On the numerical range of generators of symmetric L_∞-contractive semigroups

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Abstract. A result by Liskevich and Perelmuter from 1995 yields the optimal angle of analyticity for symmetric submarkovian semigroups on L_p, $1 < p < \infty$. C. Kriegler showed in 2011 that the result remains true without the assumption of positivity of the semigroup. Here we give an elementary proof of Kriegler's result.

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1. Introduction

Let (Ω, μ) be a measure space and let A be a positive self-adjoint operator in $L_2(\Omega, \mu)$. Then $-A$ generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ which extends analytically to a contraction semigroup on the open right half plane. Such a semigroup is called a **symmetric L_∞-contractive semigroup**[1] if, in addition, one has

$$\|T(t)f\|_\infty \leqslant \|f\|_\infty, \quad \text{for all } f \in L_2(\Omega, \mu) \cap L_\infty(\Omega, \mu).$$

Then, by symmetry, the semigroup is also L_1-contractive, and by interpolation one obtains for each $1 < p < \infty$ a consistent C_0-semigroup $(T_p(t))_{t\geqslant 0}$ of contractions on $L_p(\Omega, \mu)$. Let the generator of this semigroup be denoted by $-A_p$, with domain $\operatorname{dom}(A_p)$.

In order to state the main result we define for $p \in [1, \infty)$ the mapping

$$F_p : \mathbb{C} \to \mathbb{C}, \qquad F_p(z) := \begin{cases} z|z|^{p-2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

[1]Such semigroups are called *diffusion semigroups* in [4] and *symmetric contraction semigroups* in [2].

Note that for $f \in \mathrm{L}_p(\Omega,\mu)$ with $\|f\|_p = 1$ the function $F_p(f) := F_p \circ f$ has the properties

$$\|F_p(f)\|_{p'} = 1, \qquad \int_\Omega f \cdot \overline{F_p(f)}\,\mathrm{d}\mu = 1,$$

where p' is the dual exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. In case that $p > 1$, $F_p(f)$ is uniquely characterized by these properties, and the **numerical range** of A_p is the set of numbers

$$\int_\Omega A_p f \cdot \overline{F_p(f)}\,\mathrm{d}\mu, \qquad \text{where} \quad f \in \mathrm{dom}(A_p),\ \|f\|_p = 1.$$

For $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ we define the sector

$$\Sigma(\varphi) := \big\{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leqslant \varphi\big\} \cup \{0\}$$

and call φ its **opening angle**. For $p \in (1,\infty)$, let $\Sigma_p := \Sigma(\varphi_p)$, where

$$\varphi_p := \arcsin|1 - \tfrac{2}{p}|.$$

Note that $\varphi_{p'} = \varphi_p$. Moreover, if p passes from 2 to 1 or to ∞, then $|1-\frac{2}{p}|$ passes from 0 to 1 and the opening angle $\varphi_p = \arcsin|1-\frac{2}{p}|$ of Σ_p passes from 0 to $\frac{\pi}{2}$. We point out that φ_p is smaller than the angle $\frac{\pi}{2}|1-\frac{2}{p}|$ that arises from interpolation between 0 and $\frac{\pi}{2}$. A short computation reveals that φ_p has the (often used) alternative representation

$$\varphi_p = \arcsin|1-\tfrac{2}{p}| = \arctan\tfrac{|p-2|}{2\sqrt{p-1}}.$$

Now, here is the main result.

Theorem 1. *Let $p \in (1,\infty)$ and let $-A_p$ be the generator on L_p of a symmetric L_∞-contractive semigroup on $\mathrm{L}_2(\Omega,\mu)$. Then the numerical range of A_p is contained in the sector Σ_p, and $(T_p(t))_{t\geqslant 0}$ extends to an analytic contraction semigroup on the sector with opening angle* $\arccos|1-\frac{2}{p}|$.

