,s) \in D_J$. For $x \in X_s$ and a.e. $\tau \in J_{\geqslant t}$ we have

$$A(s)x(\tau) = B(\tau)U(\tau,t)U(t,s)x = A(t)U(t,s)x(\tau).$$

This implies that $\mathbb{1}_{J_{\geqslant t}}A(s)x = A(t)U(t,s)x$ for all $x \in X_s$ and hence for all $x \in X$. Since $U(t,\cdot)x\colon J_{\leqslant t} \to X$ is measurable, it follows that

$$J_{\leqslant t} \ni s \longmapsto \mathbb{1}_{J_{\geqslant t}}A(s)x \in L_1(J;X)$$

is measurable, for all $x \in X$ and all $t \in J$. From this we deduce the strong measurability of A, using that $\mathbb{1}_{J_{\geqslant t}}A(s)x \to A(s)x$ in $L_1(J;X)$ as $t \downarrow s$. ☐

The following result will be used to establish strong continuity of perturbed propagators that satisfy both Duhamel formulas. For the proof of the strong continuity assertion of Theorem 1.8 we will only need Lemma 1.11(a) below. In these two results, $(B(t))_{t\in J}$ is a family of linear operators in X, and Y_s is a dense subspace of X, for each $s \in J$.

1.10 Proposition. *([LVV06; Prop. 1.18]) Assume that U is strongly continuous, and let W be a locally bounded propagator on X with parameter interval J. Assume that, for all $(t,s) \in D_J$ and all $x \in Y_s$, the functions $W(t,\cdot)B(\cdot)U(\cdot,s)x$ and $B(\cdot)W(\cdot,s)x$ are integrable on $[s,t]$, and*

$$W(t,s)x = U(t,s)x + \int_s^t W(t,\tau)B(\tau)U(\tau,s)x\,d\tau, \tag{1.4}$$

$$W(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)W(\tau,s)x\,d\tau. \tag{1.5}$$

Then W is strongly continuous.

Proof. This is an immediate consequence of Lemma 1.4 and the following lemma. ☐

1.11 Lemma. *([LVV06; Lemma 1.19]) Assume that U is strongly continuous, and let W be a locally bounded propagator on X with parameter interval J.*

(a) *Assume that, for all $(t,s) \in D_J$, $x \in Y_s$, the function $W(t,\cdot)B(\cdot)U(\cdot,s)x$ is integrable on $[s,t]$, and* (1.4) *holds. Then $W(t,\cdot)$ is strongly continuous on $J_{\leqslant t}$ for all $t \in J$.*

(b) *Assume that, for all $(t,s) \in D_J$, $x \in Y_s$, the function $B(\cdot)W(\cdot,s)x$ is integrable on $[s,t]$, and* (1.5) *holds. Then $W(\cdot,s)$ is strongly continuous on $J_{\geqslant s}$ for all $s \in J$.*

Proof. (a) Fix $t \in J$ and $s_0 \in J_{\leqslant t}$. Let $r \in J_{<s_0}$ ($r = s_0$ in case $s_0 = \inf J$), $x \in Y_r$ and $x_s := U(s,r)x$ for $s \in [r,t]$. By (1.4) we have

$$x_s = W(s,r)x - \int_r^s W(s,\tau)B(\tau)U(\tau,r)x\,d\tau.$$

Applying $W(t,s)$ to this equation and using (1.4) again, we obtain that

$$\begin{aligned} W(t,s)x_s &= W(t,r)x - \int_r^s W(t,\tau)B(\tau)U(\tau,r)x\,d\tau \\ &= U(t,r)x + \int_s^t W(t,\tau)B(\tau)U(\tau,r)x\,d\tau. \end{aligned}$$

(Note that we cannot obtain this by setting $x = x_s$ in (1.4) if $U(s,r)Y_r \not\subseteq Y_s$.) Thus, by the dominated convergence theorem,

$$W(t,s)x_s \to U(t,r)x + \int_{s_0}^t W(t,\tau)B(\tau)U(\tau,r)x\,d\tau = W(t,s_0)x_{s_0}$$

as $[r,t] \ni s \to s_0$. Since $x_s \to U(s_0,r)x = x_{s_0}$, we conclude that

$$W(t,s)x_{s_0} = W(t,s)x_s + W(t,s)(x_{s_0} - x_s) \to W(t,s_0)x_{s_0}$$

as $[r,t] \ni s \to s_0$. The set of elements x_{s_0} under consideration is dense in X, so the assertion follows.

(b) Let $s \in J$ and $x \in Y_s$. Since

$$W(t,s)x = U(t,s)x + \int_J \mathbb{1}_{[s,t]}(\tau)U(t,\tau)B(\tau)W(\tau,s)x\,d\tau \qquad (t \in J_{\geqslant s})$$

and U is strongly continuous (and hence locally bounded), $W(\cdot,s)x$ is continuous on $J_{\geqslant s}$ by the dominated convergence theorem. This shows the assertion since Y_s is dense in X and W is locally bounded. □

Proof of Theorem 1.8. We will first show: If U_B is an element of the space

$$\begin{aligned} \mathcal{M} := \bigl\{V\colon D_J \to \mathcal{L}(X);\ &V \text{ is locally bounded,} \\ &V \text{ is separately and jointly strongly measurable}\bigr\} \end{aligned}$$

that satisfies the Duhamel formula (1.2), then U_B is a propagator. Then we will show that there exists a unique $U_B \in \mathcal{M}$ satisfying (1.2). The assertion about strong continuity is proved in the last step.

Let $\alpha \in (0,\infty)$, $\gamma \in [0,1)$ be as in condition (M), and let $U_B \in \mathcal{M}$ satisfy (1.2). (Note that the integral in (1.2) exists since $B(\cdot)U(\cdot,s)x$ is integrable on (s,t) by (1.1) and $U_B(t,\cdot)$ is strongly measurable and bounded on (s,t); see also Lemma 1.2.) Clearly, $U_B(t,t) = I$ for all $t \in J$. Given $(t,s) \in D_J$ we show that $U_B(t,s)U_B(s,r) = U_B(t,r)$ for all $r \in [s-\alpha,s] \cap J$; then it follows by induction that U_B is a propagator.

Let $r \in [s-\alpha,s] \cap J$ and $x \in X_r$. Then $U(s,r)x \in X_s$, so by (1.2) we obtain that

$$U_B(t,s)U(s,r)x = U(t,r)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,r)x\,d\tau. \tag{1.6}$$

Moreover,

$$U_B(s,r)x = U(s,r)x + \int_r^s U_B(s,\tau)B(\tau)U(\tau,r)x\,d\tau.$$

Applying $U_B(t,s)$ to both sides of the latter formula and using (1.6), we infer that

$$\begin{aligned} U_B(t,s)U_B(s,r)x = U(t,r)x &+ \int_s^t U_B(t,\tau)B(\tau)U(\tau,r)x\,d\tau \\ &+ \int_r^s U_B(t,s)U_B(s,\tau)B(\tau)U(\tau,r)x\,d\tau. \end{aligned}$$

We now subtract (1.2), used with r in place of s, from the preceding equation to obtain that

$$U_B(t,s)U_B(s,r)x - U_B(t,r)x = \int_r^s \big(U_B(t,s)U_B(s,\tau) - U_B(t,\tau)\big)B(\tau)U(\tau,r)x\,d\tau.$$

For the operators $D_\tau := U_B(t,s)U_B(s,\tau) - U_B(t,\tau) \in \mathcal{L}(X)$ we thus have

$$\|D_r x\| \leqslant \sup_{\tau\in[r,s]} \|D_\tau\| \int_r^s \|B(\tau)U(\tau,r)x\|\,d\tau \qquad \big(r \in [s, s-\alpha]\cap J,\ x \in X_r\big).$$

By the Miyadera condition we conclude that $\|D_r\| \leqslant \gamma \sup_{\tau\in[r,s]}\|D_\tau\|$ for all $r \in [s, s-\alpha]\cap J$. Therefore, $D_r = 0$ for all $r \in [s, s-\alpha]\cap J$, and we have shown that U_B is a propagator.

For the proof of existence and uniqueness we assume without loss of generality that J is compact; then U is bounded. Below we will show that

$$\Phi V(t,s)x := \int_s^t V(t,\tau)A(s)x(\tau)\,d\tau \qquad \big(V \in \mathcal{M},\ (t,s)\in D_J,\ x \in X\big)$$

(with A as in Lemma 1.9) defines an operator $\Phi\colon \mathcal{M} \to \mathcal{M}$, and that Φ is a strict contraction on $\mathcal{M}$ if $\mathcal{M}$ is endowed with the Morgenstern norm

$$\|V\|_\lambda := \sup\big\{e^{-\lambda(t-s)}\|V(t,s)\|;\ (t,s)\in D_J\big\},$$

for $\lambda > 0$ large enough. Since $(\mathcal{M}, \|\cdot\|_\lambda)$ is a Banach space and $U \in \mathcal{M}$, it then follows that there exists a unique $U_B \in \mathcal{M}$ such that $U_B = U + \Phi U_B$, and this identity is equivalent to the validity of (1.2) since

$$\Phi U_B(t,s)x = \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau \qquad \big((t,s)\in D_J,\ x \in X_s\big)$$

by the definition of $A(s)$.

Let $V \in \mathcal{M}$ and $t \in J$. Since $V(t,\cdot)$ is strongly measurable and bounded on $J_{\leqslant t}$, we can define a bounded operator $C_V(t)\colon L_1(J;X) \to X$ by

$$C_V(t)f := \int_{\inf J}^t V(t,\tau)f(\tau)\,d\tau.$$

Then

$$\Phi V(t,s)x = \int_{\inf J}^t V(t,\tau)A(s)x(\tau)\,d\tau = C_V(t)A(s)x$$

for all $s \in J_{\leqslant t}$ and all $x \in X$, where we have used that $A(s)x(\tau) = 0$ for $\tau \in J_{<s}$. From the (joint) strong measurability of V it follows that $C_V\colon J \to \mathcal{L}\big(L_1(J;X), X\big)$ is strongly

measurable. Since $A\colon J \to \mathcal{L}\big(X; L_1(J;X)\big)$ is strongly measurable by Lemma 1.9, we conclude that ΦV is separately and jointly strongly measurable (see Lemma 1.2).

In order to show that Φ is a strict contraction on $(\mathcal{M}, \|\cdot\|_\lambda)$, we endow $L_1(J;X)$ with the norm given by $\|f\|_{1,\lambda} := \|t \mapsto e^{-\lambda t} f(t)\|_1$. Let $M := \sup\big\{\|U(t,s)\|;\ (t,s) \in D_J\big\}$. Then for $s \in J$ and $x \in X_s$ we obtain as in Remark 1.7(b), with $t_j = s + \alpha j$, that

$$\begin{aligned} \|A(s)x\|_{1,\lambda} &= \int_s^{\sup J} e^{-\lambda\tau} \|B(\tau)U(\tau,s)x\|\, d\tau \leqslant \sum_{j=0}^{n-1} e^{-\lambda t_j} \gamma \|U(t_j,s)x\| \\ &\leqslant \gamma\Bigg(1 + M\sum_{j=1}^{n-1} e^{-\lambda\alpha j}\Bigg) e^{-\lambda s}\|x\| \leqslant \widetilde{\gamma}_\lambda e^{-\lambda s}\|x\|, \end{aligned} \tag{1.7}$$

where $\widetilde{\gamma}_\lambda := \gamma\big(1 + \frac{M}{e^{\lambda\alpha}-1}\big) \to \gamma < 1$ as $\lambda \to \infty$, and $n \in \mathbb{N}$ is the smallest number such that $t_n \notin J$. Moreover, for $V \in \mathcal{M},\ t \in J$ and $f \in L_1(J;X)$ we can estimate

$$\|C_V(t)f\| \leqslant \int_{\inf J}^t \|V(t,\tau)f(\tau)\|\, d\tau \leqslant \int_{\inf J}^t \|V\|_\lambda e^{\lambda(t-\tau)} \|f(\tau)\|\, d\tau \leqslant \|V\|_\lambda e^{\lambda t} \|f\|_{1,\lambda}.$$

It follows that $\|\Phi V(t,s)\| = \|C_V(t)A(s)\| \leqslant \widetilde{\gamma}_\lambda e^{\lambda(t-s)}\|V\|_\lambda$ for all $(t,s) \in D_J$, so that $\Phi V \in \mathcal{M}$ and $\|\Phi V\|_\lambda \leqslant \widetilde{\gamma}_\lambda \|V\|_\lambda$. We conclude that Φ is a strict contraction on $(\mathcal{M}, \|\cdot\|_\lambda)$ for λ large enough, and the proof of the first assertion is complete.

Assume now that U is strongly continuous. Then $U_B(t,\cdot)$ is strongly continuous on $J_{\leqslant t}$ for all $t \in J$, by Lemma 1.11(a). Moreover, it follows from the dominated convergence theorem that Φ maps the closed subspace

$$\mathcal{M}_1 := \big\{V \in \mathcal{M};\ V(\cdot,s) \text{ is strongly continuous on } J_{\geqslant s}\ (s \in J)\big\}$$

of $\mathcal{M}$ to itself, so $U_B \in \mathcal{M}_1$. By Lemma 1.4 we conclude that U_B is strongly continuous. □

1.12 Remarks. (a) Let $(B(t))_{t\in J}$ be a small Miyadera perturbation of U, and let $\Phi\colon \mathcal{M} \to \mathcal{M}$ be as in the above proof. Observe that for the definition of Φ, the compactness of J is actually not needed since for $V \in \mathcal{M}$ one only has to define ΦV on D_K for compact subintervals K of J. Since $(I - \Phi)U_B = U$, we obtain that the perturbed propagator U_B is given by the Dyson-Phillips series $\sum_{k=0}^\infty U_k$, where $U_k := U_{k,B} := \Phi^k U\colon D_J \to \mathcal{L}(X)$ is recursively given by $U_0 = U$ and

$$U_k(t,s)x = \int_s^t U_{k-1}(t,\tau)B(\tau)U(\tau,s)x\, d\tau \qquad \big((t,s) \in D_J,\ x \in X_s\big)$$

for $k \in \mathbb{N}$. It follows from the proof of Theorem 1.8 that $U_B(t,s) = \sum_{k=0}^\infty U_k(t,s)$ converges absolutely, uniformly for (t,s) in compact subsets of D_J.

(b) The idea of the estimation in (1.7) is not new. In [Voi77; p. 168], a similar estimate is used in the proof of the Miyadera perturbation theorem for C_0-semigroups, but the purpose there is to determine the generator of the perturbed semigroup. Here the use of the Morgenstern norm allows us to show immediately that the perturbed propagator satisfies the Duhamel formula for all $(t,s) \in D_J$, and to obtain absolute convergence of the Dyson-Phillips series on all of D_J.

In the literature, the Duhamel formula is usually first shown for $t-s \leqslant \alpha$ only; then for larger $t-s$ an induction argument is used that is based on a variant of Proposition 1.13 below. The same applies to absolute convergence of the Dyson-Phillips series; see [Rha92; Prop. 2.3] for a result in the context of C_0-semigroups. Here we reverse arguments: We use the absolute convergence of the Dyson-Phillips series on D_J for an easy proof of Proposition 1.13.

(c) Clearly, Theorem 1.8 also holds if $(B(t))_{t\in J}$ is just *locally* Miyadera small, i.e., if $(B(t))_{t\in K}$ is a small Miyadera perturbation of $U|_{D_K}$ for all compact subintervals $K \subseteq J$. This holds, e.g., if $B\colon J \to \mathcal{L}(X)$ is locally bounded and strongly measurable. In the simple case that $B(t) = cI$ for all $t \in J$, for some $c \in \mathbb{K}$, one easily checks that the unique propagator U_B satisfying the Duhamel formula (1.2) is given by $U_B(t,s) = e^{c(t-s)}U(t,s)$ for all $(t,s) \in D_J$.

Note that the constituents U_k of the Dyson-Phillips series are also defined if $(B(t))_{t\in J}$ is a Miyadera perturbation of U that is not necessarily small.

1.13 Proposition. *Let $(B(t))_{t\in J}$ be a Miyadera perturbation of U, let U_k be the constituents of the corresponding Dyson-Phillips series, and let $r,s,t \in J$ satisfy $r \leqslant s \leqslant t$. Then*

$$U_k(t,r) = \sum_{j=0}^{k} U_j(t,s)U_{k-j}(s,r)$$

for all $k \in \mathbb{N}_0$.

Proof. There exists $\theta_0 > 0$ such that $\theta B = (\theta B(t))_{t\in J}$ is a small Miyadera perturbation of U for all $|\theta| < \theta_0$. Observe that $U_{k,\theta B} = \theta^k U_{k,B}$ for all $k \in \mathbb{N}_0$ and all $\theta \in \mathbb{R}$. For $|\theta| < \theta_0$ we thus obtain that

$$\begin{aligned}\sum_{k=0}^{\infty} \theta^k U_{k,B}(t,r) &= U_{\theta B}(t,r) = U_{\theta B}(t,s)U_{\theta B}(s,r)\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \theta^j U_{j,B}(t,s)\theta^{k-j}U_{k-j,B}(s,r),\end{aligned}$$

which implies the assertion. □

In the next result we show an estimate for the norm of $U_B(t,s)$, for $(t,s) \in D_J$. One can use formula (1.7) from the proof of Theorem 1.8 to derive this estimate, but we prefer to give a separate argument that relies solely on (1.2).

1.14 Proposition. *Let $(B(t))_{t\in J}$ be a small Miyadera perturbation of U with constants (α,γ), $\gamma < 1$, and assume that $M := \sup\{\|U(t,s)\|;\, (t,s) \in D_J\} < \infty$. Then*

$$\|U_B(t,s)\| \leqslant \tfrac{M}{1-\gamma} e^{\omega(t-s)} \qquad \big((t,s) \in D_J\big),$$

where $\omega = \frac{1}{\alpha}\ln(1 + \frac{\gamma}{1-\gamma}M)$.

Proof. We are going to prove that

$$c_j := \sup\bigl\{\|U_B(t,s)\|;\,(t,s)\in D_J,\ t-s\leqslant j\alpha\bigr\} \leqslant \tfrac{M}{1-\gamma}\bigl(1+\tfrac{\gamma}{1-\gamma}M\bigr)^{j-1}$$

for all $j \in \mathbb{N}$; then the assertion follows by choosing j such that $(j-1)\alpha \leqslant t-s \leqslant j\alpha$. Assuming the above estimate for all $j < n$, for some $n \in \mathbb{N}$, we show that it also holds for $j = n$.

Let $(t,s) \in D_J$ with $t-s \leqslant n\alpha$, and let $t_j := s + \frac{j}{n}(t-s)$ for $j = 0,\ldots,n$. Similarly as in Remark 1.7(b) we obtain from

$$U_B(t,s)x = U(t,s)x + \sum_{j=1}^{n} \int_{t_{n-j}}^{t_{n-j+1}} U_B(t,\tau)B(\tau)U(\tau,t_{n-j})U(t_{n-j},s)x\,d\tau \qquad (x\in X_s)$$

and the definition of the c_j that

$$\|U_B(t,s)\| \leqslant M + \sum_{j=1}^{n-1} c_j\gamma M + c_n\gamma.$$

By the definition of c_n we infer that $c_n \leqslant M\bigl(1+\gamma\sum_{j=1}^{n-1}c_j\bigr)+\gamma c_n$. Since

$$\gamma\sum_{j=1}^{n-1}c_j \leqslant \tfrac{\gamma}{1-\gamma}M\sum_{j=0}^{n-2}\bigl(1+\tfrac{\gamma}{1-\gamma}M\bigr)^{j} = \bigl(1+\tfrac{\gamma}{1-\gamma}M\bigr)^{n-1}-1$$

by the induction hypothesis, we conclude that $(1-\gamma)c_n \leqslant M\bigl(1+\frac{\gamma}{1-\gamma}M\bigr)^{n-1}$. □

1.15 Remark. For Miyadera perturbations of C_0-semigroups, an analogous estimate is proved in [Voi77; Thm. 1c)], but with $\omega = \frac{1}{\alpha}\ln\frac{M}{1-\gamma}$. In the case $M > 1$, Proposition 1.14 yields a better estimate; in particular, the exponential growth factor ω given in the proposition tends to 0 as $\gamma \to 0$.

1.3 Closed Miyadera perturbations

Throughout this section, U is again a locally bounded strongly measurable propagator on a Banach space X with parameter interval $J \subseteq \mathbb{R}$. Given a small Miyadera perturbation $(B(t))_{t\in J}$ of U, we now want to show the second Duhamel formula

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\,d\tau \qquad \bigl((t,s)\in D_J,\ x\in X\bigr) \tag{1.8}$$

for the perturbed propagator U_B. In the autonomous case, if U comes from a C_0-semigroup and B is relatively bounded with respect to the generator, this easily follows from the fundamental theorem of calculus. Here we can only show (1.8) in the case that almost every operator $B(t)$ is closed; see Theorem 1.20(a) below. We start with some preparations.

1.16 Lemma. *Let $(B(t))_{t\in J}$ be a family of operators in X. Assume that almost every $B(t)$ is closed, that $B(\cdot)x$ is a.e. defined and measurable on J, for all $x \in X$, and that there exists $c \geqslant 0$ such that $\|B(\cdot)x\|_1 \leqslant c\|x\|$ for all $x \in X$. Let (Ω, μ) be a σ-finite measure space and $f \in L_1(\mu; X)$. Then the function $(t, s) \mapsto B(t)f(s)$ is a.e. defined and measurable on $J \times \Omega$.*

Proof. The assertion is clear if f is a simple function. For $f \in L_1(\mu; X)$ there exists a sequence (f_n) of simple functions in $L_1(\mu; X)$ such that $f_n \to f$ a.e. and in L_1. By the assumption $\|B(\cdot)x\|_1 \leqslant c\|x\|$ we obtain that $g_n(t, s) := B(t)f_n(s)$ defines a Cauchy sequence (g_n) in $L_1(J \times \Omega; X)$, so there exist $g \in L_1(J \times \Omega; X)$ and a subsequence (g_{n_k}) such that $g_{n_k} \to g$ a.e. Since a.e. $B(t)$ is closed and $f_{n_k} \to f$ a.e., we conclude that $B(t)f(s) = g(t, s)$ for a.e. $(t, s) \in J \times \Omega$, which implies the assertion. □

The next lemma says in particular that the spaces X_s in condition (M) (that also occur in the Duhamel formula (1.2)) are equal to X if $(B(t))_{t\in J}$ is a *closed Miyadera perturbation* of U, by which we mean that $(B(t))_{t\in J}$ is a Miyadera perturbation of U and a.e. $B(t)$ is closed.

1.17 Lemma. *Let $(B(t))_{t\in J}$ be a closed Miyadera perturbation of U. Then $X_s = X$ for all $s \in J$. If J is compact then $\mathcal{A}f(t, s) := B(t)U(t, s)f(s)$ defines a bounded operator $\mathcal{A}\colon L_1(J; X) \to L_1(D_J; X)$, and for all $f \in L_1(J; X)$ we have*

$$C_U(t)f = \int_{\inf J}^{t} U(t, s)f(s)\, ds \in D(B(t)), \quad B(t)C_U(t)f = \int_{\inf J}^{t} B(t)U(t, s)f(s)\, ds$$

for a.e. $t \in J$.

Proof. Without loss of generality we assume the compactness of J also for the proof of the first assertion. Let $s \in J$, and let $A(s)$ be the operator defined in Lemma 1.9. Then $A(s)|_{X_s}$ is bounded and densely defined. Moreover, from the definition of X_s and the closedness of a.e. $B(t)$ we easily infer that $A(s)|_{X_s}$ is closed and hence everywhere defined, i.e., $X_s = X$.

Let now $f \in L_1(J; X)$. Given $\tau \in J$, we apply Lemma 1.16 with $B(t)U(t, \tau)$ in place of $B(t)$ and $U(\tau, \cdot)f(\cdot)$ in place of f to obtain that $\mathcal{A}f$ is a.e. defined and measurable on $J_{\geqslant\tau} \times J_{\leqslant\tau}$. It follows that $\mathcal{A}f$ is a.e. defined and measurable on D_J. Moreover,

$$\|\mathcal{A}f\|_1 \leqslant \int_J \int_s^{\sup J} \|B(t)U(t, s)f(s)\|\, dt\, ds \leqslant \int_J \|A(s)\|\|f(s)\|\, ds \leqslant \sup_{s\in J} \|A(s)\| \cdot \|f\|,$$

by Fubini's theorem, so $\mathcal{A}$ is a bounded operator by Lemma 1.9.

Finally, the integrability of $\mathcal{A}f$ implies that $B(t)U(t, \cdot)f(\cdot)$ is integrable on $J_{\leqslant t}$, for a.e. $t \in J$, so the last assertion follows from the closedness of a.e. $B(t)$, by Hille's theorem. □

The next result expresses, roughly speaking, that one can iterate Duhamel formulas. An important feature is that the Miyadera perturbations are not assumed to be small.

1.18 Proposition. *Let V, W be locally bounded strongly measurable propagators on X with parameter interval J. Let $B = (B(t))_{t\in J}$ be a Miyadera perturbation of U and $B_1 = (B_1(t))_{t\in J}$ a closed Miyadera perturbation of V, and assume that*

$$V(t,s)x = U(t,s)x + \int_s^t V(t,\tau)B(\tau)U(\tau,s)x\,d\tau \qquad (x \in X_s),$$

$$W(t,s)x = V(t,s)x + \int_s^t W(t,\tau)B_1(\tau)V(\tau,s)x\,d\tau \qquad (x \in X)$$

for all $(t,s) \in D_J$, where the spaces X_s are from the Miyadera condition for U and B. Then $B + B_1$ is a Miyadera perturbation of U, and

$$W(t,s)x = U(t,s)x + \int_s^t W(t,\tau)(B+B_1)(\tau)U(\tau,s)x\,d\tau \tag{1.9}$$

for all $(t,s) \in D_J$ and all $x \in X_s$. (Here $B + B_1 = \big(B(t)+B_1(t)\big)_{t\in J}$, where $B(t)+B_1(t)$ is the operator sum on $D(B(t)) \cap D(B_1(t))$.)

In particular, if B is a closed Miyadera perturbation and $B_1 = -B$, then $W = U$.

Proof. Let $(t,s) \in D_J$ and $x \in X_s$; then $f := B(\cdot)U(\cdot,s)x \in L_1(s,t;X)$. By Lemma 1.17 we obtain that

$$(V-U)(\tau,s)x = \int_s^\tau V(\tau,\sigma)f(\sigma)\,d\sigma \in D(B_1(\tau))$$

and

$$B_1(\tau)(V-U)(\tau,s)x = \int_s^\tau B_1(\tau)V(\tau,\sigma)f(\sigma)\,d\sigma$$

for a.e. $\tau \in (s,t)$, and that $(\tau,\sigma) \mapsto B_1(\tau)V(\tau,\sigma)f(\sigma)$ is integrable on $D_{(s,t)}$. If (α,γ), (α,γ_1) are Miyadera constants of B, B_1, respectively, and $t-s \leqslant \alpha$, then by Fubini's theorem it follows that

$$\int_s^t \|B_1(\tau)(V-U)(\tau,s)x\|\,d\tau \leqslant \int_s^t \int_\sigma^t \|B_1(\tau)V(\tau,\sigma)f(\sigma)\|\,d\tau\,d\sigma \leqslant \int_s^t \gamma_1\|f(\sigma)\|\,d\sigma$$

and hence $\int_s^t \|B_1(\tau)U(\tau,s)x\|\,d\tau \leqslant \gamma_1\gamma\|x\| + \gamma_1\|x\|$, and we infer that $B+B_1$ is a Miyadera perturbation of U. Moreover,

$$\int_s^t W(t,\tau)B_1(\tau)\int_s^\tau V(\tau,\sigma)f(\sigma)\,d\sigma\,d\tau = \int_s^t\int_\sigma^t W(t,\tau)B_1(\tau)V(\tau,\sigma)f(\sigma)\,d\tau\,d\sigma. \tag{1.10}$$

Since

$$W(t,s)x = U(t,s)x + \int_s^t V(t,\tau)B(\tau)U(\tau,s)x\,d\tau + \int_s^t W(t,\tau)B_1(\tau)V(\tau,s)x\,d\tau,$$

we conclude from (1.10) that

$$\begin{aligned} &-W(t,s)x + U(t,s)x + \int_s^t W(t,\tau)(B+B_1)(\tau)U(\tau,s)x\,d\tau \\ &\quad = \int_s^t (W-V)(t,\sigma)B(\sigma)U(\sigma,s)x\,d\sigma - \int_s^t W(t,\tau)B_1(\tau)(V-U)(\tau,s)x\,d\tau = 0, \end{aligned}$$

and the proof is complete. ∎

As an easy consequence of Proposition 1.18 we obtain the following uniqueness result for perturbed propagators satisfying both Duhamel formulas.

1.19 Corollary. *Let $(B(t))_{t\in J}$ be a closed Miyadera perturbation of U, and let V be a locally bounded strongly measurable propagator on X with parameter interval J satisfying*

$$V(t,s)x = U(t,s)x + \int_s^t V(t,\tau)B(\tau)U(\tau,s)x\,d\tau \qquad \big((t,s)\in D_J,\ x\in X\big), \tag{1.11}$$

$$V(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)V(\tau,s)x\,d\tau \qquad \big((t,s)\in D_J,\ x\in X\big). \tag{1.12}$$

Then V is the only locally bounded strongly measurable propagator satisfying (1.11)*, and the only one satisfying* (1.12)*.*

Proof. Let $\widetilde{V}$ be another locally bounded strongly measurable propagator satisfying (1.11), with V replaced by $\widetilde{V}$. Then from (1.12), in the form

$$U(t,s)x = V(t,s)x + \int_s^t U(t,\tau)(-B)(\tau)V(\tau,s)x\,d\tau \qquad \big((t,s)\in D_J,\ x\in X\big),$$

it follows by Proposition 1.18 that $\widetilde{V} = V$. Uniqueness in (1.12) is proved in the same way. □

We now prove, for closed Miyadera perturbations, that the second Duhamel formula (1.8) holds and that one can iterate Miyadera perturbations. In the case of strongly continuous propagators, the validity of (1.8) is due to [RRSV00; Thm. 3.4(c)]; see also [LVV06; Thm. 1.16(b)]. In the autonomous case of C_0-semigroups, the first assertion of part (b) of the following theorem is due to [OSSV96; Lemma 1.1]; there the closedness assumption is not needed.

1.20 Theorem. *Let $B = (B(t))_{t\in J}$ be a closed Miyadera perturbation of U with constants (α,γ).*

(a) *Then the constituents U_k of the corresponding Dyson-Phillips series satisfy the recursion formula*

$$U_k(t,s)x = \int_s^t U(t,\tau)B(\tau)U_{k-1}(\tau,s)x\,d\tau \qquad \big(k\in\mathbb{N},\ (t,s)\in D_J,\ x\in X\big). \tag{1.13}$$

If $\gamma < 1$ then the perturbed propagator U_B from Theorem 1.8 is the unique locally bounded strongly measurable propagator satisfying the second Duhamel formula (1.8)*.*

(b) *Assume that $\gamma < 1$, and let $B_1 = (B_1(t))_{t\in J}$ be a closed Miyadera perturbation of U with constants (α,γ_1). Then B_1 is a Miyadera perturbation of U_B with constants $(\alpha,\frac{\gamma_1}{1-\gamma})$. If $\gamma+\gamma_1<1$ (so that $\frac{\gamma_1}{1-\gamma}<1$) then $B+B_1$ is a small Miyadera perturbation of U, and $U_{B+B_1} = (U_B)_{B_1}$.*

1.21 Remarks. (a) Part (a) of the above theorem says in particular that the integrals in (1.13) and (1.8) are defined for all $(t,s) \in D_J$ and all $x \in X$. We will in fact show more strongly that $B(\cdot)U_k(\cdot,s)x$ and $B(\cdot)U_B(\cdot,s)x$ are integrable on (s,t). For the latter function this also follows from part (b), which implies in particular that B is a Miyadera perturbation of U_B.

(b) We observe that the second assertion of part (a) of Theorem 1.20 can be derived from part (b): By the latter, there exists $n \in \mathbb{N}$ such that $\frac{1}{n}B$ is a small Miyadera perturbation of $U_{\frac{k}{n}B}$, and $(U_{\frac{k}{n}B})_{-\frac{1}{n}B} = U_{\frac{k-1}{n}B}$, for $k = 1, \ldots, n$. Applying Proposition 1.18 $n-1$ times, we thus obtain that

$$U(t,s)x = U_B(t,s)x + \int_s^t U(t,\tau)(-B(\tau))U_B(\tau,s)x\,d\tau$$

for all $(t,s) \in D_J$ and all $x \in X$, which proves (1.8). Below we will give an independent proof based on (1.13).

(c) In [Sch02; Thm. 4.4], in the context of strongly continuous propagators, a perturbation theorem is proved that comprises versions of both the Miyadera and Desch-Schappacher theorems. From this result it follows, by [Sch02; Rem. 4.6(a)], that the second Duhamel formula (1.8) also holds for non-closable $B(t)$ if instead of the Miyadera condition (M) one assumes more restrictively that there exist $p > 1$ and $\alpha, \gamma > 0$ such that

$$\int_s^t \|B(\tau)U(\tau,s)x\|^p\,d\tau \leqslant \gamma^p\|x\| \qquad \big((t,s) \in D_J,\ t-s \leqslant \alpha,\ x \in X_s\big).$$

In order to make this possible, the operators $B(t)$ are suitably extended such that the integral in (1.8) is defined.

Proof of Theorem 1.20. (a) Without loss of generality assume that J is compact; then U is bounded. Let $\widetilde{\mathcal{M}}$ be the space of all functions $W\colon D_J \to \mathcal{L}(X)$ for which $x \mapsto B(\cdot)W(\cdot,s)x$ defines a bounded operator from X to $L_1(J_{\geqslant s};X)$, for all $s \in J$. For $W \in \widetilde{\mathcal{M}}$ and $(t,s) \in D_J$,

$$\Psi W(t,s)x := \int_s^t U(t,\sigma)B(\sigma)W(\sigma,s)x\,d\sigma$$

defines an operator $\Psi W(t,s) \in \mathcal{L}(X)$.

