

Sharp Gaussian upper bounds for Schrödinger semigroups on the half-line

Paul Holst and Hendrik Vogt

Abstract

In 1998, V. Liskevich and Y. Semenov proved sharp Gaussian upper bounds for Schrödinger semigroups on \mathbb{R}^3 with potentials satisfying a global Kato class condition. Using similar basic ideas we show sharp Gaussian upper bounds for Schrödinger semigroups on the half-line, also assuming a suitable global Kato class condition. Our proof strategy includes a new technique of weighted ultracontractivity estimates.

MSC 2020: 35J10, 35K08, 47A55, 47D06

Keywords: Gaussian upper bounds, boundary behaviour, Schrödinger semigroups, absorption semigroups, weighted ultracontractivity estimates

1 Introduction

In this paper we show a kernel estimate for the C_0 -semigroup generated by the Schrödinger operator $\Delta + V$ on $(0, \infty)$ for potentials V satisfying the integral condition given by (1.1) below. This integral condition amounts to a global Kato class condition with respect to the Dirichlet Laplacian on $(0, \infty)$; see Remark 4.6.

Throughout the paper let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $d \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^d$ be open, and let T be the C_0 -semigroup on $L_2(\Omega)$ generated by the Dirichlet Laplacian on Ω , i.e., by the operator Δ_D defined by

$$\begin{aligned} \text{dom}(\Delta_D) &:= \{u \in H_0^1(\Omega); \Delta u \in L_2(\Omega)\} \\ \Delta_D u &:= \Delta u. \end{aligned}$$

Our main result is as follows.

Theorem 1.1. *Let $d = 1$ and $\Omega = (0, \infty)$. Let $V: (0, \infty) \rightarrow \mathbb{R}$ be measurable, and assume that there exists $\alpha \in (0, 1)$ such that*

$$\int_0^\infty x|V(x)| dx \leq \alpha. \tag{1.1}$$

Then V is T -admissible, for every $t > 0$ the operator $T_V(t)$ has an integral kernel $k_t^V \in L_\infty((0, \infty) \times (0, \infty))$, and there exists $c > 0$ such that

$$0 \leq k_t^V(x, y) \leq c \left(1 \wedge \left(\frac{xy}{t} \left(1 + \frac{(x-y)^2}{4t} \right)^{3\alpha/4} \right) \right) t^{-1/2} e^{-(x-y)^2/4t} \quad (1.2)$$

for all $t > 0$ and a.e. $x, y \in (0, \infty)$.

Recall that k_t^V being an integral kernel of the operator $T_V(t)$ means that $T_V(t)f(x) = \int_0^\infty k_t^V(x, y)f(y) dy$ for all $f \in L_2(0, \infty)$ and a.e. $x \in (0, \infty)$. In Theorem 4.10 we will show that in the case $V \leq 0$ the integral condition (1.1) for some $\alpha < \infty$ is in fact necessary for the validity of the kernel estimate (1.2).

For $V = 0$ the kernel k_t^0 satisfies

$$\frac{1}{2} \left(1 \wedge \frac{xy}{t} \right) (4\pi t)^{-1/2} e^{-(x-y)^2/4t} \leq k_t^0(x, y) \leq \left(1 \wedge \frac{xy}{t} \right) (4\pi t)^{-1/2} e^{-(x-y)^2/4t},$$

see (3.2). The term $1 \wedge \frac{xy}{t}$ describes the boundary behaviour: it is ‘small’ if x and y are ‘close’ to the boundary $\{0\}$ of $(0, \infty)$ (where the meaning of ‘close’ depends on t). Observe that the Gaussian upper bound (1.2) is ‘almost sharp’; in particular, the factor 4 in the exponential term $e^{-(x-y)^2/4t}$ is the same as in k_t^0 . In comparison with k_t^0 , the major difference is the polynomial correction factor $(1 + (x-y)^2/4t)^{3\alpha/4}$ for the term $\frac{xy}{t}$. It is an open question whether this term can be avoided.

In the general context of ultracontractive self-adjoint C_0 -semigroups on metric measure spaces with the doubling property it is known that a Davies-Gaffney estimate implies the kernel estimate

$$k_t(x, y) \leq c \mu(B(x, \sqrt{t}))^{-1/2} \mu(B(y, \sqrt{t}))^{-1/2} \left(1 + \frac{\rho(x, y)^2}{4t} \right)^{(D-1)/2} e^{-\rho(x, y)^2/4t},$$

where μ is the measure, ρ the metric and D the dimension defined via the polynomial volume growth that holds because of the doubling property; see [Sik04, Remark 3 after Theorem 5]. (Note that in [Sik04, p. 651, line -2] the exponent should read $D - 1$ and not $(D-1)/2$.) In our context, the above result would lead to a kernel estimate similar to (1.2), but with the (larger!) exponent 1 instead of $3\alpha/4$ in the polynomial correction factor since we will be working with the measure $x^2 dx$ on $(0, \infty)$, for which one obtains $D = 3$. (We refer to [Mol75] for an example showing that in general the exponent $(D-1)/2$ is sharp.)

We point out that the 1 in the boundary term $1 \wedge \frac{xy}{t}$ goes without a correction factor. To that effect, the estimate (1.2) in Theorem 1.1 is in the spirit of [LiSe98, Cor. 1], one of the few results on kernel estimates that do not include a polynomial correction factor either. There is vast literature on kernel estimates including a polynomial correction factor; see for instance [DaPa89] and [Ouh06]. In [Ouh06] it was shown that in many circumstances, Gaussian upper bounds “automatically” improve to sharp bounds with a polynomial correction factor.

In the remainder of this introduction we explain the major steps in the proof of Theorem 1.1. We will first show a kernel estimate for Schrödinger semigroups on an arbitrary open set $\Omega \subseteq \mathbb{R}^d$ under the assumption that the potential V satisfies the form smallness condition

$$\int_{\Omega} |V||u|^2 dx \leq \alpha \langle -\Delta_D u, u \rangle = \alpha \|\nabla u\|_2^2 \quad (u \in \text{dom}(\Delta_D)) \quad (1.3)$$

for some $\alpha \in (0, 1)$ and that the Schrödinger semigroup T_V satisfies a suitable exponentially weighted L_1 -estimate; see Theorem 2.11. More precisely, we show the kernel estimate

$$0 \leq k_t^V(x, y) \leq ct^{-d/2} e^{-|x-y|^2/4t} \quad (t > 0, \text{ a.e. } x, y \in \Omega). \quad (1.4)$$

Note that there is no restriction on the dimension d in this kernel estimate. It will be proved by means of weighted ultracontractivity estimates and the well-known Davies trick (see [Dav87]); in Subsection 2.3 we develop new techniques that allow us to avoid the use of (weighted) sesquilinear forms that is common in this context.

In Section 3 we consider the special case in which Ω is the positive half-space, i.e.,

$$\Omega = \Omega_0 := \begin{cases} (0, \infty) & \text{if } d = 1, \\ (0, \infty) \times \mathbb{R}^{d-1} & \text{if } d \geq 2. \end{cases}$$

In Subsection 3.1 we prove exponentially weighted L_1 - and ultracontractivity estimates for the heat semigroup T with suitable weights. They will be needed later in that section, where we show a second kernel estimate for the Schrödinger semigroup T_V ,

$$0 \leq k_t^V(x, y) \leq cx_1y_1t^{-(d/2+1)} \left(1 + \frac{|x-y|^2}{4t}\right)^{\alpha(d+2)/4} e^{-|x-y|^2/4t} \quad (1.5)$$

for all $t > 0$ and a.e. $x, y \in \Omega_0$, again using the technique of weighted ultracontractivity estimates and Davies trick. This kernel estimate will be proved under condition (1.3) and the assumption that T_V satisfies suitable weighted L_1 -estimates; see Theorem 3.4.