Under the additional assumption that the semigroup consists of positivity-preserving operators, Theorem 1 is due to Liskevich and Perelmuter [5]. The full result was established by Kriegler in [4] in the framework of non-commutative operator theory. Recently, Theorem 1 has been recovered by Carbonaro and Dragičević in [2] as a corollary of much stronger results. In [3], the first-named author streamlined and extended some of the methods used in [2] and showed that Theorem 1 can be deduced easily without making use of Bellman functions (which feature prominently in Carbonaro and Dragičević's work).

In the following, we shall present an essentially elementary proof of Theorem 1 extending the arguments from [5]. The relation to the other proofs shall be explained in Section 4 below. We note that the second assertion in Theorem 1 follows from the first by virtue of the Lumer-Phillips theorem and the exponential formula $T_p(t) = \text{s-lim}_{n\to\infty}(\mathrm{I}+\frac{t}{n}A_p)^{-n}$, cf. [1, Theorem 3.14]. Hence, it suffices to prove the first assertion.

2. A two-dimensional special case

Consider the special case $\Omega = \{1,2\}$ with measure $\mu = \delta_1 + \delta_2$ and the matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then $\mathrm{L}_p(\Omega,\mu) = \mathbb{C}^2$ with the usual p-norm and

$$\mathrm{e}^{-tA} = \tfrac{1}{2}\begin{pmatrix} 1+\mathrm{e}^{-2t} & 1-\mathrm{e}^{-2t} \\ 1-\mathrm{e}^{-2t} & 1+\mathrm{e}^{-2t} \end{pmatrix} \quad \text{for } t \geqslant 0.$$

Hence, $-A$ generates a (positivity-preserving) symmetric L_∞-contractive semigroup. For this special case, Theorem 1 reduces to the assertion

$$(w-z)\cdot\overline{F_p(w)-F_p(z)} \in \Sigma_p \quad \text{for all } z,w \in \mathbb{C}, \tag{2.1}$$

which will be established with the next lemma. Moreover, Lemma 2 also shows that the sector Σ_p in Theorem 1 is optimal already in this special case.

Lemma 2. *For all $p \in (1,\infty)$ and $z,w \in \mathbb{C}$ one has*

$$\mathrm{cl}\big\{(w-z)\cdot\overline{F_p(w)-F_p(z)} \,:\, z,w\in\mathbb{C}\big\} = \Sigma_p. \tag{2.2}$$

The inclusion $\subseteq$ in Lemma 2 has been proved originally by Liskevich and Perelmuter [5, Lemma 2.2]. We include a new proof that helps to understand the appearance of the angle φ_p.

Proof. Fix $p \in (1,\infty)$ and write $F = F_p$. To establish (2.1) we can restrict to the case that 0 is not on the line segment joining z and w; otherwise, $(w-z)\cdot\overline{F_p(w)-F_p(z)} \geq 0$. We identify, as usual, $\mathbb{C}$ with $\mathbb{R}^2$, and note that F is continuously $\mathbb{R}$-differentiable on $\mathbb{R}^2\setminus\{0\}$. Hence, abbreviating $h = w-z$, we obtain

$$F(z+h)-F(z) = \int_0^1 F'(z+th)h\,\mathrm{d}t,$$

where F' is the Jacobian matrix of F.

Since Σ_p is a closed convex cone (note that $\varphi_p \leqslant \frac{\pi}{2}$), it suffices to prove that

$$h\cdot\overline{F'(y)h} \in \Sigma_p \quad \text{for all } h\in\mathbb{R}^2,\ y\in\mathbb{R}^2\setminus\{0\}.$$

Now, a short elementary computation yields, for $0 \neq y \in \mathbb{R}^2$,

$$F'(y) = |y|^{p-2}A_y,$$

where $A_y := \mathrm{I}_2 + \frac{p-2}{|y|^2}\,yy^t$. The matrix A_y is symmetric and has eigenvalues 1 and $p-1>0$. (Indeed, $A_y y = (p-1)y$ and $A_y z = z$ for all $z \perp y$.) Thus, by Lemma 3 below,

$$h\cdot\overline{F'(y)h} \in \Sigma\big(\arcsin\tfrac{|p-2|}{p}\big) = \Sigma_p$$

for all $h\in\mathbb{R}^2$, and this concludes the proof of (2.1), i.e., the inclusion "$\subseteq$" in (2.2).