Let $W \in \widetilde{\mathcal{M}}$; we show that $\Psi W \in \widetilde{\mathcal{M}}$. Let $s \in J,\ x \in X$ and $f := B(\cdot)W(\cdot,s)x$. Then $f \in L_1(J_{\geqslant s};X)$, so $\Psi W(\tau,s)x = \int_s^\tau U(\tau,\sigma)f(\sigma)\,d\sigma \in D(B(\tau))$ and

$$B(\tau)\Psi W(\tau,s)x = \int_s^\tau B(\tau)U(\tau,\sigma)f(\sigma)\,d\sigma$$

for a.e. $\tau \in J_{\geqslant s}$, by Lemma 1.17. Moreover, $(\tau,\sigma) \mapsto B(\tau)U(\tau,\sigma)f(\sigma)$ is integrable on $D_{J_{\geqslant s}}$. Thus, with the weighted norm $\|\cdot\|_{1,\lambda}$ on $L_1(J_{\geqslant s};X)$ from the proof of Theorem 1.8

we obtain by Fubini's theorem and (1.7) that

$$\begin{aligned}\|B(\cdot)\Psi W(\cdot,s)x\|_{1,\lambda} &\leqslant \int_{J_{\geqslant s}} e^{-\lambda\tau}\int_s^\tau \|B(\tau)U(\tau,\sigma)f(\sigma)\|\,d\sigma\,d\tau \\ &= \int_{J_{\geqslant s}}\int_{J_{\geqslant\sigma}} e^{-\lambda\tau}\|B(\tau)U(\tau,\sigma)f(\sigma)\|\,d\tau\,d\sigma = \int_{J_{\geqslant s}} \|A(\sigma)f(\sigma)\|_{1,\lambda}\,d\sigma \\ &\leqslant \int_{J_{\geqslant s}} \widetilde{\gamma}_\lambda e^{-\lambda\sigma}\|f(\sigma)\|\,d\sigma = \widetilde{\gamma}_\lambda\|B(\cdot)W(\cdot,s)x\|_{1,\lambda}, \end{aligned} \tag{1.14}$$

and it follows that $\Psi W \in \widetilde{\mathcal{M}}$.

Let $\Phi\colon \mathcal{M}\to\mathcal{M}$ be as in the proof of Theorem 1.8. Let $(t,s)\in D_J$ and $x\in X$. We use Fubini's theorem as above to obtain for $V\in\mathcal{M}$ that

$$\begin{aligned}\int_s^t V(t,\tau)B(\tau)\Psi W(\tau,s)x\,d\tau &= \int_s^t\int_\sigma^t V(t,\tau)B(\tau)U(\tau,\sigma)B(\sigma)W(\sigma,s)x\,d\tau\,d\sigma \\ &= \int_s^t \Phi V(t,\sigma)B(\sigma)W(\sigma,s)x\,d\sigma.\end{aligned} \tag{1.15}$$

Observe that $U\in\widetilde{\mathcal{M}}$ since $X_\sigma = X$ for all $\sigma\in J$. For $k\in\mathbb{N}$ it thus follows that

$$\begin{aligned}U_k(t,s)x = \Phi^k U(t,s)x &= \int_s^t \Phi^{k-1}U(t,\sigma)B(\sigma)U(\sigma,s)x\,d\sigma = \dots \\ &= \int_s^t U(t,\tau)B(\tau)\Psi^{k-1}U(\tau,s)x\,d\tau = \Psi^k U(t,s)x,\end{aligned} \tag{1.16}$$

which proves the first assertion.

Assume now that $\gamma<1$, and choose $\lambda>0$ such that $\widetilde{\gamma}_\lambda<1$. From (1.14) and (1.16) we infer that

$$\|B(\cdot)U_k(\cdot,s)x\|_{1,\lambda} = \|B(\cdot)\Psi^k U(\cdot,s)x\|_{1,\lambda} \leqslant \widetilde{\gamma}_\lambda^k\|B(\cdot)U(\cdot,s)x\|_{1,\lambda} \tag{1.17}$$

for all $k\in\mathbb{N}$, so $\sum_{k=0}^\infty B(\cdot)U_k(\cdot,s)x$ converges absolutely in $L_1(J_{\geqslant s};X)$ and hence a.e. on $J_{\geqslant s}$. Since $\sum_{k=0}^\infty U_k = U_B$ and a.e. $B(\tau)$ is closed, we thus obtain that

$$U_B(\tau,s)x\in D(B(\tau)) \quad\text{and}\quad B(\tau)U_B(\tau,s)x = \sum_{k=0}^\infty B(\tau)U_k(\tau,s)x$$

for a.e. $\tau\in J_{\geqslant s}$. We conclude that $\sum_{k=0}^\infty U(t,\cdot)B(\cdot)U_k(\cdot,s)x = U(t,\cdot)B(\cdot)U_B(\cdot,s)x$ in $L_1(s,t;X)$ and hence

$$\begin{aligned}U_B(t,s)x = \sum_{k=0}^\infty \Psi^k U(t,s)x &= U(t,s)x + \sum_{k=1}^\infty\int_s^t U(t,\tau)B(\tau)U_{k-1}(\tau,s)x\,d\tau \\ &= U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\,d\tau.\end{aligned}$$

The uniqueness assertion follows from Corollary 1.19, so the proof of part (a) is complete.

(b) Let $(t,s) \in D_J$ with $t-s \leqslant \alpha$, and let $x \in X$. By an argument analogous to the one that leads to (1.14), we obtain for $W \in \widetilde{\mathcal{M}}$ that

$$\int_s^t \|B_1(\tau)\Psi W(\tau,s)x\|\,d\tau \leqslant \gamma_1 \int_s^t \|B(\tau)W(\tau,s)x\|\,d\tau. \tag{1.18}$$

Applying (1.18) with $B_1 = B$, we infer that

$$\int_s^t \|B(\tau)\Psi^{k-1}U(\tau,s)x\|\,d\tau \leqslant \gamma^{k-1}\int_s^t \|B(\tau)U(\tau,s)x\|\,d\tau \leqslant \gamma^k\|x\|$$

for all $k \in \mathbb{N}$. With $W = U_{k-1} = \Psi^{k-1}U$, (1.18) thus yields

$$\int_s^t \|B_1(\tau)U_k(\tau,s)x\|\,d\tau \leqslant \gamma_1 \int_s^t \|B(\tau)U_{k-1}(\tau,s)x\|\,d\tau \leqslant \gamma_1\gamma^k\|x\|.$$

The latter inequality (without the middle term) is clear for $k=0$, so as in (a) we conclude that $B_1(\cdot)U_B(\cdot,s)x \in L_1(s,t;X)$ and

$$\int_s^t \|B_1(\tau)U_B(\tau,s)x\|\,d\tau \leqslant \gamma_1\sum_{k=0}^{\infty}\gamma^k\|x\| = \frac{\gamma_1}{1-\gamma}\|x\|,$$

which proves the first assertion.

Since both B and B_1 satisfy condition (M) with $X_s = X$, we obtain that $B + B_1$ is a Miyadera perturbation of U with constants $(\alpha, \gamma+\gamma_1)$. Now assume that $\gamma + \gamma_1 < 1$, so that $(B_1(t))_{t\in J}$ is a small Miyadera perturbation of U_B. Then by Proposition 1.18 we obtain that

$$(U_B)_{B_1}(t,s)x = U(t,s)x + \int_s^t (U_B)_{B_1}(t,\tau)(B+B_1)(\tau)U(\tau,s)x\,d\tau$$

for all $(t,s) \in D_J$ and all $x \in X$, so the last assertion $U_{B+B_1} = (U_B)_{B_1}$ follows from uniqueness in the first Duhamel formula. □

1.4 The Trotter product formula

In this section let U be a bounded strongly measurable propagator on a Banach space X with parameter interval $J \subseteq \mathbb{R}$. We are going to study the special case of bounded perturbations in more detail. In particular, in Theorem 1.26 below, we present a Trotter product formula for perturbed propagators. Throughout the section we assume that $B\colon J \to \mathcal{L}(X)$ is bounded and strongly measurable.

1.22 Remark. Let $c := \|B\|_\infty$, and choose $M \geqslant 1$ such that $\|U(t,s)\| \leqslant M$ for all $(t,s) \in D_J$. Then from the recursion formula (1.13) we obtain by induction that

$$\|U_k(t,s)\| \leqslant M(Mc)^k\frac{(t-s)^k}{k!} \qquad (k \in \mathbb{N}_0)$$

and hence $\|U_B(t,s)\| \leqslant Me^{Mc(t-s)}$ for all $(t,s) \in D_J$.

The following convergence result for perturbed propagators is a generalisation of [RäSc99; Prop. 2.3(b)] and [LVV06; Prop. 1.8]. For simplicity it is formulated for bounded propagators and bounded perturbations, the generalisation to locally bounded propagators with locally bounded perturbations is straightforward (cf. Remark 1.12(c)).

1.23 Proposition. *For each $n \in \mathbb{N}$ let U^n be a bounded strongly measurable propagator on X with parameter interval J, and let $B_n \colon J \to \mathcal{L}(X)$ be bounded and strongly measurable. Assume that*

$$M := \sup\bigl\{\|U^n(t,s)\|;\, n \in \mathbb{N},\ (t,s) \in D_J\bigr\} < \infty$$

and $c := \sup_{n\in\mathbb{N}} \|B_n\|_\infty < \infty$, that $U^n(t,s) \to U(t,s)$ strongly as $n \to \infty$, for all $(t,s) \in D_J$, and that $B_n(\cdot)x \to B(\cdot)x$ in $L_{1,\mathrm{loc}}(J;X)$ as $n \to \infty$, for all $x \in X$. Then

$$U^n_{k,B_n}(t,s) \to U_{k,B}(t,s) \quad \textit{and} \quad U^n_{B_n}(t,s) \to U_B(t,s) \tag{1.19}$$

strongly for all $k \in \mathbb{N}_0$ and all $(t,s) \in D_J$.

If, in addition, U is strongly continuous and the strong convergence $U^n(t,s) \to U(t,s)$ is uniform for (t,s) in compact subsets of D_J, then the strong convergences in (1.19) *are also uniform for (t,s) in compact subsets of D_J.*

Proof. Without loss of generality assume that J is compact. As an initial step we define operators $\mathcal{B}, \mathcal{B}_n \in \mathcal{L}(L_1(J;X))$ by $\mathcal{B}f(t) := B(t)f(t)$ and $\mathcal{B}_nf(t) := B_n(t)f(t)$, for all $n \in \mathbb{N}$. The assumption on convergence of the B_n implies that $\mathcal{B}_nf \to \mathcal{B}f$ in $L_1(J;X)$ for all simple functions $f \colon J \to X$. Since $(\mathcal{B}_n)$ is a bounded sequence in $\mathcal{L}(L_1(J;X))$ and the simple functions are dense in $L_1(J;X)$, it follows that $\mathcal{B}_n \to \mathcal{B}$ strongly as $n \to \infty$.

We now prove the first convergence assertion in (1.19) by induction on k. For $k = 0$ there is nothing to show. Assume that the assertion holds for some $k \in \mathbb{N}_0$. Let $(t,s) \in D_J$ and $x \in X$. Using the initial observation, the induction hypothesis and the estimate

$$\|U^n_{k,B_n}(t,\tau)\| \leqslant M(Mc)^k \frac{(t-s)^k}{k!} \qquad \bigl(n \in \mathbb{N},\ \tau \in [s,t]\bigr) \tag{1.20}$$

from Remark 1.22, we obtain that

$$f^n_k(t,s) := U^n_{k,B_n}(t,\cdot)B_n(\cdot)U^n(\cdot,s)x \to U_{k,B}(t,\cdot)B(\cdot)U(\cdot,s)x =: f_k(t,s)$$

in $L_1(s,t;X)$ as $n \to \infty$. Therefore,

$$U^n_{k+1,B_n}(t,s)x = \int_s^t f^n_k(t,s)(\tau)\,d\tau \to \int_s^t f_k(t,s)(\tau)\,d\tau = U_{k+1,B}(t,s)x \tag{1.21}$$

as $n \to \infty$, and the first part of (1.19) is proved. From this we infer, taking into account (1.20) with $\tau = s$, that

$$U^n_{B_n}(t,s)x = \sum_{k=0}^\infty U^n_{k,B_n}(t,s)x \to \sum_{k=0}^\infty U_{k,B}(t,s)x = U_B(t,s)x \tag{1.22}$$

as $n \to \infty$.

For the second part of the proof assume that U is strongly continuous and that $U^n(t,s) \to U(t,s)$ strongly as $n \to \infty$, uniformly for $(t,s) \in D_J$. We prove by induction on k that the strong convergence $U^n_{k,B_n}(t,s) \to U_{k,B}(t,s)$ is uniform for $(t,s) \in D_J$. Again, for $k=0$ there is nothing to show. Assuming uniform strong convergence for some $k \in \mathbb{N}_0$, we proceed with the notation from the first part of the proof. We are going to show that the convergence $f^n_k(t,s) \to f_k(t,s)$ is uniform for $(t,s) \in D_J$; then it follows that the convergences in (1.21) and (1.22) are uniform for $(t,s) \in D_J$, too. Here and in the following, we tacitly extend functions defined on subintervals of J by zero to functions on J.

The uniform strong convergence $U^n \to U$ implies that

$$g_n(s) := U^n(\cdot,s)x \to U(\cdot,s)x =: g(s)$$

in $L_1(J;X)$ as $n \to \infty$, uniformly for $s \in J$. Moreover, the function $g\colon J \to L_1(J;X)$ thus defined is continuous since U is strongly continuous. By the initial observation, the induction hypothesis and estimate (1.20) we obtain as in the first part of the proof, for all $f \in L_1(J;X)$, that

$$T_n(t)f := U^n_{k,B_n}(t,\cdot)B_n(\cdot)f \to U_{k,B}(t,\cdot)B(\cdot)f =: T(t)f$$

in $L_1(J;X)$, uniformly for $t \in J$. By Lemma 1.24(b) below, applied with $K = L = J$ and $X = Y = L_1(J;X)$, we conclude that

$$f^n_k(t,s) = T_n(t)g_n(s) \to T(t)g(s) = f_k(t,s)$$

as $n \to \infty$, uniformly for $(t,s) \in D_J$, and the proof is complete. □

1.24 Lemma. *Let K be a compact topological space, and let X, Y be Banach spaces. Let (g_n) be a sequence in $\ell_\infty(K;X)$ such that $g_n \to g \in C(K;X)$ uniformly.*

(a) *Let (T_n) be a sequence in $\mathcal{L}(X,Y)$ such that $T_n \to T \in \mathcal{L}(X,Y)$ strongly as $n \to \infty$. Then $T_n \circ g_n \to T \circ g$ uniformly as $n \to \infty$.*

(b) *Let L be a non-empty set, let $T, T_n\colon L \to \mathcal{L}(X,Y)$ be bounded functions, for all $n \in \mathbb{N}$, and assume that $T_n(t) \to T(t)$ strongly as $n \to \infty$, uniformly for $t \in L$. Then $T_n(t)g_n(s) \to T(t)g(s)$ as $n \to \infty$, uniformly for $t \in L$ and $s \in K$.*

Proof. (a) The sequence (T_n) is bounded by the uniform boundedness theorem. Since $g(K)$ is compact, we obtain that the convergence $T_nx \to Tx$ is uniform for $x \in g(K)$. It follows that

$$T_ng_n(t) - Tg(t) = T_n\big(g_n(t) - g(t)\big) + (T_n - T)g(t) \to 0$$

as $n \to \infty$, uniformly for $t \in K$.

(b) follows from (a), applied with $\mathcal{Y} := \ell_\infty(L;Y)$ and $\mathcal{T}, \mathcal{T}_n\colon X \to \mathcal{Y}$,

$$\mathcal{T}_nx := [t \mapsto T_n(t)x] \to [t \mapsto T(t)x] =: \mathcal{T}x$$

in $\mathcal{Y}$ as $n \to \infty$, for all $x \in X$. □

For the remainder of the section we assume that U is strongly continuous.

1.25 Remarks. (a) An important application of the above convergence result is the approximation of strongly continuous propagators by "discrete" propagators: For $n \in \mathbb{N}$ let P_n be a discrete subset of J with $\inf P_n = \inf J$ and $\sup P_n = \sup J$, and define $\varphi_n \colon J \to J$ by $\varphi_n(s) := \min\{t \in P_n;\, t \geqslant s\}$. Assume that $\varphi_n \to \mathrm{id}_J$ uniformly as $n \to \infty$. Then for $n \in \mathbb{N}$ we define a propagator U^n on X with parameter interval J by

$$U^n(t,s) := U\big(\varphi_n(t), \varphi_n(s)\big).$$

It follows from the strong continuity of U that $U^n(t,s) \to U(t,s)$ strongly as $n \to \infty$, uniformly for (t,s) in compact subsets of D_J.

For $n \in \mathbb{N}$ we now define $B_n \colon J \to \mathcal{L}(X)$ as follows. If K is one of the connected components of $J \setminus P_n$ and $t \in K \cup \{\sup K\}$, then we set

$$B_n(t) := \frac{1}{|K|} \int_K B(\tau)\, d\tau \quad \text{(strong integral)},$$

where $|K|$ denotes the length of K, and $B_n(\inf J) := 0$ if $\inf J \in J$. Then $B_n(\cdot)x \to B(\cdot)x$ in $L_{1,\mathrm{loc}}(J;X)$ as $n \to \infty$, for all $x \in X$, so we can apply Proposition 1.23 to obtain (1.19) with uniform strong convergence for (t,s) in compact subsets of D_J.

(b) We keep the notation from part (a). Let $s_0 < t_0$ be two consecutive points in P_n. Then $U^n(t,\tau) = I$ and $B_n(\tau) = B_n(t_0)$ for all $(t,\tau) \in D_{(s_0,t_0]}$. For the constituents of the Dyson-Phillips series for $U^n_{B_n}$ we thus obtain by the recursion formula (1.13) that

$$U^n_{k+1,B_n}(t,s_0)x = \int_{s_0}^t U^n(t,\tau) B_n(\tau) U^n_{k,B_n}(\tau,s_0)x\, d\tau = B_n(t_0) \int_{s_0}^t U^n_{k,B_n}(\tau,s_0)x\, d\tau$$

for all $k \in \mathbb{N}_0$, $t \in (s_0,t_0]$ and all $x \in X$. By induction we infer that

$$U^n_{k,B_n}(t,s_0) = \frac{(t-s_0)^k}{k!} B_n(t_0)^k U(t_0,s_0) = \frac{(t-s_0)^k}{k!} B_n(t)^k U(t,s_0) \qquad (k \in \mathbb{N}_0)$$

and hence $U^n_{B_n}(t,s_0) = e^{(t-s_0)B_n(t)} U(t,s_0)$ for all $t \in (s_0,t_0]$. Therefore, if $t_0 < t_1 < \ldots < t_m$ are $m+1$ consecutive points in P_n, then we obtain by the propagator property that

$$U^n_{B_n}(t_m,t_0) = e^{(t_m-t_{m-1})B_n(t_m)} U(t_m,t_{m-1}) \cdots e^{(t_1-t_0)B_n(t_1)} U(t_1,t_0).$$

From the above application of Proposition 1.23 we immediately obtain the following Trotter product formula for perturbed propagators.

1.26 Theorem. *Assume that U is strongly continuous, and let $s,t \in J$ with $s < t$. For $n \in \mathbb{N}$ let $t_0^n, \ldots, t_n^n \in [s,t]$ satisfy $s = t_0^n < t_1^n < \cdots < t_n^n = t$, and assume that $\sup\{t_k^n - t_{k-1}^n;\, k = 1,\ldots,n\} \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$ and $k = 1,\ldots,n$ let*

$$B_k^n := \int_{t_{k-1}^n}^{t_k^n} B(\tau)\, d\tau \quad \textit{(strong integral)}.$$

Then

$$e^{B_n^n} U(t_n^n, t_{n-1}^n) e^{B_{n-1}^n} U(t_{n-1}^n, t_{n-2}^n) \cdots e^{B_1^n} U(t_1^n, t_0^n) \to U_B(t,s)$$

strongly as $n \to \infty$.

Chapter 2

Absorption propagators

In this chapter we further develop the theory of absorption propagators that was initiated in [RäSc99] and pushed forward in [LVV06]. The basic ideas of the approach are the same as in the theory of absorption semigroups; see [Voi86] and [Voi88].

Throughout the section let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We do not require the measure μ to be σ-finite, which raises a few technical issues. In particular, $L_\infty(\mu)$ will denote the space of all locally measurable functions from Ω to $\mathbb{K}$ that are bounded outside a local null set, and functions are identified if they coincide locally a.e.

We are going to study perturbation of positive propagators on $L_p(\mu)$ by locally measurable potentials $V\colon J \times \Omega \to \mathbb{R}$. Here, $J \times \Omega$ is endowed with the σ-algebra generated by $\mathcal{B}_{\mathrm{fin}} \otimes \mathcal{A}_{\mathrm{fin}}$, where $\mathcal{B}_{\mathrm{fin}}$ denotes the system of all Lebesgue measurable subsets of J of finite measure, and $\mathcal{A}_{\mathrm{fin}} \subseteq \mathcal{A}$ is the subsystem of all μ-measurable sets of finite measure. On this σ-algebra, the product measure of Lebesgue measure and the measure μ is uniquely defined. If $A \in \mathcal{A}_{\mathrm{fin}}$ then V is measurable on $J \times A$, so $V(t, \cdot)$ is measurable on A for all $t \in J$. It follows that $V(t, \cdot)$ is locally measurable for all $t \in J$.

The chapter is organised as follows. In Section 2.1 we develop the abstract theory of absorption propagators, and we present a new version of the notions of admissible and regular potentials. Several of the results are straightforward generalisations of the corresponding results for C_0-semigroups, but some of the proofs are substantially simplified. We point out the following two new aspects: We study perturbation of strongly measurable propagators (as opposed to strongly continuous propagators), and we introduce a larger class of perturbations that includes highly oscillating potentials (see Example 2.14). Section 2.2 deals with the connection between admissibility and the Miyadera perturbation theorem. In Section 2.3 we present the backward Miyadera condition that can be used to obtain local L_∞-boundedness of perturbed propagators.

In Section 2.4 we introduce the notion of logarithmically convex operator-valued functions, and we prove an interpolation inequality that is similar to the Stein interpolation theorem. The concept of logarithmic convexity is used in Section 2.5 to prove interpolation inequalities for absorption propagators and for the corresponding Dyson-Phillips series; the latter lead to a deeper understanding of the relation between the Miyadera condition and admissibility. The Trotter product formula is a crucial ingredient in the proofs, so from this point on we will only consider perturbations of strongly continuous propagators. In Section 2.6 we apply the results of Section 2.5 to the interpolation of admissibility in different weighted L_p-spaces.

2.1 Abstract theory of absorption propagators

Let $1 \leqslant p < \infty$, and let U be a positive locally bounded strongly measurable propagator on $L_p(\mu)$ with parameter interval $J \subseteq \mathbb{R}$. (A positive operator-valued function is, by definition, a function taking its values in the positive operators on $L_p(\mu)$.) We are going to apply the perturbation results of Section 1.2 to bounded potentials and to extend the definition of a perturbed propagator to suitable unbounded real-valued potentials.

Let $V\colon J \times \Omega \to \mathbb{R}$ be locally measurable and bounded. As explained in the introduction to the chapter, $V(t) := V(t, \cdot)$ is locally measurable (and bounded) and can hence be considered as a bounded multiplication operator on $L_p(\mu)$, for all $t \in J$. We define a strongly measurable function $B\colon J \to \mathcal{L}(L_p(\mu))$ by

$$B(t) := -V(t) \qquad (t \in J). \tag{2.1}$$

With this B we define

$$U_V := U_B,$$

in the sense of Section 1.2. We choose the negative sign in order to stay compatible with the notation usually chosen for Schrödinger operators; positive V corresponds to absorption, negative V to excitation.

2.1 Remarks. (a) It is important to notice that for $V \in L_\infty(J \times \Omega)$, the perturbed propagator U_V is defined independently of the chosen representative of V. Indeed, let $V, \widetilde{V}\colon J \times \Omega \to \mathbb{R}$ be locally measurable and bounded, and assume that $V = \widetilde{V}$ locally a.e. Then $V(\cdot)U(\cdot, s)f = \widetilde{V}(\cdot)U(\cdot, s)f$ a.e. on $J_{\geqslant s}$, for all $s \in J$ and all $f \in L_p(\mu)$ (cf. part (b) below), so by uniqueness in the first Duhamel formula (1.2) we obtain that $U_V = U_{\widetilde{V}}$. Nevertheless it can happen that $V(t) \neq \widetilde{V}(t)$ (as elements of $L_\infty(\mu)$) for all $t \in J$:

As in Example 1.3 let $J := \Omega := [0, 1]$, and let μ be the counting measure on Ω. We define $V\colon J \times \Omega \to \mathbb{R}$ as the indicator function of the diagonal $\big\{(t, t) \in J \times \Omega;\ 0 \leqslant t \leqslant 1\big\}$. Then V is locally measurable and $V = 0$ locally a.e., but $V(t) = \mathbb{1}_{\{t\}} \neq 0$ for all $t \in J$.

(b) If $h\colon J \to L_p(\mu)$ is measurable then there exists a σ-finite set $\Omega_h \subseteq \Omega$ such that $h(t) = \mathbb{1}_{\Omega_h} h(t)$ a.e. for a.e. $t \in J$. Indeed, let $h_n\colon J \to L_p(\mu)$ be simple functions such that $h_n \to h$ a.e. as $n \to \infty$. Then $\Omega_h := \bigcup\big\{[h_n(t) \neq 0];\ n \in \mathbb{N},\ t \in J\big\}$ is σ-finite, and for all $t \in J$ such that $h_n(t) \to h(t)$ in $L_p(\mu)$ we obtain that $h(t) = \mathbb{1}_{\Omega_h} h(t)$ a.e. Here, for a function $f\colon \Omega \to \mathbb{K}$, we denote $[f \neq 0] := \big\{x \in \Omega;\ f(x) \neq 0\big\}$.

2.2 Remark. Recall from Theorem 1.20(b) that $(U_{V_1})_{V_2} = U_{V_1+V_2}$ for $V_1, V_2 \in L_\infty(J \times \Omega)$. Moreover, if $V = c$ for some $c \in \mathbb{R}$ then $U_V(t, s) = e^{-c(t-s)}U(t, s)$ for all $(t, s) \in D_J$, by Remark 1.12(c). As a consequence we obtain that

$$U_V(t, s) = e^{-c(t-s)}U_{V-c}(t, s) \quad \big((t, s) \in D_J\big)$$

for all $c \in \mathbb{R}$ and all $V \in L_\infty(J \times \Omega)$.

The next proposition, which is a complete analogue of the corresponding statement for C_0-semigroups [Voi88; Prop. 1.3], is the corner stone to the whole approach. For strongly continuous propagators, part (a) has already been proved in [RäSc99; Prop. 2.3(c)], part (b) in [LVV06; Prop. 2.1(b)].

2.3 Proposition. *Let U_1 and U_2 be positive locally bounded strongly measurable propagators on $L_p(\mu)$ with parameter interval J, and let $V, V_1, V_2 \in L_\infty(J\times\Omega)$.*

(a) *If $V_1 \leqslant V_2$ then $U_{V_1} \geqslant U_{V_2} \geqslant 0$.*

(b) *If $U_1 \leqslant U_2$ then $(U_1)_V \leqslant (U_2)_V$. If additionally $V \geqslant 0$ then $(U_2)_V - (U_1)_V \leqslant U_2 - U_1$.*

Proof. (a) By Remark 2.2 we can assume without loss of generality that $V_1 \leqslant V_2 \leqslant 0$. Then for the constituents of the Dyson-Phillips series for U_{V_1} and U_{V_2} (see Remark 1.12(a)) we obtain by induction that $U_{k,-V_1} \geqslant U_{k,-V_2} \geqslant 0$ for all $k \in \mathbb{N}_0$. Summation over k yields the assertion.

(b) The first assertion follows as in (a) from Remark 2.2 and the Dyson-Phillips series expansions of $(U_1)_V$ and $(U_2)_V$. The second, in the form $U_1 - (U_1)_V \leqslant U_2 - (U_2)_V$, then follows from the Duhamel formula (1.2). $\square$

We are now going to define U_V for locally measurable $V\colon J\times\Omega \to \mathbb{R}$ that are possibly unbounded. Our approach differs from the classical theory of absorption semigroups ([Voi86], [Voi88]) and from the approach in [LVV06; Sec. 2.1], where V is approximated by means of cut-offs $V^{(n)} := (V\wedge n)\vee(-n)$. We use this approximation only for the cases $V \geqslant 0$ and $V \leqslant 0$. For the link with the classical theory see Remark 2.22(c) below.

2.4 Definition. (a) If $V \geqslant 0$ then $U_{V\wedge n}(t,s) \geqslant U_{V\wedge(n+1)}(t,s) \geqslant 0$ for all $n\in\mathbb{N}$ and all $(t,s)\in D_J$, by Proposition 2.3(a), so the dominated convergence theorem implies that

$$U_V(t,s) := \operatorname*{s-lim}_{n\to\infty} U_{V\wedge n}(t,s) \tag{2.2}$$

exists for all $(t,s)\in D_J$. Clearly, this defines a locally bounded strongly measurable propagator U_V.

(b) If $V \leqslant 0$ then $\big(U_{V\vee(-n)}(t,s)\big)$ is an increasing sequence of positive operators, for all $(t,s)\in D_J$, and we say that V is *weakly U-admissible* if

$$U_V(t,s) := \operatorname*{s-lim}_{n\to\infty} U_{V\vee(-n)}(t,s) \tag{2.3}$$

exists for all $(t,s)\in D_J$ and defines a locally bounded function $U_V\colon D_J \to \mathcal{L}(X)$. Again, U_V is a strongly measurable propagator in this case.

(c) In the general case let V^+ and V^- denote the positive and negative parts of V, respectively, i.e., $V^+ = V\vee 0$ and $V^- = (-V)^+$. We say that V is *weakly U-admissible* if $-V^-$ is weakly U_{V^+}-admissible. (In particular, any $V\geqslant 0$ is weakly U-admissible.) Then we set

$$U_V := (U_{V^+})_{-V^-}. \tag{2.4}$$

By Remark 2.2, this is consistent with the definition of U_V in the case of bounded V.

(d) If U is strongly continuous then we say that V is *U-admissible* if U_V is strongly continuous. In [LVV06; Def. 2.2], parallel to [Voi88; Def. 2.5], it was required more restrictively for U-admissibility of V that V^+ and $-V^-$ are U-admissible (cf. Remark 2.18(b) below).

2.5 Remarks. (a) Observe that from (2.3) and (2.4) it follows that (2.3) holds for any weakly U-admissible potential V, not only for negative V.

(b) Let V be weakly U-admissible, and let $\widetilde{V} = V$ locally a.e. Then it follows from Remark 2.1(a) that $\widetilde{V}$ is weakly U-admissible, and $U_{\widetilde{V}} = U_V$.

We now show that the inequalities stated in Proposition 2.3 for bounded potentials carry over to weakly admissible potentials.

2.6 Proposition. *Let U_1 and U_2 be positive locally bounded strongly measurable propagators on $L_p(\mu)$ with parameter interval J, and let V, V_1, V_2 be locally measurable potentials.*

(a) *If V_1 is weakly U-admissible and $V_1 \leqslant V_2$, then V_2 is weakly U-admissible and $U_{V_1} \geqslant U_{V_2} \geqslant 0$.*

(b) *If V is weakly U_2-admissible and $U_1 \leqslant U_2$, then V is weakly U_1-admissible and $(U_1)_V \leqslant (U_2)_V$. If additionally $V \geqslant 0$ then $(U_2)_V - (U_1)_V \leqslant U_2 - U_1$.*

Proof. The assertions are clear from (2.2), (2.3) and Proposition 2.3 if the potentials are of one sign. As a consequence we obtain the first assertion of (b) in the general case: From $(U_1)_{V^+} \leqslant (U_2)_{V^+}$ and the weak $(U_2)_{V^+}$-admissibility of $-V^-$ it follows that $-V^-$ is weakly $(U_1)_{V^+}$-admissible and

$$(U_1)_V = \big((U_1)_{V^+}\big)_{-V^-} \leqslant \big((U_2)_{V^+}\big)_{-V^-} = (U_2)_V.$$

Using part (b) we also obtain (a) in the general case: For all $n \in \mathbb{N}$, the inequalities $U_{V_1^+} \geqslant U_{V_2^+}$ and $V_1^- \wedge n \geqslant V_2^- \wedge n$ imply that

$$U_{V_1} \geqslant (U_{V_1^+})_{-V_1^- \wedge n} \geqslant (U_{V_2^+})_{-V_1^- \wedge n} \geqslant (U_{V_2^+})_{-V_2^- \wedge n} \geqslant 0,$$

and the assertions of (a) follow. $\square$

In the next chapter we will study propagators that consist of integral operators. The next result implies that then the corresponding absorption propagators consist of integral operators, too.

2.7 Proposition. *Let V be weakly U-admissible. Let $t, s \in J$ satisfy $t > s$, and assume that $U(t,s)$ is an integral operator. Then $U_V(t,s)$ is an integral operator.*

Proof. For $n \in \mathbb{N}$ let $V_n := V \vee (-n)$. Then $0 \leqslant U_{V_n}(t,s) \leqslant e^{n(t-s)}U(t,s)$ by Proposition 2.6(a) and Remark 2.2, so $U_{V_n}(t,s)$ is an integral operator by a theorem of Bukhvalov and Schep, [AbAl02; Thm. 5.9]. By the monotone convergence theorem, this implies the assertion since $U_{V_n}(t,s) \uparrow U_V(t,s)$ by Remark 2.5(a). $\square$

The major issues for the remainder of the section will be to investigate the following two questions. Firstly, to what extent does the definition of U_V depend on the decomposition of V into a difference of positive potentials (see Theorem 2.17 and Remark 2.18(a) below), and secondly, when does convergence a.e. of a sequence of weakly U-admissible potentials imply convergence of the corresponding perturbed propagators (Theorem 2.21 below)?

For this we first show, for perturbations of one sign, that one can use different approximations than in (2.2) and (2.3), and that one can iterate perturbations. In the context of C_0-semigroups, the former problem has already been addressed in [Voi86; Props. A.1 and A.2], the latter in [Man01; Prop. 4.1.35]. For a sequence (U_n) of propagators on $L_p(\mu)$ we write $U_n \to U$ strongly or $U = \operatorname{s-lim}_{n\to\infty} U_n$ if $U_n(t,s) \to U(t,s)$ strongly for all $(t,s) \in D_J$.

2.8 Proposition. (a) *Let $V, V_n \colon J \times \Omega \to \mathbb{R}$ be locally measurable with $V \leqslant V_n \leqslant 0$ for all $n \in \mathbb{N}$, and assume that $V_n \to V$ locally a.e. as $n \to \infty$. Then the following are equivalent:*

(i) *V is weakly U-admissible;*

(ii) *the V_n are weakly U-admissible, and $\operatorname{s-lim}_{n\to\infty} U_{V_n}$ exists and is locally bounded.*

If V is weakly U-admissible then $U_{V_n} \to U_V$ strongly.