Taking the minimum of (1.4) and (1.5) yields the kernel estimate

$$0 \leq k_t^V(x, y) \leq c \left(1 \wedge \left(\frac{x_1y_1}{t} \left(1 + \frac{|x-y|^2}{4t}\right)^{\alpha(d+2)/4}\right)\right) t^{-d/2} e^{-|x-y|^2/4t} \quad (1.6)$$

for all $t > 0$ and a.e. $x, y \in \Omega_0$, see Corollary 3.6. In the case $d = 1$, (1.6) is exactly the kernel estimate stated in Theorem 1.1 above. Therefore, for the proof of Theorem 1.1 it remains to show that (1.1) implies the assumptions indicated above for the validity of (1.4) and (1.5); this will be done in Section 4.

The basic idea for the proof of (1.4) is similar as in [LiSe98]. There the case of Schrödinger semigroups on $\Omega = \mathbb{R}^3$ is treated, based on the following main observation: for every $\xi \in \mathbb{R}^3$ the Green kernel G_ξ of the resolvent $(|\xi|^2 - \Delta_D)^{-1}$ of the Dirichlet Laplacian satisfies the weighted estimate $e^{\xi \cdot x} G_\xi(x, y) e^{-\xi \cdot y} \leq \lim_{\epsilon \rightarrow 0} G_\epsilon(x, y)$, for all $x, y \in \mathbb{R}^3$. In

our case $\Omega = (0, \infty) \subseteq \mathbb{R}^1$ an analogous weighted estimate is valid, as we will show in Subsection 4.2.

Notation. For $x, y \in [-\infty, \infty]$ we denote by $x \wedge y$ and $x \vee y$ the minimum and the maximum of x and y , respectively. Similarly, if A is a set and $f, g: A \rightarrow [-\infty, \infty]$, then $f \wedge g$ and $f \vee g$ is the pointwise minimum and maximum of f and g , respectively. Moreover we write $[f \geq g] := \{x \in A; f(x) \geq g(x)\}$; the sets $[f \leq g]$, $[f > g]$ and $[f = g]$ are defined in a similar way.

We write $\mathbb{C}_+ := \{z \in \mathbb{C}; \operatorname{Re} z > 0\}$ for the right complex half-plane.

If X, Y are two Banach spaces, then we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y . In the case $X = Y$ we simply write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

Now let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We write $\mathbf{1}_A$ for the indicator function of a set $A \in \mathcal{A}$. For a measurable function $\rho: \Omega \rightarrow [0, \infty)$ we denote by $\rho\mu$ the measure that has density ρ with respect to μ . If $p \in [1, \infty)$ and $V: \Omega \rightarrow \mathbb{K}$ is measurable, then we denote by V the associated multiplication operator on $L_p(\mu)$ as well. Furthermore, if $p, q, r \in [1, \infty]$ and $A: L_p(\mu) \rightarrow L_p(\mu)$ is linear, then we write

$$\|A\|_{q \rightarrow r} := \|A\|_{L_q(\mu) \rightarrow L_r(\mu)} := \sup\{\|Af\|_r; f \in L_p \cap L_q(\mu), \|f\|_q \leq 1\} \ (\in [0, \infty]).$$

More generally, if $\rho_1, \rho_2: \Omega \rightarrow (0, \infty)$ are measurable, we also write

$$\begin{aligned} & \|\rho_1^{-1}A\rho_1\|_{L_q(\rho_2\mu) \rightarrow L_r(\rho_2\mu)} \\ & := \sup\{\|\rho_1^{-1}A\rho_1f\|_{L_r(\rho_2\mu)}; f \in L_q(\rho_2\mu), \|f\|_{L_q(\rho_2\mu)} \leq 1, \rho_1f \in L_p(\mu)\}. \end{aligned}$$

In the case $\rho_2 = 1$ we simply write $\|\rho_1^{-1}A\rho_1\|_{q \rightarrow r} := \|\rho_1^{-1}A\rho_1\|_{L_q(\mu) \rightarrow L_r(\mu)}$.

Finally, let H be a Hilbert space. Then we denote the scalar product of x, y in H by $\langle x, y \rangle_H$. We also write $\langle x, y \rangle$ if it is obvious from the context that this is the scalar product on H .

2 Kernel estimates for Schrödinger semigroups via weighted ultracontractivity estimates

The ultimate goal of this section is the proof of the kernel estimate (1.4). To achieve this goal, we will show that the Schrödinger semigroup T_V provided with exponential weights is ultracontractive and then use the well-known Davies trick. The main tool to prove this ultracontractivity will be given in Subsection 2.3, where we show more generally for a positive self-adjoint C_0 -semigroup T on an arbitrary measure space and a suitable T -admissible potential V that a weighted ultracontractivity estimate for T implies a similar estimate for the perturbed C_0 -semigroup T_V ; see Theorem 2.7. The kernel estimate (1.4) will then be proved in Subsection 2.4. In Subsection 2.1 we first recall some important properties of admissible potentials. Subsection 2.2 provides an interpolation inequality for positive C_0 -semigroups, which we will prove using logarithmically convex functions. It will be important for the proof of the perturbation results in Subsection 2.3.

2.1 Admissibility of real-valued measurable potentials

The purpose of this subsection is to recall the notion of admissibility for real-valued measurable potentials, which was introduced in [Voi86, Sec. 2]. For this let (Ω, μ) be a measure space, $p \in [1, \infty)$ and T a positive C_0 -semigroup on $L_p(\mu)$ with generator A .

For $V \in L_\infty(\mu)$, the C_0 -semigroup generated by $A - V$ is denoted by T_V . If $V, W \in L_\infty(\mu)$, $V \geq W$, then $0 \leq T_V(t) \leq T_W(t)$ for all $t \geq 0$ (see [Voi86, Rem. 2.1(a)]); in particular, T_V is positive.

If now $V: \Omega \rightarrow [0, \infty)$ is measurable, then we have

$$0 \leq T_{V \wedge (n+1)}(t) \leq T_{V \wedge n}(t) \quad (t \geq 0)$$

and, therefore, $T_V(t) := \text{s-lim}_{n \rightarrow \infty} T_{V \wedge n}(t)$ exists for all $t \geq 0$ by dominated convergence. The function V is called *T-admissible* if $T_V: [0, \infty) \rightarrow \mathcal{L}(L_p(\mu))$ thus defined is a C_0 -semigroup.

Similarly, a measurable function $V: \Omega \rightarrow (-\infty, 0]$ is called *T-admissible* if $T_V(t) := \text{s-lim}_{n \rightarrow \infty} T_{V \vee (-n)}(t)$ exists for all $t \geq 0$ and $T_V: [0, \infty) \rightarrow \mathcal{L}(L_p(\mu))$ is a C_0 -semigroup.

More generally, a measurable function $V: \Omega \rightarrow \mathbb{R}$ is called *T-admissible* if V^+ and $-V^-$ are *T-admissible*. In this case $T_V(t) := \text{s-lim}_{n \rightarrow \infty} T_{(V \wedge n) \vee (-n)}(t)$ exists and $T_V: [0, \infty) \rightarrow \mathcal{L}(L_p(\mu))$ thus defined is a C_0 -semigroup; see [Voi88, Thm. 2.6]. (Note that for $V \in L_\infty(\mu)$ (resp. $V \geq 0$, $V \leq 0$) the two definitions of T_V given above coincide.)

The following basic properties of admissible potentials will be used throughout without further notice. Let $V: \Omega \rightarrow \mathbb{R}$ be *T-admissible*. Then T_V is positive because $T_{(V \wedge n) \vee (-n)}$ is positive for all $n \in \mathbb{N}$. If $W: \Omega \rightarrow \mathbb{R}$ is *T-admissible* and $W \leq V$, then $T_V(t) \leq T_W(t)$ for all $t \geq 0$; see [Voi88, Remark 2.7]. Finally, if $p = 2$ and T is self-adjoint, then T_V is self-adjoint. Indeed, $A - (V \wedge n) \vee (-n)$ is self-adjoint for all $n \in \mathbb{N}$, and thus $T_V(t) = \text{s-lim}_{n \rightarrow \infty} T_{V \wedge n}(t)$ is self-adjoint for all $t \geq 0$.