For the converse inclusion we denote

$$\Sigma := \mathrm{cl}\big\{(w-z)\cdot\overline{F(w)-F(z)} \,:\, z,w\in\mathbb{C}\big\}.$$

Since $F(tz) = t^{p-1}F(z)$ for all $t > 0$ and $z \in \mathbb{C}$ the set Σ is invariant under multiplication with $t > 0$, i.e., a cone. Hence, for all $z, h \in \mathbb{C} \setminus \{0\}$ and $t > 0$ we obtain

$$\frac{1}{t} h \cdot \overline{F(z + th) - F(z)} = \frac{1}{t^2}(th) \cdot \overline{F(z + th) - F(z)} \in \Sigma.$$

Letting $t \searrow 0$ we arrive at $h \cdot \overline{F'(z)h} \in \Sigma$, and another application of Lemma 3 completes the proof. $\square$

Lemma 3. *Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix with eigenvalues 1 and $\lambda > 0$. Then*

$$\big\{h \cdot \overline{Ah} \,:\, h \in \mathbb{C}\big\} = \Sigma\big(\arcsin \tfrac{|\lambda-1|}{\lambda+1}\big).$$

Proof. Note that $\big\{h \cdot \overline{Ah} \,:\, h \in \mathbb{C}\big\}$ is a cone in $\mathbb{C}$. Thus it suffices to show that

$$I := \big\{\arg\big(h \cdot \overline{Ah}\big) \,:\, h \in \mathbb{C} \setminus \{0\}\big\} = \big[-\arcsin \tfrac{|\lambda-1|}{\lambda+1}, \arcsin \tfrac{|\lambda-1|}{\lambda+1}\big]. \qquad (2.3)$$

Now observe that for $h \neq 0$, $\arg\big(h \cdot \overline{Ah}\big)$ equals $\sphericalangle(Ah, h)$, the signed angle between Ah and h. As a consequence, one may suppose without loss of generality that $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$. Then $I = \big\{\sphericalangle\big(A\binom{1}{x}, \binom{1}{x}\big) \,:\, x \in \mathbb{R}\big\}$ since $\sphericalangle\big(A\binom{0}{1}, \binom{0}{1}\big) = 0$. Setting $a := \arctan x$ and $b := \arctan(\lambda x)$ we obtain

$$\alpha_x := \sphericalangle\big(A\tbinom{1}{x}, \tbinom{1}{x}\big) = \sphericalangle\big(\tbinom{1}{\lambda x}, \tbinom{1}{x}\big) = a - b$$

and, by virtue of the addition formula for the sine,

$$(1 \pm \lambda)x = \tan a \pm \tan b = \frac{\sin a}{\cos a} \pm \frac{\sin b}{\cos b} = \frac{\sin(a \pm b)}{\cos a \cos b}.$$

Hence,

$$\sin \alpha_x = \sin(a - b) = \sin(a + b) \cdot \frac{1 - \lambda}{1 + \lambda}.$$

Note that the angle $a + b$ passes from $-\pi$ to π as x passes from $-\infty$ to ∞. Thus we obtain the identity (2.3), and the proof is complete. $\square$

Remark 4. A more geometric way to prove Lemma 3 consists in applying the law of the sines in the triangles $\triangle OBC$ and $\triangle OB'C$, where the points A, B, B', C and O are defined as $A := (1, 0)$, $B := (1, x)$, $B' := (1, -x)$, $C := (1, \lambda x)$ and $O := (0, 0)$. (The angle of interest α_x appears at O in the triangle $\triangle BOC$.)

3. Proof of Theorem 1

Let us now turn to the proof of Theorem 1.

Proof of Theorem 1. Fix $p \in (1, \infty)$ and write $\langle f, g\rangle = \int_\Omega f\overline{g}\,\mathrm{d}\mu$ for $f \in \mathrm{L}_p$ and $g \in \mathrm{L}_{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$. As above, we abbreviate $F(z) = F_p(z)$.