(b) *Let $V_1, V_2 \colon J \times \Omega \to (-\infty, 0]$ be locally measurable, and assume that V_1 is weakly U-admissible. Then $V_1 + V_2$ is weakly U-admissible if and only if V_2 is weakly U_{V_1}-admissible, and $U_{V_1+V_2} = (U_{V_1})_{V_2}$ in this case.*

Proof. (a) First observe that by Remark 2.5(b) we can assume without loss of generality that $V_n \to V$ pointwise: Just replace each of the V_n with V on the local null set on which the convergence does not hold.

Assume that (ii) holds, and let $m \in \mathbb{N}$. For $n \in \mathbb{N}$ let $V_n^m := V_n \vee (-m)$, and let $V^m := V \vee (-m)$. Then $V_n^m(t,\cdot) \to V^m(t,\cdot)$ strongly as $n \to \infty$, for all $t \in J$, where we consider $V_n^m(t,\cdot)$ and $V^m(t,\cdot)$ as multiplication operators in $\mathcal{L}(L_p(\mu))$. By Proposition 1.23 we thus obtain that

$$U_{V^m} = \operatorname{s-lim}_{n\to\infty} U_{V_n^m} \leqslant \operatorname{s-lim}_{n\to\infty} U_{V_n}. \tag{2.5}$$

Since the right-hand side is locally bounded, it follows that V is weakly U-admissible.

Now assume that (i) holds; then the V_n are weakly U-admissible by Proposition 2.6(a). To complete the proof of (a), we show that $U_{V_n} \to U_V$ strongly. Let $(t,s) \in D_J$ and $f \in L_p(\mu)$. Let $\varepsilon > 0$. Then there exists $m \in \mathbb{N}$ such that

$$\|U_V(t,s)f - U_{V^m}(t,s)f\|_p \leqslant \varepsilon. \tag{2.6}$$

Moreover, by (2.5) we obtain $N \in \mathbb{N}$ such that

$$\|U_{V^m}(t,s)f - U_{V_n^m}(t,s)f\|_p \leqslant \varepsilon \qquad (n \geqslant N). \tag{2.7}$$

From $U_V(t,s)f \geqslant U_{V_n}(t,s)f \geqslant U_{V_n^m}(t,s)f$ we conclude that

$$\|U_V(t,s)f - U_{V_n}(t,s)f\|_p \leqslant \|U_V(t,s)f - U_{V_n^m}(t,s)f\|_p \leqslant 2\varepsilon$$

for all $n \geqslant N$. Therefore, $U_{V_n}(t,s)f \to U_V(t,s)f$ as $n \to \infty$.

(b) First assume that V_2 is bounded. Then we only have to show the identity $U_{V_1+V_2} = (U_{V_1})_{V_2}$. By Remark 2.2 we have $U_{V_1\vee(-n)+V_2} = (U_{V_1\vee(-n)})_{V_2}$ for all $n \in \mathbb{N}$. Moreover, by part (a) we obtain that $U_{V_1\vee(-n)+V_2} \to U_{V_1+V_2}$ strongly as $n \to \infty$, and from Proposition 1.23 it follows that $(U_{V_1\vee(-n)})_{V_2} \to (U_{V_1})_{V_2}$ strongly as $n \to \infty$. This implies the desired identity.

If V_2 is unbounded then by the above we have $U_{V_1+V_2\vee(-n)} = (U_{V_1})_{V_2\vee(-n)}$ for all $n \in \mathbb{N}$. Moreover, $\operatorname*{s-lim}_{n\to\infty} (U_{V_1})_{V_2\vee(-n)}$ exists (and equals $(U_{V_1})_{V_2}$) if and only if V_2 is weakly U_{V_1}-admissible, and by part (a), $\operatorname*{s-lim}_{n\to\infty} U_{V_1+V_2\vee(-n)}$ exists (and equals $U_{V_1+V_2}$) if and only if $V_1 + V_2$ is weakly U-admissible. This completes the proof of (b). □

2.9 Remark. In the situation of Proposition 2.8(a) assume that U is strongly continuous and that V is U-admissible, i.e., U_V is strongly continuous. We show that then the strong convergence $U_{V_n} \to U_V$ is uniform on compact subsets of D_J. Assume without loss of generality that J is compact. Observe that the left-hand side of (2.6) is monotone in m if $f \geqslant 0$, and that it is a continuous function of $(t, s) \in D_J$ since U_{V^m} is strongly continuous by Theorem 1.8. From Dini's theorem we thus infer that m can be chosen independently of $(t, s) \in D_J$ in (2.6). By Proposition 1.23 we obtain the same for the choice of N in (2.7), and the asserted uniform strong convergence follows.

For positive perturbations we obtain an analogous result; only the weak admissibility is not an issue in this case, which makes the formulation simpler. The proof is also analogous (but simpler) and therefore omitted. As in Remark 2.9 one sees that the convergence in part (a) is uniform on compact subsets of D_J if U is strongly continuous and V is U-admissible.

2.10 Proposition. (a) *(cf. [RäSc99; Lemma 3.1(c)]) Let $V, V_n \colon J \times \Omega \to \mathbb{R}$ be locally measurable with $0 \leqslant V_n \leqslant V$ for all $n \in \mathbb{N}$, and assume that $V_n \to V$ locally a.e. as $n \to \infty$. Then $U_{V_n} \to U_V$ strongly.*

(b) *Let $V_1, V_2 \colon J \times \Omega \to [0, \infty)$ be locally measurable. Then $U_{V_1+V_2} = (U_{V_1})_{V_2}$.*

The next lemma says in particular that one can interchange the order of perturbations in (2.4) if $-V^-$ is weakly U-admissible.

2.11 Lemma. *(cf. [LVV06; Prop. 2.3(b)]) Let $V_\pm \geqslant 0$, and assume that $-V_-$ is weakly U-admissible. Then $-V_-$ is weakly U_{V^+}-admissible, and $(U_{V_+})_{-V_-} = (U_{-V_-})_{V_+}$.*

Proof. Let $n \in \mathbb{N}$. By Proposition 2.3(b) we have

$$0 \leqslant (U_{-V_-})_{V_+\wedge m} - (U_{-V_-\wedge n})_{V_+\wedge m} \leqslant U_{-V_-} - U_{-V_-\wedge n} \tag{2.8}$$

for all $m \in \mathbb{N}$. Moreover, $(U_{-V_-\wedge n})_{V_+\wedge m} = (U_{V_+\wedge m})_{-V_-\wedge n} \to (U_{V_+})_{-V\wedge n}$ strongly as $m \to \infty$, by Remark 2.2 and Proposition 1.23. Thus, letting $m \to \infty$ in (2.8) yields

$$0 \leqslant (U_{-V_-})_{V_+} - (U_{V_+})_{-V_-\wedge n} \leqslant U_{-V_-} - U_{-V_-\wedge n}. \tag{2.9}$$

This implies the assertion since $U_{-V_-\wedge n} \to U_{-V_-}$ strongly as $n \to \infty$. □

2.12 Remark. Let V be weakly U-admissible; we show that then $-V$ is weakly U_V-admissible and

$$(U_V)_{-V} = (U_{V^+})_{-V^+} = \operatorname*{s-lim}_{n\to\infty} U_{V^+ - V^+\wedge n} \leqslant U.$$

If $V \geqslant 0$ then for $n \in \mathbb{N}$ we obtain by Proposition 2.10(b) that $U_V = (U_{V-V\wedge n})_{V\wedge n}$, so from Remark 2.2 it follows that $(U_V)_{-V\wedge n} = U_{V-V\wedge n} \leqslant U$. This implies the claim in the case $V \geqslant 0$. In the general case, $-V^-$ is weakly U_{V^+}-admissible, so

$$(U_V)_{(-V)^+} = \big((U_{V^+})_{-V^-}\big)_{V^-} = U_{V^+}$$

by Proposition 2.13 below. It follows that $-(-V)^- = -V^+$ is weakly $(U_V)_{(-V)^+}$-admissible (i.e., $-V$ is weakly U_V-admissible) and $(U_V)_{-V} = (U_{V^+})_{-V^+}$.

The above observation leads to the notion of regularity that is crucial for the subsequent results. We say that V is *U-regular* if V is weakly U-admissible and $(U_V)_{-V} = U$. This is in analogy with [Voi88; Sec. 3], where regularity is defined for positive potentials in the context of C_0-semigroups. Observe that, by Remark 2.12, V is U-regular if and only if V is weakly U-admissible and V^+ is U-regular. We point out that in [Voi88], for V being U-regular it was also required that V is U-admissible. We do not know if, in the case of strongly continuous propagators, U-regularity of V implies U-admissibility.

2.13 Proposition. *(cf. [Voi88; Prop. 3.3(b)]) Let $V \geqslant 0$ be such that $-V$ is weakly U-admissible. Then $(U_V)_{-V} = (U_{-V})_V = U$, i.e., V and $-V$ are U-regular.*

Proof. The first identity is a special case Lemma 2.11. For the second identity we use (2.8) with $n = m$: Since $(U_{-V\wedge n})_{V\wedge n} = U$ by Remark 2.2, it follows from (2.8) that $(U_{-V})_{V\wedge n} \to U$ strongly as $n \to \infty$, i.e., $(U_{-V})_V = U$. $\square$

2.14 Example. Let $n \geqslant 2$, and let U be the heat propagator on $L_1(\mathbb{R}^n)$ with parameter interval $\mathbb{R}$, i.e., $U(t,s) = e^{(t-s)\Delta}$ for all $(t,s) \in D_{\mathbb{R}}$. For the oscillating potential V defined by

$$V(t,x) := e^{1/|x|} \sin e^{1/|x|}$$

it follows from [Stu92a; Cor. 5.5] that cV is U-admissible for all $c \in \mathbb{R}$ and hence also U-regular, by Proposition 2.13. We point out that in this example V is admissible although $-V^-$ is far from being (weakly) admissible.

Next we compare regularity with respect to different propagators. The corresponding results for C_0-semigroups are shown in [LiMa97; Cor. 1.15] and [Voi88; Prop. 3.4].

2.15 Lemma. (a) *Let $\widetilde{U}$ be a strongly measurable propagator on $L_p(\mu)$ with parameter interval J, and assume that $0 \leqslant \widetilde{U} \leqslant U$. Then any U-regular potential V is also $\widetilde{U}$-regular.*

(b) *Let $V_\pm \geqslant 0$ be such that $-V_-$ is weakly U-admissible. Then V_+ is U-regular if and only if V_+ is U_{-V_-}-regular.*

Proof. (a) By Proposition 2.6(b), V is weakly $\widetilde{U}$-admissible, so we only have to show that V^+ is $\widetilde{U}$-regular. In other words, we can assume without loss of generality that $V \geqslant 0$. Then by Proposition 2.6 we have

$$0 \leqslant \widetilde{U} - \widetilde{U}_{V-V\wedge n} \leqslant U - U_{V-V\wedge n},$$

and from Remark 2.12 it follows that $0 \leqslant \widetilde{U} - (\widetilde{U}_V)_{-V} \leqslant U - (U_V)_{-V} = 0$.

(b) Assume that V_+ is U-regular. Then by Lemma 2.11 and Proposition 2.8(b) we obtain that

$$\big((U_{-V_-})_{V_+}\big)_{-V_+} = \big((U_{V_+})_{-V_-}\big)_{-V_+} = \big((U_{V_+})_{-V_+}\big)_{-V_-} = U_{-V_-},$$

so V_+ is U_{-V_-}-regular. The other implication follows from part (a). $\square$

2.16 Corollary. *(cf. [Voi88; Prop. 3.3(a)]) The set $\{V \geqslant 0;\ V$ is U-regular$\}$ is a solid convex cone.*

Proof. Let $V_1, V_2 \geqslant 0$ be U-regular. Then V_2 is U_{V_1}-regular by Lemma 2.15(a), so from Propositions 2.8(b) and 2.10(b) we deduce that $(U_{V_1+V_2})_{-V_1-V_2} = (U_{V_1})_{-V_1} = U$.

Let now $V_1 \geqslant 0$ be U-regular, and let $0 \leqslant V_2 \leqslant V_1$. Then $V_2 - V_2 \wedge n \leqslant V_1 - V_1 \wedge n$ and hence $U_{V_1 - V_1\wedge n} \leqslant U_{V_2 - V_2 \wedge n} \leqslant U$ for all $n \in \mathbb{N}$, so it follows from Remark 2.12 that V_2 is U-regular. $\square$

We can now characterise which decompositions $V = V_+ - V_-$ lead to $U_V = (U_{V_+})_{-V_-}$.

2.17 Theorem. *Let $V_\pm \geqslant 0$ and $V := V_+ - V_-$.*
(a) *If V is weakly U-admissible then $-V_-$ is weakly U_{V_+}-admissible.*
(b) *The following are equivalent:*
 (i) *V is weakly U-admissible and $U_V = (U_{V_+})_{-V_-}$;*
 (ii) *$-V_-$ is weakly U_{V_+}-admissible and $V_+ \wedge V_-$ is U_{V^+}-regular.*

Proof. Let $W := V_+ \wedge V_-$; then $V_+ = V^+ + W$ and $V_- = V^- + W$.

(a) By Proposition 2.10(b) and Lemma 2.11 we obtain that

$$(U_{V_+})_{-V^-} = \big((U_{V^+})_W\big)_{-V^-} = \big((U_{V^+})_{-V^-}\big)_W = (U_V)_W.$$

In particular, $-W$ is weakly $(U_{V_+})_{-V^-}$-admissible. From Proposition 2.8(b) we conclude that $-V_-$ is weakly U_{V_+}-admissible and $(U_{V_+})_{-V_-} = \big((U_{V_+})_{-V^-}\big)_{-W} = \big((U_V)_W\big)_{-W}$.

(b) Assume that (i) holds. Then by the above, $-V_-$ is weakly U_{V_+}-admissible and $U_V = \big((U_V)_W\big)_{-W}$. Therefore, W is U_V-regular and hence U_{V^+}-regular by Lemma 2.15(a).

Conversely, if (ii) holds then by Proposition 2.10(b) we obtain that

$$U_{V^+} = \big((U_{V^+})_W\big)_{-W} = (U_{V_+})_{-W}.$$

Since $-V_- = -V^- - W$ is weakly U_{V_+}-admissible, we infer by Proposition 2.8(b) that $-V^-$ is weakly $(U_{V_+})_{-W}$-admissible and

$$(U_{V_+})_{-V_-} = \big((U_{V_+})_{-W}\big)_{-V^-} = (U_{V^+})_{-V^-} = U_V. \qquad \square$$

2.18 Remarks. (a) Let $V_\pm \geqslant 0$ be such that $-V_-$ is weakly U_{V_+}-admissible, and let $V = V_+ - V_-$. Then by Proposition 2.13, Corollary 2.16 and Lemma 2.15(a) we obtain that

$$\begin{aligned} -V_- \text{ is weakly } U\text{-admissible} &\implies V_- \text{ is } U\text{-regular} \\ &\implies V_+ \wedge V_- \text{ is } U\text{-regular} \implies V_+ \wedge V_- \text{ is } U_{V^+}\text{-regular}, \end{aligned}$$

and by Theorem 2.17, each of these properties implies that V is weakly U-admissible and $U_V = (U_{V_+})_{-V_-}$. If $-V_-$ is weakly U-admissible then by Lemma 2.11 we also have $U_V = (U_{-V_-})_{V_+}$.

(b) Assume that U is strongly continuous and that V_+ and $-V_-$ are U-admissible. Then by (2.9) and Remark 2.9 one obtains that $(U_{V_+})_{-V_- \wedge n} \to (U_{-V_-})_{V_+}$ strongly, uniformly on compact subsets of D_J. Thus, $(U_{-V_-})_{V_+} = U_V$ is strongly continuous, i.e., V is U-admissible.

Next we show that the two notions of U-regularity that were introduced in [Voi86; Def. 2.12] and [Voi88; Def. 3.1] are equivalent.

2.19 Proposition. *(cf. [Man01; Satz 4.1.54]) Let $V \geqslant 0$. Then $U_{\varepsilon V} \to (U_V)_{-V}$ strongly as $\varepsilon \to 0$. In particular, V is U-regular if and only if $U_{\varepsilon V} \to U$ strongly as $\varepsilon \to 0$.*

Proof. Let $\varepsilon \in (0,1)$. Then $-\varepsilon V$ is weakly $U_{\varepsilon V}$-admissible by Remark 2.12. From Proposition 2.13 and Corollary 2.16 it follows that $(1-\varepsilon)V$ is $U_{\varepsilon V}$-regular. By Propositions 2.10(b) and 2.8(a) we conclude that

$$U_{\varepsilon V} = \big((U_{\varepsilon V})_{(1-\varepsilon)V}\big)_{-(1-\varepsilon)V} = (U_V)_{-(1-\varepsilon)V} \to (U_V)_{-V}$$

strongly as $\varepsilon \to 0$. □

For later use we note the following auxiliary result.

2.20 Lemma. *Let V be U-regular. Then V^+ and V^- are U-regular.*

Proof. Remark 2.12 yields the U-regularity of V^+. Since $-V^-$ is weakly U_{V^+}-admissible, V^- is U_{V^+}-regular by Proposition 2.13. By Lemma 2.15(b) it follows that V^- is $(U_{V^+})_{-V^+}$-regular and hence U-regular. □

We conclude the section with an analogue of the non-monotone convergence result for semigroups [Voi88; Thm. 3.5].

2.21 Theorem. *Let (V_n) be a sequence of potentials satisfying $V_n \to V$ locally a.e. as $n \to \infty$ and $-V_- \leqslant V_n \leqslant V + V_+$ for all $n \in \mathbb{N}$, where $V_\pm \geqslant 0$, $-V_-$ is weakly U-admissible and V_+ is U_V-regular. Then $U_{V_n} \to U_V$ strongly as $n \to \infty$.*

(Note that V and the V_n are weakly U-admissible since $V_n \geqslant -V_-$ for all $n \in \mathbb{N}$ and hence $V \geqslant -V_-$.)

2.22 Remarks. (a) In [Voi88; Thm. 3.5], in the context of C_0-semigroups, it was assumed more restrictively that V_+ is U-regular. This implies U_V-regularity of V_+ by Lemma 2.15. If V is not U-regular then $V_+ = cV^+$, with $c > 0$, is an example of a U_V-regular potential that is not U-regular.

(b) The assumption $-V_- \leqslant V_n \leqslant V + V_+$ can be replaced with the seemingly weaker assumption $-V_- \leqslant V_n \leqslant V^+ + V_+$. Indeed, as in Remark 2.18(a) we obtain from the weak U-admissibility of $-V_-$ that V^- is U_V-regular. Thus $\widetilde{V}_+ := V_+ + V^-$ is U_V-regular by Corollary 2.16, and $V + \widetilde{V}_+ = V^+ + V_+$.

(c) If $-V^-$ is weakly U-admissible and $V_n = V^{(n)} := (V \wedge n) \vee (-n)$, then the assumptions of Theorem 2.21 are satisfied with $V_- = V^-$ and $V_+ = 0$. It follows that in this case, our definition of the absorption propagator coincides with the classical definition $U_V = \operatorname{s-lim}_{n\to\infty} U_{V^{(n)}}$.

(d) Assume that U is strongly continuous, that $-V_-$ and V are U-admissible and that V_+ is U_V-admissible. Then a close inspection of the proof below shows that the strong convergence $U_{V_n} \to U_V$ is uniform on compact subsets of D_J; cf. Remark 2.9.

Proof of Theorem 2.21. In a first step we prove the assertion for the case $V = V_- = 0$. Then $V_n \geqslant 0$ and $0 \leqslant W_n := V_+ - V_n \leqslant V_+$ for all $n \in \mathbb{N}$. Moreover, the W_n are U-regular by Corollary 2.16, and $W_n \to V_+$ locally a.e. as $n \to \infty$. Therefore, by Remark 2.18(a) and Proposition 2.8(a) we obtain that

$$U_{V_n} = (U_{V_+})_{-W_n} \to (U_{V_+})_{-V_+} = U \qquad (n \to \infty).$$

In a second step we prove the assertion for the case $V_- = 0$. Then $V, V_n \geqslant 0$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $W_n := V_n - V_n \wedge V$ it follows from Propositions 2.10(b) and 2.6(b) that

$$0 \leqslant U_{V_n} - (U_V)_{W_n} = (U_{V_n \wedge V})_{W_n} - (U_V)_{W_n} \leqslant U_{V_n \wedge V} - U_V. \tag{2.10}$$

Observe that $U_{V_n \wedge V} \to U_V$ strongly as $n \to \infty$ by Proposition 2.10(a). Moreover, $W_n \to 0$ locally a.e. as $n \to \infty$, $0 \leqslant W_n \leqslant V_+$ for all $n \in \mathbb{N}$ and V_+ is U_V-regular, so by the first step we obtain that $(U_V)_{W_n} \to U_V$ strongly as $n \to \infty$. From (2.10) we thus infer that $U_{V_n} \to U_V$ strongly as $n \to \infty$.

To conclude the proof, we show that the general case can be reduced to the case $V_- = 0$. Let $\widetilde{V} := V + V_-$ and $\widetilde{V}_n := V_n + V_-$ for all $n \in \mathbb{N}$. Then $\widetilde{V}_n \to \widetilde{V}$ locally a.e. as $n \to \infty$ and $0 \leqslant \widetilde{V}_n \leqslant \widetilde{V} + V_+$ for all $n \in \mathbb{N}$. Moreover, $(U_{-V_-})_{\widetilde{V}_n} = U_{V_n}$ for all $n \in \mathbb{N}$ and $(U_{-V_-})_{\widetilde{V}} = U_V$ by Remark 2.18(a); in particular, V_+ is $(U_{-V_-})_{\widetilde{V}}$-regular. Thus, $(U_{-V_-})_{\widetilde{V}_n} \to (U_{-V_-})_{\widetilde{V}}$ strongly as $n \to \infty$ by the second step, and the proof is complete. $\square$

2.2 Miyadera class for potentials

Let (Ω, μ) be a measure space, let $1 \leqslant p < \infty$, and let U be a locally bounded positive strongly measurable propagator on $L_p(\mu)$ with parameter interval $J \subseteq \mathbb{R}$. Let $V\colon J \times \Omega \to \mathbb{R}$ be locally measurable. Then $V(t) = V(t, \cdot)$ is locally measurable for all $t \in J$ and hence defines a (closed) multiplication operator in $L_p(\mu)$. Thus, V is a (closed) Miyadera perturbation of U if and only if there exist $\alpha > 0$ and $\gamma \geqslant 0$ such that

$$\int_s^t \|V(\tau)U(\tau, s)f\|_p \, d\tau \leqslant \gamma \|f\|_p \tag{2.11}$$

for all $(t, s) \in D_J$ with $t - s \leqslant \alpha$ and all $f \in L_p(\mu)$. (Recall from Lemma 1.17 that $X_s = X$ for closed Miyadera perturbations; moreover, by the monotone convergence theorem one sees that $V(\cdot)U(\cdot, s)f$ is measurable on $J_{\geqslant s}$ if it is a.e. defined.) Note that it suffices to require (2.11) for $f \geqslant 0$ since $\|V(\tau)U(\tau, s)f\|_p \leqslant \|V(\tau)U(\tau, s)|f|\|_p$ for all $(\tau, s) \in D_J$.

In the following theorem, which extends [LVV06; Thm. 2.8], we formulate a sufficient condition under which a potential $V \geqslant 0$ is U-regular, and U-admissible in case that U is strongly continuous. The latter generalises the strong continuity assertion in [RäSc99; Thm. 3.3].

2.23 Theorem. *Let $V \geqslant 0$. Assume that for each $s \in J$ there exists a dense sublattice X_s of $L_p(\mu)$ such that $\tau \mapsto U(t,\tau)V(\tau)U(\tau,s)f \in L_p(\mu)$ is a.e. defined and integrable on (s,t), for all $t \in J_{>s}$ and all $f \in X_s$. (This assumption is satisfied with $X_s = L_p(\mu)$ if V is a Miyadera perturbation of U.) Then V is U-regular, and*

$$U_V(t,s)f = U(t,s)f - \int_s^t U_V(t,\tau)V(\tau)U(\tau,s)f\,d\tau \tag{2.12}$$

$$= U(t,s)f - \int_s^t U(t,\tau)V(\tau)U_V(\tau,s)f\,d\tau \tag{2.13}$$

for all $(t,s) \in D_J$ and all $f \in X_s$. Moreover, if U is strongly continuous then V is U-admissible.

Proof. Let $(t,s) \in D_J$ and $f \in X_s$. Then $|f| \in X_s$. For $n \in \mathbb{N}$ and $V_n := V \wedge n$ we have $U_{V_n} \leqslant U$ and hence

$$\begin{aligned}\big|U_{V_n}(t,\tau)V_n(\tau)U(\tau,s)f\big| &\leqslant U(t,\tau)V(\tau)U(\tau,s)|f|,\\ \big|U(t,\tau)V_n(\tau)U_{V_n}(\tau,s)f\big| &\leqslant U(t,\tau)V(\tau)U(\tau,s)|f|\end{aligned}$$

for all $\tau \in (s,t)$. Thus, by the dominated convergence theorem we can pass to the limit in

$$\begin{aligned}U_{V_n}(t,s)f &= U(t,s)f - \int_s^t U_{V_n}(t,\tau)V_n(\tau)U(\tau,s)f\,d\tau\\ &= U(t,s)f - \int_s^t U(t,\tau)V_n(\tau)U_{V_n}(\tau,s)f\,d\tau\end{aligned}$$

to obtain (2.12) and (2.13).

The assumption of the theorem is also satisfied for εV in place of V, and $U_{\varepsilon V} \leqslant U$ for all $\varepsilon > 0$. By (2.12) we thus obtain that

$$U_{\varepsilon V}(t,s)f = U(t,s)f - \varepsilon \int_s^t U_{\varepsilon V}(t,\tau)V(\tau)U(\tau,s)f\,d\tau \to U(t,s)f$$

as $\varepsilon \to 0$, for all $(t,s) \in D_J$ and all $f \in X_s$, so Proposition 2.19 implies that V is U-regular. Finally, if U is strongly continuous, then from Proposition 1.10 we conclude that U_V is strongly continuous, i.e., V is U-admissible. □

In the case of a small Miyadera perturbation V of U, a perturbed propagator can be defined either by Theorem 1.8 or by using absorption propagators. The following result shows that these two approaches lead to the same object.

2.24 Theorem. *Let V be a Miyadera perturbation of U, with V^- Miyadera small. Then V is U-regular,* (2.12) *and* (2.13) *hold for all $(t,s) \in D_J$ and all $f \in L_p(\mu)$, and U_V is the unique locally bounded strongly measurable propagator satisfying either of these Duhamel formulas. Moreover, if U is strongly continuous then V is U-admissible.*

Proof. We only have to show that V is weakly U-admissible, that V is a Miyadera perturbation of U_V, and that the two Duhamel formulas hold. Then the U-regularity of V^+ and hence of V follows from Theorem 2.23, the uniqueness assertion results from Corollary 1.19, and, as in the proof of Theorem 2.23, Proposition 1.10 implies that V is U-admissible if U is strongly continuous.

Assume first that $V \leqslant 0$. Then V is Miyadera small, so by Theorems 1.8 and 1.20(a) there exists a locally bounded strongly measurable propagator $\widetilde{U}_V$ satisfying (2.12) and (2.13) with $\widetilde{U}_V$ in place of U_V, for all $(t,s) \in D_J$ and all $f \in L_p(\mu)$. Moreover, V is a Miyadera perturbation of $\widetilde{U}_V$ by Theorem 1.20(b). From Corollary 1.19 we infer that U is the unique locally bounded strongly measurable propagator satisfying

$$U(t,s)f = \widetilde{U}_V(t,s)f - \int_s^t U(t,\tau)(-V)(\tau)\widetilde{U}_V(\tau,s)f\,d\tau \qquad \big((t,s) \in D_J,\ f \in L_p(\mu)\big).$$

By Theorem 2.23, the latter also holds with $(\widetilde{U}_V)_{-V}$ in place of U, so we obtain that $(\widetilde{U}_V)_{-V} = U$. Therefore, V is weakly U-admissible, and $U_V = \widetilde{U}_V$ since $-V$ is $\widetilde{U}_V$-regular by Theorem 2.23. This proves the theorem in the case $V \leqslant 0$.

If $V \geqslant 0$ then V is a Miyadera perturbation of U_V since $U_V \leqslant U$; the case $V \geqslant 0$ is thus already covered by Theorem 2.23. In the general case, the above implies that $-V^-$ and hence V is weakly U-admissible. Moreover, it follows from Theorem 1.20(b) that V is a Miyadera perturbation of U_{-V^-} and hence of U_V since $U_V \leqslant U_{-V^-}$. By Proposition 1.18 we conclude from the cases $V \geqslant 0$ and $V \leqslant 0$ that $U_V = (U_{V^+})_{-V^-}$ satisfies (2.12) and (2.13) for all $(t,s) \in D_J$ and all $f \in L_p(\mu)$, and the proof is complete. ☐

We conclude this section by investigating under what conditions a weakly U-admissible potential is a Miyadera perturbation of U or of U_V. The results are limited to the case that $p = 1$ and that V is of one sign, and they demonstrate that Miyadera perturbations are particularly interesting in this case. Part (b) of the following proposition is a propagator version of [Voi86; Lemma 4.1]. We use the notation $D_\alpha := \big\{(t,s) \in D_J;\ t - s \leqslant \alpha\big\}$, for $\alpha \in (0,\infty]$.

2.25 Proposition. *Let $p = 1$ and $c > 0$, and let $V \geqslant 0$. Assume that there exist $M \geqslant 1$ and $\alpha \in (0,\infty]$ such that $\|U(t,s)\| \leqslant M$ for all $(t,s) \in D_\alpha$.*

(a) *Assume that $\|U_V(t,s)f\|_1 \geqslant c\|f\|_1$ for all $(t,s) \in D_\alpha$ and all $0 \leqslant f \in L_1(\mu)$. Then V is a Miyadera perturbation of U with constants $(\alpha, \frac{M}{c} - 1)$.*

(b) *Assume that $\|U(t,s)f\|_1 \geqslant c\|f\|_1$ for all $(t,s) \in D_\alpha$ and all $0 \leqslant f \in L_1(\mu)$. Then V is a Miyadera perturbation of U_V with constants $(\alpha, \frac{M}{c})$.*

Proof. Let $0 \leqslant f \in L_1(\mu)$ and $(t,s) \in D_\alpha$, and for $n \in \mathbb{N}$ let $V_n := V \wedge n$.

(a) For $n \in \mathbb{N}$ we have $U_V \leqslant U_{V_n}$, so we can estimate

$$\begin{aligned} c\int_s^t \|V_n(\tau)U(\tau,s)f\|_1\,d\tau &\leqslant \int_s^t \|U_{V_n}(t,\tau)V_n(\tau)U(\tau,s)f\|_1\,d\tau \\ &= \Big\|\int_s^t U_{V_n}(t,\tau)V_n(\tau)U(\tau,s)f\,d\tau\Big\|_1 = \|U(t,s)f - U_{V_n}(t,s)f\|_1. \end{aligned}$$

Letting $n \to \infty$ we infer that

$$c \int_s^t \|V(\tau)U(\tau,s)f\|_1 \, d\tau \leqslant \|U(t,s)f\|_1 - \|U_V(t,s)f\|_1 \leqslant M\|f\|_1 - c\|f\|_1,$$

and the assertion follows.

(b) As above we obtain that

$$\begin{aligned} c \int_s^t \|V_n(\tau)U_V(\tau,s)f\|_1 \, d\tau &\leqslant \int_s^t \|U(t,\tau)V_n(\tau)U_{V_n}(\tau,s)f\|_1 \, d\tau \\ &= \|U(t,s)f - U_{V_n}(t,s)f\|_1 \leqslant M\|f\|_1 \end{aligned}$$

for all $n \in \mathbb{N}$, and the assertion follows for $n \to \infty$. □

Recall from Proposition 2.13 that $(U_V)_{-V} = U$ if $V \leqslant 0$ is weakly U-admissible. Thus, applying Proposition 2.25 with U_V and $-V$ in place of U and V, we immediately obtain the following result.

2.26 Corollary. *Let $p = 1$ and $c > 0$, and let $V \leqslant 0$ be weakly U-admissible. Assume that there exist $M \geqslant 1$ and $\alpha \in (0,\infty]$ such that $\|U_V(t,s)\| \leqslant M$ for all $(t,s) \in D_\alpha$.*

(a) *Assume that $\|U(t,s)f\|_1 \geqslant c\|f\|_1$ for all $(t,s) \in D_\alpha$ and all $0 \leqslant f \in L_1(\mu)$. Then V is a Miyadera perturbation of U_V with constants $(\alpha, \frac{M}{c} - 1)$.*

(b) *Assume that $\|U_V(t,s)f\|_1 \geqslant c\|f\|_1$ for all $(t,s) \in D_\alpha$ and all $0 \leqslant f \in L_1(\mu)$. Then V is a Miyadera perturbation of U with constants $(\alpha, \frac{M}{c})$.*

2.27 Remarks. (a) Let $p = 1$ and assume that U is *stochastic*, i.e., $\|U(t,s)f\|_1 = \|f\|_1$ for all $(t,s) \in D_J$ and all $0 \leqslant f \in L_1(\mu)$. Let $V \leqslant 0$ be weakly U-admissible and assume that U_V is exponentially bounded. Then $M_\alpha := \sup\big\{\|U_V(t,s)\|;\, (t,s) \in D_\alpha\big\} < \infty$ for all $\alpha > 0$, and by Corollary 2.26(a), V is a Miyadera perturbation of U_V with constants $(\alpha, M_\alpha - 1)$. In the context of C_0-semigroups, an analogous result has already been shown in [Voi86; Prop. 4.6].

Since $U \leqslant U_V$, the above implies that V is also a Miyadera perturbation of U with constants $(\alpha, M_\alpha - 1)$. The constant $M_\alpha - 1$ will be considerably improved in Remark 2.44(b) below, but for this we will have to assume that U is strongly continuous.

(b) Proposition 2.25(b) implies in particular that any $V \geqslant 0$ is a Miyadera perturbation of U_V with constants $(\infty, 1)$ if U is stochastic.