Note that for $p \in [1, \infty)$ and a measurable function $m: \Omega \rightarrow (0, \infty)$, the mapping $L_p(m^p \mu) \ni f \mapsto mf \in L_p(\mu)$ is an isometric lattice isomorphism. Thus, if T is a positive C_0 -semigroup on $L_p(\mu)$, then we can define a positive C_0 -semigroup T^m on $L_p(m^p \mu)$ by

$$T^m(t)f := m^{-1}T(t)mf \quad (t \geq 0, f \in L_p(m^p \mu)).$$

In the next lemma we characterize T^m -admissibility of a potential V by means of the C_0 -semigroup T .

Lemma 2.1. *Let $p \in [1, \infty)$, and let T be a positive C_0 -semigroup on $L_p(\mu)$ with generator A . Let $m: \Omega \rightarrow (0, \infty)$ and $V: \Omega \rightarrow \mathbb{R}$ be measurable. Then V is T^m -admissible if and only if V is T -admissible. Moreover, in this case one has $(T^m)_V = (T_V)^m$.*

Proof. Observe that the generator A^m of T^m is given by

$$\begin{aligned} \text{dom}(A^m) &:= \{f \in L_p(m^p \mu); mf \in \text{dom}(A)\}, \\ A^m f &:= m^{-1}A mf \quad (f \in \text{dom}(A^m)). \end{aligned}$$

For $W \in L_\infty(\mu) = L_\infty(m^p\mu)$ it is straightforward to show that $A^m + W = (A + W)^m$ and hence $(T^m)_W = (T_W)^m$. From this identity and the definition of admissibility one easily infers that $\pm V^\pm$ is T -admissible if and only if $\pm V^\pm$ is T^m -admissible, using that $L_p(m^p\mu) \ni f \mapsto \rho f \in L_p(\mu)$ is an isometry. This proves the first assertion. The second assertion then follows from the identity $(T^m)_{(V \wedge n) \vee (-n)} = (T_{(V \wedge n) \vee (-n)})^m$. \square

To conclude this subsection, we show that in the case $p = 2$, a potential V is T -admissible if it is form small with respect to the generator of T .

Proposition 2.2. *Let T be a positive C_0 -semigroup on $L_2(\mu)$ and A its generator. Let $V: \Omega \rightarrow \mathbb{R}$ be measurable, and assume that*

$$\int_{\Omega} |V||u|^2 d\mu \leq \operatorname{Re}\langle -Au, u \rangle \quad (u \in \operatorname{dom}(A)). \quad (2.1)$$

Then V is T -admissible. Moreover, T_V is contractive.

Proof. For every $n \in \mathbb{N}$, (2.1) implies that

$$\operatorname{Re}\langle (A + (V^\pm \wedge n))u, u \rangle \leq \operatorname{Re}\langle Au, u \rangle + \int_{\Omega} |V||u|^2 d\mu \leq 0 \quad (u \in \operatorname{dom}(A)),$$

so $A + (V^\pm \wedge n)$ is dissipative. It follows that $T_{-(V^\pm \wedge n)}$ is contractive for all $n \in \mathbb{N}$, so [Voi88, Prop. 2.2] shows that $-V^\pm$ is T -admissible. By [Voi88, Prop. 3.3(b)] it follows that V^+ and hence also V is T -admissible. The contractivity of T_V follows from $T_V \leq T_{-V^-}$ since $T_{-(V^- \wedge n)}$ is contractive for all $n \in \mathbb{N}$. \square

2.2 Logarithmically convex functions

Throughout let (Ω, μ) be a measure space. The aim of this subsection is to prove the following interpolation inequality.

Theorem 2.3. *Let $p \in (1, \infty)$, let T be a positive C_0 -semigroup on $L_p(\mu)$, and let $V: \Omega \rightarrow \mathbb{R}$ be T -admissible. Let $t > 0$, $p_0, p_1, q_0, q_1 \in [1, \infty]$, and assume that $\|T_{jV}(t)\|_{p_j \rightarrow q_j} < \infty$ for $j = 0, 1$. Let $\theta \in (0, 1)$, and define $p_\theta, q_\theta \in [1, \infty]$ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \quad (2.2)$$

Then

$$\|T_{\theta V}(t)\|_{p_\theta \rightarrow q_\theta} \leq \|T(t)\|_{p_0 \rightarrow q_0}^{1-\theta} \|T_V(t)\|_{p_1 \rightarrow q_1}^\theta.$$

For bounded potentials V this inequality is well-known and can be proved by means of the Stein interpolation theorem. The problem is that the inequality does not easily carry over from bounded to admissible potentials since one only has strong convergence $T_{V_n}(t) \rightarrow T_V(t)$, which does not imply convergence of the operator norms. To remedy this

problem we introduce the notion of logarithmic convexity that is motivated by [Haa07]. Our presentation largely follows [Vog10, Sec. 2.3].

Let X be an ordered vector space, i.e., X is a (real or complex) vector space endowed with a proper convex cone X_+ of positive elements, where X_+ being proper means that $X_+ \cap (-X_+) = \{0\}$. Let $I \subseteq \mathbb{R}$ be an interval. We say that a function $f: I \rightarrow X$ is *logarithmically convex* if

$$f((1-\theta)t_0 + \theta t_1) \leq (1-\theta)r^{-\theta}f(t_0) + \theta r^{1-\theta}f(t_1) \quad (r > 0) \quad (2.3)$$

for all $t_0, t_1 \in I$ and all $\theta \in (0, 1)$. By choosing $t_0 = t_1$ and $r \neq 1$ (so that $(1-\theta)r^{-\theta} + \theta r^{1-\theta} > 1$) we see that a logarithmically convex function f takes its values in X_+ .

The next lemma implies in particular that $f: I \rightarrow \mathbb{R}$ is logarithmically convex if and only if $f \geq 0$ and $\ln \circ f$ is convex, where we use the convention $\ln 0 := -\infty$. Thus, the assertion of Theorem 2.3 is that $[0, 1] \ni \theta \mapsto \|T_{\theta V}(t)\|_{p_\theta \rightarrow q_\theta} \in \mathbb{R}$ is logarithmically convex.

Lemma 2.4. *Let $M(\mu)$ be the ordered vector space of all scalar-valued measurable functions on Ω , where functions are identified if they coincide a.e., and $M(\mu)_+ = \{f \in M(\mu); f \geq 0 \text{ a.e.}\}$.*

(a) *A function $f: I \rightarrow M(\mu)_+$ is logarithmically convex if and only if*

$$f((1-\theta)t_0 + \theta t_1) \leq f(t_0)^{1-\theta} f(t_1)^\theta$$

a.e. for all $t_0, t_1 \in I$ and all $\theta \in (0, 1)$.

(b) *If $f, g: I \rightarrow M(\mu)_+$ are logarithmically convex, then $t \mapsto f(t)g(t)$ is logarithmically convex as well.*

Proof. (a) follows from Young's inequality: For $a, b \geq 0$ and $\theta \in (0, 1)$ we have

$$a^{1-\theta}b^\theta = (r^{-\theta}a)^{1-\theta}(r^{1-\theta}b)^\theta \leq (1-\theta)r^{-\theta}a + \theta r^{1-\theta}b \quad (r > 0)$$

and

$$a^{1-\theta}b^\theta = \inf\{(1-\theta)r^{-\theta}a + \theta r^{1-\theta}b; 0 < r \in \mathbb{Q}\}.$$

(b) is an immediate consequence of part (a). □

The next result, though being elementary, is the basis of the proof of Theorem 2.3. We assume that $L_p(\mu)$ and $\mathcal{L}(L_p(\mu))$ are endowed with their natural orderings; in particular, $\mathcal{L}(L_p(\mu))_+$ consists of the positive operators on $L_p(\mu)$. Let $\langle \cdot, \cdot \rangle_{p, p'}$ denote the natural bilinear map associated with the the dual pairing $\langle L_p(\mu), L_{p'}(\mu) \rangle$.