As noted already, the second assertion of Theorem 1 follows from the first by virtue of the Lumer–Phillips theorem. Hence, we have to show that

$$\langle A_p f, F(f)\rangle \in \Sigma_p, \quad \text{for all} \quad f \in \mathrm{dom}(A_p).$$

For this it suffices[2] to show

$$\big\langle(\mathrm{I}-T_p(t))f, F(f)\big\rangle \in \Sigma_p, \quad \text{for all} \quad f \in \mathrm{L}_p(\Omega,\mu),\ t>0,$$

since one can divide by t and let $t \searrow 0$. Moreover, it is sufficient to check this for the dense subset $\mathcal{D}$ of step functions

$$f = \sum_{j=1}^{n} c_j \mathbb{1}_{B_j} \tag{3.1}$$

where the sets B_j are pairwise disjoint measurable sets of positive and finite measure and $c_j \in \mathbb{C} \setminus \{0\}$. (In order to see this, take an arbitrary $f \in \mathrm{L}_p$ and a sequence $(f_n)_n$ of step functions with $\|f_n - f\|_p \to 0$, $f_n \to f$ almost everywhere and absolutely dominated by some $0 \leqslant g \in \mathrm{L}_p$. Then $F(f_n) \to F(f)$ almost everywhere and absolutely dominated by g^{p-1}, hence in $\mathrm{L}_{p'}$-norm.)[3]

Fix $t > 0$ and f as in (3.1), so that $F(f) = \sum_k F(c_k)\mathbb{1}_{B_k}$. Define $d_j := \langle \mathbb{1}_{B_j}, \mathbb{1}_{B_j}\rangle = \mu(B_j)$ and $a_{kj} = \langle T(t)\mathbb{1}_{B_j}, \mathbb{1}_{B_k}\rangle$ for $1 \leqslant j,k \leqslant n$. Then

$$\begin{aligned}
\big\langle(I-T(t))f, F(f)\big\rangle &= \sum\nolimits_{jk} c_j\overline{F(c_k)}\big\langle(I-T(t))\mathbb{1}_{B_j}, \mathbb{1}_{B_k}\big\rangle \\
&= \sum\nolimits_j d_j c_j \overline{F(c_j)} - \sum\nolimits_{jk} c_j \overline{F(c_k)} a_{kj} \\
&= \sum\nolimits_j \Big(d_j - \sum\nolimits_k |a_{kj}|\Big)\, c_j\overline{F(c_j)} \,+\, \sum\nolimits_{jk}\Big(c_j\overline{F(c_j)}|a_{kj}| - c_j\overline{F(c_k)}a_{kj}\Big).
\end{aligned}$$

We claim that the first sum satisfies

$$\sum\nolimits_j \Big(d_j - \sum\nolimits_k |a_{kj}|\Big)\, c_j\overline{F(c_j)} \geqslant 0.$$

Since $c_j\overline{F(c_j)} = |c_j|^p \geqslant 0$, it suffices to show that $\sum_k |a_{kj}| \leqslant d_j$. Choose λ_{kj} such that $|\lambda_{kj}| = 1$ and $a_{kj} = \lambda_{kj}|a_{kj}|$. Then

$$\sum\nolimits_k |a_{kj}| = \sum\nolimits_k \overline{\lambda_{kj}}\big\langle T(t)\mathbb{1}_{B_j}, \mathbb{1}_{B_k}\big\rangle = \big\langle T(t)\mathbb{1}_{B_j}, \sum\nolimits_k \lambda_{kj}\mathbb{1}_{B_k}\big\rangle,$$

and hence $\sum_k |a_{kj}| \leqslant \|T(t)\mathbb{1}_{B_j}\|_1 \|\sum_k \lambda_{kj}\mathbb{1}_{B_k}\|_\infty \leqslant \|\mathbb{1}_{B_j}\|_1 = d_j$, since $T(t)$ is an L_1-contraction.

In order to deal with the second sum, we note that, by symmetry,

$$a_{jk} = \langle T(t)\mathbb{1}_{B_k}, \mathbb{1}_{B_j}\rangle = \langle \mathbb{1}_{B_k}, T(t)\mathbb{1}_{B_j}\rangle = \overline{a_{kj}}.$$

[2] It is also necessary since $-(\mathrm{I}-T(t))$ is again the generator of a symmetric L_∞-contractive semigroup on $\mathrm{L}_2(\Omega,\mu)$, see [3, Section 3.1].