(c) It follows from Corollary 2.26(a) that there are no "borderline" weakly U-admissible potentials $V \leqslant 0$ if U is stochastic. More precisely, if K is a compact subinterval of J then $U_V|_{D_K}$ is bounded, so there exists $\varepsilon_K > 0$ such that $\varepsilon_K V|_{K\times\Omega}$ is a small Miyadera perturbation of $U_V|_{D_K}$. Using Theorem 2.24, we infer that there exists $\varphi\colon J \to (0,\infty)$ such that $V_\varphi(t,x) := \varphi(t)V(t,x)$ defines a weakly U_V-admissible potential, and from Proposition 2.10(b) we obtain that $V + V_\varphi$ is weakly U-admissible.

2.3 The backward Miyadera condition

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $1 \leqslant p < \infty$. In the case $p = 1$ we assume that μ is localisable, which is equivalent to $L_1(\mu)' = L_\infty(\mu)$. Let U be a locally bounded positive strongly measurable propagator on $L_p(\mu)$ with parameter interval $J \subseteq \mathbb{R}$, and let $V\colon J \times \Omega \to \mathbb{R}$ be locally measurable.

2.28 Remark. Let $A \in \mathcal{A}$ be σ-finite, let $t \in J$, and let $0 \leqslant f \in L_{p'}(\mu)$. Then the function

$$J_{\leqslant t} \ni \tau \mapsto \varphi(\tau) := \|\mathbb{1}_A V(\tau) U(t,\tau)' f\|_1 \in [0,\infty]$$

is measurable. Indeed, let (A_n) be a sequence of sets of finite measure such that $\mathbb{1}_{A_n} \uparrow \mathbb{1}_A$ a.e. Then $g_n(\tau) := |V(\tau)| \wedge (n\mathbb{1}_{A_n})$ defines a measurable function on $g_n\colon J_{\leqslant t} \to L_p(\mu)$, for all $n \in \mathbb{N}$. It follows that

$$J_{\leqslant t} \ni \tau \mapsto \|g_n(\tau) U(t,\tau)' f\|_1 = \int f\, U(t,\tau) g_n(\tau)\, d\mu$$

is measurable, so the monotone convergence theorem implies that φ is measurable.

2.29 Proposition. *Let $\gamma \geqslant 0$, and let $t, s \in J$ satisfy $t > s$. Then the following conditions on V are equivalent.*

(i) *Let $h\colon (s,t) \to L_p(\mu)$ be such that $\tau \mapsto |V(\tau)| h(\tau) \in L_p(\mu)$ is integrable on (s,t) and $0 \leqslant h(\tau) \leqslant 1$ a.e. for all $\tau \in (s,t)$. Then*

$$\Big\| \int_s^t U(t,\tau) |V(\tau)| h(\tau)\, d\tau \Big\|_\infty \leqslant \gamma.$$

(ii) *For all $0 \leqslant g \in L_p(\mu)$ one has*

$$\Big\| \int_s^t U(t,\tau) \big(|V(\tau)| \wedge g\big)\, d\tau \Big\|_\infty \leqslant \gamma.$$

(As in (i)*, the integral on the left-hand side is an L_p-valued Bochner integral.)*

(iii) *For all $0 \leqslant f \in L_1(\mu) \cap L_{p'}(\mu)$ and all σ-finite sets $A \in \mathcal{A}$ one has*

$$\int_s^t \|\mathbb{1}_A V(\tau) U(t,\tau)' f\|_1\, d\tau \leqslant \gamma \|f\|_1.$$

Proof. Without loss of generality assume that $V \geqslant 0$. Let $h\colon (s,t) \to L_p(\mu)$ be as in condition (i). Then, for all $0 \leqslant f \in L_1(\mu) \cap L_{p'}(\mu)$, the function $\varphi_f\colon (s,t) \to \mathbb{R}$,

$$\varphi_f(\tau) := \|V(\tau) h(\tau) U(t,\tau)' f\|_1 = \int_\Omega f\, U(t,\tau) V(\tau) h(\tau)\, d\mu$$

is integrable, and

$$\int_s^t \varphi_f(\tau)\, d\tau = \int_\Omega f \int_s^t U(t,\tau) V(\tau) h(\tau)\, d\tau\, d\mu.$$

It follows that the inequality in (i) holds if and only if

$$\int_s^t \|V(\tau)h(\tau)U(t,\tau)'f\|_1\,d\tau \leqslant \gamma\|f\|_1 \qquad \big(0 \leqslant f \in L_1(\mu)\cap L_{p'}(\mu)\big). \tag{2.14}$$

In the same way we obtain that the inequality in (ii) is equivalent to

$$\int_s^t \|(V(\tau)\wedge g)U(t,\tau)'f\|_1\,d\tau \leqslant \gamma\|f\|_1 \qquad \big(0 \leqslant f \in L_1(\mu)\cap L_{p'}(\mu)\big) \tag{2.15}$$

(use the integrability of $(s,t) \ni \tau \mapsto V(\tau)\wedge g \in L_p(\mu)$).

Now we derive the equivalence of conditions (i) to (iii) from the monotone convergence theorem.

(i) ⇒ (iii). Let $A \in \mathcal{A}$ be σ-finite, and let (A_n) be a sequence of sets of finite measure such that $\mathbb{1}_{A_n} \uparrow \mathbb{1}_A$ a.e. For $n \in \mathbb{N}$ we define $h_n\colon (s,t) \to L_p(\mu)$ by $h_n(\tau) := \big(nV(\tau)^{-1}\big) \wedge \mathbb{1}_{A_n}$. Then $V(\tau)h_n(\tau) = n \wedge \big(\mathbb{1}_{A_n}V(\tau)\big) \uparrow \mathbb{1}_A V(\tau)$ as $n \to \infty$, for all $\tau \in (s,t)$, and $\tau \mapsto V(\tau)h_n(\tau) \in L_p(\mu)$ is integrable for all $n \in \mathbb{N}$. Thus, (iii) follows from (2.14).

(iii) ⇒ (ii). Let $0 \leqslant g \in L_p(\mu)$. Then $A := [g > 0] = \big\{x \in \Omega;\, g(x) > 0\big\}$ is σ-finite and $V(\tau)\wedge g \leqslant \mathbb{1}_A V(\tau)$ for all $\tau \in (s,t)$, so (2.15) follows from (iii).

(ii) ⇒ (i). Let $h\colon (s,t) \to L_p(\mu)$ be as in condition (i). By Remark 2.1(b) there exists a σ-finite set $\Omega_h \subseteq \Omega$ such that $V(\tau)h(\tau) = \mathbb{1}_{\Omega_h}V(\tau)h(\tau)$ a.e. for a.e. $\tau \in (s,t)$. Choose $0 \leqslant g \in L_p(\mu)$ such that $[g > 0] = \Omega_h$. Then $V(\tau)\wedge(ng) \uparrow \mathbb{1}_{\Omega_h}V(\tau) \geqslant V(\tau)h(\tau)$ a.e. as $n \to \infty$, for a.e. $\tau \in (s,t)$, and hence (2.15) implies (2.14). □

For $p,q \in [1,\infty]$ and a linear operator B in $L_1(\mu) + L_\infty(\mu)$, we denote the norm of B as an operator from $L_p(\mu)$ to $L_q(\mu)$ by

$$\|B\|_{p\to q} := \sup\big\{\|Bf\|_q;\, f \in L_p(\mu)\cap D(B),\, \|f\|_p \leqslant 1\big\} \in [0,\infty].$$

In the following we assume that the propagator U is locally L_∞-bounded, i.e., for all compact subintervals K of J there exists $M_K \geqslant 1$ such that $\|U(t,s)\|_{\infty\to\infty} \leqslant M_K$ for all $(t,s) \in D_K$.

We say that V is a *backward Miyadera perturbation* of U (*with constants* $(\alpha,\gamma) \in (0,\infty]\times[0,\infty)$) if the conditions of Proposition 2.29 are satisfied for all $(t,s) \in D_J$ with $t - s \leqslant \alpha$. This is a Desch-Schappacher type condition; in the case of the heat equation one easily shows that it coincides with the backward Kato condition mentioned in the introduction (see also Remark 3.21(d)). Observe that for exponentially bounded U any $V \in L_\infty(J\times\Omega)$ is an *infinitesimally small* backward Miyadera perturbation (which is defined in the obvious way).

2.30 Theorem. *Assume that V is weakly U-admissible and that V^- is a backward Miyadera perturbation of U with constants (α,γ), $\gamma < 1$. Then U_V is locally L_∞-bounded. If $\|U(t,s)\|_{\infty\to\infty} \leqslant M$ for all $(t,s) \in D_J$ then*

$$\|U_V(t,s)\|_{\infty\to\infty} \leqslant \tfrac{M}{1-\gamma}e^{\omega(t-s)} \qquad \big((t,s) \in D_J\big),$$

where $\omega = \frac{1}{\alpha}\ln(1 + \frac{\gamma}{1-\gamma}M)$.

Proof. We only have to show the second assertion since for the first assertion we can assume without loss of generality that J is compact. Moreover, $U_{V^+} \leqslant U$ and hence V^- is a backward Miyadera perturbation of U_{V^+} with constants (α, γ), so we can assume without loss of generality that $V \leqslant 0$. Then $V_n := V \vee (-n)$ is a backward Miyadera perturbation of U with constants (α, γ), for each $n \in \mathbb{N}$. Since the operators $U_{V_n}(t,s)$ are positive and $U_{V_n}(t,s)f \uparrow U_V(t,s)f$ as $n \to \infty$, for all $(t,s) \in D_J$ and all $0 \leqslant f \in L_p(\mu) \cap L_\infty(\mu)$, we only have to estimate $\|U_V(t,s)f\|_\infty$ for $f \geqslant 0$, and we can assume without loss of generality that V is bounded.

By Proposition 2.3(a) we have $\|U_V(t,s)\|_{\infty\to\infty} \leqslant Me^{\|V\|_\infty(t-s)}$ for all $(t,s) \in D_J$. Given $f \in L_p(\mu)$ such that $0 \leqslant f \leqslant 1$ a.e., we are going to prove that

$$c_j := \sup\big\{\|U_V(t,s)f\|_\infty;\ (t,s) \in D_J,\ t-s \leqslant j\alpha\big\} \leqslant \tfrac{M}{1-\gamma}\big(1 + \tfrac{\gamma}{1-\gamma}M\big)^{j-1} \tag{2.16}$$

for all $j \in \mathbb{N}$. As in the proof of Proposition 1.14, the assertion then follows by choosing j such that $(j-1)\alpha \leqslant t-s \leqslant j\alpha$. Let $n \in \mathbb{N}$ and assume that (2.16) holds for all $j < n$. Let $(t,s) \in D_J$ with $t-s \leqslant n\alpha$, and let $t_j := s + \frac{j}{n}(t-s)$ for $j = 0, \ldots, n$. Applying condition (i) of Proposition 2.29 with $h = U_V(\cdot,s)f\colon (t_{j-1},t_j) \to L_p(\mu),\ 0 \leqslant h(\tau) \leqslant c_j$ a.e. for all $\tau \in (t_{j-1},t_j)$, we obtain that

$$\Big\|\int_{t_{j-1}}^{t_j} U(t_j,\tau)V(\tau)U_V(\tau,s)f\,d\tau\Big\|_\infty \leqslant \gamma c_j$$

for $j = 1, \ldots, n$. From the second Duhamel formula (2.13) we infer that

$$\begin{aligned}\|U_V(t,s)f\|_\infty &\leqslant \|U(t,s)f\|_\infty + \sum_{j=1}^{n}\Big\|\int_{t_{j-1}}^{t_j} U(t,t_j)U(t_j,\tau)V(\tau)U_V(\tau,s)f\,d\tau\Big\|_\infty \\ &\leqslant M + M\sum_{j=1}^{n-1}\gamma c_j + \gamma c_n,\end{aligned}$$

and as in the proof of Proposition 1.14 we conclude that (2.16) holds for $j = n$. ☐

2.31 Remarks. (a) The local L_∞-boundedness of U implies that $U(t,s)'|_{L_1(\mu)\cap L_{p'}(\mu)}$ extends to a bounded operator $U'(s,t)$ on $L_1(\mu)$, for each $(t,s) \in D_J$. We thus obtain a locally bounded *backward propagator* U' on $L_1(\mu)$ with parameter interval J, i.e., a function

$$U'\colon \big\{(s,t) \in J \times J;\ s \leqslant t\big\} \to \mathcal{L}(L_1(\mu))$$

that satisfies

$$U'(t,t) = I, \quad U'(r,t) = U'(r,s)\,U'(s,t) \qquad (r \leqslant s \leqslant t \text{ in } J).$$

(We refer to [Gul02; Def. 3] for this definition.) Note that taking the adjoint of a propagator is connected with time reversal. Moreover, observe that U' is (separately and jointly) weakly measurable. Therefore, U' is a strongly measurable backward propagator if $L_1(\mu)$

is separable. If one only assumes μ to be σ-finite then U' need not be strongly measurable; see Example 2.32(a) below.

(b) If μ is σ-finite then condition (iii) in Proposition 2.29 simplifies to

$$\int_s^t \|V(\tau)U'(\tau,t)f\|_1\,d\tau \leqslant \gamma\|f\|_1 \qquad \big(0 \leqslant f \in L_1(\mu)\cap L_{p'}(\mu)\big). \tag{2.17}$$

It follows from Remark 2.1(b) that the same is true if $U'(\cdot,t)$ is strongly measurable. In Example 2.32(b) below we will show that one cannot omit the indicator function $\mathbb{1}_A$ in the general case. We point out that in [LVV06; Sec. 3.1], strong continuity of U' was required for the backward Miyadera condition.

(c) Assume that U' is strongly measurable. As observed in part (b), V is a backward Miyadera perturbation of U with constants (α,γ) if and only if (2.17) holds for all $(t,s)\in D_J$ with $t-s\leqslant\alpha$. This can also be expressed as V being a Miyadera perturbation of the backward propagator U' with constants (α,γ) (cf. (2.11)), or equivalently, as $(-J)\times\Omega \ni (t,x)\mapsto V(-t,x)$ being a Miyadera perturbation of the associated (forward) propagator

$$D_{-J}\ni(t,s)\mapsto U'(-t,-s).$$

If $\gamma<1$ then from the above it is clear how the perturbed backward propagator $(U')_V$ is defined. Following the last step in the proof of [LVV06; Prop. 3.6] one can show that $(U')_V=(U_V)'$ if additionally V is a small Miyadera perturbation of U. For a weakly U-admissible potential V it follows that U_V is locally L_∞-bounded if and only if V is weakly U'-admissible, and again $(U')_V=(U_V)'$ in this case.

2.32 Example. (a) Let $\Omega := [0,1]^{[0,1]}$, and let μ be the canonical product probability measure on Ω. Then $L_1(\mu)$ is not separable since any function $y\mapsto f(y)$ in $L_1(\mu)$ depends only on countably many of the variables y_s, $s\in[0,1]$. Let $J:=[0,1]$. We define a propagator $U\colon D_J\to\mathcal{L}(L_1(\mu))$ by $U(t,t):=I$ for all $t\in J$ and

$$U(t,s)f := \int_\Omega 2y_s f(y)\,d\mu(y)\cdot\mathbb{1}_\Omega \qquad \big(0\leqslant s<t\leqslant 1,\ f\in L_1(\mu)\big).$$

Then $U(t,s)\mathbb{1}_\Omega=\mathbb{1}_\Omega$ for all $(t,s)\in D_J$, and it follows that U is a propagator. Moreover, $U(t,s)f=\int_\Omega f(y)\,d\mu(y)\cdot\mathbb{1}_\Omega =: Pf$ if $t>s$ and $f\in L_1(\mu)$ does not depend on the variable y_s, so for $t\in J$ we obtain that $U(t,s)f=Pf$ for a.e. $s\in J_{\leqslant t}$. This implies that U is a strongly measurable propagator. Note, however, that the function $(s,y)\mapsto 2y_s$ is not measurable on $J\times\Omega$ since it depends on all the variables y_s.

Now observe that $\|U(t,s)\|_{\infty\to\infty}\leqslant 1$ for all $(t,s)\in D_J$ and that U' is given by $U'(t,t)=I$ for all $t\in J$ and

$$U'(s,t)f(x)=2x_s\int_\Omega f(y)\,d\mu(y) \qquad \big(0\leqslant s<t\leqslant 1,\ f\in L_1(\mu),\ x\in\Omega\big).$$

Therefore, if $t>0$ and $\int_\Omega f(y)\,dy\neq 0$, then $U'(\cdot,t)f$ is not separably valued and hence not measurable.

(b) As in Remark 2.1(a) let $J := \Omega := [0,1]$, let μ be the counting measure on Ω, and let $V\colon J\times\Omega \to \mathbb{R}$ be the indicator function of the diagonal $\bigl\{(t,t)\in J\times\Omega;\ 0\leqslant t\leqslant 1\bigr\}$. We define $U\colon D_J \to \mathcal{L}(L_1(\mu))$ by $U(t,t) := I$ for all $t\in J$ and

$$U(t,s)f := \bigl(f(s)+f(0)\bigr)\delta_0 \qquad \bigl(0\leqslant s<t\leqslant 1,\ f\in L_1(\mu)\bigr).$$

One easily checks that U is a propagator on $L_1(\mu)$. Moreover, $\|U(t,s)\|_{\infty\to\infty}\leqslant 2$ for all $(t,s)\in D_J$.

Since $V=0$ locally a.e., V is a backward Miyadera perturbation of U with constants $(\infty,0)$; in fact $U_V = U$ by Remark 2.1(a). Let now $t\in J$ and $0\leqslant f\in L_1(\mu)\cap L_\infty(\mu)$. Then $V(\tau)U(t,\tau)'f = V(\tau)f(0)(\delta_0+\delta_\tau) = f(0)\delta_\tau$ for all $0<\tau<t$, so

$$\int_0^t \|V(\tau)U(t,\tau)'f\|_1\,d\tau = \int_0^t |f(0)|\,d\tau = t|f(0)|.$$

This shows that in general one cannot omit the indicator function $\mathbb{1}_A$ in condition (iii) of Proposition 2.29.

2.4 Logarithmically convex functions

The ideas presented in this section play a central role for this thesis; they are the basis for the interpolation inequalities of the next section. The following definition is motivated by the paper [Haa07].

Let X be an ordered vector space, i.e., X is a (real or complex) vector space endowed with a proper convex cone X_+ of positive elements, where X_+ being proper means that $X_+\cap(-X_+)=\{0\}$. Let D be a non-empty subset of $\mathbb{R}$. We say that a function $f\colon D\to X$ is *logarithmically convex* if

$$f(\xi_\theta)\leqslant (1-\theta)r^{-\theta}f(\xi_0)+\theta r^{1-\theta}f(\xi_1) \qquad (r>0) \tag{2.18}$$

for all $\xi_0,\xi_1\in D$ and all $\theta\in(0,1)$ with $\xi_\theta := (1-\theta)\xi_0+\theta\xi_1\in D$. By choosing $\xi_\theta=\xi_0=\xi_1$ and $r\neq 1$ (so that $(1-\theta)r^{-\theta}+\theta r^{1-\theta}>1$) we see that a logarithmically convex function f takes its values in X_+.

In the following let (Ω,μ) be a measure space. The next lemma implies in particular that $f\colon D\to[0,\infty)$ is logarithmically convex if and only if $\ln\circ f$ is convex, where we use the convention $\ln 0 := -\infty$.

2.33 Lemma. *Let $M(\mu)$ be the ordered vector space of all scalar-valued locally measurable functions on Ω, where functions are identified if they coincide locally a.e., and $M(\mu)_+ = \{f\in M(\mu);\ f\geqslant 0$ locally a.e.$\}$.*

(a) *A function $f\colon D\to M(\mu)_+$ is logarithmically convex if and only if*

$$f(\xi_\theta)\leqslant f(\xi_0)^{1-\theta}f(\xi_1)^\theta$$

locally a.e. for all $\xi_0,\xi_1\in D$ and all $\theta\in(0,1)$ with $\xi_\theta := (1-\theta)\xi_0+\theta\xi_1\in D$.

(b) *For $h\in M(\mu)$ the function $\xi\mapsto e^{\xi h}$ is logarithmically convex. If $f,g\colon D\to M(\mu)_+$ are logarithmically convex, then $\xi\mapsto f(\xi)g(\xi)$ is logarithmically convex, too.*

Proof. (a) follows from Young's inequality: For $a, b \geqslant 0$ and $\theta \in (0,1)$ we have

$$a^{1-\theta}b^{\theta} = \bigl(r^{-\theta}a\bigr)^{1-\theta}\bigl(r^{1-\theta}b\bigr)^{\theta} \leqslant (1-\theta)r^{-\theta}a + \theta r^{1-\theta}b \qquad (r>0)$$

and

$$a^{1-\theta}b^{\theta} = \inf\bigl\{(1-\theta)r^{-\theta}a + \theta r^{1-\theta}b;\ 0 < r \in \mathbb{Q}\bigr\}.$$

(b) is an immediate consequence of part (a). □

The next result, though being elementary, is important for the kernel estimates in the next chapter. We assume that $L_p(\mu)$ and $\mathcal{L}(L_p(\mu))$ are endowed with their natural orderings, in particular, $\mathcal{L}(L_p(\mu))_+$ consists of the positive operators on $L_p(\mu)$.

2.34 Proposition. *Let $p \in [1,\infty)$, and let $T\colon D \to \mathcal{L}(L_p(\mu))$. Then the following are equivalent:*

(i) *T is logarithmically convex;*
(ii) *$\xi \mapsto T(\xi)f$ is logarithmically convex for all $f \in L_p(\mu)_+$;*
(iii) *$\xi \mapsto \langle T(\xi)f, g\rangle$ is logarithmically convex for all $f \in L_p(\mu)_+$, $g \in L_{p'}(\mu)_+$.*

Assume, in addition, that μ is σ-finite and that the operators $T(\xi)$ are integral operators with integral kernels k_ξ. Then (i)–(iii) *hold if and only if $D \ni \xi \mapsto k_\xi \in M(\mu\otimes\mu)$ is logarithmically convex.*

Proof. The equivalences are immediate from the following two facts: A function $h \in L_p(\mu)$ is in $L_p(\mu)_+$ if and only if $\langle h, g\rangle \geqslant 0$ for all $g \in L_{p'}(\mu)_+$, and an integral operator is positive if and only if its integral kernel is $\geqslant 0$ a.e. □

2.35 Remarks. (a) We find it worth emphasizing that the conclusion from the logarithmic convexity of T to the logarithmic convexity of $\xi \mapsto k_\xi$ is entirely elementary. In the literature, δ-sequences (f_n), (g_n) and continuity of the kernel or Lebesgue points of the kernel are sometimes used instead; see, e.g., [Sim82; proof of Thm. B.6.7] and [LiSo03; proof of Prop. 2.8].

(b) Clearly, if $f\colon D \to X$ is logarithmically convex and $T\colon X \to X$ is a positive operator, then $T \circ f$ is logarithmically convex. In the case $X = L_p(\mu)$ we can prove the following more general result.

2.36 Lemma. *Let $p \in [1,\infty)$, and let $f\colon D \to L_p(\mu)$ and $T\colon D \to \mathcal{L}(L_p(\mu))$ be logarithmically convex. Then $D \ni \xi \mapsto T(\xi)f(\xi) \in L_p(\mu)$ is logarithmically convex, too.*

Proof. Let $\xi_0, \xi_1 \in D$ and $\theta \in (0,1)$ with $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1 \in D$. Let $g := f(\xi_0)$ and $h := f(\xi_1)$. Then $f(\xi_\theta) \leqslant g^{1-\theta}h^{\theta}$ by Lemma 2.33(a), so the assertion follows if we show that

$$T(\xi_\theta)(g^{1-\theta}h^{\theta}) \leqslant (1-\theta)r^{-\theta}T(\xi_0)g + \theta r^{1-\theta}T(\xi_1)h \qquad (r>0). \tag{2.19}$$

For the proof of this inequality we can assume without loss of generality that g and h are simple functions,

$$g = \sum_{j=1}^{m} a_j \mathbb{1}_{A_j}, \quad h = \sum_{j=1}^{m} b_j \mathbb{1}_{A_j},$$

with $m \in \mathbb{N}$, $a_j, b_j \geqslant 0$ and pairwise disjoint measurable sets A_j of finite measure ($j = 1, \dots, m$). By Proposition 2.34, (i)⇒(ii), and Lemma 2.33(a), the logarithmic convexity of T implies that

$$\begin{aligned} a_j^{1-\theta} b_j^{\theta} T(\xi_\theta) \mathbb{1}_{A_j} &\leqslant \big(a_j T(\xi_0)\mathbb{1}_{A_j}\big)^{1-\theta} \big(b_j T(\xi_1)\mathbb{1}_{A_j}\big)^{\theta} \\ &\leqslant (1-\theta) r^{-\theta} a_j T(\xi_0)\mathbb{1}_{A_j} + \theta r^{1-\theta} b_j T(\xi_1)\mathbb{1}_{A_j} \end{aligned}$$

for all $r > 0$ and $j = 1, \dots, m$. Summation over j yields (2.19), and the proof is complete. $\square$

The following consequence of Lemma 2.36 will be applied in Section 3.1 to show ultracontractivity of perturbed propagators. It replaces the use of the Stein interpolation theorem in [LVV06]. As before, if B is a linear operator in $L_1(\mu) + L_\infty(\mu)$, then $\|B\|_{p\to q}$ denotes the norm of B considered as as operator from $L_p(\mu)$ to $L_q(\mu)$.

2.37 Corollary. *Let $p \in [1, \infty)$, and let $T\colon D \to \mathcal{L}(L_p(\mu))$ be logarithmically convex. Let $\xi_0, \xi_1 \in D$ and $p_0, p_1, q_0, q_1 \in [1, \infty]$, and assume that $\|T(\xi_j)\|_{p_j \to q_j} < \infty$ for $j = 0, 1$. Let $\theta \in (0, 1)$ be such that $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1 \in D$, and define $p_\theta, q_\theta \in [1, \infty]$ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.20}$$

Then

$$\|T(\xi_\theta)\|_{p_\theta \to q_\theta} \leqslant \|T(\xi_0)\|_{p_0\to q_0}^{1-\theta} \|T(\xi_1)\|_{p_1 \to q_1}^{\theta}.$$

If, in addition, $\rho_1, \rho_2\colon D \to M(\mu)_+$ are logarithmically convex then

$$\|\rho_1(\xi_\theta)T(\xi_\theta)\rho_2(\xi_\theta)\|_{p_\theta\to q_\theta} \leqslant \|\rho_1(\xi_0)T(\xi_0)\rho_2(\xi_0)\|_{p_0\to q_0}^{1-\theta} \|\rho_1(\xi_1)T(\xi_1)\rho_2(\xi_1)\|_{p_1\to q_1}^{\theta},$$

where $\rho_j(\xi)$ is considered as a multiplication operator, for $j = 1, 2$ and $\xi \in D$.

Proof. Let $g \in L_1(\mu)_+$ be bounded, $\mu(\operatorname{spt} g) < \infty$ and $\|g\|_1 \leqslant 1$. Define $f\colon [0,1] \to L_p(\mu)$ by $f(\tau) := g^{1/p_\tau}\mathbb{1}_{\operatorname{spt} g}$, where p_τ is defined as in (2.20). (The indicator function is only needed for the case $p_\tau = \infty$.) Then f is logarithmically convex, so from Lemma 2.36 we infer that

$$T(\xi_\theta) f(\theta) \leqslant \big(T(\xi_0)f(0)\big)^{1-\theta} \big(T(\xi_1)f(1)\big)^{\theta}.$$

By Hölder's inequality and the estimate $\|f(j)\|_{p_j} \leqslant 1$ for $j = 0, 1$ it follows that

$$\|T(\xi_\theta)f(\theta)\|_{q_\theta} \leqslant \|T(\xi_0)f(0)\|_{q_0}^{1-\theta} \|T(\xi_1)f(1)\|_{q_1}^{\theta} \leqslant \|T(\xi_0)\|_{p_0\to q_0}^{1-\theta} \|T(\xi_1)\|_{p_1\to q_1}^{\theta} =: c.$$

In the case $p_\theta < \infty$ we thus have shown $\|T(\xi_\theta)h\|_{q_\theta} \leqslant c$ for all bounded functions $h \in L_p(\mu)_+$ with $\mu(\operatorname{spt} h) < \infty$ and $\|h\|_{p_\theta} \leqslant 1$; in the case $p_\theta = \infty$ we have shown $\|T(\xi_\theta)\mathbb{1}_A\|_{q_\theta} \leqslant c$ for all measurable sets A with $\mu(A) < \infty$. This implies the first assertion since $T(\xi_\theta)$ is a positive operator.

For the proof of the second assertion let

$$A_n := \big\{x \in \Omega;\ \rho_k(\xi_j)(x) \leqslant n\ (k = 1, 2,\ j = 0, 1)\big\}$$

for all $n \in \mathbb{N}$. Then $[0,1] \ni \theta \mapsto \mathbb{1}_{A_n}\rho_1(\xi_\theta)T(\xi_\theta)\rho_2(\xi_\theta)\mathbb{1}_{A_n} \in \mathcal{L}(L_p(\mu))$ is logarithmically convex for all $n \in \mathbb{N}$, by Lemma 2.36 and Proposition 2.34, and $\mathbb{1}_{A_n} \uparrow \mathbb{1}_\Omega$ as $n \to \infty$. Thus, by the monotone convergence theorem, the second assertion follows from the first one. $\square$

2.5 Interpolation inequalities for absorption propagators

In this section we assume that U is a positive strongly continuous propagator on $L_p(\mu)$ with parameter interval $J \subseteq \mathbb{R}$, where (Ω, μ) is a measure space and $p \in [1, \infty)$. We do not know if the subsequent results are true without the strong continuity assumption. The following is the first main result of the section; for a discussion of its assumptions see Remark 2.41(a) below.

2.38 Theorem. *Let $V_0, V_1 \colon J \times \Omega \to \mathbb{R}$ be weakly U-admissible potentials, and assume that $V_0^+ \wedge V_1^-$ and $V_0^- \wedge V_1^+$ are U-regular. Then $V_\theta := (1-\theta)V_0 + \theta V_1$ is weakly U-admissible, for every $\theta \in (0,1)$, and $[0,1] \ni \theta \mapsto U_{V_\theta}$ is logarithmically convex.*

By Propositions 2.7 and 2.34 we immediately obtain the following consequence of Theorem 2.38, which will be crucial for the kernel estimates in the next chapter.

2.39 Corollary. *In addition to the assumptions of Theorem 2.38 suppose that $U(t,s)$ is an integral operator, for some $t, s \in J$ with $t > s$. Then $U_{V_\theta}(t,s)$ is an integral operator with kernel $p_{t,s}^{V_\theta}$, for every $\theta \in [0,1]$, and $[0,1] \ni \theta \mapsto p_{t,s}^{V_\theta}$ is logarithmically convex.*

2.40 Remark. One of the first results on interpolation of integral kernels is proved in [HeSl78; Thm. 8.2], for translation invariant ultracontractive positive selfadjoint C_0-semigroups. There the perturbation V_1 is allowed to contain a form small distributional part, and $V_0 = 0$. The proof is based on the Trotter product formula, as our proof below, but it additionally involves kernel techniques.

Several different methods have since been used to prove versions of the above results. For Schrödinger semigroups it is well-known that the Feynman-Kac formula and Hölder's inequality in path space can be used to obtain interpolation inequalities; see, e.g., [Sim82; Lemma B.4.1]. In the rather general framework of perturbation of positive C_0-semigroups on a Banach lattice X by operators from the centre of X, logarithmic convexity of $\theta \mapsto U_{V_\theta}$ is implicit in [Voi88; top of p. 121]; the proof is based on the three lines theorem. In [ArDe06; Thm. 2.4], the case of positive C_0-semigroups on L_p-spaces and negative admissible perturbations is treated by means of the Gelfand-Naimark theorem.

The method of [HeSl78] is generalised to the non-autonomous situation in [MiSe03; Sec. 2], under the assumption that the propagator U has local infinitesimal generators that are sectorial, that $V_0 = 0$, and that $-V_1^-$ is a small Miyadera perturbation of U. General positive strongly continuous propagators are investigated in [LVV06; Prop. 3.8]; the proof given there is based on the three lines theorem as in [Voi88], and again the V_j^- are assumed to be Miyadera small.

All the above results require at least some smallness of the perturbations or admissibility of the negative parts, and mostly $V_0 = 0$ is assumed. Much more general perturbations are allowed in [Stu93; Lemma 2.20], which deals with Schrödinger semigroups on Riemannian manifolds with (possibly highly oscillating) signed smooth measures. (We refer to [Stu93; top of p. 326] for an explanation where the smoothness assumption comes into play.) The assumption that the measures are smooth corresponds to our regularity assumption. Our definitions of weak admissibility and of regularity in Section 2.1 were motivated by the desire to cover, in the context of potentials, the same generality in the perturbations.

Proof of Theorem 2.38. First assume that V_0 and V_1 are bounded. Then we only have to show that $[0,1] \ni \theta \mapsto U_{V_\theta}(t,s) \in \mathcal{L}(L_p(\mu))$ is logarithmically convex for all $(t,s) \in D_J$, without loss of generality $t > s$. Let $t_0, \ldots, t_n \in [s,t]$ satisfy $s = t_0 < t_1 < \cdots < t_n = t$, and for $k = 1, \ldots, n$ let

$$V_{\theta,k} := \int_{t_{k-1}}^{t_k} V_\theta(\tau)\, d\tau = \int_{t_{k-1}}^{t_k} V_0(\tau)\, d\tau + \theta \int_{t_{k-1}}^{t_k} (V_1 - V_0)(\tau)\, d\tau \qquad (\theta \in [0,1]).$$

Recall from Remark 2.35(b) that logarithmic convexity is stable under composition with positive operators. By Lemma 2.33(b) we thus obtain that

$$[0,1] \ni \theta \mapsto e^{V_{\theta,n}} U(t_n, t_{n-1}) e^{V_{\theta,n-1}} U(t_{n-1}, t_{n-2}) \cdots e^{V_{\theta,1}} U(t_1, t_0)$$

is logarithmically convex, so the assertion follows from the Trotter product formula (Theorem 1.26) since logarithmic convexity is also stable under strong convergence.

Next assume that $V_0 \geqslant c$ and $V_1 \geqslant c$ for some $c \in \mathbb{R}$. Let $\theta \in [0,1]$. Then $V_\theta^n := (1-\theta)(V_0 \wedge n) + \theta(V_1 \wedge n) \to V_\theta$ as $n \to \infty$, and $c \leqslant V_\theta^n \leqslant V_\theta$ for all $n \in \mathbb{N}$. Therefore, $U_{V_\theta^n} \to U_{V_\theta}$ strongly as $n \to \infty$ by Theorem 2.21, and the assertion follows from the first step.