Proposition 2.5. *Let $p \in [1, \infty)$, $f: I \rightarrow L_p(\mu)$ and $S: I \rightarrow \mathcal{L}(L_p(\mu))$. Then*

(a) *f is logarithmically convex if and only if $t \mapsto \langle f(t), g \rangle_{p, p'}$ is logarithmically convex for all $g \in L_{p'}(\mu)_+$;*

(b) *S is logarithmically convex if and only if $t \mapsto S(t)h$ is logarithmically convex for all $h \in L_p(\mu)_+$.*

Proof. (a) is immediate from the following fact: A function $h \in L_p(\mu)$ is in $L_p(\mu)_+$ if and only if $\langle h, g \rangle_{p,p'} \geq 0$ for all $g \in L_{p'}(\mu)_+$.

(b) is clear. \square

Clearly, if $f: I \rightarrow X$ is logarithmically convex and $B: X \rightarrow X$ is a positive operator, then $B \circ f$ is logarithmically convex. In the case $X = L_p(\mu)$ we can prove the following more general result.

Lemma 2.6. *Let $p \in (1, \infty)$, and let $f: I \rightarrow L_p(\mu)$, $g: I \rightarrow L_{p'}(\mu)$ and $S: I \rightarrow \mathcal{L}(L_p(\mu))$ be logarithmically convex. Then*

(a) $I \ni t \mapsto \langle f(t), g(t) \rangle_{p,p'} \in \mathbb{R}$ is logarithmically convex.

(b) $I \ni t \mapsto S(t)f(t) \in L_p(\mu)$ is logarithmically convex.

(c) $I \ni t \mapsto \langle S(t)f(t), g(t) \rangle_{p,p'} \in \mathbb{R}$ is logarithmically convex.

Proof. (a) follows from Lemma 2.4(b) and the positivity of the linear operator $L_1(\mu) \ni f \mapsto \int f d\mu \in \mathbb{R}$.

(b) It follows from Proposition 2.5 that $I \ni t \mapsto S(t)' \in \mathcal{L}(L_{p'}(\mu))$ is logarithmically convex. Therefore, by part (a), $I \ni t \mapsto \langle S(t)f(t), h \rangle_{p,p'} = \langle f(t), S(t)'h \rangle_{p,p'} \in \mathbb{R}$ is logarithmically convex, for all $h \in L_{p'}(\mu)_+$, and the assertion follows by Proposition 2.5(a).

(c) follows immediately from parts (a) and (b). \square

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Fix $t > 0$. For $n \in \mathbb{N}$ let $V_n := (V \wedge n) \vee (-n)$. It follows from [Voi88, top of p. 121] and Proposition 2.5 that $[0, 1] \ni \theta \mapsto T_{\theta V_n}(t) \in \mathcal{L}(L_p(\mu))$ is logarithmically convex. Therefore, $[0, 1] \ni \theta \mapsto T_{\theta V}(t) = \text{s-lim}_{n \rightarrow \infty} T_{\theta V_n}(t) =: S(\theta) \in \mathcal{L}(L_p(\mu))$ is logarithmically convex. (Note that θV is T -admissible for all $\theta \in [0, 1]$, by [Voi88, Prop. 2.3].)

To prove the theorem we now show that $[0, 1] \ni \theta \mapsto \|S(\theta)\|_{p_\theta \rightarrow q_\theta} \in \mathbb{R}$ is logarithmically convex for any logarithmically convex function $S: [0, 1] \rightarrow \mathcal{L}(L_p(\mu))$ with $\|S(j)\|_{p_j \rightarrow q_j} < \infty$ for $j = 0, 1$. Let $\varphi, \psi \in S(\mu)_+$, where $S(\mu)$ denotes the vector space of simple functions on (Ω, μ) . Define $f: [0, 1] \rightarrow L_p(\mu)$ by $f(\theta) := \varphi^{1/p_\theta} \mathbf{1}_{[\varphi \neq 0]}$ and $g: [0, 1] \rightarrow L_{p'}(\mu)$ by $g(\theta) := \psi^{1/q'_\theta} \mathbf{1}_{[\psi \neq 0]}$. (The indicator functions are only needed for the cases $p_\theta = \infty$, $q'_\theta = \infty$, respectively.) Then f, g are logarithmically convex, so Lemma 2.6(c) implies that

$$[0, 1] \ni \theta \mapsto \langle S(\theta)f(\theta), g(\theta) \rangle_{p,p'} \in \mathbb{R}$$

is logarithmically convex. It easily follows that

$$[0, 1] \ni \theta \mapsto \sup_{\varphi, \psi \in M} \langle S(\theta)\varphi^{1/p_\theta} \mathbf{1}_{[\varphi \neq 0]}, \psi^{1/q'_\theta} \mathbf{1}_{[\psi \neq 0]} \rangle_{p,p'} \in \mathbb{R}$$

is logarithmically convex as well, where $M := \{\varphi \in S(\mu)_+; \|\varphi\|_1 \leq 1\}$.

To complete the proof, it remains to observe that for $q, r \in [1, \infty]$ and a positive operator $B \in \mathcal{L}(L_p(\mu))$ one has

$$\|B\|_{q \rightarrow r} = \sup_{\varphi, \psi \in M} \langle B\varphi^{1/q} \mathbf{1}_{[\varphi \neq 0]}, \psi^{1/r'} \mathbf{1}_{[\psi \neq 0]} \rangle_{p,p'}.$$

To see this identity, note that for $q < \infty$ the set $\{\varphi^{1/q}\mathbf{1}_{[\varphi \neq 0]}; \varphi \in M\} = \{h \in S(\mu)_+; \|h\|_q \leq 1\}$ is dense in $\{h \in L_p(\mu)_+; \|h\|_q \leq 1\}$. If $q = \infty$ then $\{\varphi^{1/\infty}\mathbf{1}_{[\varphi \neq 0]}; \varphi \in M\} = \{\mathbf{1}_A; A \text{ measurable, } \mu(A) < \infty\}$, and for all $h \in L_p(\mu)_+$ with $\|h\|_\infty \leq 1$ there exists an increasing sequence (A_n) of measurable sets of finite measure such that $h \leq \lim_{n \rightarrow \infty} \mathbf{1}_{A_n}$. Moreover

$$\|Bf\|_r = \sup_{\psi \in M} \langle Bf, \psi^{1/r'} \mathbf{1}_{[\psi \neq 0]} \rangle_{p,p'} \quad (f \in L_p(\mu)_+),$$

where in the case $r = \infty$ we use the fact that the set $[Bf \neq 0]$ is σ -finite; note that for $r = 1$ the above equality reads $\|Bf\|_1 = \sup\{\int_A Bf d\mu; A \text{ measurable, } \mu(A) < \infty\}$. \square

2.3 Weighted ultracontractivity estimates for perturbed C_0 -semigroups on L_2

In this subsection let $\mathbb{K} = \mathbb{C}$, and let (Ω, μ) be a measure space. The main result is the next theorem; we state and prove it in greater generality than actually needed for showing Theorem 1.1.

Note that, for a positive C_0 -semigroup T on $L_2(\mu)$ and measurable $V: \Omega \rightarrow \mathbb{R}$, the potential V is T -admissible if pV is T -admissible for some $p > 1$, by [Voi88, Prop. 2.3].