[3] Combining this with an argument involving subsequences shows that the mapping $f \mapsto F(f)$ is continuous from L_p to $\mathrm{L}_{p'}$.

Therefore and since $\lambda_{kj}F(c_j) = F(\lambda_{kj}c_j)$,

$$\begin{aligned}
&\sum_{j,k}\Big(c_j\overline{F(c_j)}|a_{kj}| - c_j\overline{F(c_k)}a_{kj}\Big)\\
&\quad = \frac{1}{2}\sum_{j,k}\Big(c_j\overline{F(c_j)}|a_{kj}| - c_j\overline{F(c_k)}a_{kj} + c_k\overline{F(c_k)}|a_{kj}| - c_k\overline{F(c_j)}\overline{a_{kj}}\Big)\\
&\quad = \frac{1}{2}\sum_{j,k}|a_{kj}|(\lambda_{kj}c_j - c_k)\big(\overline{F(\lambda_{kj}c_j)} - \overline{F(c_k)}\big) \in \Sigma_p
\end{aligned}$$

by Lemma 2. This concludes the proof. □

4. Relation to the Existing Proofs

Our elementary proof proceeds basically along the same reduction lines as the proof in [3]. In fact, the main ingredient in the proof given above was the fact (established in Lemma 2) that

$$(\lambda w - z) \cdot \overline{\lambda F(w) - F(z)} \in \Sigma_p$$

for all $w, z, \lambda \in \mathbb{C}$ with $|\lambda| = 1$. A short computation reveals that this is actually equivalent to Theorem 1 being valid for the special cases

$$A = \begin{pmatrix} 1 & -\overline{\lambda} \\ -\lambda & 1 \end{pmatrix}.$$

where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

The main difference to the paper [3] is that here we perform an immediate reduction to a finite atomic measure space similar as in [5], where [3], following [2], takes the detour via a compact model. To make this precise, let us consider as above the function f as in (3.1) and define the atomic measure space $\Omega' := \{1, \dots, n\}$ with $\mu' = \sum_{j=1}^{n} \mu(B_j)\delta_{\{j\}}$. On $\mathrm{L}_2(\Omega', \mu')$ consider the matrix

$$S = \big(\tfrac{a_{jk}}{\mu(B_j)}\big)_{j,k}$$

where $a_{jk} = \langle T(t)\mathbb{1}_{B_k}, \mathbb{1}_{B_j}\rangle$ as above. Let $v := (c_1 \ \dots \ c_n)^t$; then a short computation reveals that

$$\big\langle (\mathrm{I} - S)v, F(v)\big\rangle_{\mathrm{L}_2(\Omega',\mu')} = \big\langle (\mathrm{I} - T(t))f, F(f)\big\rangle_{\mathrm{L}_2(\Omega,\mu)}. \tag{4.1}$$

The operator S can be written as $J^*T(t)J$, where

$$J : \mathrm{L}_2(\Omega', \mu') \to \mathrm{L}_2(\Omega, \mu), \quad (c_j)_j \mapsto \sum_{j=1}^{n} c_j \mathbb{1}_{B_j}$$

is the natural isometric lattice embedding and J is its Hilbert space adjoint. (Note that from this observation it is straightforward that S is an L_1-contraction, a fact that has been proved in Section 3 by direct computation.)

Identity (4.1) implies that Theorem 1 is true in general if it is true for finite atomic measure spaces. Such spaces are in particular compact, and the remaining part of the proof in the previous section is nothing but an adaptation of the proof of [3, Theorem 4.15] to this special situation.

Remark 5. It is straightfoward to conjecture that also the general results of [3], Theorem 2.2-2.4, can be proved by a direct reduction to finite atomic measure spaces and avoiding the use of compact models and the sophisticated operator theory presented in Section 4 of [3]. This is indeed true, and will be the topic of a future publication.

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