For the proof of the general case let $V_{\theta,\pm} := (1-\theta)V_0^\pm + \theta V_1^\pm$,

$$V_{\theta,-}^n := (1-\theta)(V_0^- \wedge n) + \theta(V_1^- \wedge n) \quad \text{and} \quad V_\theta^n := V_{\theta,+} - V_{\theta,-}^n$$

for all $\theta \in [0,1]$ and all $n \in \mathbb{N}$. Then for $\theta \in [0,1]$ we infer from Theorem 2.17(b) and the second step that

$$(U_{V_{\theta,+}})_{-V_{\theta,-}^n} = U_{V_\theta^n} \leqslant (1-\theta)U_{V_0^n} + \theta U_{V_1^n} \leqslant (1-\theta)U_{V_0} + \theta U_{V_1}$$

for all $n \in \mathbb{N}$. By Proposition 2.8(a) it follows that $-V_{\theta,-}$ is weakly $U_{V_{\theta,+}}$-admissible since $V_{\theta,-}^n \uparrow V_{\theta,-}$ as $n \to \infty$ and $(1-\theta)U_{V_0} + \theta U_{V_1}$ is locally bounded. Since $V_\theta = V_{\theta,+} - V_{\theta,-}$, and the potential

$$V_{\theta,+} \wedge V_{\theta,-} = (\theta V_1^+) \wedge \big((1-\theta)V_0^-\big) + (\theta V_1^-) \wedge \big((1-\theta)V_0^+\big)$$

is U-regular by Corollary 2.16, we conclude from Remark 2.18(a) that V_θ is weakly U-admissible and $U_{V_\theta} = (U_{V_{\theta,+}})_{-V_{\theta,-}} = \operatorname*{s-lim}_{n\to\infty} (U_{V_{\theta,+}})_{-V_{\theta,-}^n}$. Finally, $\theta \mapsto (U_{V_{\theta,+}})_{-V_{\theta,-}^n} = U_{V_\theta^n}$ is logarithmically convex by the second step, so the proof is complete. $\square$

2.41 Remarks. (a) The assumption that $V_0^+ \wedge V_1^-$ and $V_0^- \wedge V_1^+$ are U-regular is a rather technical condition. It is trivially satisfied if $V_0 = 0$, so as a special case of Theorem 2.38 we obtain: If V is weakly U-admissible then θV is weakly U-admissible for all $\theta \in [0,1]$, and $[0,1] \ni \theta \mapsto U_{V_\theta}$ is logarithmically convex.

For the lower bounds in the next chapter we will need another application of Theorem 2.38: There we assume that $V_1 = V$ is U-regular and that $V_0 := -\varepsilon V$ is weakly U-admissible, for some $\varepsilon > 0$ (see Theorem 3.12); then it follows from Lemma 2.20 that the assumptions on V_0 and V_1 are satisfied.

(b) We use part (a) to show that Proposition 2.19 is valid for any weakly U-admissible potential V (not just for $V \geqslant 0$) if U is strongly continuous. For $\varepsilon > 0$ we have

$$U_{\varepsilon V^+} \leqslant U_{\varepsilon V} \leqslant (1-\varepsilon)U + \varepsilon U_V, \tag{2.21}$$

where the second inequality is due to Theorem 2.38. If V is U-regular then both the right-hand side and the left-hand side of (2.21) converge to U strongly, by Proposition 2.19, so $U_{\varepsilon V} \to U$ strongly as $\varepsilon \to 0$.

Assume now that V is weakly U-admissible, and let $U_{0,V} := (U_V)_{-V} = (U_{V^+})_{-V^+}$. Then it follows from Proposition 2.13 and Lemma 2.15(b) that V^+ and hence also V is $U_{0,V}$-regular. Moreover, using Theorem 2.17 one shows that $(U_{0,V})_{\varepsilon V^+} = U_{\varepsilon V^+}$ for all $\varepsilon > 0$, so by the above we conclude that $U_{\varepsilon V} = (U_{0,V})_{\varepsilon V} \to U_{0,V} = (U_V)_{-V}$ as $\varepsilon \to 0$.

Whereas versions of Theorem 2.38 have long been known (see Remark 2.40), our second main result on logarithmic convexity and its consequences seem to be new. As in Remark 1.12(a) we denote by $U_k = U_{k,-V}$ the constituents of the Dyson-Phillips series for U and V if V is a Miyadera perturbation of U; recall that the minus sign is due to (2.1).

2.42 Theorem. *Let $V \leqslant 0$ be a Miyadera perturbation of U. Then $\mathbb{N}_0 \ni k \mapsto k!U_k$ is logarithmically convex.*

By Proposition 2.34 and Lemma 2.33(a), the logarithmic convexity of $k \mapsto k!U_k$ can be expressed as follows: If $(t,s) \in D_J$ and $f \in L_p(\mu)_+$ then

$$\bigl(\ell!U_\ell(t,s)f\bigr)^{m-k} \leqslant \bigl(k!U_k(t,s)f\bigr)^{m-\ell}\bigl(m!U_m(t,s)f\bigr)^{\ell-k} \qquad (0 \leqslant k \leqslant \ell \leqslant m). \tag{2.22}$$

The proof of Theorem 2.42 will be given at the end of the section. We first present an application that is particularly interesting for the perturbation of stochastic propagators on $L_1(\mu)$; see Remark 2.44(b) below.

2.43 Corollary. *Let $V \leqslant 0$ be a small Miyadera perturbation of U, and let $(t,s) \in D_J$.*

(a) *Let $f \in L_p(\mu)_+$ and $\Omega_0 := [U(t,s)f = 0]$. Then $U_1(t,s)f = U_V(t,s)f = 0$ a.e. on Ω_0 and*

$$\frac{U_1(t,s)f}{U(t,s)f} \leqslant \ln \frac{U_V(t,s)f}{U(t,s)f}$$

a.e. on $\Omega \setminus \Omega_0$.

(b) *If $p = 1$ then*

$$\frac{\|U_1(t,s)f\|_1}{\|U(t,s)f\|_1} \leqslant \ln \frac{\|U_V(t,s)f\|_1}{\|U(t,s)f\|_1}$$

for all $f \in L_1(\mu)_+$ with $\|U(t,s)f\|_1 > 0$.

Proof. (a) By Theorem 2.42, $k \mapsto k!U_k(t,s)f$ is logarithmically convex. By (2.22) it follows that $U_k(t,s)f = 0$ a.e. on Ω_0 for all $k \in \mathbb{N}$ and hence also $U_V(t,s)f = 0$ a.e. on Ω_0. Moreover,

$$\bigl(U_1(t,s)f\bigr)^k \leqslant \bigl(U(t,s)f\bigr)^{k-1} \cdot k!U_k(t,s)f \tag{2.23}$$

for all $k \in \mathbb{N}$ and thus

$$\frac{1}{k!}\left(\frac{U_1(t,s)f}{U(t,s)f}\right)^k \leqslant \frac{U_k(t,s)f}{U(t,s)f}$$

a.e. on $\Omega \setminus \Omega_0$. The latter inequality is trivially true for $k = 0$, so we conclude that

$$\exp\left(\frac{U_1(t,s)f}{U(t,s)f}\right) \leqslant \frac{U_V(t,s)f}{U(t,s)f}$$

a.e. on $\Omega \setminus \Omega_0$.

(b) By Hölder's inequality it follows from (2.23) that

$$\|U_1(t,s)f\|_1^k \leqslant \|U(t,s)f\|_1^{k-1} \cdot k!\|U_k(t,s)f\|_1.$$

Since $\|U_V(t,s)f\|_1 = \|\sum_{k=0}^\infty U_k(t,s)f\|_1 = \sum_{k=0}^\infty \|U_k(t,s)f\|_1$, we obtain as in (a) that

$$\exp\left(\frac{\|U_1(t,s)f\|_1}{\|U(t,s)f\|_1}\right) \leqslant \frac{\|U_V(t,s)f\|_1}{\|U(t,s)f\|_1},$$

and the proof is complete. □

2.44 Remarks. (a) The estimate in Corollary 2.43(a) is sharp as one sees in the trivial example that $U(t,s) = I$ for all $(t,s) \in D_J$: Then

$$U_1(t,s)f = -\int_s^t V(\tau)\,d\tau \cdot f \quad \text{and} \quad U_V(t,s)f = \exp\left(-\int_s^t V(\tau)\,d\tau\right) \cdot f$$

for all $(t,s) \in D_J$ and all $f \in L_p(\mu)$.

(b) In the situation of Corollary 2.43(b) assume that U is stochastic (as defined in Remark 2.27(a)). Let $(t,s) \in D_J$. Then for $f \in L_1(\mu)_+$ with $\|f\|_1 = 1$ we obtain by Corollary 2.43(b) that $\|U_1(t,s)f\|_1 \leqslant \ln\|U_V(t,s)f\|_1 \leqslant \ln\|U_V(t,s)\|$, so

$$\|U_1(t,s)\| \leqslant \ln\|U_V(t,s)\|. \tag{2.24}$$

Moreover,

$$\|U_1(t,s)f\|_1 = \int_s^t \|U(t,\tau)V(\tau)U(\tau,s)f\|_1\,d\tau = \int_s^t \|V(\tau)U(\tau,s)f\|_1\,d\tau$$

for all $f \in L_1(\mu)_+$. Given $\alpha > 0$, it follows that

$$\gamma_\alpha := \sup\bigl\{\|U_1(t,s)\|;\,(t,s) \in D_J,\, t-s \leqslant \alpha\bigr\}$$

is the smallest constant such that V is a Miyadera perturbation of U with constants (α, γ_α). With $M_\alpha := \sup\bigl\{\|U_V(t,s)\|;\,(t,s) \in D_J,\, t-s \leqslant \alpha\bigr\}$ we infer from (2.24) that

$$\gamma_\alpha \leqslant \ln M_\alpha.$$

This estimate is considerably better that the estimate $\gamma_\alpha \leqslant M_\alpha - 1$ from Remark 2.27(a). On the other hand, by Proposition 1.14 we know that

$$M_\alpha \leqslant \frac{1}{1-\gamma_\alpha}$$

if $\gamma_\alpha < 1$, so we obtain the two-sided estimate $1 - \frac{1}{M_\alpha} \leqslant \gamma_\alpha \leqslant \ln M_\alpha$.

(c) We want to show that an estimate analogous to the one in part (b) also applies to backward Miyadera perturbations of Markovian propagators. For simplicity we assume that the measure μ is σ-finite. We will say that an operator $B \in \mathcal{L}(L_p(\mu))$ *extrapolates* to a bounded operator on $L_q(\mu)$, for some $q \in [1,\infty)$, if $B|_{L_p(\mu)\cap L_q(\mu)}$ extends to an operator in $\mathcal{L}(L_q(\mu))$. We will also use this notion for operator-valued functions and for the case $q = \infty$ and weakly* continuous extensions; the space of weakly* continuous operators on $L_\infty(\mu)$ will be denoted by $\mathcal{L}_{\mathrm{w}^*}(L_\infty(\mu))$.

We start with the following observation. Let B be a positive operator on $L_p(\mu)$, and assume that $\|B\|_{\infty\to\infty} < \infty$. Then B extrapolates to an operator $B_\infty \in \mathcal{L}_{\mathrm{w}^*}(L_\infty(\mu))$, and if $L_p(\mu)_+ \ni f_n \uparrow \mathbb{1}$ a.e. then $Bf_n \uparrow B_\infty\mathbb{1}$ a.e. Indeed, B' extrapolates to a bounded operator $\widetilde{B}'$ on $L_1(\mu)$, and $B_\infty := \big(\widetilde{B}'\big)'$ is a weakly* continuous extension of $B|_{L_p(\mu)\cap L_\infty(\mu)}$. Moreover, $f_n \uparrow \mathbb{1}$ a.e. implies that $f_n \to \mathbb{1}$ weakly*, so $Bf_n \to B_\infty\mathbb{1}$ weakly*, and from the positivity of B it follows that $Bf_n \uparrow B_\infty\mathbb{1}$ a.e.

If the propagator U is locally L_∞-bounded then by the above, U extrapolates to a locally bounded propagator $\widetilde{U}\colon D_J \to \mathcal{L}_{\mathrm{w}^*}(L_\infty(\mu))$. We assume that U is *Markovian*, i.e., U is L_∞-contractive and $\widetilde{U}(t,s)\mathbb{1} = \mathbb{1}$ for all $(t,s) \in D_J$. Let $V \in L_\infty(J\times\Omega)$, $V \leqslant 0$. Then U_1 and U_V are locally L_∞-bounded and hence extrapolate to functions $\widetilde{U}_1$ and $\widetilde{U}_V$ from D_J to $\mathcal{L}_{\mathrm{w}^*}(L_\infty(\mu))$. By the above observation we infer from Corollary 2.43(a) that $\widetilde{U}_1(t,s)\mathbb{1} \leqslant \ln \widetilde{U}_V(t,s)\mathbb{1}$ a.e. for all $(t,s) \in D_J$. Now it is easy to see that

$$\int_s^t U(t,\tau)\big(|V(\tau)| \wedge g\big)\,d\tau = \int_s^t U(t,\tau)\big(|V(\tau)| \wedge g\big)\widetilde{U}(\tau,s)\mathbb{1}\,d\tau \leqslant \widetilde{U}_1(t,s)\mathbb{1}$$

a.e. for all $(t,s) \in D_J$ and all $g \in L_1(\mu)$, so we obtain the following.

Let $V \leqslant 0$ be weakly U-admissible, let $\alpha > 0$, and assume that

$$M_\alpha := \sup\big\{\|U_V(t,s)\|_{\infty\to\infty};\, (t,s) \in D_J,\, t-s \leqslant \alpha\big\} < \infty.$$

Then from the above, applied with $V_n := V \vee (-n)$ in place of V, we conclude that

$$\int_s^t U(t,\tau)\big(|V_n(\tau)| \wedge g\big)\,d\tau \leqslant \ln \widetilde{U}_{V_n}(t,s)\mathbb{1} \leqslant \ln \|U_V(t,s)\|_{\infty\to\infty} \leqslant \ln M_\alpha$$

a.e. for all $n \in \mathbb{N}$, $g \in L_1(\mu)$ and all $(t,s) \in D_J$ with $t-s \leqslant \alpha$, so V is a backward Miyadera perturbation of U with constants $(\alpha, \ln M_\alpha)$.

We point out that the above argument requires strong continuity of U but not of the (stochastic) backward propagator U' defined in Remark 2.31(a).

(d) If U is only strongly measurable then one can use the observation in part (c) to show the following analogues of Proposition 2.25(a) and of Corollary 2.26(a). Let $\alpha \in (0,\infty]$

and $M \geqslant c > 0$. If $V \geqslant 0$ and $c\mathbb{1} \leqslant \widetilde{U}_V(t,s)\mathbb{1} \leqslant \widetilde{U}(t,s)\mathbb{1} \leqslant M\mathbb{1}$ for all $(t,s) \in D_J$ with $t-s \leqslant \alpha$, then V is a backward Miyadera perturbation of U with constants $(\alpha, \frac{M}{c}-1)$. On the other hand, if $V \leqslant 0$ is weakly U-admissible and $c\mathbb{1} \leqslant \widetilde{U}(t,s)\mathbb{1} \leqslant \widetilde{U}_V(t,s)\mathbb{1} \leqslant M\mathbb{1}$ for all $(t,s) \in D_J$ with $t-s \leqslant \alpha$, then V is a backward Miyadera perturbation of U_V and hence also of U with constants $(\alpha, \frac{M}{c}-1)$.

We now turn to the proof of Theorem 2.42. The crucial ingredient is the following result on logarithmically convex sequences in $[0,\infty)$. Observe that a sequence $x\colon \mathbb{N}_0 \to [0,\infty)$ is logarithmically convex if and only if $x_k^2 \leqslant x_{k-1}x_{k+1}$ for all $k \in \mathbb{N}$.

2.45 Proposition. *Let $x\colon \mathbb{N}_0 \to [0,\infty)$ be logarithmically convex. Then*

$$\mathbb{N}_0 \ni k \longmapsto \sum_{j=0}^{k} \binom{k}{j} x_j \in [0,\infty)$$

is logarithmically convex, too.

Proof. Let $\ell \in \mathbb{N}$. We have to show that

$$L := \Big(\sum_{j=0}^{\ell} \binom{\ell}{j} x_j\Big)^2 \leqslant \Big(\sum_{j=0}^{\ell-1} \binom{\ell-1}{j} x_j\Big)\Big(\sum_{j=0}^{\ell+1} \binom{\ell+1}{j} x_j\Big) =: R.$$

For $k = 0,\dots,2\ell$ we are going to prove that

$$L_k := \sum_{j=0}^{k} \binom{\ell}{j}\binom{\ell}{k-j} x_j x_{k-j} \leqslant \sum_{j=0}^{k} \binom{\ell-1}{j}\binom{\ell+1}{k-j} x_j x_{k-j} =: R_k \tag{2.25}$$

(where $\binom{\ell}{j} = 0$ if $j > \ell$); then the assertion follows since $L = \sum_{k=0}^{2\ell} L_k$ and $R = \sum_{k=0}^{2\ell} R_k$.

Observe that for $y \in \mathbb{R}$ we have

$$\begin{aligned}\sum_{k=0}^{2\ell} y^k \sum_{j=0}^{k} \binom{\ell}{j}\binom{\ell}{k-j} &= \Big(\sum_{j=0}^{\ell} \binom{\ell}{j} y^j\Big)^2 = (1+y)^{2\ell} \\ &= \Big(\sum_{j=0}^{\ell-1} \binom{\ell-1}{j} y^j\Big)\Big(\sum_{j=0}^{\ell+1} \binom{\ell+1}{j} y^j\Big) = \sum_{k=0}^{2\ell} y^k \sum_{j=0}^{k} \binom{\ell-1}{j}\binom{\ell+1}{k-j}.\end{aligned}$$

For a fixed $k \in \{0,\dots,2\ell\}$ we thus obtain, with

$$\alpha_j := \binom{\ell}{j}\binom{\ell}{k-j} - \binom{\ell-1}{j}\binom{\ell+1}{k-j} \qquad (j = 0,\dots,k),$$

that $\sum_{j=0}^{k} \alpha_j = 0$. With this notation, (2.25) simplifies to $\sum_{j=0}^{k} \alpha_j x_j x_{k-j} \leqslant 0$. The latter is equivalent to $\sum_{j=0}^{k} \beta_j x_j x_{k-j} \leqslant 0$, where $\beta_j := \alpha_j + \alpha_{k-j}$. From the logarithmic convexity of x it follows that $j \mapsto x_j x_{k-j}$ is decreasing on $[0, \frac{k}{2}] \cap \mathbb{N}_0$ and increasing on $[\frac{k}{2}, k] \cap \mathbb{N}_0$. Below we will show that there exists $j_0 \in [0, \frac{k}{2}] \cap \mathbb{N}_0$ such that $\beta_j \geqslant 0$ for $j = j_0,\dots,k-j_0$

and $\beta_j \leqslant 0$ otherwise. Combining these properties with $\sum_{j=0}^k \beta_j = 0$, we then conclude that $\sum_{j=0}^k \beta_j x_j x_{k-j} \leqslant 0$, which proves (2.25) and hence the assertion.

Let now $j \in \{0, \dots, k\}$, and let $i := k - j$. Assume that $i, j \leqslant \ell + 1$; otherwise $\beta_j = \alpha_j + \alpha_i = 0$. The inequality $\beta_j \geqslant 0$ holds if and only if

$$2\binom{\ell}{j}\binom{\ell}{i} \geqslant \binom{\ell-1}{j}\binom{\ell+1}{i} + \binom{\ell-1}{i}\binom{\ell+1}{j}.$$

Since $\binom{\ell-1}{j} = \frac{\ell-j}{\ell}\binom{\ell}{j} = \frac{\ell-j}{\ell}\frac{\ell+1-j}{\ell+1}\binom{\ell+1}{j}$ and $\binom{\ell+1}{j}, \binom{\ell+1}{i} \neq 0$, the latter is equivalent to

$$2\,\frac{\ell+1-j}{\ell+1}\,\frac{\ell+1-i}{\ell+1} \geqslant \frac{\ell-j}{\ell}\,\frac{\ell+1-j}{\ell+1} + \frac{\ell-i}{\ell}\,\frac{\ell+1-i}{\ell+1}$$

and thus to

$$a_j := \tfrac{2\ell}{\ell+1}(\ell+1-j)(\ell+1-i) \geqslant (\ell-j)(\ell+1-j) + (\ell-i)(\ell+1-i) =: b_j.$$

Note that $j \mapsto a_j$ is concave on $\{0, \dots, k\}$ since $i = k - j$. Moreover,

$$b_j = 2\ell(\ell+1) - (2\ell+1)j + j^2 - (2\ell+1)i + i^2 = 2\ell(\ell+1) - (2\ell+1)k + j^2 + (k-j)^2,$$

so $j \mapsto b_j$ is convex on $\{0, \dots, k\}$. Therefore, there exists $j_0 \in [0, \frac{k}{2}] \cap \mathbb{N}_0$ such that $a_j \geqslant b_j$ for $j = j_0, \dots, k - j_0$ and $a_j < b_j$ otherwise. Then $\beta_j \geqslant 0$ for $j = j_0, \dots, k - j_0$ and $\beta_j \leqslant 0$ otherwise, and the proof is complete. □

2.46 Remark. Let $x\colon \mathbb{N}_0 \to [0, \infty)$ be logarithmically convex, and let $q \geqslant 0$. Then $k \mapsto q^k x_k$ is logarithmically convex. By Proposition 2.45 we thus obtain that

$$k \longmapsto \sum_{j=0}^{k} \binom{k}{j} q^{k-j} x_j =: z_k$$

is logarithmically convex. (For the case $q = 0$ note that then $z_k = x_k$.) This might lead one to the conjecture that more generally $k \mapsto \sum_{j=0}^k \binom{k}{j} x_j y_{k-j}$ is logarithmically convex if $y\colon \mathbb{N}_0 \to [0, \infty)$ is a second logarithmically convex sequence. We do not know whether this is true, but the idea of the proof of Proposition 2.45 does not work: One can show, e.g., that the analogue of the estimate (2.25) does not hold for $\ell = k = 2$.

Proof of Theorem 2.42. Let $(t, s) \in D_J$ and $f \in L_p(\mu)_+$; we show logarithmic convexity of $k \mapsto k! U_{k,-V}(t, s) f$. Using the monotone convergence theorem, one easily proves by induction that $U_{k,(-V)\wedge n}(t, s) \to U_{k,-V}(t, s)$ strongly as $n \to \infty$, for all $k \in \mathbb{N}_0$, so we can assume without loss of generality that V is bounded. Then by Remark 1.25 we can assume without loss of generality that U and V are "discrete" on $[s, t]$, i.e., there exist $m \in \mathbb{N}$ and $s = t_0 < t_1 < \cdots < t_m = t$ such that

$$U(\tau, \sigma) = I \quad \text{and} \quad V(\tau) = V(t_n)$$

for all $(\tau, \sigma) \in D_{(t_{n-1}, t_n]}$ and $n = 1, \dots, m$.

We show by induction on n that $k \mapsto k!U_k(t_n,t_0)f$ is logarithmically convex for $n = 0,\ldots,m$. For $n=0$ this is trivial since $U_k(t_0,t_0)=0$ for all $k\in\mathbb{N}$. Assume that logarithmic convexity is valid for some $n\in\{0,\ldots,m-1\}$. Then

$$k \longmapsto g_k := k!U(t_{n+1},t_n)U_k(t_n,t_0)f$$

is logarithmically convex by Remark 2.35(b). As explained in Remark 1.25(b), we have $U_k(t_{n+1},t_n) = \frac{1}{k!}W^kU(t_{n+1},t_n)$ for all $k\in\mathbb{N}_0$, where $W := -(t_{n+1}-t_n)V(t_{n+1}) \geqslant 0$. By Proposition 1.13 we infer that

$$k!U_k(t_{n+1},t_0)f = k!\sum_{j=0}^{k} U_{k-j}(t_{n+1},t_n)U_j(t_n,t_0)f = \sum_{j=0}^{k}\frac{k!}{(k-j)!j!}W^{k-j}g_j,$$

so from Remark 2.46 it follows that $k\mapsto k!U_k(t_{n+1},t_0)f$ is logarithmically convex. $\square$

2.6 Consistent propagators on weighted L_p-spaces

Let (Ω,μ) be a measure space. Let $\rho\colon \Omega\to(0,\infty)$ be locally measurable, and let $p\in[1,\infty)$. Then the operator of multiplication with $\rho^{1/p}$ is an isometry $\rho^{1/p}\colon L_p(\rho\mu)\to L_p(\mu)$. Let U be a locally bounded positive strongly measurable propagator on $L_p(\rho\mu)$ with parameter interval J. Then $\widetilde{U}(t,s) := \rho^{1/p}U(t,s)\rho^{-1/p}$ defines a locally bounded positive strongly measurable propagator on $L_p(\mu)$.

2.47 Lemma. *A potential V is weakly U-admissible if and only if it is weakly $\widetilde{U}$-admissible, and $\widetilde{U}_V(t,s) = \rho^{1/p}U_V(t,s)\rho^{-1/p}$ for all $(t,s)\in D_J$ in this case.*

Proof. Since $\rho^{1/p}V\rho^{-1/p} = V$, the assertion for bounded V follows from uniqueness in Duhamel's formula. Then for general V one uses the definition of absorption propagators to complete the proof. $\square$

For $j=0,1$ let now $\rho_j\colon\Omega\to(0,\infty)$ be locally measurable, let $p_j\in[1,\infty)$, and let U_j be a positive strongly continuous propagator on $L_{p_j}(\rho_j\mu)$ with parameter interval J. We assume that U_0 and U_1 are *consistent*, i.e., $U_0(t,s)f = U_1(t,s)f$ for all $(t,s)\in D_J$ and all $f\in L_{p_0}(\rho_0\mu)\cap L_{p_1}(\rho_1\mu)$. Observe that $L_{p_0}(\rho_0\mu)\cap L_{p_1}(\rho_1\mu)$ is dense in $L_{p_j}(\rho_j\mu)$, for $j=0,1$, so U_0 and U_1 determine each other uniquely.

2.48 Lemma. *(cf. [Voi86; Prop. 3.1]) Let V be weakly U_0- and U_1-admissible.*

(a) *Then $(U_0)_V$ and $(U_1)_V$ are consistent.*

(b) *The potential V is U_0-regular if and only if it is U_1-regular.*

Proof. (a) For bounded V one obtains by induction that the constituents of the Dyson-Phillips series for $(U_0)_V$ and $(U_1)_V$ are consistent, which implies the assertion in this case. The general case follows since strong limits of consistent propagators are again consistent.

(b) It follows from part (a) that $\big((U_0)_V\big)_{-V}$ and $\big((U_1)_V\big)_{-V}$ are consistent. This yields the assertion. $\square$

For $\theta \in (0,1)$ define $p_\theta \in [1,\infty)$ by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and let $\rho_\theta := \rho_0^{(1-\theta)p_\theta/p_0}\rho_1^{\theta p_\theta/p_1}$. We conclude the chapter with a result on interpolation of weak admissibility in the spaces $L_{p_\theta}(\rho_\theta\mu)$. In the application in Section 3.2 we will only use the case $p_0 = p_1$; note that then $\rho_\theta = \rho_0^{1-\theta}\rho_1^\theta$.

2.49 Proposition. *For $j = 0,1$ let V_j be weakly U_j-admissible. Assume that $V_0^+ \wedge V_1^-$ and $V_0^- \wedge V_1^+$ are U_0-regular. Then U_0 extrapolates to a strongly continuous propagator U_θ on $L_{p_\theta}(\rho_\theta\mu)$, for each $\theta \in (0,1)$, the potential $V_\theta := (1-\theta)V_0 + \theta V_1$ is weakly U_θ-admissible, and*

$$\|(U_\theta)_{V_\theta}(t,s)\| \leqslant \|(U_0)_{V_0}(t,s)\|^{1-\theta}\|(U_1)_{V_1}(t,s)\|^\theta \qquad \big((t,s) \in D_J\big). \tag{2.26}$$

Proof. First assume that V_0 and V_1 are bounded from below. Then $[0,1] \ni \theta \mapsto (U_0)_{V_\theta}$ is logarithmically convex by Theorem 2.38. Moreover, $[0,1] \ni \theta \mapsto \rho_\theta^{1/p_\theta} \in M(\mu)_+$ is logarithmically convex. Given $\theta \in (0,1)$, we thus we obtain by Corollary 2.37 that

$$\begin{aligned}\|\rho_\theta^{1/p_\theta}(U_0)_{V_\theta}(t,s)\rho_\theta^{-1/p_\theta}\|_{p_\theta\to p_\theta} &\leqslant \|\rho_0^{1/p_0}(U_0)_{V_0}(t,s)\rho_0^{-1/p_0}\|_{p_0\to p_0}^{1-\theta}\|\rho_1^{1/p_1}(U_0)_{V_1}(t,s)\rho_1^{-1/p_1}\|_{p_1\to p_1}^\theta \\ &= \|(U_0)_{V_0}(t,s)\|^{1-\theta}\|(U_1)_{V_1}(t,s)\|^\theta\end{aligned}$$

for all $(t,s) \in D_J$, where we have used that $(U_0)_{V_1}$ and $(U_1)_{V_1}$ are consistent by Lemma 2.48(a). By considering $V_\theta = V_0 = V_1 = 0$, we obtain in particular that U_0 extrapolates to a propagator U_θ on $L_{p_\theta}(\rho_\theta\mu)$; the strong continuity of U_θ is easily proved by means of the inequality $\|g\|_{L_{p_\theta}(\rho_\theta\mu)} \leqslant \|g\|_{L_{p_0}(\rho_0\mu)}^{1-\theta}\|g\|_{L_{p_1}(\rho_1\mu)}^\theta$ that follows from Hölder's inequality and is valid for all $g \in L_{p_0}(\rho_0\mu) \cap L_{p_1}(\rho_1\mu)$. This completes the proof in the case that V_0 and V_1 are bounded from below since $(U_0)_{V_\theta}$ and $(U_\theta)_{V_\theta}$ are consistent by Lemma 2.48(a).

In the case of general V_0, V_1 we proceed as in the proof of Theorem 2.38: Let $\theta \in (0,1)$ and $V_{\theta,\pm} := (1-\theta)V_0^\pm + \theta V_1^\pm$. Using that the right-hand side of (2.26) defines a locally bounded function on D_J, we infer from the first part of the proof that $-V_{\theta,-}$ is weakly $(U_\theta)_{V_{\theta,+}}$-admissible and that

$$\big\|\big((U_\theta)_{V_{\theta,+}}\big)_{-V_{\theta,-}}(t,s)\big\| \leqslant \|(U_0)_{V_0}(t,s)\|^{1-\theta}\|(U_1)_{V_1}(t,s)\|^\theta$$

for all $(t,s) \in D_J$. As in the proof of Theorem 2.38 we obtain that $V_{\theta,+} \wedge V_{\theta,-}$ is U_0-regular, and hence U_θ-regular by Lemma 2.48(b). Since $V_\theta = V_{\theta,+} - V_{\theta,-}$, we conclude from Remark 2.18(a) that V_θ is weakly U-admissible and $(U_\theta)_{V_\theta} = \big((U_\theta)_{V_{\theta,+}}\big)_{-V_{\theta,-}}$. ☐

Chapter 3

Bounds for the integral kernels of propagators

In this chapter we apply the results of the previous two chapters and prove stability of kernel estimates for strongly continuous propagators under perturbation by weakly admissible potentials.

Section 3.1, which deals with rather general kernel estimates for ultracontractive propagators, is adapted from [LVV06; Sec. 3]. The main difference is that the assumptions on the perturbation V are formulated in terms of weak admissibility and regularity, not in terms of Miyadera conditions. Moreover, we avoid the technical assumption of strong continuity of the backward propagator from Remark 2.31(a).

In Section 3.2 we investigate Gaussian type estimates with boundary terms, still in the setting of ultracontractive propagators. The main issue here is stability of the boundary terms under perturbations. In Section 3.3 we drop the ultracontractivity assumption and prove stability of a type of kernel estimate that includes heat kernel estimates on complete Riemannian manifolds with Ricci curvature bounded below.

3.1 Stability of kernel bounds for ultracontractive propagators

Throughout this section we assume that (Ω, μ) is a σ-finite measure space and that

(A1) U is a positive strongly continuous propagator on $L_1(\mu)$ with parameter interval $J \subseteq \mathbb{R}$, and there is a constant $L \geqslant 1$ such that

$$\|U(t,s)\|_{1\to1} \leqslant L \qquad \big((t,s) \in D_J\big).$$

3.1 Remark. If U is only exponentially bounded then U can be rescaled to a propagator $(t,s) \mapsto e^{-\omega(t-s)}U(t,s)$ satisfying the bound in (A1). Observe that a potential V is (weakly) U-admissible if and only if V is (weakly) admissible with respect to the rescaled propagator. Moreover, a small Miyadera perturbation of U is also a small Miyadera perturbation of the rescaled propagator. (Make α in the definition smaller if necessary.) Thus, all the qualitative assertions of the subsequent results remain true in the more general case of exponentially bounded U.

We will also use the ultracontractivity assumption that

(A2) there exist constants $K, \nu > 0$ and $A \in \mathbb{R}$ such that

$$\|U(t,s)\|_{1\to\infty} \leqslant K(t-s)^{-\nu} e^{A(t-s)} \qquad \big((t,s) \in D'_J\big),$$

where $D'_J := \big\{(t,s) \in J \times J;\, t > s\big\}$.

3.2 Remarks. (a) In assumption (A2), only the case $A \leqslant 0$ is of actual interest. In fact, if (A1) holds and $\|U(t,s)\|_{1\to\infty} \leqslant K(t-s)^{-\nu}$ for all $(t,s) \in D'_J$ with $t-s \leqslant 1$, then for $(t,s) \in D'_J$ with $t-s > 1$ we have

$$\|U(t,s)\|_{1\to\infty} \leqslant \|U(t,t-1)\|_{1\to\infty} \|U(t-1,s)\|_{1\to 1} \leqslant KL.$$

With $C := KL$ and $f(t) := t^{-\nu} \vee 1$ we thus obtain that

(A2') $\|U(t,s)\|_{1\to\infty} \leqslant Cf(t-s)$ for all $(t,s) \in D'_J$

(which is slightly weaker than (A2) with $A = 0$, but much stronger than (A2) with $A > 0$).