Theorem 2.7. *Let $\rho: \Omega \rightarrow (0, \infty)$ be measurable, and let T be a positive self-adjoint C_0 -semigroup on $L_2(\mu)$ satisfying*

$$\|\rho^{-\alpha} T(t) \rho^\alpha\|_{1 \rightarrow \infty} \leq ct^{-\nu} e^{\alpha^2 t}, \quad \|\rho^{-\alpha} T(t) \rho^\alpha\|_{2 \rightarrow 2} \leq e^{\alpha^2 t} \quad (t > 0, \alpha \in \mathbb{R})$$

for some $c, \nu > 0$. Let $V: \Omega \rightarrow \mathbb{R}$ be measurable, and assume that there exist $p > 1$, $M \geq 1$ and $r \geq 1$ such that pV is T -admissible, $\|T_{pV}(t)\|_{2 \rightarrow 2} \leq 1$,

$$\|\rho^{-\alpha} T_V(t) \rho^\alpha\|_{1 \rightarrow 1} \leq M e^{\alpha^2 t}, \quad \|\rho^{-\alpha} T_V(t) \rho^\alpha\|_{\infty \rightarrow \infty} \leq M e^{r\alpha^2 t} \quad (t > 0, \alpha > 0).$$

Then there exists $\tilde{c} > 0$ such that

$$\|\rho^{-\alpha} T_V(t) \rho^\alpha\|_{1 \rightarrow \infty} \leq \tilde{c} t^{-\nu} (1 + (r-1)\alpha^2 t)^{\nu/2p} e^{\alpha^2 t} \quad (t > 0, \alpha \in \mathbb{R}).$$

We first show an extrapolation result that extends [Cou90, Lemme 1], for semigroups that are not necessarily self-adjoint; in the proof of Theorem 2.7 it will be applied to the semigroup $t \mapsto e^{-\alpha^2 t} \rho^{-\alpha} T_V(t) \rho^\alpha$.

Proposition 2.8. *Let T be a one-parameter semigroup on $L_2(\mu)$. Assume that there exist $1 \leq p < q \leq \infty$, $c, \nu > 0$, $M \geq 1$ and $\omega \geq 0$ such that*

$$\|T(t)\|_{1 \rightarrow 1} \leq M, \quad \|T(t)\|_{\infty \rightarrow \infty} \leq M e^{\omega t}, \quad \|T(t)\|_{p \rightarrow q} \leq ct^{-\nu(\frac{1}{p} - \frac{1}{q})} \quad (t > 0).$$

Then T is ultracontractive,

$$\|T(t)\|_{1 \rightarrow \infty} \leq \tilde{c} t^{-\nu} (1 + \omega t)^{\nu/q} \quad (t > 0),$$

with a constant $\tilde{c} > 0$ depending only on p, q, c, ν and M .

Proof. (i) In the first step we show that

$$\|T(t)\|_{1 \rightarrow q} \leq c_0 t^{-\nu/q'} \quad (t > 0), \quad (2.4)$$

with $c_0 = 2^{\alpha\nu/q'} c^\alpha M$, $\alpha = \frac{1}{q' \frac{1}{p} - \frac{1}{q}}$. (Actually this is the first step in the proof of the extrapolation result [Cou90, Lemme 1], except that there no explicit constant c_0 is given. Since it will be crucial for us that c_0 depends only on p, q, c, ν and M , we present the extrapolation argument in full detail.)

Fix $f \in L_1 \cap L_\infty(\mu)$, $\|f\|_1 \leq 1$ and $t_0 > 0$. Since by assumption $\|T(t)\|_{p \rightarrow q} \leq ct^{-\nu(\frac{1}{p} - \frac{1}{q})}$ for all $t > 0$, there exists $\tilde{c}_0 \geq 0$ (depending on f and t_0 !) such that

$$\|T(t)f\|_q \leq \tilde{c}_0 t^{-\nu/q'} \quad (0 < t \leq t_0);$$

we choose the minimal constant \tilde{c}_0 making this estimate valid. Let $\theta \in [0, 1]$ be such that $\frac{1-\theta}{1} + \frac{\theta}{q} = \frac{1}{p}$, i.e. $\frac{1}{p'} = \frac{\theta}{q'}$. Then by Hölder's inequality

$$\|T(t/2)f\|_p \leq \|T(t/2)f\|_1^{1-\theta} \|T(t/2)f\|_q^\theta \leq M^{1-\theta} \tilde{c}_0^\theta (t/2)^{-\theta\nu/q'}$$

and thus

$$\begin{aligned} \|T(t)f\|_q &\leq \|T(t/2)\|_{p \rightarrow q} \|T(t/2)f\|_p \\ &\leq c(t/2)^{-\nu(\frac{1}{p} - \frac{1}{q})} \cdot M^{1-\theta} \tilde{c}_0^\theta (t/2)^{-\nu/p'} = cM^{1-\theta} \tilde{c}_0^\theta 2^{\nu/q'} t^{-\nu/q'} \end{aligned}$$

for all $0 < t \leq t_0$. By the choice of \tilde{c}_0 it follows that $\tilde{c}_0 \leq cM^{1-\theta} \tilde{c}_0^\theta 2^{\nu/q'}$ and hence $\tilde{c}_0^{1-\theta} \leq cM^{1-\theta} 2^{\nu/q'}$. Thus (2.4) is valid with $c_0 = 2^{\alpha\nu/q'} c^\alpha M$, $\alpha = \frac{1}{1-\theta} = \frac{1}{q' \frac{1}{p} - \frac{1}{q}}$ as asserted.

(ii) The one-parameter semigroup S on $L_2(\mu)$ defined by $S(t) := e^{-\omega t} T(t)^*$ satisfies $\|S(t)\|_{1 \rightarrow 1} \leq M$ and $\|S(t)\|_{q' \rightarrow p'} \leq ct^{-\nu(\frac{1}{p} - \frac{1}{q})}$ for all $t > 0$. Applying step (i) to S (and noting $\frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'}$) we obtain

$$\|T(t)\|_{p \rightarrow \infty} = \|T(t)^*\|_{1 \rightarrow p'} \leq c_1 t^{-\nu/p} e^{\omega t} \quad (t > 0),$$

with $c_1 = 2^{\beta\nu/p} c^\beta M$, $\beta = \frac{1}{p \frac{1}{p} - \frac{1}{q}}$. By the Riesz-Thorin interpolation theorem we infer that

$$\|T(t)\|_{q \rightarrow \infty} \leq \|T(t)\|_{p \rightarrow \infty}^{p/q} \|T(t)\|_{\infty \rightarrow \infty}^{1-p/q} \leq c_2 t^{-\nu/q} e^{\omega t} \quad (t > 0), \quad (2.5)$$

with $c_2 = c_1^{p/q} M^{1-p/q}$. Combining (2.4) and (2.5) we conclude that

$$\|T(t)\|_{1 \rightarrow \infty} \leq \|T((1-\varepsilon)t)\|_{1 \rightarrow q} \|T(\varepsilon t)\|_{q \rightarrow \infty} \leq c_0 ((1-\varepsilon)t)^{-\nu/q'} \cdot c_2 (\varepsilon t)^{-\nu/q} e^{\omega \varepsilon t}$$

for all $t > 0$, $\varepsilon \in (0, 1)$. For $\varepsilon := 1/(2 + \omega t)$ ($\leq 1/2$) we obtain

$$\|T(t)\|_{1 \rightarrow \infty} \leq c_0 2^{\nu/q'} t^{-\nu} \cdot c_2 (2 + \omega t)^{\nu/q} e \quad (t > 0),$$

and the asserted ultracontractivity estimate follows, with $\tilde{c} = 2^\nu e c_0 c_2$. \square

In the proof of Theorem 2.7 we will use the following result to show an L_2 -bound for the operators $\rho^{-\alpha}T_{pV}(t)\rho^\alpha$.