More generally, let us briefly discuss the assumption (A2') with

$$f(t) := t^{-\nu_0} \ (0 < t \leqslant 1), \quad f(t) := t^{-\nu_1} \ (t > 1), \tag{3.1}$$

where $\nu_0, \nu_1 \geqslant 0$. (Above we had $\nu_0 = \nu,\ \nu_1 = 0$.) For heat propagators on manifolds, the different t-exponents for $t \leqslant 1$ and $t > 1$ are important; e.g., for a compact complete Riemannian manifold one has to take $\nu_1 = 0$. Our subsequent results can be adapted to this more general setting, but we confine ourselves to assumption (A2) for the sake of simplicity.

(b) By the Dunford-Pettis theorem, assumption (A2) implies that $U(t,s)$ is an integral operator, for each $(t,s) \in D'_J$, with a positive kernel $p_{t,s}$ bounded by $K(t-s)^{-\nu} e^{A(t-s)}$. Here, $p_{t,s}$ being a kernel of $U(t,s)$ means that

$$U(t,s)f(x) = \int_\Omega p_{t,s}(x,y) f(y)\, d\mu(y)$$

for all $f \in L_1(\mu)$ and a.e. $x \in \Omega$.

We start with the following result that requires only assumptions (A1) and (A2).

3.3 Proposition. *Assume (A1) and (A2). Let $p \in (1,\infty)$, and let V be a potential such that pV is weakly U-admissible. Assume that U_{pV} is exponentially bounded. Then V is weakly U-admissible, and there exist $c > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|U_V(t,s)\|_{1\to p} \leqslant c(t-s)^{-\nu/p'} e^{\omega(t-s)} \qquad \big((t,s) \in D'_J\big). \tag{3.2}$$

If V^- is a small Miyadera perturbation of U with constants (α,γ), $\gamma < 1$, then for all $1 \leqslant p < \frac{1}{\gamma}$ there exists $c > 0$ such that (3.2) holds with $\omega = \frac{1}{p\alpha} \ln\big(1 + \frac{p\gamma L}{1-p\gamma}\big) + (1 - \frac{1}{p})A$.

Proof. By Theorem 2.38 and Corollary 2.37 we obtain that V is weakly U-admissible and that

$$\|U_V(t,s)\|_{1\to p} \leqslant \|U(t,s)\|_{1\to\infty}^{1-1/p} \|U_{pV}(t,s)\|_{1\to 1}^{1/p} \tag{3.3}$$

for all $(t,s) \in D_J'$. The estimate (3.2) thus follows from (A2) and the exponential boundedness of U_{pV}.

Let now V^- be a small Miyadera perturbation of U with constants (α,γ), $\gamma < 1$, and let $1 \leqslant p < \frac{1}{\gamma}$. Then $(pV)^-$ is a small Miyadera perturbation of U with constants $(\alpha, p\gamma)$, so Proposition 1.14 yields

$$\|U_{pV}(t,s)\|_{1\to 1} \leqslant \tfrac{L}{1-p\gamma} e^{\omega_p (t-s)} \qquad \big((t,s) \in D_J\big),$$

where $\omega_p = \frac{1}{\alpha} \ln(1 + \frac{p\gamma}{1-p\gamma} L)$. By (3.3) this implies the second assertion. ☐

In order to obtain L_1–L_∞-estimates, we will need in addition that the propagator U is L_∞-bounded,

(A3) $\|U(t,s)\|_{\infty\to\infty} \leqslant L$ for all $(t,s) \in D_J$.

3.4 Remarks. (a) Let assumptions (A1) and (A3) be satisfied. Then $\|U(t,s)\|_{p\to p} \leqslant L$ for all $1 \leqslant p \leqslant \infty$ and all $(t,s) \in D_J$ by Riesz-Thorin interpolation. Thus, U extrapolates to a consistent family of bounded strongly continuous propagators U_p on $L_p(\mu)$, $1 \leqslant p < \infty$ (for $f \in L_1(\mu) \cap L_\infty(\mu)$, the continuity of $(t,s) \mapsto U_p(t,s)f$ can be obtained from the inequality $\|g\|_p \leqslant \|g\|_1^{1/p} \|g\|_\infty^{1-1/p}$, valid for all $g \in L_1(\mu) \cap L_\infty(\mu)$).

(b) In addition, let V be a potential such that V^- is a small Miyadera perturbation of U, with bound $\gamma < 1$. For $1 \leqslant p < \frac{1}{\gamma}$ and $(t,s) \in D_J$ we obtain by Theorem 2.38 and Corollary 2.37 that

$$\|U_V(t,s)\|_{p\to p} \leqslant \|U_{pV}(t,s)\|_{1\to 1}^{1/p} \|U(t,s)\|_{\infty\to\infty}^{1/p'}.$$

Thus, U_V extrapolates to a consistent family of exponentially bounded strongly continuous propagators $U_{p,V}$ on $L_p(\mu)$, $1 \leqslant p < \frac{1}{\gamma}$. It follows from Lemma 2.48(a) that V is U_p-admissible and $U_{p,V} = (U_p)_V$, for $1 \leqslant p < \frac{1}{\gamma}$. If additionally (A2) holds, then by Riesz-Thorin interpolation the above inequality together with (3.2) implies the estimates

$$\|U_{p,V}(t,s)\|_{p\to q} \leqslant C(t-s)^{-\nu(\frac{1}{p}-\frac{1}{q})} e^{\omega(t-s)} \qquad \big((t,s) \in D_J',\ 1 \leqslant p \leqslant q < \tfrac{1}{\gamma}\big), \tag{3.4}$$

where $C > 0$ and $\omega \in \mathbb{R}$ are constants depending on q.

In particular, if V^- is infinitesimally Miyadera small then the above holds with $\frac{1}{\gamma}$ replaced by ∞. This observation generalises [Gul02; Thm. 2]. In [Gul05; Thm. 3(b)], an example is presented showing that (3.4) need not hold for $p = q = \infty$, for infinitesimally Miyadera small V.

The following extrapolation lemma is a propagator version of [Cou90; Lemme 1], which deals with semigroups. The analogous proof is included for the reader's convenience.

3.5 Lemma. *Let W be a propagator on $L_1(\mu)$ with parameter interval J. Let $p \in (1, \infty)$ and $\nu > 0$, and assume that there exist $c, M > 0$ and $\omega, \omega_\infty \in \mathbb{R}$ such that*

$$\|W(t,s)\|_{1\to p} \leqslant c(t-s)^{-\nu/p'} e^{\omega(t-s)}, \quad \|W(t,s)\|_{\infty\to\infty} \leqslant M e^{\omega_\infty(t-s)} \qquad \big((t,s) \in D'_J\big).$$

Then there exists $c_1 > 0$ such that

$$\|W(t,s)\|_{1\to\infty} \leqslant c_1(t-s)^{-\nu} e^{\omega_1(t-s)} \qquad \big((t,s) \in D'_J\big), \tag{3.5}$$

where $\omega_1 = \max\{\omega, \omega_\infty\}$.

Proof. By rescaling W we can assume without loss of generality that $\omega, \omega_\infty \leqslant 0$. Let $t \in J$, $T > 0$ and $f \in L_1(\mu) \cap L_\infty(\mu)$ with $\|f\|_1 \leqslant 1$. Then

$$\varphi(s) := \|W(t,s)'f\|_\infty \leqslant \|W(t,s)'\|_{p'\to\infty} \|f\|_{p'} \leqslant c(t-s)^{-\nu/p'} \|f\|_{p'}$$

for all $s \in J_{<t}$, by the assumption on $\|W(t,s)\|_{1\to p}$, so there exists $c_0 \geqslant 0$ (depending on the quantities fixed above) such that

$$\varphi(s) \leqslant c_0(t-s)^{-\nu} \qquad \big(s \in [t-T, t) \cap J\big).$$

We choose the minimal c_0 making this estimate valid. Let $r \in [t-T, t) \cap J$, and let $s := \frac{r+t}{2}$. For $g := W(t,s)'f$ we can estimate

$$\|g\|_{p'} \leqslant \|g\|_1^{1/p'} \|g\|_\infty^{1/p} \leqslant \|W(t,s)'\|_{1\to1}^{1/p'} \varphi(s)^{1/p} \leqslant M^{1/p'} \big(c_0(t-s)^{-\nu}\big)^{1/p}.$$

Moreover, $\|W(s,r)'\|_{p'\to\infty} \leqslant c(s-r)^{-\nu/p'}$. Since $t-s = s-r = \frac{1}{2}(t-r)$, we infer that

$$\varphi(r) = \|W(s,r)'W(t,s)'f\|_\infty \leqslant \|W(s,r)'\|_{p'\to\infty} \|g\|_{p'} \leqslant cM^{1/p'} c_0^{1/p} 2^\nu (t-r)^{-\nu}.$$

From the choice of c_0 it follows that $c_0 \leqslant cM^{1/p'} c_0^{1/p} 2^\nu$ and hence $c_0 \leqslant c^{p'} M 2^{\nu p'}$. We thus have shown

$$\|W(t,s)'f\|_\infty \leqslant c^{p'} M 2^{\nu p'} (t-s)^{-\nu}$$

for all $(t,s) \in D'_J$ and all $f \in L_1(\mu) \cap L_\infty(\mu)$ with $\|f\|_1 \leqslant 1$, which implies the assertion. $\square$

3.6 Remark. In the situation of Lemma 3.5 assume that $\omega \neq \omega_\infty$. Then by a suitable modification of the above proof one can show that for any $\omega_1 > \min\{\omega, \omega_\infty\}$ there exists $c_1 > 0$ such that (3.5) holds. The following simple argument yields a better estimate for large $t - s$.

By Lemma 3.5 and the Riesz-Thorin interpolation theorem we obtain $C > 0$ such that $\|W(t, t-1)\|_{p\to\infty} \leqslant C$ for all $t \in J$ with $t - 1 \in J$. It follows that there exists $\widetilde{C} > 0$ such that for $(t,s) \in D_J$ with $t - s \geqslant 2$ we have

$$\begin{aligned} \|W(t,s)\|_{1\to\infty} &\leqslant \|W(t,t-1)\|_{p\to\infty} \|W(t-1,s)\|_{1\to p} \\ &\leqslant Cc(t-1-s)^{-\nu/p'} e^{\omega(t-s-1)} \leqslant \widetilde{C}(t-s)^{-\nu/p'} e^{\omega(t-s)}. \end{aligned}$$

If $\omega_\infty < \omega$ then the estimate

$$\|W(t,s)\|_{1\to\infty} \leqslant \|W(t,s+1)\|_{\infty\to\infty} \|W(s+1,s)\|_{1\to\infty} \leqslant C_1 M e^{-\omega_\infty} e^{\omega_\infty(t-s)}$$

is better for large $t - s$, where $C_1 := \sup\big\{\|W(s+1,s)\|_{1\to\infty};\ s \in J,\ s+1 \in J\big\}$. In the same way, if $\omega_1 \in \mathbb{R}$ is such that $\|W(t,s)\|_{1\to1} \leqslant Me^{\omega_1(t-s)}$ for all $(t,s) \in D_J$, then $\|W(t,s)\|_{1\to\infty} \leqslant C_1 M e^{-\omega_1} e^{\omega_1(t-s)}$ for $t - s \geqslant 1$.

As an immediate consequence of Proposition 3.3 and Lemma 3.5 we obtain the following result which strengthens [LVV06; Prop. 3.6]. Besides the weaker qualitative assumptions, the difference is that the condition on the L_∞-bound is posed on the propagator U_V itself, not on U_{pV} for some $p > 1$ as the condition on the L_1-bound. This is parallel to [ArDe06; Thm. 3.3], where also a variant of [Cou90; Lemme 1] is used.

3.7 Proposition. *Assume (A1) to (A3). Let $p \in (1, \infty)$, and let V be a potential such that pV is weakly U-admissible. Assume that U_{pV} is exponentially bounded and that U_V is exponentially L_∞-bounded. Then there exist $c > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|U_V(t,s)\|_{1\to\infty} \leqslant c(t-s)^{-\nu} e^{\omega(t-s)} \qquad \big((t,s) \in D_J'\big). \tag{3.6}$$

3.8 Remark. Assume that V^- is a small Miyadera perturbation and a small backward Miyadera perturbation of U, both with constants (α, γ), $\gamma < 1$. Then one can use Proposition 3.3 and an analogous estimate for $\|U_V(t,s)\|_{p'\to\infty}$ to show as in [LVV06; Prop. 3.6] that for $\theta \in (\gamma, 1)$ there exists $c > 0$ such that (3.6) holds with $\omega = \frac{\theta}{\alpha} \ln\big(1 + \frac{\gamma L}{\theta - \gamma}\big) + (1-\theta)A$.

More strongly than in (A2) we now assume that the integral kernels $p_{t,s}$ of the operators $U(t,s)$ satisfy the Gaussian type upper bound

$$p_{t,s}(x,y) \leqslant K(t-s)^{-\nu} e^{A(t-s) - \psi_{t,s}(x,y)} \qquad \big((t,s) \in D_J',\ x, y \in \Omega\big), \tag{3.7}$$

with constants $K, \nu > 0$, $A \in \mathbb{R}$ and measurable functions $\psi_{t,s} \colon \Omega \times \Omega \to [0, \infty)$.

3.9 Remark. If $\Omega = \mathbb{R}^n$, $\nu = \frac{n}{2}$, $A = 0$ and $\psi_{t,s}(x,y) = a\frac{d(x,y)^2}{t-s}$ for some $a > 0$, then (3.7) becomes the classical Gaussian bound. By the results of [Aro68], this bound holds for the fundamental solution of (0.1).

Let now Ω be a complete n-dimensional Riemannian manifold with Ricci curvature bounded below, μ the Riemannian volume and d the Riemannian distance. Assume that $\inf_{x\in\Omega} \mu(B(x,1)) > 0$. Then it follows from [Dav93; Thm. 3 and bottom of p. 3] that a modified form of the above Gaussian bound also holds for the heat kernel on Ω: The term $(t-s)^{-n/2}$ has to be replaced by $(t-s)^{-n/2} \vee 1$ (cf. Remark 3.2(a)), and one can choose $A = -\lambda_0$ and any $0 < a < \frac{1}{4}$, where $\lambda_0 \geqslant 0$ is the bottom of the spectrum of the Laplace-Beltrami operator in $L_2(\Omega, \mu)$. In particular, for a negatively curved manifold one can have $A < 0$. However, without a uniform lower bound on the volume of unit balls, not even the ultracontractivity bound (A2) is valid. We will show in Section 3.3 how our method can be adapted to cover this more general case, too.

After the above preparations we easily obtain the following result on stability of upper kernel bounds.

3.10 Theorem. *Let U satisfy (A1), (A3) and the Gaussian type upper bound* (3.7)*. Let $p \in (1, \infty)$, and let V be a potential such that pV is weakly U-admissible. Assume that U_{pV} is exponentially bounded and exponentially L_∞-bounded. Then the operators $U_V(t,s)$ are integral operators, and for $\frac{1}{p} < \beta < 1$ there exist $c > 0$ and $\omega \in \mathbb{R}$ such that the kernels $p_{t,s}^V$ satisfy*

$$p_{t,s}^V \leqslant c(t-s)^{-\nu} e^{\omega(t-s) - (1-\beta)\psi_{t,s}} \qquad \big((t,s) \in D_J'\big).$$

Proof. We have $\frac{1}{\beta} < p$, so by Proposition 3.7 there exist $c_1 > 0$ and $\omega_1 \in \mathbb{R}$ such that

$$\|U_{V/\beta}(t,s)\|_{1\to\infty} \leqslant c_1(t-s)^{-\nu}e^{\omega_1(t-s)} \qquad \big((t,s)\in D_J'\big).$$

From Corollary 2.39 we obtain that $p_{t,s}^V \leqslant \big(p_{t,s}^{V/\beta}\big)^\beta (p_{t,s})^{1-\beta}$. Thus, by (3.7) the assertion follows with $\omega = \beta\omega_1 + (1-\beta)A$. □

In the following, a Miyadera perturbation V of U will also be called a *forward* Miyadera perturbation.

3.11 Remarks. (a) Assume that V^- is a small forward and backward Miyadera perturbation of U, both with constants (α,γ), $\gamma < 1$. Then the assumptions on V in Theorem 3.10 are satisfied for all $p \in (1, \frac{1}{\gamma})$, by Theorem 2.24, Proposition 1.14 and Theorem 2.30. In this situation one can use Remark 3.8 to show that for $\gamma < \theta < \beta < 1$ there exists $c > 0$ such that the kernel estimate of Theorem 3.10 holds with $\omega = \frac{\theta}{\alpha}\ln\big(1+\frac{\gamma L}{\theta-\gamma}\big) + (1-\theta)A$; cf. [LVV06; Thm. 3.10].

Assume that $\Omega = \mathbb{R}^n$, that U is stochastic and Markovian and that U satisfies a classical Gaussian upper bound as in Remark 3.9. This is the case, e.g., if U corresponds to the Cauchy problem for (0.1). Then for $V \leqslant 0$ the conditions on V in the previous paragraph are close to necessary for the assertion of Theorem 3.10. Indeed, if U_V satisfies a Gaussian upper bound, then U_V is exponentially bounded and exponentially L_∞-bounded, so from Corollary 2.26(a) and Remark 2.44(c) it follows that V is a (not necessarily small) forward and backward Miyadera perturbation of U.

(b) If V^- is an infinitesimally small forward and backward Miyadera perturbation of U then one can choose any $\beta > 0$ in Theorem 3.10, and we obtain sharp upper bounds for $t - s \leqslant 1$. In applications to second-order parabolic equations this observation shows the difference between the non-autonomous Kato class $\mathbf{NK}$ and the enlarged non-autonomous Kato class $\widehat{\mathbf{NK}}$.

(c) Assume that (3.7) holds with $A = 0$; we call this a global upper bound. For questions of long time behaviour it is important to know under what conditions the global upper bound is stable under perturbation by a potential V. By part (a) this is the case if V^- is a forward and backward Miyadera perturbation of U with constants (∞,γ), $\gamma < 1$.

With the same methods as above we obtain the following result about stability of the lower bound of the integral kernels, in which regularity of the perturbation V plays a crucial role.

3.12 Theorem. *Let U satisfy (A1) to (A3) and the Gaussian type lower bound*

$$p_{t,s}(x,y) \geqslant K_1(t-s)^{-\nu}e^{-A_1(t-s)-\psi_{t,s}(x,y)} \qquad \big((t,s)\in D_J',\ x,y\in\Omega\big), \tag{3.8}$$

with some $K_1 > 0$, $A_1 \in \mathbb{R}$ and measurable functions $\psi_{t,s}\colon \Omega\times\Omega \to [0,\infty)$. Let V be a U-regular potential. Let $\varepsilon > 0$, and assume that $-\varepsilon V$ is weakly U-admissible and that $U_{-\varepsilon V}$ is exponentially bounded and exponentially L_∞-bounded. Then for $\beta > \frac{1}{\varepsilon}$ there are constants $c > 0$ and $\omega \in \mathbb{R}$ such that the kernels $p_{t,s}^V$ of the operators $U_V(t,s)$ satisfy

$$p_{t,s}^V \geqslant c(t-s)^{-\nu}e^{-\omega(t-s)-(1+\beta)\psi_{t,s}} \qquad \big((t,s)\in D_J'\big).$$

Proof. We have $\frac{1}{\beta} < \varepsilon$, so by Proposition 3.7 there exist $c_1 > 0$ and $\omega_1 \in \mathbb{R}$ such that

$$\|U_{-V/\beta}(t,s)\|_{1\to\infty} \leqslant c_1(t-s)^{-\nu}e^{\omega_1(t-s)} \qquad \big((t,s)\in D_J'\big). \tag{3.9}$$

Recall from Lemma 2.20 that V^+ and V^- are U-regular. Therefore, $V_0^+ \wedge V^-$ and $V_0^- \wedge V^+$ are U-regular by Corollary 2.16, and Corollary 2.39 yields

$$p_{t,s} \leqslant \big(p_{t,s}^{-V/\beta}\big)^{\frac{\beta}{1+\beta}}\big(p_{t,s}^{V}\big)^{\frac{1}{1+\beta}}, \quad \text{so} \quad p_{t,s}^{V} \geqslant \big(p_{t,s}^{-V/\beta}\big)^{-\beta}(p_{t,s})^{1+\beta}$$

for all $(t,s) \in D_J'$. The asserted lower bound thus follows from (3.8) and (3.9), with $\omega = \beta\omega_1 + (1+\beta)A_1$. ☐

Observations similar to those in Remark 3.11 also apply to the lower bounds.

3.13 Remarks. (a) If V is weakly U-admissible and V^+ is a forward and backward Miyadera perturbation of U, both with constants (α,γ), $\gamma \geqslant 0$, then the assumptions on V in Theorem 3.12 are satisfied for all $\varepsilon \in (0,\frac{1}{\gamma})$, and one can use Remark 3.8 to show that for $\beta > \theta > \gamma$ there exists $c > 0$ such that the kernel estimate of Theorem 3.12 holds with $\omega = \frac{\theta}{\alpha}\ln\big(1+\frac{\gamma L}{\theta-\gamma}\big) + (\beta-\theta)A + (1+\beta)A_1$; cf. [LVV06; Thm. 3.12].

Assume that $\Omega = \mathbb{R}^n$ and that U satisfies a classical Gaussian lower bound (cf. Remark 3.9). Then for $V \geqslant 0$ the above conditions on V are necessary for the validity of a Gaussian lower bound for U_V. Indeed, one easily sees that there exists $c \in (0,1]$ such that $\|U_V(t,s)f\|_1 \geqslant c\|f\|_1$ and $\widetilde{U}_V(t,s)\mathbb{1} \geqslant c\mathbb{1}$ for all $(t,s) \in D_J$ with $t-s \leqslant 1$ and all $0 \leqslant f \in L_1(\mu)$, where $\widetilde{U}_V$ is defined as in Remark 2.44(c). From Proposition 2.25(a) and Remark 2.44(d) it thus follows that V is a forward and backward Miyadera perturbation of U. We point out that this argument does not require U to be stochastic or Markovian.

(b) If V is weakly U-admissible and V^+ is an infinitesimally small forward and backward Miyadera perturbation of U, then one can choose any $\beta > 0$ in Theorem 3.12, and we obtain sharp lower bounds for $t-s \leqslant 1$.

(c) Assume that $A = 0$ in the ultracontractivity bound (A2) and $A_1 = 0$ in (3.8). Then by part (a), the global lower bound is stable under perturbation by the potential V if V^+ is a forward and backward Miyadera perturbation of U with constants (∞,γ), with any $\gamma \geqslant 0$.

3.2 Stability of the boundary behaviour

Let Ω be a bounded connected open subset of $\mathbb{R}^n$ with $C^{1,\alpha}$-boundary, let $T > 0$, and let U be the propagator with parameter interval $[0,T)$ associated with the uniformly parabolic equation (0.4) with Dini continuous coefficients. Recall from the introduction that by [Cho06] there exist $c, a > 0$ such that the kernels $p_{t,s}$ of $U(t,s)$ satisfy the Gaussian type upper bound

$$p_{t,s}(x,y) \leqslant c(t-s)^{-n/2}\bigg(\frac{d(x)d(y)}{t-s}\wedge 1\bigg)\cdot\exp\Big(-a\frac{|x-y|^2}{t-s}\Big) \tag{3.10}$$

for all $(t,s) \in D_J$ and a.e. $x, y \in \Omega$, where $d(x)$ denotes the distance from x to the boundary of Ω, and also an analogous lower bound.

The propagator U is ultracontractive, so one can apply Theorem 3.10 in this situation; however, this does not yield stability of the kernel estimate: It leads to the boundary term $\frac{d(x)d(y)}{t-s} \wedge 1$ being raised to the power $1-\beta$. In fact, one can show that without stronger assumptions on the perturbation V, this change of the boundary behaviour of the kernel is possible. It will turn out that for stability of (3.10) one has to assume weak admissibility of V in a suitable weighted space. We will now study this type of kernel estimate in a more general framework.

For the remainder of the section let (Ω, μ) be a σ-finite measure space and $d\colon \Omega \times \Omega \to [0,\infty)$ a measurable semi-metric on Ω (see also Remark 3.14(a) below). We assume that the volume growth condition

$$\mu(B(x,r)) \leqslant Cr^n e^{\kappa r} \qquad (x \in \Omega,\ r > 0) \tag{3.11}$$

is satisfied for some $C, n > 0$ and $\kappa \geqslant 0$, where $B(x,r)$ denotes the open ball with centre x and radius r with respect to d. Moreover, let U be a positive strongly measurable propagator on $L_1(\mu)$ with parameter interval $J \subseteq \mathbb{R}$. We assume that for each $(t,s) \in D_J'$ the operator $U(t,s)$ is an integral operator whose kernel $p_{t,s}$ satisfies the Gaussian type estimate

$$p_{t,s}(x,y) \leqslant c_0(t-s)^{-n/2} \prod_{j=1}^m \Big(\frac{d_j(x)d_j(y)}{t-s} \wedge 1 \Big)^{\alpha_j} \cdot \exp\Big(-a\frac{d(x,y)^2}{t-s}\Big) \tag{3.12}$$

for a.e. $x, y \in \Omega$, where n is the constant from (3.11), $m \in \mathbb{N}$, $c_0, a, \alpha_j > 0$ for $j = 1, \dots, m$, and $d_j\colon \Omega \to (0,\infty)$ is a d-Lipschitz continuous function with Lipschitz constant 1 for each $j = 1, \dots, m$.

3.14 Remarks. (a) The measurability of the semi-metric d is meant with respect to the product σ-algebra on $\Omega \times \Omega$, so it implies that $d(x, \cdot)$ is measurable on Ω for all $x \in \Omega$. Therefore, the open balls $B(x,r)$ in (3.11) are measurable sets. The setting we chose here is more flexible than that of metric measure spaces, i.e., metric spaces with a measure on the Borel σ-algebra. It allows, e.g., to take a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lebesgue measure and the discrete metric. In this (admittedly pathological) example, the Borel σ-algebra is the power set of Ω, so one does not obtain a metric measure space.

(b) For the validity of (3.11) it is necessary that μ is σ-finite.

(c) Condition (3.11) is chosen in such a way that it comprises both the volume growth of $\mathbb{R}^n$ and exponential volume growth. For the case of exponential volume growth it would be more natural to assume $\mu(B(x,r)) \leqslant Ce^{\kappa r}$ for $r > 1$, but we want to keep the notational effort to a minimum.

(d) We have assumed for simplicity that the d_j are everywhere strictly positive. The case that $d_j > 0$ a.e. can easily be accommodated by replacing Ω with $\bigcap_{j=1}^m [d_j > 0]$.

(e) We observe that the upper bound in (3.12) is not changed essentially if the terms $\frac{d_j(x)d_j(y)}{t-s} \wedge 1$ are replaced with $\big(\frac{d_j(x)}{\sqrt{t-s}} \wedge 1\big)\big(\frac{d_j(y)}{\sqrt{t-s}} \wedge 1\big)$. Indeed, for $a, b \geqslant 0$ one easily shows the estimate

$$(a \wedge 1)(b \wedge 1) \leqslant (ab) \wedge 1 \leqslant (1 + |a-b|)(a \wedge 1)(b \wedge 1)$$

(in the case $a < 1 < b$ use that $ab \wedge 1 \leqslant a(b-a+1)$), and for $a = \frac{d_j(x)}{\sqrt{t-s}}$ and $b = \frac{d_j(y)}{\sqrt{t-s}}$ one obtains that $|a-b| \leqslant \frac{d(x,y)}{\sqrt{t-s}} \leqslant c_\varepsilon \exp\big(\varepsilon \frac{d(x,y)^2}{t-s}\big)$.

3.15 Example. Let $n \geqslant 2$, let Ω be an open subset of $\mathbb{R}^n$, and let T be the C_0-semigroup on $L_1(\Omega)$ associated with the heat equation $\partial_t u = \Delta u$ on $[0,\infty) \times \Omega$ with homogeneous Dirichlet boundary condition at $\partial\Omega$.

(a) Let $\Omega := (0,\infty)^n$. Then it is easy to see that, for $t > 0$, the kernel k_t of $T(t)$ is given by

$$k_t(x,y) = (4\pi t)^{-n/2} \prod_{j=1}^{n} \bigl(1 - e^{-x_j y_j/t}\bigr) \cdot \exp\Bigl(-\frac{|x-y|^2}{4t}\Bigr) \qquad (x, y \in \Omega)$$

(cf. [Dav89; Examples 4.1.1 and 4.1.2]). Since $\frac{1}{2}(r \wedge 1) \leqslant 1 - e^{-r} \leqslant r \wedge 1$ for all $r > 0$, the kernels k_t satisfy (an autonomous version of) the kernel bound (3.12) and an analogous lower bound, with $m = n$, $\alpha_j = 1$ and $d_j(x) = x_j$ for $j = 1, \ldots, n$ and $x \in \Omega$.

(b) Let Ω be a conical domain, i.e., $\Omega = \bigl\{r\xi;\, r > 0,\ \xi \in \Sigma\bigr\}$, where Σ is a proper open subset of the unit sphere S_{n-1}. We assume that Σ has C^2-boundary; then the ground state $\varphi_0 \geqslant 0$ of the spherical Dirichlet Laplacian on Σ satisfies $c^{-1} d_\Sigma \leqslant \varphi_0 \leqslant c d_\Sigma$ for some $c \geqslant 1$, where $d_\Sigma(\xi)$ denotes the distance of $\xi \in \Sigma$ from $\partial\Sigma$; see, e.g., [Var99; formula (0.2.1)]. If $\lambda_0 > 0$ denotes the ground state energy and α is the positive solution of $\alpha(\alpha + n - 2) = \lambda_0$, then $h(x) := |x|^\alpha \varphi_0(\frac{x}{|x|})$ defines a positive harmonic function on Ω that vanished at $\partial\Omega$, the so-called réduite of Ω; cf. [Var99; p. 335]. By the above, there exists $C \geqslant 1$ such that

$$C^{-1}|x|^{\alpha-1} d(x) \leqslant h(x) \leqslant C|x|^{\alpha-1} d(x) \qquad (x \in \Omega),$$

where $d(x)$ denotes the distance from x to $\partial\Omega$.

We further suppose that $\alpha \geqslant 1$ (which is satisfied, e.g., if Ω is convex) and that $e_n \in \Omega$ and $-e_n \notin \overline{\Omega}$, where e_n denotes the n-th unit vector. The latter implies that $\partial\Omega$ is the graph of a Lipschitz function $\Phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$. Under these assumptions it is straightforward to show the estimate

$$C_1^{-1}\Bigl(\frac{|x|}{\sqrt{t}} \wedge 1\Bigr)^{\alpha-1}\Bigl(\frac{d(x)}{\sqrt{t}} \wedge 1\Bigr) \leqslant \frac{h(x)}{h(x + \sqrt{t}\, e_n)} \leqslant C_1\Bigl(\frac{|x|}{\sqrt{t}} \wedge 1\Bigr)^{\alpha-1}\Bigl(\frac{d(x)}{\sqrt{t}} \wedge 1\Bigr)$$

for all $x \in \Omega$ and all $t > 0$, with a constant $C_1 \geqslant 1$. From [GySa09; Cor. 6.14 and Sec. 6.5.2] we now conclude, taking into account Remark 3.14(e), that the kernel k_t of $T(t)$ satisfies (an autonomous version of) the kernel bound (3.12) and an analogous lower bound, for all $t > 0$. Here one takes $m = 2$, $\alpha_1 = \alpha - 1$, $d_1(x) = |x|$, $\alpha_2 = 1$ and $d_2(x) = d(x)$, and $d(x,y)$ is the intrinsic geodesic distance of $x, y \in \Omega$ ($d(x,y) = |x-y|$ if Ω is convex).

We are going to study under what conditions the kernel estimate (3.12) is stable under perturbation by a potential $V\colon J \times \Omega \to \mathbb{R}$; for a precise statement see Theorem 3.20 below. As a preparation we show that U extrapolates to a scale of weighted L_1-spaces. In the following, a measurable function $\rho\colon \Omega \to (0,\infty)$ will be called a *weight*.

3.16 Proposition. *Let a and the α_j be as in* (3.12)*, and let ρ be a weight satisfying*

$$\frac{\rho(x)}{\rho(y)} \leqslant c \prod_{j=1}^{m} \Bigl(\frac{d_j(x)}{d_j(y)} \vee \frac{d_j(y)}{d_j(x)}\Bigr)^{\alpha_j} \qquad (x, y \in \Omega) \qquad (3.13)$$

for some $c \geqslant 1$. Then for all $a_1 < a$ there exists $M \geqslant 1$ such that

$$\|\rho U(t,s)\rho^{-1}\|_{p\to p} \leqslant Me^{\omega(t-s)} \qquad \big((t,s) \in D_J,\ 1 \leqslant p \leqslant \infty\big)$$

holds with $\omega = \frac{\kappa^2}{4a_1}$. Moreover, U extrapolates to an exponentially bounded strongly measurable propagator U_ρ on $L_1(\rho\mu)$, and U_ρ is bounded if $\kappa = 0$. If U is strongly continuous then so is U_ρ.

The above proposition implies in particular that U is exponentially bounded, and bounded if $\kappa = 0$. In the proof of the proposition we will use the following lemma in which the volume growth condition (3.11) is essentially reformulated as an integral condition.

3.17 Lemma. *Let $y \in \Omega$ and $n > 0$. If*

$$\mu(B(y,r)) \leqslant c_1 r^n e^{\kappa r} \qquad (r > 0) \tag{3.14}$$

for some $c_1 > 0$ and $\kappa \geqslant 0$, then for all $\theta \in (0,1)$ there exists $c_2 > 0$ such that

$$\int_\Omega e^{-d(x,y)^2/t}\, d\mu(x) \leqslant c_2 t^{n/2} e^{\omega t} \qquad (t > 0) \tag{3.15}$$

holds with $\omega = \frac{\kappa^2}{4\theta}$, in particular, $\omega = 0$ if $\kappa = 0$. Conversely, if (3.15) holds for some $c_2 > 0$ and $\omega \geqslant 0$, then (3.14) holds with $c_1 = ec_2$ and $\kappa = 2\sqrt{\omega}$.

Proof. Assume that (3.14) holds, without loss of generality with $c_1 = 1$, and let $m(r) := r^n e^{\kappa r}$ for all $r \geqslant 0$. Then for any decreasing function $f\colon [0,\infty) \to [0,\infty)$ we can estimate

$$\int_\Omega f(d(x,y))\, d\mu(x) \leqslant \int_0^\infty f(r)\, dm(r).$$

(This is clear if $f = \mathbb{1}_{[0,r)}$ for some $r > 0$; for general f it follows by a superposition argument.) Given $t > 0$ we thus obtain, substituting $r = \sqrt{ts}$ and integrating by parts, that

$$\int_\Omega e^{-d(x,y)^2/t}\, d\mu(x) \leqslant \int_0^\infty e^{-r^2/t}\, dm(r) = \int_0^\infty e^{-s}\, dm(\sqrt{ts}) = \int_0^\infty m(\sqrt{ts})e^{-s}\, ds.$$

Let $\theta \in (0,1)$. Then $\kappa\sqrt{ts} \leqslant \frac{\kappa^2}{4\theta}t + \theta s$ for all $s > 0$ and hence $m(\sqrt{ts}) \leqslant (ts)^{n/2}e^{\omega t + \theta s}$, with $\omega = \frac{\kappa^2}{4\theta}$. We conclude that

$$\int_\Omega e^{-d(x,y)^2/t}\, d\mu(x) \leqslant t^{n/2}e^{\omega t}\int_0^\infty s^{n/2}e^{-(1-\theta)s}\, ds = \Gamma(\tfrac{n}{2}+1)(1-\theta)^{-n/2-1}t^{n/2}e^{\omega t}.$$

Assume now that (3.15) holds, and let $r > 0$. Then for all $t > 0$ we have

$$\mu(B(y,r)) \leqslant \int_\Omega e^{(r^2 - d(x,y)^2)/t}\, d\mu(x) \leqslant e^{r^2/t}\cdot c_2 t^{n/2}e^{\omega t}.$$

In the case $r < \omega^{-1/2}$ we choose $t = r^2$. Then $\omega t = \omega r^2 \leqslant \sqrt{\omega}r$ and hence

$$\mu(B(y,r)) \leqslant ec_2 r^n e^{\sqrt{\omega}r}.$$

In the case $\omega > 0$ and $r \geqslant \omega^{-1/2}$ we choose $t = r\omega^{-1/2}$. Then $t \leqslant r^2$, $r^2/t + \omega t = 2\sqrt{\omega}r$, and thus

$$\mu(B(y,r)) \leqslant c_2 r^n e^{2\sqrt{\omega}r}. \qquad \square$$

3.18 Remarks. (a) Let $k\colon \Omega \times \Omega \to [0,\infty)$ be measurable. It is easy to see that

$$Bf(x) := \int_\Omega k(x,y)f(y)\,d\mu(y) \qquad \big(f \in L_1(\mu)\big)$$

defines a bounded operator on $L_1(\mu)$ if and only if

$$b := \operatorname*{ess\,sup}_{y\in\Omega} \int_\Omega k(x,y)\,d\mu(x) < \infty,$$

and that then $\|B\| = b$. Moreover,

$$\|B\|_{\infty\to\infty} = \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega k(x,y)\,d\mu(y) \ \big(\in [0,\infty]\big).$$

For $\theta \in (0,1)$ we thus obtain by Lemma 3.17 that there exists $c_2 > 0$ such that the following holds: If $a, t > 0$ and $k(x,y) \leqslant t^{-n/2}e^{-ad(x,y)^2/t}$ for a.e. $x, y \in \Omega$, then $\|B\| \leqslant c_2 a^{-n/2}e^{\omega t}$ and $\|B\|_{\infty\to\infty} \leqslant c_2 a^{-n/2}e^{\omega t}$, with $\omega = \frac{\kappa^2}{4\theta a}$.

(b) Let k and B be as above, and let ρ be a weight. If B extrapolates to a bounded operator B_ρ on the weighted space $L_1(\rho\mu)$, then the kernel k_ρ of B_ρ is given by $k_\rho(x,y) = k(x,y)\rho(y)^{-1}$. It follows that

$$b_\rho := \|B\colon L_1(\rho\mu) \to L_1(\rho\mu)\| = \operatorname*{ess\,sup}_{y\in\Omega} \int_\Omega \frac{\rho(x)}{\rho(y)} k(x,y)\,d\mu(x),$$

and as in (a) we obtain that $b_\rho \leqslant c_2 a^{-n/2}e^{\omega t}$ if $\frac{\rho(x)}{\rho(y)}k(x,y) \leqslant t^{-n/2}e^{-ad(x,y)^2/t}$ for a.e. $x, y \in \Omega$ and some $a, t > 0$.

Proof of Proposition 3.16. Let $(t,s) \in D_J'$. First observe that for all $x, y \in \Omega$ and $j = 1, \dots, m$ we have

$$\frac{d_j(x)}{d_j(y)}\Big(\frac{d_j(x)d_j(y)}{t-s} \wedge 1\Big) = \frac{d_j(x)^2}{t-s} \wedge \frac{d_j(x)}{d_j(y)} \leqslant 3\Big(1 + \frac{d(x,y)^2}{t-s}\Big).$$

Indeed, in the case $d_j(x) \leqslant 3d_j(y)$ the inequality is clear, and in the case $d_j(x) > 3d_j(y)$ the Lipschitz continuity of d_j yields $d(x,y) \geqslant d_j(x) - d_j(y) \geqslant \frac{2}{3}d_j(x)$ and hence $\frac{d_j(x)^2}{t-s} \leqslant \frac{9}{4}\frac{d(x,y)^2}{t-s}$. Interchanging the roles of x and y we obtain that

$$\Big(\frac{d_j(x)}{d_j(y)} \vee \frac{d_j(y)}{d_j(x)}\Big)\Big(\frac{d_j(x)d_j(y)}{t-s} \wedge 1\Big) \leqslant 3\Big(1 + \frac{d(x,y)^2}{t-s}\Big).$$

From (3.12) and (3.13) we conclude that for $\theta \in (0,1)$ there exists $c_\theta > 0$ such that

$$\begin{aligned}\frac{\rho(x)}{\rho(y)}p_{t,s}(x,y) &\leqslant c \cdot c_0(t-s)^{-n/2}\prod_{j=1}^m 3^{\alpha_j}\Big(1 + \frac{d(x,y)^2}{t-s}\Big)^{\alpha_j} \cdot \exp\Big(-a\frac{d(x,y)^2}{t-s}\Big) \\ &\leqslant c_\theta(t-s)^{-n/2}\exp\Big(-\theta a\frac{d(x,y)^2}{t-s}\Big)\end{aligned} \tag{3.16}$$

for a.e. $x, y \in \Omega$ and all $(t,s) \in D_J'$. The first assertion now follows from Remark 3.18(a) and the Riesz-Thorin interpolation theorem. In particular, U extrapolates to an exponentially bounded propagator U_ρ on $L_1(\rho\mu)$ (cf. Remark 3.18(b)), and U_ρ is bounded if $\kappa = 0$.

Given $f \in L_1(\rho\mu) \cap L_1(\mu)$ we now show that $(t,s) \mapsto U_\rho(t,s)f \in L_1(\rho\mu)$ is separately measurable; then it follows that U_ρ is a strongly measurable propagator. For $(t,s) \in D_J$ we have $U(t,s)f \in L_1(\rho\mu)$ and hence $\rho U(t,s)f \in L_1(\mu)$, so $(\rho \wedge n)U(t,s)f \to \rho U(t,s)f$ in $L_1(\mu)$ as $n \to \infty$. Since U is separately strongly measurable, this implies the separate measurability of $(t,s) \mapsto \rho U(t,s)f \in L_1(\mu)$ and hence of $(t,s) \mapsto U(t,s)f \in L_1(\rho\mu)$.

Now assume that U is strongly continuous. We first show that U_ρ is strongly continuous in (t_0,t_0), for all $t_0 \in J$. Observe that the set

$$F := \big\{ f \in L_1(\mu);\ \operatorname{spt} f \text{ is bounded},\ d_j^{\pm 1} \text{ is bounded on } \operatorname{spt} f\ (j = 1,\dots,m) \big\}$$

is dense in $L_1(\rho\mu)$. It thus suffices to show that $U(t,s)f \to U(t_0,t_0)f = f$ in $L_1(\rho\mu)$ as $(t,s) \to (t_0,t_0)$, for all $f \in F$. Given $f \in F$, we obtain from the Lipschitz continuity of the d_j that there exists $\varepsilon > 0$ such that ρ is bounded on $B := \big\{x \in \Omega;\ \operatorname{dist}(x, \operatorname{spt} f) \leqslant \varepsilon\big\}$. Therefore, the strong continuity of U implies that $\mathbb{1}_B U(t,s)f \to \mathbb{1}_B f = f$ in $L_1(\rho\mu)$ as $(t,s) \to (t_0,t_0)$, and it remains to show that $\mathbb{1}_{\Omega\setminus B} U(t,s)f \to 0$ in $L_1(\rho\mu)$.

For $x \in \Omega \setminus B$ and $y \in \operatorname{spt} f$ we have $d(x,y) \geqslant \varepsilon$ and hence $d(x,y)^2 \geqslant \frac{1}{2}d(x,y)^2 + \frac{1}{2}\varepsilon^2$, so by (3.16) we obtain that

$$\frac{\rho(x)}{\rho(y)} \mathbb{1}_{\Omega\setminus B}(x) p_{t,s}(x,y) \mathbb{1}_{\operatorname{spt} f}(y) \leqslant \exp\Big(-\frac{a\varepsilon^2}{4(t-s)}\Big) \cdot c_{1/2}(t-s)^{-n/2} \exp\Big(-\frac{a}{4}\frac{d(x,y)^2}{t-s}\Big).$$

By Remark 3.18(b) we infer that there exist $M > 0$ and $\omega \geqslant 0$ such that

$$\|\mathbb{1}_{\Omega\setminus B} U(t,s)f\|_{L_1(\rho\mu)} \leqslant \exp\Big(-\frac{a\varepsilon^2}{4(t-s)}\Big) \cdot Me^{\omega(t-s)} \|f\|_{L_1(\rho\mu)} \to 0$$

as $t - s \to 0$.

Finally, given $\varepsilon \in (0,1)$, we show that U_ρ is strongly continuous on

$$D_J^\varepsilon := \big\{(t,s) \in D_J;\ \varepsilon \leqslant t - s \leqslant \tfrac{1}{\varepsilon}\big\}.$$

Let $f \in L_1(\rho\mu) \cap L_1(\mu)$. Then the continuity of $D_J^\varepsilon \ni (t,s) \mapsto U(t,s)f$ in $L_1(\rho\mu)$ can be deduced from the continuity in $L_1(\mu)$ by means of the dominated convergence theorem if there exists $g \in L_1(\rho\mu)$ such that $|U(t,s)f| \leqslant g$ for all $(t,s) \in D_J^\varepsilon$. By (3.16) we obtain that

$$\frac{\rho(x)}{\rho(y)} p_{t,s}(x,y) \leqslant c_{1/2}\varepsilon^{-n/2} \exp\big(-\tfrac{\varepsilon a}{2} d(x,y)^2\big) =: k(x,y)$$

for all $(t,s) \in D_J^\varepsilon$ and a.e. $x, y \in \Omega$, so

$$|U(t,s)f(x)| \leqslant \rho(x)^{-1} \cdot \int_\Omega k(x,y)|\rho f|(y)\, d\mu(y) =: g(x)$$

for all $(t,s) \in D_J^\varepsilon$ and a.e. $x \in \Omega$. This completes the proof of the strong continuity of U_ρ since $g \in L_1(\rho\mu)$ by Remark 3.18(a). □

From now on we assume that U is strongly continuous. Moreover, let $\rho_0 := \prod_{j=1}^m d_j^{\alpha_j}$ in the following. Then ρ_0 is a weight that satisfies the assumption of Proposition 3.16.

3.19 Corollary. *Let V be weakly U_{ρ_0}-admissible, and let ρ be a weight satisfying*

$$\frac{\rho(x)}{\rho(y)} \leqslant c \prod_{j=1}^m \Big(\frac{d_j(x)}{d_j(y)} \vee 1 \Big)^{\alpha_j} \qquad (x, y \in \Omega)$$

for some $c \geqslant 1$. Then $\frac{1}{2}V$ is weakly U_ρ-admissible. Moreover, if $(U_{\rho_0})_V$ is exponentially bounded then so is $(U_\rho)_{\frac{1}{2}V}$.

Proof. Observe that the weight $\rho_1 := \rho^2 \rho_0^{-1}$ satisfies the assumption of Proposition 3.16. Indeed,

$$\frac{\rho_1(x)}{\rho_1(y)} \leqslant c^2 \prod_{j=1}^m \Big(\frac{d_j(x)^2}{d_j(y)^2} \vee 1 \Big)^{\alpha_j} \cdot \frac{\rho_0(y)}{\rho_0(x)} = c^2 \prod_{j=1}^m \Big(\frac{d_j(x)}{d_j(y)} \vee \frac{d_j(y)}{d_j(x)} \Big)^{\alpha_j}$$

for all $x, y \in \Omega$. The assertions thus follow from Proposition 2.49, applied with $U_j = U_{\rho_j}$ for $j = 0, 1$, $V_0 = V$, $V_1 = 0$ and $\theta = \frac{1}{2}$. $\square$

After these preparations we are ready to prove our main result on stability of the boundary behaviour of kernel estimates.

3.20 Theorem. *Let V be a potential such that qV is weakly U_{ρ_0}-admissible, for some $q > 2$. Assume that $(U_{\rho_0})_{qV}$ is exponentially bounded and that*

$$\|\rho_0^{-1} (U_{\rho_0})_{qV}(t, s) \rho_0\|_{\infty \to \infty} \leqslant M e^{\omega(t-s)} \qquad \big((t, s) \in D_J\big) \tag{3.17}$$

for some $M \geqslant 1$ and $\omega \in \mathbb{R}$. Then for $0 < a_2 < (1 - \frac{2}{q})a$ there exist $c_2 > 0$ and $\omega_2 \in \mathbb{R}$ such that the kernels $p_{t,s}^V$ of the operators $U_V(t, s)$ satisfy

$$p_{t,s}^V(x, y) \leqslant c_2 (t - s)^{-n/2} \prod_{j=1}^m \Big(\frac{d_j(x) d_j(y)}{t - s} \wedge 1 \Big)^{\alpha_j} \cdot \exp\Big(\omega_2 (t - s) - a_2 \frac{d(x, y)^2}{t - s} \Big)$$

for all $(t, s) \in D_J'$ and a.e. $x, y \in \Omega$.

Proof. Let $F \subseteq \{1, \ldots, m\}$ and $\rho_F := \prod_{j \in F} d_j^{\alpha_j}$. Then ρ_F satisfies the assumption of Corollary 3.19, so $\frac{q}{2}V$ is weakly U_{ρ_F}-admissible and $(U_{\rho_F})_{\frac{q}{2}V}$ is exponentially bounded. The operator of multiplication with ρ_F is an isometry $\rho_F \colon L_1(\rho_F^2 \mu) \to L_1(\rho_F \mu)$, so

$$U_F(t, s) := \rho_F^{-1} U_{\rho_F}(t, s) \rho_F \qquad \big((t, s) \in D_J\big)$$

defines a strongly continuous propagator U_F on $L_1(\rho_F^2 \mu)$. The kernel $p_{t,s}^F$ of $U_F(t, s)$ is given by $p_{t,s}^F(x, y) = \rho_F(x)^{-1} p_{t,s}(x, y) \rho_F(y) \rho_F(y)^{-2}$, for all $(t, s) \in D_J$ (cf. Remark 3.18(b)), so it satisfies

$$p_{t,s}^F(x, y) = \prod_{j \in F} \big(d_j(x) d_j(y) \big)^{-\alpha_j} p_{t,s}(x, y) \leqslant c_0 (t - s)^{-n/2 - \sum_{j \in F} \alpha_j} \exp\big(-a \tfrac{d(x,y)^2}{t-s} \big)$$

for a.e. $x, y \in \Omega$. In particular we obtain that U_F is ultracontractive,

$$\big\|U_F(t,s)\colon L_1(\rho_F^2\mu) \to L_\infty(\rho_F^2\mu)\big\| \leqslant c_0(t-s)^{-n/2-\sum_{j\in F}\alpha_j} \qquad \big((t,s)\in D_J'\big).$$

Let $p := \frac{q}{2}$, and let $(t,s) \in D_J$. By Lemma 2.47, the weak U_{ρ_F}-admissibility of pV implies that pV is weakly U_F-admissible, and with $(U_{\rho_F})_{pV}$ also $(U_F)_{pV}$ is exponentially bounded. Moreover,

$$(U_F)_{pV}(t,s)f = \rho_F^{-1}(U_{\rho_F})_{pV}(t,s)\rho_F f = \rho_F^{-1}(U_{\rho_0})_{pV}(t,s)\rho_F f \tag{3.18}$$

for all $f \in L_1(\rho_F^2\mu)$ with $\rho_F f \in L_1(\rho_0\mu)$ since $(U_{\rho_F})_{pV}$ and $(U_{\rho_0})_{pV}$ are consistent by Lemma 2.48(a). Let $\rho_1 := \rho_F^2\rho_0^{-1}$, and observe that the weight ρ_1^{-1} satisfies the assumption of Proposition 3.16. By Theorem 2.38 and Corollary 2.37 we obtain that

$$\|\rho_F^{-1}(U_{\rho_0})_{pV}(t,s)\rho_F\|_{\infty\to\infty} \leqslant \|\rho_1^{-1}U_{\rho_0}(t,s)\rho_1\|_{\infty\to\infty}^{1/2}\|\rho_0^{-1}(U_{\rho_0})_{qV}(t,s)\rho_0\|_{\infty\to\infty}^{1/2}.$$

From (3.18), Proposition 3.16 and the assumption (3.17) we thus infer that $(U_F)_{pV}$ is exponentially L_∞-bounded.

Let now $a_2 < (1-\frac{1}{p})a$. Then by the above we can apply Theorem 3.10 to obtain $c_F > 0$ and $\omega_F \in \mathbb{R}$ such that, for all $(t,s) \in D_J'$, the kernel $p_{t,s}^{F,V}$ of $(U_F)_V(t,s)$ satisfies

$$p_{t,s}^{F,V}(x,y) \leqslant c_F(t-s)^{-n/2-\sum_{j\in F}\alpha_j}\exp\big(\omega_F(t-s) - a_2\tfrac{d(x,y)^2}{t-s}\big)$$

for a.e. $x, y \in \Omega$ (see also Remark 3.1 for the case that U_F is not bounded). Moreover, $(U_F)_V(t,s) = \rho_F^{-1}(U_{\rho_F})_V(t,s)\rho_F$ for all $(t,s) \in D_J$ by Lemma 2.47, and $(U_{\rho_F})_V$ and U_V are consistent by Lemma 2.48(a), so we conclude that

$$\begin{aligned} p_{t,s}^{V}(x,y) &= \rho_F(x)p_{t,s}^{F,V}(x,y)\rho_F(y) \\ &\leqslant c_F(t-s)^{-n/2}\prod_{j\in F}\left(\frac{d_j(x)d_j(y)}{t-s}\right)^{\alpha_j}\exp\Big(\omega_F(t-s) - a_2\frac{d(x,y)^2}{t-s}\Big), \end{aligned}$$

for all $(t,s) \in D_J'$ and a.e. $x, y \in \Omega$. The assertion thus follows by taking the minimum over $F \subseteq \{1,\dots,m\}$. $\square$

3.21 Remarks. (a) Observe that the assumptions of Theorem 3.20 are necessary except for the factor $q > 2$. Indeed, if U_V satisfies the asserted upper bound, then by Proposition 3.16 there exist $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|\rho_0^{\pm1}U_V(t,s)\rho_0^{\mp1}\|_{p\to p} \leqslant Me^{\omega(t-s)}$ for all $(t,s) \in D_J$ and $p = 1, \infty$. From this it follows that V is weakly U_{ρ_0}-admissible, that $(U_{\rho_0})_V$ is exponentially bounded and that (3.17) is satisfied.

(b) The assumption that qV is weakly U_{ρ_0}-admissible for some $q > 2$, and that $(U_{\rho_0})_{qV}$ is exponentially bounded (the "forward" condition), is in particular satisfied if V^- is a Miyadera perturbation of U_{ρ_0} with constants (α,γ), $\gamma < \frac{1}{2}$.

Let $F_m := \{1,\dots,m\}$; then $\rho_{F_m} = \rho_0$. Let the propagator U_{F_m} on $L_1(\rho_0^2\mu)$ be defined as in the proof of Theorem 3.20, and assume that V^- is a backward Miyadera perturbation of U_{F_m} with constants (α,γ), $\gamma < \frac{1}{2}$. Then $(U_{F_m})_{qV}$ is exponentially L_∞-bounded for all $q \in (2, \frac{1}{\gamma})$, and this implies that (3.17) (the "backward" condition) is satisfied since $(U_{F_m})_{qV}(t,s) = \rho_0^{-1}(U_{\rho_0})_{qV}(t,s)\rho_0$ for all $(t,s) \in D_J$.

(c) Assume that the backward propagator U' defined in Remark 2.31(a) is strongly measurable. Then it is possible to reformulate condition (3.17) so that it becomes analogous to the forward condition. Since $\|\rho_0 U'(s,t)\rho_0^{-1}\|_{1\to 1} = \|\rho_0^{-1}U(t,s)\rho_0\|_{\infty\to\infty}$ for all $(t,s) \in D_J$, it follows from Proposition 3.16 that U' extrapolates to an exponentially bounded strongly measurable backward propagator $(U')_{\rho_0}$ on $L_1(\rho_0\mu)$. By Remark 2.31(c), Lemma 2.47 and Lemma 2.48(a) we obtain, for $(t,s) \in D_J$, that the estimate in (3.17) holds if and only if

$$\big\|\big((U')_{\rho_0}\big)_{qV\vee(-n)}(s,t)\big\| = \big\|\rho_0 U_{qV\vee(-n)}(t,s)'\rho_0^{-1}\big\|_{1\to 1} \leqslant Me^{\omega(t-s)}$$

for all $n \in \mathbb{N}$. Therefore, (3.17) is satisfied if and only if qV is $(U')_{\rho_0}$-admissible and $\big((U')_{\rho_0}\big)_{qV}$ is exponentially bounded.

(d) In the following we explain how the Miyadera conditions occurring in (b) can be formulated analogously to the non-autonomous Kato class conditions discussed in the introduction, as inequalities for integrals involving V, ρ and the kernels $p_{t,s}$.

We assume that the kernels $p_{t,s}$ can be chosen in such a way that $(t,s,x,y) \mapsto p_{t,s}(x,y)$ is measurable on $D_J' \times \Omega \times \Omega$, and we fix such a choice. As in [ArBu94; proof of Thm. 2.1] one can show that this is always possible if $L_1(\mu)$ is separable. (The argument given there requires only strong measurability of U; Example 2.32(a) demonstrates that then σ-finiteness of μ is not sufficient, as was claimed in [ArBu94].)

Let ρ be a weight satisfying the assumption of Proposition 3.16 (or, more generally, a weight such that U extrapolates to a locally bounded strongly measurable propagator U_ρ on $L_1(\rho\mu)$). Observe that a potential V is a Miyadera perturbation of U_ρ with constants (α,γ) if and only if the operators $A_{t,s}^n \in \mathcal{L}(L_1(\mu))$ that are defined by

$$A_{t,s}^n f := \int_s^t (|V| \wedge n)(\tau)U(\tau,s)f\,d\tau$$

satisfy $\|A_{t,s}^n \colon L_1(\rho\mu) \to L_1(\rho\mu)\| \leqslant \gamma$ for all $n \in \mathbb{N}$ and all $(t,s) \in D_J'$ with $t - s \leqslant \alpha$. Moreover, the kernel $k_{t,s}^n$ of $A_{t,s}^n$ is given by

$$k_{t,s}^n(x,y) = \int_s^t (|V| \wedge n)(\tau,x)p_{\tau,s}(x,y)\,d\tau,$$

for $n \in \mathbb{N}$ and $(t,s) \in D_J'$. Therefore, by Remark 3.18(b) and the monotone convergence theorem, V is a Miyadera perturbation of U_ρ with constants (α,γ) if and only if

$$\gamma_\alpha^+(\rho,V) := \sup_{0<t-s\leqslant\alpha}\ \operatorname*{ess\,sup}_{x\in\Omega} \int_s^t \int_\Omega \frac{\rho(y)}{\rho(x)} p_{\tau,s}(y,x)\big|V(\tau,y)\big|\,d\mu(y)\,d\tau \leqslant \gamma.$$

Let $\widetilde{U}_\rho$ be the strongly measurable propagator on $L_1(\rho^2\mu)$ defined by $\widetilde{U}_\rho(t,s) := \rho^{-1}U_\rho(t,s)\rho$. By condition (ii) of Proposition 2.29 and the monotone convergence theorem we obtain that V is a backward Miyadera perturbation of $\widetilde{U}_\rho$ with constants (α,γ)

if and only if

$$\gamma_\alpha^-(\rho, V) := \sup_{0<t-s\leqslant\alpha} \operatorname*{ess\,sup}_{x\in\Omega} \int_s^t \int_\Omega \frac{\rho(y)}{\rho(x)} p_{t,\tau}(x,y) \big|V(\tau,y)\big| \, d\mu(y)\, d\tau \leqslant \gamma.$$

Note that the only difference to the definition of $\gamma_\alpha^+(\rho, V)$ is that there the term $p_{\tau,s}(y,x)$ occurs in the integral.

Now assume that $\mu(B(x,r)) > 0$ for all $x \in X$ and all $r > 0$. Observe that then any lower semicontinuous function h on (Ω, d) satisfies $\operatorname{ess\,sup}_{x\in\Omega} h(x) = \sup_{x\in\Omega} h(x)$. Therefore, the ess sup in the definition of $\gamma_\alpha^+(\rho, V)$ can be replaced with a sup if the function

$$x \mapsto \int_s^t \int_\Omega \frac{\rho(y)}{\rho(x)} p_{\tau,s}(y,x) \big|V(\tau,y)\big| \, d\mu(y)\, d\tau$$

is lower semicontinuous on (Ω, d) for all $(t,s) \in D_J'$. By Fatou's lemma, this is the case if $p_{t,s}(y,\cdot)$ is lower semicontinuous for all $(t,s) \in D_J'$ and a.e. $y \in \Omega$. An analogous observation applies to the ess sup in the definition of $\gamma_\alpha^-(\rho, V)$.

For the last result of this section we assume, in addition to the Gaussian type upper bound (3.12), that the kernels $p_{t,s}$ satisfy a matching lower bound

$$p_{t,s}(x,y) \geqslant K_1 (t-s)^{-n/2} \prod_{j=1}^m \Big(\frac{d_j(x)d_j(y)}{t-s} \wedge 1\Big)^{\alpha_j} \cdot \exp\Big(-A_1(t-s) - a_1 \frac{d(x,y)^2}{t-s}\Big)$$

for all $(t,s) \in D_J'$ and a.e. $x, y \in \Omega$, where $K_1, a_1 > 0$ and $A_1 \geqslant 0$, and the other quantities are the same as in the upper bound.

3.22 Theorem. *Let V be a U-regular potential. Let $\varepsilon > 0$, and assume that $-\varepsilon V$ is weakly U_{ρ_0}-admissible, that $(U_{\rho_0})_{-\varepsilon V}$ is exponentially bounded, and that*

$$\|\rho_0^{-1}(U_{\rho_0})_{-\varepsilon V}(t,s)\rho_0\|_{\infty\to\infty} \leqslant M e^{\omega(t-s)} \qquad \big((t,s) \in D_J\big)$$

for some $M \geqslant 1$ and $\omega \in \mathbb{R}$. Then for $a_2 > (1+\frac{2}{\varepsilon})a_1$ there exist $c_2 > 0$ and $\omega_2 \in \mathbb{R}$ such that

$$p_{t,s}^V(x,y) \geqslant c_2 (t-s)^{-n/2} \prod_{j=1}^m \Big(\frac{d_j(x)d_j(y)}{t-s} \wedge 1\Big)^{\alpha_j} \cdot \exp\Big(-\omega_2(t-s) - a_2 \frac{d(x,y)^2}{t-s}\Big)$$

for all $(t,s) \in D_J'$ and a.e. $x, y \in \Omega$.

Proof. Let $a_2 > (1+\frac{2}{\varepsilon})a_1$, and let $\beta > \frac{2}{\varepsilon}$ be such that $a_2 = (1+\beta)a$. Then $q(-V/\beta) = -\varepsilon V$ with $q = \varepsilon\beta > 2$, so by Theorem 3.20 there exist $c_1 > 0$ and $\omega_1 \in \mathbb{R}$ such that

$$p_{t,s}^{-V/\beta}(x,y) \leqslant c_1 (t-s)^{-n/2} \prod_{j=1}^m \Big(\frac{d_j(x)d_j(y)}{t-s} \wedge 1\Big)^{\alpha_j} \cdot e^{\omega_1(t-s)}$$

for all $(t,s) \in D_J'$ and a.e. $x, y \in \Omega$. Moreover, as in the proof of Theorem 3.12 we obtain that

$$p_{t,s}^V \geqslant \big(p_{t,s}^{-V/\beta}\big)^{-\beta} (p_{t,s})^{1+\beta} \qquad \big((t,s) \in D_J'\big).$$

The assertion thus follows from the lower bound on the kernels $p_{t,s}$. $\square$

3.3 Kernel estimates on manifolds

Throughout this section let (Ω,μ) be a σ-finite measure space, and let U be a positive strongly continuous propagator on $L_1(\mu)$ with parameter interval $J\subseteq\mathbb{R}$. We assume that the operators $U(t,s)$ have integral kernels $p_{t,s}$ satisfying

$$p_{t,s}(x,y)\leqslant h_{t-s}(x)h_{t-s}(y)e^{A(t-s)-\psi_{t,s}(x,y)} \tag{3.19}$$

for all $(t,s)\in D_J'$ and a.e. $x,y\in\Omega$, with $A\in\mathbb{R}$ and measurable functions $h_t\colon\Omega\to(0,\infty)$ and $\psi_{t,s}\colon\Omega\times\Omega\to[0,\infty)$. We do not need a multiplicative constant c in the estimate since this can be incorporated into the functions h_t.

In Section 3.1, h_t was the constant function $h_t=K^{1/2}t^{-\nu/2}$, for $t>0$. For the heat kernel of a complete Riemannian manifold with non-negative Ricci curvature it is proved in [LiYa86] that the bound (3.19) and a matching lower bound hold with $A=0$, $h_t(x)=c\mu\big(B(x,\sqrt{t})\big)^{-1/2}$ and $\psi_{t,s}(x,y)=a\frac{d(x,y)^2}{t-s}$, where μ is the Riemannian volume and d the Riemannian distance. This is the so-called Li-Yau estimate. Another possible choice for heat kernel estimates on Riemannian manifolds is $h_t(x)=c\mu\big(B(x,\sqrt{t}\wedge 1)\big)^{-1/2}$. Note that in the above mentioned cases, $t\mapsto h_t(x)$ is decreasing for all $x\in\Omega$; this property of the h_t will be needed in some of the results below.

For our first result on stability of the kernel estimate (3.19) we only require the following asymmetric bound. For all $k\in\mathbb{N}$ there exist $c_k>0$ and $r_k\in\mathbb{R}$ such that

$$p_{t,s}(x,y)\leqslant c_k h_{k(t-s)}(x)^{1+q}h_{k(t-s)}(y)^{1-q}e^{r_k(t-s)} \tag{3.20}$$

for all $(t,s)\in D_J'$, a.e. $x,y\in\Omega$ and $q=\pm(k-1)$. This is analogous to assumption (IV) in [Stu93; Sec. 4], where similar results for Schrödinger semigroups with measures on manifolds are proved. (By [Stu93; Cor. 4.4], the heat kernel of a complete Riemannian manifold with Ricci curvature bounded below satisfies (3.20) with $h_t(x)=c\mu\big(B(x,\sqrt{t})\big)^{-1/2}$; see also (3.30) below.) We point out that we do not require any quantitative assumptions on the measure μ such as volume growth conditions.

3.23 Theorem. *Assume* (3.19) *and* (3.20). *Let $\gamma\in(0,1)$, and let V be a potential such that V/γ is weakly U-admissible. Assume that there exist $M\geqslant 1$ and $\omega_\gamma\in\mathbb{R}$ such that $\|U_{V/\gamma}(t,s)\|_{p\to p}\leqslant Me^{\omega_\gamma(t-s)}$ for all $(t,s)\in D_J$ and $p=1,\infty$. Then for $\beta\in(\gamma,1)$ there exist $c>0$ and $\omega\in\mathbb{R}$ such that, for all $(t,s)\in D_J'$, the kernel $p^V_{t,s}$ of $U_V(t,s)$ satisfies*

$$p^V_{t,s}(x,y)\leqslant ch_{t-s}(x)h_{t-s}(y)e^{\omega(t-s)-(1-\beta)\psi_{t,s}(x,y)}\qquad (x,y\in\Omega). \tag{3.21}$$

More precisely, if $k\in\mathbb{N}$ satisfies $\frac{1}{k}\leqslant 1-\frac{\gamma}{\beta}$, then there exists $c>0$ such that the above holds with $\omega=\gamma\omega_\gamma+k'\gamma\frac{r_k}{k}+(1-k'\gamma)A$, where $\frac{1}{k}+\frac{1}{k'}=1$.

Proof. Observe that the condition on k is equivalent to $\frac{\gamma}{\beta}\leqslant\frac{1}{k'}$ and hence to $\beta\geqslant k'\gamma$. We can thus assume without loss of generality that $\beta=k'\gamma$. Then for $(t,s)\in D_J'$ we obtain by Theorem 2.38, Corollary 2.37 and the bound (3.20) that

$$\begin{aligned}\big\|h^{1-2/k}_{k(t-s)}U_{V/\beta}(t,s)h^{-1}_{k(t-s)}\big\|_{1\to k'}&\leqslant\big\|h^{k-2}_{k(t-s)}U(t,s)h^{-k}_{k(t-s)}\big\|^{1/k}_{1\to\infty}\|U_{V/\gamma}(t,s)\|^{1/k'}_{1\to 1}\\&\leqslant\big(c_ke^{r_k(t-s)}\big)^{1/k}\big(Me^{\omega_\gamma(t-s)}\big)^{1/k'}=:\varphi(t-s),\end{aligned}$$

and in the same way

$$\big\|h_{k(t-s)}^{-1}U_{V/\beta}(t,s)h_{k(t-s)}^{1-2/k}\big\|_{k\to\infty} \leqslant \varphi(t-s).$$

Another application of Corollary 2.37 yields

$$\big\|h_{k(t-s)}^{-1+2/q}U_{V/\beta}(t,s)h_{k(t-s)}^{1-2/p}\big\|_{p\to q} \leqslant \varphi(t-s) \tag{3.22}$$

for all $1 \leqslant p < q \leqslant \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{1}{k}$.