Proposition 2.9. *Let $\rho: \Omega \rightarrow (0, \infty)$ be measurable, and let T be a positive self-adjoint C_0 -semigroup on $L_2(\mu)$ satisfying*

$$\|\rho^{-\alpha}T(t)\rho^\alpha\|_{2 \rightarrow 2} \leq Me^{\alpha^2 t + \omega t} \quad (t \geq 0, \alpha \in \mathbb{R})$$

for some $M \geq 1$, $\omega \in \mathbb{R}$. Let $V: \Omega \rightarrow \mathbb{R}$ be T -admissible, and assume that T_V is contractive. Then

$$\|\rho^{-\alpha}T_V(t)\rho^\alpha\|_{2 \rightarrow 2} \leq e^{\alpha^2 t} \quad (t \geq 0, \alpha \in \mathbb{R}).$$

Proof. Let $n \in \mathbb{N}$ and $V_n := V \vee (-n)$; then the assumption implies that

$$\|\rho^{-\alpha}T_{V_n}(t)\rho^\alpha\|_{2 \rightarrow 2} \leq Me^{\alpha^2 t + (n + \omega)t} \quad (t \geq 0, \alpha \in \mathbb{R}).$$

Moreover, T_{V_n} is contractive because $T_{V_n} \leq T_V$, and T_{V_n} is self-adjoint. Thus T_{V_n} is an analytic semigroup of angle $\frac{\pi}{2}$, $\|T_{V_n}(z)\| \leq 1$ for all $z \in \mathbb{C}_+$. Now [Vog19, Prop. 2.3] yields the estimate $\|\rho^{-\alpha}T_{V_n}(t)\rho^\alpha\|_{2 \rightarrow 2} \leq e^{\alpha^2 t}$ for all $t \geq 0$ (independent of n !), and letting $n \rightarrow \infty$ we obtain the assertion. \square

The next result deals with strong continuity and admissibility for weighted semigroups.

Proposition 2.10. *Let $\rho: \Omega \rightarrow (0, \infty)$ be measurable, and let T be a positive C_0 -semigroup on $L_2(\mu)$ satisfying*

$$\|\rho^{-1}T(t)\rho\|_{2 \rightarrow 2} \leq Me^{\omega t} \quad (t \geq 0) \tag{2.6}$$

for some $M \geq 1$, $\omega \in \mathbb{R}$.

Then $\rho^{-1}T(t)\rho$ extends to a bounded operator $T^\rho(t)$ on $L_2(\mu)$, for all $t \geq 0$, and the family $(T^\rho(t))_{t \geq 0}$ thus defined is a C_0 -semigroup on $L_2(\mu)$. Moreover, if $V: \Omega \rightarrow \mathbb{R}$ is bounded from below and T -admissible, then V is T^ρ -admissible, $(T^\rho)_V = (T_V)^\rho$.

Proof. For $t \geq 0$ the operator $\rho^{-1}T(t)\rho$ is defined on the dense subspace $\text{dom}(\rho) = \{f \in L_2(\mu); \rho f \in L_2(\mu)\}$ and extends by continuity to a bounded operator $T^\rho(t)$ on $L_2(\mu)$, by (2.6). It is easy to see that T^ρ is a semigroup; we show that it is strongly continuous, arguing as in [Voi92, proof of Prop. 1]. If $f \in \text{dom}(\rho)$ and $g \in \text{dom}(\rho^{-1})$, then $t \mapsto \langle T^\rho(t)f, g \rangle (= \langle T(t)\rho f, \rho^{-1}g \rangle)$ is continuous. The bound (2.6) implies that the continuity carries over to all $f, g \in L_2(\mu)$. Thus, T^ρ is weakly continuous and hence strongly continuous; see [EnNa00, Thm. I.5.8].

For bounded V the identity $(T^\rho)_V = (T_V)^\rho$ follows from the Trotter product formula stated in [EnNa00, Exercise III.5.11]. In the case of unbounded $V \geq 0$ we infer that

$$(T^\rho)_V(t) = \text{s-lim}_{n \rightarrow \infty} (T^\rho)_{V \wedge n}(t) = \text{s-lim}_{n \rightarrow \infty} (T_{V \wedge n})^\rho(t) = (T_V)^\rho(t) \quad (t \geq 0);$$

then as above, the strong continuity of T_V implies that $(T_V)^\rho$ (and hence $(T^\rho)_V$) is strongly continuous, so V is T^ρ -admissible. The assertion in the general case follows similarly. \square

Now we can turn to the proof of Theorem 2.7.

Proof of Theorem 2.7. By Proposition 2.9, the assumed estimates $\|T_{pV}(t)\|_{2 \rightarrow 2} \leq 1$ and $\|\rho^{-\alpha}T(t)\rho^\alpha\|_{2 \rightarrow 2} \leq e^{\alpha^2 t}$ ($t > 0$, $\alpha \in \mathbb{R}$) imply that

$$\|\rho^{-\alpha}T_{pV}(t)\rho^\alpha\|_{2 \rightarrow 2} \leq e^{\alpha^2 t} \quad (t > 0, \alpha \in \mathbb{R}). \quad (2.7)$$

Now fix $\alpha > 0$. Let $n \in \mathbb{N}$ and $V_n := V \vee (-n)$. Applying Proposition 2.10 with ρ^α in place of ρ , we see that

$$T_{\rho, \alpha}(t) := e^{-\alpha^2 t} \rho^{-\alpha} T(t) \rho^\alpha \quad (t \geq 0)$$

defines a C_0 -semigroup $T_{\rho, \alpha}$ on $L_2(\mu)$. Moreover, for $s \in \{1, p\}$ the potential sV_n is $T_{\rho, \alpha}$ -admissible, and $(T_{\rho, \alpha})_{sV_n}(t) = e^{-\alpha^2 t} \rho^{-\alpha} T_{sV_n}(t) \rho^\alpha$ on $\text{dom}(\rho^\alpha)$ for all $t \geq 0$.

Let $q := 2p$ and observe that

$$\frac{1}{q'} = \frac{1/p'}{1} + \frac{1/p}{2}, \quad \frac{1}{q} = \frac{1/p'}{\infty} + \frac{1/p}{2}.$$

Note that $\|(T_{\rho, \alpha})(t)\|_{1 \rightarrow \infty} \leq ct^{-\nu}$ by assumption. The inequalities $T_{pV_n}(t) \leq T_{pV}(t)$ and (2.7) imply that $\|(T_{\rho, \alpha})_{pV_n}(t)\|_{2 \rightarrow 2} \leq 1$. Thus, using Theorem 2.3 we obtain

$$\|(T_{\rho, \alpha})_{V_n}(t)\|_{q' \rightarrow q} \leq \|(T_{\rho, \alpha})(t)\|_{1 \rightarrow \infty}^{1/p'} \|(T_{\rho, \alpha})_{pV_n}(t)\|_{2 \rightarrow 2}^{1/p} \leq c^{1/p'} t^{-\nu/p'} \quad (t > 0).$$

Moreover, since $T_{V_n} \leq T_V$, the assumptions imply

$$\|(T_{\rho, \alpha})_{V_n}(t)\|_{1 \rightarrow 1} \leq M, \quad \|(T_{\rho, \alpha})_{V_n}(t)\|_{\infty \rightarrow \infty} \leq M e^{(r-1)\alpha^2 t} \quad (t > 0).$$

Thus, applying Proposition 2.8 to the semigroup $(T_{\rho, \alpha})_{V_n}$ we obtain

$$\|e^{-\alpha^2 t} \rho^{-\alpha} T_{V_n}(t) \rho^\alpha\|_{1 \rightarrow \infty} \leq \tilde{c} t^{-\nu} (1 + (r-1)\alpha^2 t)^{\nu/2p} \quad (t > 0),$$

with a constant $\tilde{c} > 0$ depending only on p, c, ν and M . Now the assertion follows by taking the limit $n \rightarrow \infty$. \square

2.4 Kernel estimates for Schrödinger semigroups with exponentially weighted L_1 -bounds

In this subsection we show the kernel estimate (1.4). Throughout let $\Omega \subseteq \mathbb{R}^d$ be open, Δ_D the Dirichlet Laplacian on Ω and T the generated C_0 -semigroup on $L_2(\Omega)$. It is well-known that T is dominated by the free heat semigroup $T^{\mathbb{R}^d}$ on $L_2(\mathbb{R}^d)$, i.e., $T(t)(\mathbf{1}_\Omega f) \leq T^{\mathbb{R}^d}(t)f$ for all $t \geq 0$ and all $0 \leq f \in L_2(\mathbb{R}^d)$; see, e.g., [Ouh05, Prop. 4.23]. For $\xi \in \mathbb{R}^d$ let $\rho_\xi: \Omega \rightarrow (0, \infty)$ be defined by $\rho_\xi(x) := e^{\xi \cdot x}$ ($x \in \Omega$). Let $V: \Omega \rightarrow \mathbb{R}$ be measurable and assume that there exists $\alpha \in (0, 1)$ such that

$$\int_\Omega |V||u|^2 dx \leq \alpha \langle -\Delta_D u, u \rangle \quad (u \in \text{dom}(\Delta_D)); \quad (2.8)$$

note that then V is T -admissible by Proposition 2.2.

Further, let $\xi \in \mathbb{R}^d$, and for $t > 0$ let k_t be the integral kernel of $T(t)$. Then the integral kernel $k_{\xi,t}$ of $\rho_\xi T(t) \rho_\xi^{-1}$ is given by $k_{\xi,t}(x, y) = e^{\xi \cdot (x-y)} k_t(x, y)$, and since T is dominated by the free heat semigroup, we obtain

$$k_{\xi,t}(x, y) \leq (4\pi t)^{-d/2} e^{\xi \cdot (x-y) - |x-y|^2/4t} = (4\pi t)^{-d/2} e^{|\xi|^2 t - |x-y-2t\xi|^2/4t} \quad (x, y \in \Omega). \quad (2.9)$$

It easily follows that

$$\|\rho_\xi T(t) \rho_\xi^{-1}\|_{p \rightarrow p} \leq e^{|\xi|^2 t} \quad (t \geq 0, p \in [1, \infty]). \quad (2.10)$$

Now we can prove the kernel estimate (1.4).

Theorem 2.11. *Let (2.8) be true, and assume that there exists $M \geq 1$ such that*

$$\|\rho_\xi T_V(t) \rho_\xi^{-1}\|_{1 \rightarrow 1} \leq M e^{|\xi|^2 t} \quad (t \geq 0, \xi \in \mathbb{R}^d).$$

Then for every $t > 0$ the operator $T_V(t)$ has an integral kernel $k_t^V \in L_\infty(\Omega \times \Omega)$ such that

$$0 \leq k_t^V(x, y) \leq c t^{-d/2} e^{-|x-y|^2/4t} \quad (\text{a.e. } x, y \in \Omega)$$

for some $c > 0$ independent of t .

Proof. Without loss of generality let $\mathbb{K} = \mathbb{C}$.

It follows from (2.9) that

$$\|\rho_\xi T(t) \rho_\xi^{-1}\|_{1 \rightarrow \infty} \leq (4\pi t)^{-d/2} e^{|\xi|^2 t} \quad (t > 0, \xi \in \mathbb{R}^d);$$

Expressed differently,

$$\|\rho_\xi^{-\beta} T(t) \rho_\xi^\beta\|_{1 \rightarrow \infty} \leq (4\pi t)^{-d/2} e^{\beta^2 t} \quad (t > 0, \beta \in \mathbb{R})$$

for all $\xi \in \mathbb{R}^d$ with $|\xi| = 1$. Moreover, by (2.10) we have

$$\|\rho_\xi^{-\beta} T(t) \rho_\xi^\beta\|_{2 \rightarrow 2} \leq e^{\beta^2 t} \quad (t \geq 0, \beta \in \mathbb{R})$$

for all $\xi \in \mathbb{R}^d$ with $|\xi| = 1$.

Let $p := 1/\alpha (> 1)$. Then, by (2.8) and Proposition 2.2, the potential pV is T -admissible and T_{pV} is contractive. Since

$$\|\rho_\xi^{-\beta} T_V(t) \rho_\xi^\beta\|_{1 \rightarrow 1} \leq M e^{\beta^2 t} \quad (t \geq 0, \beta \in \mathbb{R})$$

for all $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ by hypothesis and thus also

$$\|\rho_\xi^{-\beta} T_V(t) \rho_\xi^\beta\|_{\infty \rightarrow \infty} \leq M e^{\beta^2 t} \quad (t \geq 0, \beta \in \mathbb{R})$$

by duality and the self-adjointness of T_V , we can now apply Theorem 2.7 (with $r = 1$) and conclude that there exists $c > 0$ such that

$$\|\rho_\xi^{-\beta} T_V(t) \rho_\xi^\beta\|_{1 \rightarrow \infty} \leq ct^{-d/2} e^{\beta^2 t} \quad (t > 0, \beta \in \mathbb{R})$$

for all $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, i.e., $\|\rho_\xi T_V(t) \rho_\xi^{-1}\|_{1 \rightarrow \infty} \leq ct^{-d/2} e^{|\xi|^2 t}$ for all $t > 0$ and all $\xi \in \mathbb{R}^d$. Hence, the Dunford-Pettis theorem implies that for every $t > 0$ the operator $T_V(t)$ has an integral kernel $k_t^V \in L_\infty(\Omega \times \Omega)$ such that

$$0 \leq k_t^V(x, y) \leq ct^{-d/2} e^{-\xi \cdot (x-y)} e^{|\xi|^2 t} \quad (\text{a.e. } x, y \in \Omega)$$

for all $\xi \in \mathbb{R}^d$. The assertion now follows from the well-known Davies trick by putting $\xi = (x - y)/2t$. \square

3 Kernel estimates for Schrödinger semigroups: the boundary term

In this section we prove the kernel estimate (1.5). We apply Theorem 2.7 to a weighted Schrödinger semigroup on the positive half-space provided with the weighted measure $x_1^2 dx$ and then use Davies trick. In order to be able to apply Theorem 2.7, we need to show an exponentially weighted ultracontractivity estimate for this semigroup, which is topic of Subsection 3.2. The kernel estimate (1.5) is also shown in Subsection 3.2.

3.1 L_1 -contractivity and ultracontractivity of the heat semigroup on the positive half-space with weight x_1

In this subsection let

$$\Omega_0 := \begin{cases} (0, \infty) & \text{if } d = 1, \\ (0, \infty) \times \mathbb{R}^{d-1} & \text{if } d \geq 2. \end{cases}$$

Let T be the C_0 -semigroup on $L_2(\Omega_0)$ generated by the Dirichlet Laplacian on Ω_0 , and let $m: \Omega_0 \rightarrow (0, \infty)$ be defined by $m(x) := x_1$ ($x \in \Omega_0$). As in Lemma 2.1 we define the unitarily transformed semigroup T^m on $L_2(\Omega_0, m^2 \lambda^d)$ by $T^m(t)f := m^{-1}T(t)mf$, where λ^d is the Lebesgue measure on Ω_0 . We show that the semigroup T^m satisfies exponentially weighted L_1 - and ultracontractivity estimates on $(\Omega_0, m^2 \lambda^d)$. Later, in Section 4, we will use these estimates to show corresponding estimates for the perturbed semigroup $(T^m)_V$, in dimension $d = 1$.

We will use the fact that for every $t > 0$ the operator $T(t)$ has the integral kernel $k_t: \Omega_0 \times \Omega_0 \rightarrow (0, \infty)$ defined by

$$k_t(x, y) := (4\pi t)^{-d/2} (e^{-|x-y|^2/4t} - e^{-|Sx-y|^2/4t}) \quad (x, y \in \Omega_0),$$

where $S : \Omega_0 \rightarrow \mathbb{R}^d$ is defined by $Sx := (-x_1, x_2, \dots, x_d)$. This can be seen by a reflection principle. Observe that

$$k_t(x, y) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t} (1 - e^{-x_1 y_1/t}) \quad (x, y \in \Omega_0). \quad (3.1)$$

Thus, by the elementary inequality $\frac{1}{2}(1 \wedge r) \leq 1 - e^{-r} \leq 1 \wedge r$ ($r \geq 0$), we have

$$\frac{1}{2} \left(1 \wedge \frac{x_1 y_1}{t}\right) (4\pi t)^{-d/2} e^{-|x-y|^2/4t} \leq k_t(x, y) \leq \left(1 \wedge \frac{x_1 y_1}{t}\right) (4\pi t)^{-d/2} e^{-|x-y|^2/4t} \quad (3.2)$$

for all $x, y \in \Omega_0$.