Let now $(t,s) \in D_J'$, and let $1 = p_0 < p_1 < \cdots < p_k = \infty$ and $s = t_0 < t_1 < \cdots < t_k = t$ satisfy $\frac{1}{p_{j-1}} - \frac{1}{p_j} = \frac{1}{k}$ and $t_j - t_{j-1} = \frac{t-s}{k}$ for $j = 1, \ldots, k$. Then from (3.22) we infer that

$$\begin{aligned}\|h_{t-s}^{-1}U_{V/\beta}(t,s)h_{t-s}^{-1}\|_{1\to\infty} &\leqslant \prod_{j=1}^{k} \big\|h_{t-s}^{-1+2/p_j}U_{V/\beta}(t_j,t_{j-1})h_{t-s}^{1-2/p_{j-1}}\big\|_{p_{j-1}\to p_j} \\ &\leqslant \varphi\big(\tfrac{t-s}{k}\big)^k = c_k M^{k-1}e^{\omega_1(t-s)},\end{aligned}$$

where $\omega_1 = \frac{r_k}{k} + \frac{\omega_\gamma}{k'}$, so $p_{t,s}^{V/\beta}(x,y) \leqslant c_k M^{k-1} h_{t-s}(x)h_{t-s}(y)e^{\omega_1(t-s)}$ for a.e. $x, y \in \Omega$. By Corollary 2.39 and the upper bound (3.19) for $p_{t,s}$ we conclude that

$$p_{t,s}^V(x,y) \leqslant p_{t,s}^{V/\beta}(x,y)^\beta p_{t,s}(x,y)^{1-\beta} \leqslant c h_{t-s}(x)h_{t-s}(y)e^{\omega(t-s)-(1-\beta)\psi_{t,s}(x,y)}$$

for all a.e. $x, y \in \Omega$, with $c = c_k^\beta M^{\beta(k-1)}$ and $\omega = \beta\omega_1 + (1-\beta)A$. This implies the assertion since $\beta = k'\gamma$ and hence $\beta\omega_1 = k'\gamma\frac{r_k}{k} + \gamma\omega_\gamma$. □

3.24 Remark. (a) The basic interpolation and iteration idea behind the L_1–L_∞-estimate in the above proof already appears in [Voi86; proof of Prop. 6.3], in the context of Schrödinger semigroups on $\mathbb{R}^n$. In this case one does not need the weight h_{t-s} in the estimates. It seems that for kernel estimates on manifolds, this type of estimate is hardly ever used. The argument in [Stu93; proof of Prop. 4.7] is similar, but not quite as straightforward.

(b) As explained in the previous two sections, theorems on upper bounds can also be used to prove stability of lower bounds. Assume, e.g., that the propagator U is L_1- and L_∞-bounded, that (3.20) holds with $r_k = 0$ for all $k \in \mathbb{N}$, and that $V \geqslant 0$ is a forward and backward Miyadera perturbation of U with constants (∞, γ), for some $\gamma \geqslant 0$. Then $U_{-V/\beta}$ is L_1- and L_∞-bounded for $\beta > \gamma$, so $p_{t,s}^{-V/\beta}(x,y) \leqslant c h_{t-s}(x)h_{t-s}(y)$ by Theorem 3.23 (applied with $A = 0$ and $\psi_{t,s} = 0$). From the estimate $p_{t,s}^V \geqslant \big(p_{t,s}^{-V/\beta}\big)^{-\beta}(p_{t,s})^{1+\beta}$ it thus follows that the lower bound

$$p_{t,s}(x,y) \geqslant c_1 h_{t-s}(x)h_{t-s}(y)e^{-\psi_{t,s}(x,y)}$$

is, up to a factor $1+\beta$ in front of $\psi_{t,s}$, stable under perturbation by V.

In the context of complete Riemannian manifolds with non-negative Ricci curvature, the above is a generalisation to the time-dependent situation of the fact that the Li-Yau estimate is stable under perturbation by a positive Green bounded (time-independent) potential. This stability theorem is due to [Zha00; Thm. C]; see also [Gri06; Thm. 10.5] for a different proof and [Tak07; Thm. 2] for a generalisation to perturbation by measures.

(c) Assume that one has good control of the exponential growth bound of $U_{V/\beta}$ in $L_2(\mu)$, for some $\beta \in (\gamma, 1)$. Moreover, suppose that $h_t \leqslant h_1$ a.e. for all $t > 1$. Then one can proceed as follows to obtain an alternative bound on $p_{t,s}$ for large $t-s$. As in the above proof one shows that there exists $C > 0$ such that

$$\|U_{V/\beta}(s+1,s)h_1^{-1}\|_{1\to 2} \leqslant C \quad \text{and} \quad \|h_1^{-1}U_{V/\beta}(s+1,s)\|_{2\to\infty} \leqslant C$$

for all $s \in J$ satisfying $s+1 \in J$. Therefore, if $M \geqslant 1$ and $\omega_{2,\beta} \in \mathbb{R}$ are such that $\|U_{V/\beta}(t,s)\|_{2\to 2} \leqslant Me^{\omega_{2,\beta}(t-s)}$ for all $(t,s) \in D_J$, then

$$\|h_1^{-1}U_{V/\beta}(t,s)h_1^{-1}\|_{1\to\infty} \leqslant C^2 Me^{\omega_{2,\beta}(t-s-2)}$$

and hence

$$p_{t,s}^V(x,y) \leqslant p_{t,s}^{V/\beta}(x,y)^\beta p_{t,s}(x,y)^{1-\beta} \leqslant (C^2Me^{-2\omega_{2,\beta}})^\beta h_1(x)h_1(y)e^{\omega(t-s)-(1-\beta)\psi_{t,s}(x,y)}$$

for all $(t,s) \in D_J$ with $t-s \geqslant 2$ and a.e. $x,y \in \Omega$, with $\omega = \beta\omega_{2,\beta} + (1-\beta)A$, where we have used that $h_{t-s} \leqslant h_1$ a.e.

In the remainder of the section we discuss the case of classical Gaussian bounds in more detail, i.e., we assume that $\psi_{t,s}(x,y) = a\frac{d(x,y)^2}{t-s}$ with $a > 0$ and a measurable metric d on Ω. Moreover, we assume that (Ω, d) is separable. Then one can use the following version of Davies' trick to refine the estimate of Theorem 3.23. We will use the weight functions ρ_z defined by $\rho_z(x) := e^{-d(x,z)}$, for $z \in \Omega$.

3.25 Lemma. *Let $A, \alpha, \beta \in \mathbb{R}$ and $a > 0$, and let $(t,s) \in D_J'$. Then the estimate*

$$p_{t,s}(x,y) \leqslant h_{t-s}(x)^\alpha h_{t-s}(y)^\beta \exp\Big(A(t-s) - a\frac{d(x,y)^2}{t-s}\Big)$$

holds for a.e. $x,y \in \Omega$ if and only if

$$\|h_{t-s}^{-\alpha}\rho_z^\lambda U(t,s)\rho_z^{-\lambda}h_{t-s}^{-\beta}\|_{1\to\infty} \leqslant \exp\big(A(t-s) + \tfrac{\lambda^2}{4a}(t-s)\big) \tag{3.23}$$

for all $z \in \Omega$ and all $\lambda \geqslant 0$.

Proof. By the Dunford-Pettis theorem, (3.23) holds if and only if

$$p_{t,s}(x,y) \leqslant h_{t-s}(x)^\alpha h_{t-s}(y)^\beta \exp\big(A(t-s) + \tfrac{\lambda^2}{4a}(t-s) + \lambda d(x,z) - \lambda d(y,z)\big) \tag{3.24}$$

for a.e. $x,y \in \Omega$. Observe that

$$\inf\big\{\tfrac{\lambda^2}{4a}(t-s) + \lambda d(x,z) - \lambda d(y,z);\, z \in \Omega,\ \lambda \geqslant 0\big\} = -a\frac{d(x,y)^2}{t-s},$$

the infimum being attained for $z = x$ and $\lambda = \frac{2ad(x,y)}{t-s}$. This implies the assertion since the right-hand side of (3.24) is continuous in z and λ, and both Ω and $[0,\infty)$ are separable. $\square$

It is important to get a grip on the constants ω and β in the kernel bound (3.21). Remark 3.24(c) only gives some information about the constant ω. We now show that a suitable weighted L_2-estimate for U_V leads to good control of both ω and β. We formulate this as a result on the unperturbed propagator U since it does not involve perturbation theory.

In fact, Gaussian estimates have the following "self-improving" property that is particularly useful in the time-independent symmetric situation. In [Ouh06; Thm. 1], a similar result is proved in the context of Schrödinger semigroups on manifolds, and it is already noted there that the method also works in greater generality. Thus, the main feature of our result is not the more general context but the simpler and more transparent proof.

3.26 Proposition. *Assume that $t \mapsto h_t(x)$ is decreasing for all $x \in \Omega$ and that the kernels $p_{t,s}$ of the operators $U(t,s)$ satisfy*

$$p_{t,s}(x,y) \leqslant \big(h_{t-s}(x)^2 \wedge h_{t-s}(y)^2\big) \exp\Big(A(t-s) - a\frac{d(x,y)^2}{t-s}\Big) \tag{3.25}$$

for all $(t,s) \in D'_J$ and a.e. $x, y \in \Omega$, with constants $A \in \mathbb{R}$ and $a > 0$. Moreover, assume that there exist $M \geqslant 1$ and $\omega, \omega_2 \in \mathbb{R}$ such that $\|U(t,s)\|_{p\to p} \leqslant Me^{\omega(t-s)}$ and

$$\|\rho_z^{\lambda} U(t,s)\rho_z^{-\lambda}\|_{2\to 2} \leqslant e^{(\omega_2+\lambda^2)(t-s)} \tag{3.26}$$

for all $(t,s) \in D_J$, $p = 1, \infty$, $z \in \Omega$ and $\lambda \geqslant 0$. Then

$$p_{t,s}(x,y) \leqslant eMh_{\varepsilon(t-s)}(x)h_{\varepsilon(t-s)}(y)\exp\Big(\omega_2(t-s) - \frac{d(x,y)^2}{4(t-s)}\Big)$$

for all $(t,s) \in D'_J$ and a.e. $x, y \in \Omega$, where

$$\varepsilon = \frac{1}{2} \wedge \left((\omega + A - 2\omega_2)(t-s) + (\tfrac{1}{a} - 2)\frac{d(x,y)^2}{4(t-s)}\right)^{-1}.$$

(We assume without loss of generality that $\omega + A \geqslant 2\omega_2$ and $a < \frac{1}{2}$.)

Proof. Let $z \in \Omega$ and $\lambda \geqslant 0$. For $(t,s) \in D'_J$ we obtain by (3.25) and Lemma 3.25 that

$$\|\rho_z^{2\lambda}U(t,s)\rho_z^{-2\lambda}h_{t-s}^{-2}\|_{1\to\infty} \leqslant \exp\big(A(t-s) + \tfrac{(2\lambda)^2}{4a}(t-s)\big)$$

and hence, by Corollary 2.37,

$$\begin{aligned}\|\rho_z^{\lambda}U(t,s)\rho_z^{-\lambda}h_{t-s}^{-1}\|_{1\to 2} &\leqslant \|U(t,s)\|_{1\to 1}^{1/2}\|\rho_z^{2\lambda}U(t,s)\rho_z^{-2\lambda}h_{t-s}^{-2}\|_{1\to\infty}^{1/2} \\ &\leqslant M^{1/2}\exp\big(\tfrac{\omega+A}{2}(t-s) + \tfrac{\lambda^2}{2a}(t-s)\big).\end{aligned}$$

An analogous estimate holds for $\|h_{t-s}^{-1}\rho_z^{\lambda}U(t,s)\rho_z^{-\lambda}\|_{2\to\infty}$. Let now $(t,s) \in D'_J$ and $\varepsilon \leqslant \frac{1}{2}$, and let $s_1, t_1 \in (s,t)$ be such that $s_1 - s = t - t_1 = \varepsilon(t-s)$. Then the above, together with (3.26), yields

$$\begin{aligned}
&\big\|h_{\varepsilon(t-s)}^{-1}\rho_z^{\lambda}U(t,s)\rho_z^{-\lambda}h_{\varepsilon(t-s)}^{-1}\big\|_{1\to\infty}\\
&\quad\leqslant \|h_{t-t_1}^{-1}\rho_z^{\lambda}U(t,t_1)\rho_z^{-\lambda}\|_{2\to\infty}\|\rho_z^{\lambda}U(t_1,s_1)\rho_z^{-\lambda}\|_{2\to2}\|\rho_z^{\lambda}U(s_1,s)\rho_z^{-\lambda}h_{s_1-s}^{-1}\|_{1\to2}\\
&\quad\leqslant M\exp\big((\omega+A)\varepsilon(t-s)+\tfrac{\lambda^2}{a}\varepsilon(t-s)+(\omega_2+\lambda^2)(1-2\varepsilon)(t-s)\big)\\
&\quad\leqslant M\exp\big((\omega+A-2\omega_2+(\tfrac{1}{a}-2)\lambda^2)\varepsilon(t-s)+(\omega_2+\lambda^2)(t-s)\big).
\end{aligned}$$

Choosing $\varepsilon=\varepsilon_\lambda := \frac{1}{2}\wedge\big(\omega+A-2\omega_2+(\frac{1}{a}-2)\lambda^2\big)^{-1}(t-s)^{-1}$, we infer that

$$\big\|h_{\varepsilon(t-s)}^{-1}\rho_z^{\lambda}U(t,s)\rho_z^{-\lambda}h_{\varepsilon(t-s)}^{-1}\big\|_{1\to\infty}\leqslant M\exp\big(1+(\omega_2+\lambda^2)(t-s)\big).$$

Now observe that $\lambda\mapsto h_{\varepsilon_\lambda}(x)$ is increasing for all $x\in\Omega$ since $t\mapsto h_t(x)$ is decreasing. We thus obtain the assertion as in Lemma 3.25, letting $\mathbb{Q}_{\geqslant0}\ni\lambda\uparrow\frac{d(x,y)}{2(t-s)}$. ☐

3.27 Remark. (a) The factor $\big(h_{t-s}(x)^2\wedge h_{t-s}(y)^2\big)$ in assumption (3.25) is chosen instead of the (larger) factor $h_{t-s}(x)h_{t-s}(y)$ in order to simplify the result. If the kernel estimate (3.19) is satisfied with $\psi_{t,s}(x,y)=a\frac{d(x,y)^2}{t-s}$, and similarly as in (3.20) one has $p_{t,s}(x,y)\leqslant h_{t-s}(x)^{1+q}h_{t-s}(y)^{1-q}e^{r_3(t-s)}$ for a.e. $x,y\in\Omega$ and $q=\pm2$, then (3.25) holds with $\tilde{A}=\frac{1}{2}(A+r_3)$ and $\tilde{a}=\frac{a}{2}$ in place of A and a.

There is no factor in front of λ^2 in the weighted L_2-estimate (3.26) since we assume that the metric d is defined in such a way that (3.26) holds. In the context of C_0-semigroups generated by elliptic operators this is satisfied by the Davies metric.

Note that the above proof does not require strong continuity (or strong measurability) of U. Moreover, the result is also valid for propagators that are not positive: Then one has to use Stein interpolation instead of Corollary 2.37 in the proof, and one obtains an estimate for $|p_{t,s}(x,y)|$.

(b) Assume that there exist $C,n>0$ such that $h_{\varepsilon t}\leqslant C\varepsilon^{-n/4}h_t$ a.e. for all $t>0$ and all $0<\varepsilon<1$. Then Proposition 3.26 implies that

$$p_{t,s}(x,y)\leqslant eMC^2h_{t-s}(x)h_{t-s}(y)g\big(t-s,d(x,y)\big)$$

for all $(t,s)\in D_J'$ and a.e. $x,y\in\Omega$, where

$$g(t,r)=\Big[2+(\omega+A-2\omega_2)t+(\tfrac{1}{a}-2)\tfrac{r^2}{4t}\Big]^{n/2}\exp\Big(\omega_2 t-\frac{r^2}{4t}\Big).$$

(c) The assumption on the h_t in part (b) holds, e.g., if $h_t(x)=c\mu\big(B(x,\sqrt{t})\big)^{-1/2}$ and μ satisfies the volume growth condition $\mu(B(x,R))\leqslant C^2\big(\frac{R}{r}\big)^n\mu(B(x,r))$ for all $R>r>0$. It follows from [Sik04; Thm. 4] that in this situation the exponent $\frac{n}{2}$ in the estimate of part (b) can be replaced with $\frac{n-1}{2}$ if, in addition, $\omega_2=0$ and the propagator U comes from a selfadjoint C_0-semigroup.

Even if the propagator U comes from a selfadjoint C_0-semigroup: The perturbed situation is usually a non-symmetric one since the potential is time-dependent. Suppose, e.g., that V is an infinitesimally small Miyadera perturbation of U, with backward Miyadera constant close to 1. Then the kernel estimate from Theorem 3.23 is rather poor since $1-\beta$ is close to 0. If one has no good control of $\|\rho_z^\lambda U_V(t,s)\rho_z^{-\lambda}\|_{2\to2}$, then instead of Proposition 3.26 one can use the following theorem to obtain a better estimate. The result may seem rather technical; in Remark 3.29 below we explain how it can be applied in the context of Riemannian manifolds with Ricci curvature bounded below. For this application it is important that the constants in the estimate are explicit.

3.28 Theorem. *Assume that for all $\theta\in(0,1]$ there exist $c_\theta, a_\theta>0$ and $A_\theta\in\mathbb{R}$ such that*

$$p_{t,s}(x,y)\vee p_{t,s}(y,x)\leqslant c_\theta h_{(t-s)/\theta}(x)^{1/\theta}h_{(t-s)/\theta}(y)^{2-1/\theta}\exp\Big(A_\theta(t-s)-a_\theta\frac{d(x,y)^2}{t-s}\Big)$$

for all $(t,s)\in D_J'$ and a.e. $x,y\in\Omega$. Let $\gamma_1,\gamma_\infty>0$ satisfy $\sigma:=\gamma_1+\gamma_\infty\leqslant 1$, and let V be a potential such that V/γ_1 is weakly U-admissible. Assume that there exist $M_p\geqslant 1$ and $\omega_p\in\mathbb{R}$ such that

$$\big\|U_{(V/\gamma_p)\vee(-n)}(t,s)\big\|_{p\to p}\leqslant M_pe^{\omega_p(t-s)}\qquad\big((t,s)\in D_J,\ n\in\mathbb{N},\ p=1,\infty\big).$$

Then

$$p_{t,s}^V(x,y)\leqslant ch_{t-s}(x)h_{t-s}(y)\exp\Big(\omega(t-s)-a\frac{d(x,y)^2}{t-s}\Big)$$

for all $(t,s)\in D_J'$ and a.e. $x,y\in\Omega$, where $c=\big(M_1^\theta M_\infty^{1-\theta}c_\theta^\theta c_{1-\theta}^{1-\theta}\big)^\sigma c_1^{1-\sigma}$ (with $\theta:=\frac{\gamma_1}{\sigma}$), $\omega=\sigma\theta(1-\theta)(\omega_1+\omega_\infty)+\sigma\theta^2A_\theta+\sigma(1-\theta)^2A_{1-\theta}+(1-\sigma)A_1$ and $a=\sigma\frac{a_\theta a_{1-\theta}}{a_\theta+a_{1-\theta}}+(1-\sigma)a_1$.

Note that the slightly sophisticated formulation of the condition on the L_p-norm bound of U_{V/γ_p} is only needed for the case $p=\infty$ and $\gamma_\infty<\gamma_1$: Then the potential V/γ_∞ need not be weakly U-admissible. In the case $p=1$ or $\gamma_1\leqslant\gamma_\infty$ one can simply require $\|U_{V/\gamma_p}(t,s)\|_{p\to p}\leqslant M_pe^{\omega_p(t-s)}$ for all $(t,s)\in D_J$. The estimate assumed for the $p_{t,s}$ can be viewed as a combined version of the bounds (3.19) and (3.20).

Proof of Theorem 3.28. Without loss of generality assume that V is bounded from below, and observe that $1-\theta=\frac{\gamma_\infty}{\sigma}$. Let $z\in\Omega$ and $\lambda\geqslant 0$. Then Lemma 3.25 yields

$$N_\theta:=\big\|h_{(t-s)/\theta}^{-1/\theta}\rho_z^{\lambda/\theta}U(t,s)\rho_z^{-\lambda/\theta}h_{(t-s)/\theta}^{1/\theta-2}\big\|_{1\to\infty}\leqslant c_\theta\exp\big(A_\theta(t-s)+\tfrac{(\lambda/\theta)^2}{4a_\theta}(t-s)\big)$$

and hence, due to Theorem 2.38 and Corollary 2.37,

$$\begin{aligned}\big\|h_{(t-s)/\theta}^{-1}\rho_z^\lambda U_{V/\sigma}(t,s)\rho_z^{-\lambda}h_{(t-s)/\theta}^{1-2\theta}\big\|_{1/\theta\to\infty}&\leqslant\|U_{V/\gamma_\infty}(t,s)\|_{\infty\to\infty}^{1-\theta}N_\theta^\theta\\&\leqslant c_{\infty,\theta}\exp\big(\omega_{\infty,\theta}(t-s)+\tfrac{\lambda^2}{4\theta a_\theta}(t-s)\big)\end{aligned}$$

for all $(t,s)\in D_J'$, where $c_{\infty,\theta}:=M_\infty^{1-\theta}c_\theta^\theta$ and $\omega_{\infty,\theta}:=(1-\theta)\omega_\infty+\theta A_\theta$. In the same way,

$$\big\|h_{(t-s)/(1-\theta)}^{1-2(1-\theta)}\rho_z^\lambda U_{V/\sigma}(t,s)\rho_z^{-\lambda}h_{(t-s)/(1-\theta)}^{-1}\big\|_{1\to1/\theta}\leqslant c_{1,\theta}\exp\big(\omega_{1,\theta}(t-s)+\tfrac{\lambda^2}{4(1-\theta)a_{1-\theta}}(t-s)\big),$$

with $c_{1,\theta} := M_1^\theta c_{1-\theta}^{1-\theta}$ and $\omega_{1,\theta} := \theta\omega_1 + (1-\theta)A_{1-\theta}$. Let now $(t,s) \in D_J'$, and let $r \in (s,t)$ be such that $t-r = \theta(t-s)$. Then $r-s = (1-\theta)(t-s)$, so by the above we obtain that

$$\begin{aligned}
&\|h_{t-s}^{-1}\rho_z^\lambda U_{V/\sigma}(t,s)\rho_z^{-\lambda}h_{t-s}^{-1}\|_{1\to\infty}\\
&\quad\leqslant \|h_{t-s}^{-1}\rho_z^\lambda U_{V/\sigma}(t,r)\rho_z^{-\lambda}h_{t-s}^{1-2\theta}\|_{1/\theta\to\infty}\|h_{t-s}^{2\theta-1}\rho_z^\lambda U_{V/\sigma}(r,s)\rho_z^{-\lambda}h_{t-s}^{-1}\|_{1\to 1/\theta}\\
&\quad\leqslant c_{1,\theta}c_{\infty,\theta}\exp\bigl(\omega_0(t-s) + \tfrac{\lambda^2}{4a_0}(t-s)\bigr),
\end{aligned}$$

with $\omega_0 := \theta\omega_{\infty,\theta} + (1-\theta)\omega_{1,\theta}$ and $a_0 > 0$ defined by $\frac{1}{a_0} = \frac{1}{a_\theta} + \frac{1}{a_{1-\theta}}$, so $a_0 = \frac{a_\theta a_{1-\theta}}{a_\theta + a_{1-\theta}}$. Therefore,

$$p_{t,s}^{V/\sigma}(x,y) \leqslant c_{1,\theta}c_{\infty,\theta}h_{t-s}(x)h_{t-s}(y)\exp\bigl(\omega_0(t-s) - a_0\tfrac{d(x,y)^2}{t-s}\bigr)$$

for a.e. $x,y \in \Omega$ by Lemma 3.25. Using the estimate $p_{t,s}^V \leqslant \bigl(p_{t,s}^{V/\sigma}\bigr)^\sigma(p_{t,s})^{1-\sigma}$, we conclude that the asserted kernel bound holds with $c = (c_{1,\theta}c_{\infty,\theta})^\sigma c_1^{1-\sigma}$,

$$\omega = \sigma\omega_0 + (1-\sigma)A_1 = \sigma\bigl(\theta(1-\theta)(\omega_1+\omega_\infty) + \theta^2 A_\theta + (1-\theta)^2 A_{1-\theta}\bigr) + (1-\sigma)A_1$$

and $a = \sigma a_0 + (1-\sigma)a_1$. □

We conclude by demonstrating how Theorem 3.28 can be applied to stability of heat kernel estimates on Riemannian manifolds.

3.29 Remark. (a) Assume that Ω is a complete n-dimensional Riemannian manifold with non-negative Ricci curvature. Then by Bishop's comparison principle there exists $C_0 \geqslant 1$ such that

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant C_0\Bigl(\frac{R}{r}\Bigr)^n \qquad (x\in\Omega,\ R \geqslant r > 0),$$

and it follows that

$$\frac{\mu(B(x,r))}{\mu(B(y,r))} \leqslant \frac{\mu\bigl(B(y,r+d(x,y))\bigr)}{\mu(B(y,r))} \leqslant C_0\Bigl(1+\frac{d(x,y)}{r}\Bigr)^n \qquad (x,y\in\Omega,\ r>0).$$

Thus, if $h_t(x) = c\mu\bigl(B(x,\sqrt{t})\bigr)^{-1/2}$ then there exists $C \geqslant 1$ such that

$$\frac{h_t(x)}{h_t(y)} \leqslant C\Bigl(1+\frac{d(x,y)^2}{4t}\Bigr)^{n/4} \qquad (x,y\in\Omega,\ t>0), \tag{3.27}$$

and $h_{\theta t} \leqslant C_0^{1/2}\theta^{-n_0/4}h_t$ pointwise for all $t > 0$ and all $\theta \in (0,1]$, where $n_0 = n$. In the following we will only use these two properties of the functions h_t, without requiring $n_0 = n$. Note that (3.27) holds with $n = 0$ if the functions h_t are constant, as in the setting of Section 3.1. This is the reason why we have introduced the quantity n_0 in the second property of the h_t.

Let $r \geqslant 0$. Observe that $1 + r \leqslant \frac{1}{\varepsilon}(1 + \varepsilon r) \leqslant \frac{1}{\varepsilon} e^{\varepsilon r}$ for all $\varepsilon \in (0, 1]$ and $1 + r \leqslant e^{\varepsilon r}$ for all $\varepsilon > 1$, and hence $(1 + r)^\nu \leqslant (\frac{\nu}{\varepsilon} \vee 1)^\nu e^{\varepsilon r}$ for all $\varepsilon, \nu > 0$. Thus, from (3.27) we infer that

$$\frac{h_t(x)}{h_t(y)} \leqslant C\bigl(\tfrac{n}{4\varepsilon} \vee 1\bigr)^{n/4} \exp\Bigl(\varepsilon \frac{d(x,y)^2}{4t}\Bigr) \qquad (x, y \in \Omega,\ t, \varepsilon > 0). \tag{3.28}$$

Now assume that the kernel bound (3.19) holds with $\psi_{t,s}(x, y) = \frac{d(x,y)^2}{4(t-s)}$, i.e.,

$$p_{t,s}(x, y) \leqslant h_{t-s}(x) h_{t-s}(y) \exp\Bigl(A(t - s) - \frac{d(x, y)^2}{4(t - s)}\Bigr)$$

for all $(t, s) \in D_J'$ and a.e. $x, y \in \Omega$. Let $\varepsilon > 0$. Then for $\theta \in (0, 1)$ we obtain, using (3.28), that

$$\begin{aligned} p_{t,s}(x, y) \vee p_{t,s}(y, x) &\leqslant C_0 \theta^{-n_0/2} h_{(t-s)/\theta}(x) h_{(t-s)/\theta}(y) \exp\Bigl(A(t - s) - \frac{d(x, y)^2}{4(t - s)}\Bigr) \\ &\leqslant c_{\theta,\varepsilon} h_{(t-s)/\theta}(x)^{1/\theta} h_{(t-s)/\theta}(y)^{2-1/\theta} \exp\Bigl(A(t - s) - a_{\theta,\varepsilon} \frac{d(x, y)^2}{t - s}\Bigr), \end{aligned}$$

with $c_{\theta,\varepsilon} = C_0 \theta^{-n_0/2} \bigl(C(\frac{n}{4\varepsilon} \vee 1)^{n/4}\bigr)^{1/\theta - 1}$ and $a_{\theta,\varepsilon} = \frac{1}{4} - \frac{\varepsilon\theta}{4}(1/\theta - 1) \geqslant \frac{1}{4}(1 - \varepsilon)$.

One easily sees that the function $(0, 1) \ni \theta \mapsto \theta^\theta (1 - \theta)^{1-\theta}$ attains it minimum at $\theta = \frac{1}{2}$, so

$$c_{\theta,\varepsilon}^\theta c_{1-\theta,\varepsilon}^{1-\theta} = C_0 C \bigl(\theta^\theta (1 - \theta)^{1-\theta}\bigr)^{-n_0/2} \bigl(\tfrac{n}{4\varepsilon} \vee 1\bigr)^{n/4} \leqslant C_1 \bigl(1 + \tfrac{n}{4\varepsilon}\bigr)^{n/4},$$

where $C_1 = C_0 C 2^{n_0/2}$. Moreover, $\frac{1}{a_{\theta,\varepsilon}} + \frac{1}{a_{1-\theta,\varepsilon}} \leqslant \frac{8}{1-\varepsilon}$. Thus, if the potential V satisfies the assumptions of Theorem 3.28, then

$$p_{t,s}^V(x, y) \leqslant c\bigl(1 + \tfrac{n}{4\varepsilon}\bigr)^{\sigma n/4} h_{t-s}(x) h_{t-s}(y) \exp\Bigl(\omega(t - s) - a_\varepsilon \frac{d(x, y)^2}{t - s}\Bigr) \tag{3.29}$$

for all $(t, s) \in D_J'$ and a.e. $x, y \in \Omega$, where $c = \bigl(M_1^\theta M_\infty^{1-\theta} C_1\bigr)^\sigma$ (with $\theta := \frac{\gamma_1}{\sigma}$),

$$\begin{aligned} \omega &= \sigma\theta(1 - \theta)(\omega_1 + \omega_\infty) + \sigma(2\theta^2 - 2\theta + 1)A + (1 - \sigma)A \\ &= \sigma\theta(1 - \theta)(\omega_1 + \omega_\infty) + \bigl(1 - 2\sigma\theta(1 - \theta)\bigr)A \end{aligned}$$

and $a_\varepsilon = \sigma\frac{1-\varepsilon}{8} + (1 - \sigma)\frac{1}{4} = \frac{1-\sigma/2}{4} - \sigma\frac{\varepsilon}{8}$. Since the right-hand side of the estimate (3.29) is continuous in ε, we can choose ε such that $\frac{n}{\varepsilon} = \frac{d(x,y)^2}{t-s}$, so $\sigma\frac{\varepsilon}{8}\frac{d(x,y)^2}{t-s} = \frac{\sigma n}{8}$, and with $\beta := \frac{\sigma}{2} = \frac{1}{2}(\gamma_1 + \gamma_\infty)$ we conclude that

$$p_{t,s}^V(x, y) \leqslant c e^{\beta n/4} h_{t-s}(x) h_{t-s}(y) \Bigl(1 + \frac{d(x, y)^2}{4(t - s)}\Bigr)^{\beta n/2} \exp\Bigl(\omega(t - s) - (1 - \beta)\frac{d(x, y)^2}{4(t - s)}\Bigr)$$

for all $(t, s) \in D_J'$ and a.e. $x, y \in \Omega$.

Summarising, we have achieved the following improvement of the bound (3.21) on the perturbed kernel. An application of Theorem 3.23 leads to the condition $\beta > \gamma_1 \vee \gamma_\infty$

for the bound to be valid, whereas the above shows that the bound is also valid for $\beta > \frac{1}{2}(\gamma_1 + \gamma_\infty)$, or even for $\beta = \frac{1}{2}(\gamma_1 + \gamma_\infty)$ if one allows for the polynomial correction factor $\bigl(1 + \frac{d(x,y)^2}{4(t-s)}\bigr)^{\beta n/4}$. If V is an infinitesimally small Miyadera perturbation of U, and a backward Miyadera perturbation of U with bound $\gamma < 1$, then the condition $\beta > \gamma$ is replaced with the condition $\beta > \frac{\gamma}{2}$. This is a considerable improvement since $1 - \gamma$ may be close to 0, but $1 - \frac{\gamma}{2} > \frac{1}{2}$.

(b) Now assume, more generally, that Ω is a complete n-dimensional Riemannian manifold with lower bounded Ricci curvature. We indicate how the application of Theorem 3.28 presented in part (a) can be adapted to this situation; we point out that then the constants A_θ in the estimation will also depend on θ.

As in [Stu92b; proof of Lemma 2.2(a)] one obtains from Bishop's comparison principle and the explicit formula for the volume of balls in hyperbolic space there exists $\kappa > 0$ (depending on the bound for the Ricci curvature) such that

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant \frac{f_n(\kappa R)}{f_n(\kappa r)} \qquad (x \in \Omega,\ R \geqslant r > 0),$$

where $f_n(r) := \int_0^r \bigl(\sinh \frac{t}{n-1}\bigr)^{n-1} dt$ for all $r > 0$ (we assume that $n \geqslant 2$). One easily shows that there exists a constant $c_n > 1$ such that $\frac{1}{c_n} g_n \leqslant f_n \leqslant c_n g_n$, where $g_n(r) := (r \wedge 1)^n e^r$. From this one deduces that $\frac{f_n(R)}{f_n(r)} \leqslant c_n^2 \bigl(\frac{R}{r}\bigr)^n e^{R-r}$ for all $R \geqslant r > 0$, and as in (a) it follows that

$$\frac{\mu(B(x,r))}{\mu(B(y,r))} \leqslant \frac{f_n(\kappa r + \kappa d(x,y))}{f_n(\kappa r)} \leqslant c_n^2 \left(1 + \frac{d(x,y)}{r}\right)^n e^{\kappa d(x,y)} \qquad (x, y \in \Omega,\ r > 0) \quad (3.30)$$

(which improves on the estimate provided in [Stu92b; Lemma 2.2(a)]). Using the inequality $\kappa d(x,y) \leqslant \varepsilon \frac{d(x,y)^2}{4(t-s)} + \frac{1}{\varepsilon}\kappa^2(t-s)$, one can then proceed as in part (a) to verify the assumptions of Theorem 3.28 with suitable constants c_θ, A_θ and a_θ.

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