To show that T^m satisfies exponentially weighted L_1 -estimates, we will use the following properties of the kernel k_t .

Lemma 3.1. (a) *Let $t > 0$ and $y \in \Omega_0$. Then*

$$\int_{\Omega_0} \frac{x_1}{y_1} k_t(x, y) dx = 1.$$

(b) *Let $t > 0$, $y \in \Omega_0$ and $\xi \in \mathbb{R}^d$. Then*

$$\int_{\Omega_0} \frac{x_1}{y_1} e^{\xi \cdot (x-y)} k_t(x, y) dx \leq 2^{d/2+1} e^{2|\xi|^2 t}.$$

Moreover, if $\xi_1 \leq 0$, then

$$\int_{\Omega_0} \frac{x_1}{y_1} e^{\xi \cdot (x-y)} k_t(x, y) dx \leq e^{|\xi|^2 t}.$$

Proof. (a) Note that in the case $d > 1$ we have

$$\begin{aligned} & \int_{\Omega_0} \frac{x_1}{y_1} k_t(x, y) dx \\ &= \int_0^\infty \frac{x_1}{y_1} (4\pi t)^{-1/2} e^{-(x_1 - y_1)^2/4t} (1 - e^{-x_1 y_1/t}) \int_{\mathbb{R}^{d-1}} (4\pi t)^{-(d-1)/2} e^{-|\widehat{x} - \widehat{y}|^2/4t} d\widehat{x} dx_1 \\ &= \int_0^\infty \frac{x_1}{y_1} (4\pi t)^{-1/2} e^{-(x_1 - y_1)^2/4t} (1 - e^{-x_1 y_1/t}) dx_1 \end{aligned}$$

for all $t > 0$ and all $y = (y_1, \widehat{y}) \in (0, \infty) \times \mathbb{R}^{d-1} = \Omega_0$. Hence, we may assume that $d = 1$.

Now let $t, y \in (0, \infty)$. Then

$$\int_0^\infty \frac{x}{y} k_t(x, y) dx = (4\pi t)^{-1/2} y^{-1} \left(\int_0^\infty x e^{-(x-y)^2/4t} dx - \int_0^\infty x e^{-(-x-y)^2/4t} dx \right).$$

Since

$$- \int_0^\infty x e^{-(-x-y)^2/4t} dx = \int_{-\infty}^0 x e^{-(x-y)^2/4t} dx,$$

we conclude that

$$\begin{aligned} \int_0^\infty \frac{x}{y} k_t(x, y) dx &= (4\pi t)^{-1/2} y^{-1} \int_{-\infty}^\infty x e^{-(x-y)^2/4t} dx = (4\pi t)^{-1/2} y^{-1} \int_{-\infty}^\infty (z+y) e^{-z^2/4t} dz \\ &= (4\pi t)^{-1/2} \int_{-\infty}^\infty e^{-z^2/4t} dz = 1. \end{aligned}$$

(b) We observe that the Peter-Paul inequality (i.e. $ab \leq \frac{1}{2}(\gamma a^2 + \gamma^{-1} b^2)$ for all $a, b \in \mathbb{R}$, $\gamma > 0$) implies

$$\xi \cdot (x - y) \leq 2|\xi|^2 t + \frac{1}{8}|x - y|^2 \quad (x \in \Omega_0).$$

Moreover, since the function $[0, \infty) \ni r \mapsto 1 - e^{-r} \in [0, \infty)$ is concave and takes the value 0 for $r = 0$, we have that

$$1 - e^{-r} \geq \frac{1}{2}(1 - e^{-2r}) \quad (r \geq 0).$$

Using these inequalities we estimate

$$\begin{aligned} (4\pi t)^{-d/2} \int_{\Omega_0} \frac{x_1}{y_1} e^{\xi \cdot (x-y)} e^{-|x-y|^2/4t} (1 - e^{-x_1 y_1/t}) dx \\ \leq 2^{d/2} (8\pi t)^{-d/2} e^{2|\xi|^2 t} \int_{\Omega_0} \frac{x_1}{y_1} e^{-|x-y|^2/8t} 2(1 - e^{-x_1 y_1/2t}) dx \\ = 2^{d/2+1} e^{2|\xi|^2 t} \int_{\Omega_0} \frac{x_1}{y_1} k_{2t}(x, y) dx. \end{aligned}$$

This completes the proof of the first assertion of (b), by part (a).

For the proof of the second assertion we put

$$G(\xi) := (4\pi t)^{-d/2} \int_{\Omega_0} \frac{x_1}{y_1} e^{-|x-y|^2/4t + \xi \cdot (x-y) - |\xi|^2 t} (1 - e^{-x_1 y_1/t}) dx \quad (\xi \in \mathbb{R}^d);$$

we have to show that $G(\xi) \leq 1$ if $\xi_1 \leq 0$. Let $F: \mathbb{R}^d \rightarrow [0, \infty)$ be defined by $F(x) := (4\pi t)^{-d/2} \frac{x_1}{y_1} (1 - e^{-x_1 y_1/t})$ if $x \in \Omega_0$ and $F(x) := 0$ if $x \in \mathbb{R} \setminus \Omega_0$. Then, using the identity

$$-\frac{1}{4t}|x - y|^2 + \xi \cdot (x - y) - |\xi|^2 t = -\frac{1}{4t}|x - y - 2\xi t|^2 \quad (3.3)$$

and a change of variables, we can rewrite $G(\xi)$ as

$$G(\xi) = \int_{\Omega_0} F(x) e^{-|x-y-2\xi t|^2/4t} dx = \int_{\mathbb{R}^d} F(z + y + 2\xi t) e^{-|z|^2/4t} dz \quad (\xi \in \mathbb{R}^d).$$

Therefore, since F is monotone increasing in the x_1 -variable and $F(x) = F(x_1, 0, \dots, 0)$ for all $x \in \mathbb{R}^d$, $G: \mathbb{R}^d \rightarrow [0, \infty)$ is increasing in the ξ_1 -variable and $G(\xi) = G(\xi_1, 0, \dots, 0)$ for all $\xi \in \mathbb{R}^d$, which in turn implies that $G(\xi) \leq G(0)$ for all $\xi \in \mathbb{R}^d$, $\xi_1 \leq 0$. Now $G(0) = \int_{\Omega_0} \frac{x_1}{y_1} k_t(x, y) dx = 1$ by part (a), so we conclude that the asserted estimate $G(\xi) \leq 1$ holds for $\xi_1 \leq 0$. \square

Now we turn to the proof of the exponentially weighted L_1 - and ultracontractivity estimates for T^m .

Proposition 3.2. (a) *Let $\xi \in \mathbb{R}^d$ and $t \geq 0$. Then*

$$\|m^{-1}\rho_\xi T(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda^d) \rightarrow L_1(m^2\lambda^d)} \leq 2^{d/2+1}e^{2|\xi|^2t}.$$

Moreover, if $\xi_1 \leq 0$, then

$$\|m^{-1}\rho_\xi T(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda^d) \rightarrow L_1(m^2\lambda^d)} \leq e^{|\xi|^2t}.$$

(b) *Let $\xi \in \mathbb{R}^d$ and $t > 0$. Then*

$$\|m^{-1}\rho_\xi T(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda^d) \rightarrow L_\infty(m^2\lambda^d)} \leq (4\pi)^{-d/2}t^{-(d/2+1)}e^{|\xi|^2t}.$$

Proof. (a) For $t = 0$ there is nothing to show, so we assume that $t > 0$. Then k_t is the integral kernel of $T(t)$; hence for $f \in L_1(m^2\lambda^d)$ with $\rho_\xi^{-1}mf \in L_2(\Omega_0)$ we obtain

$$\begin{aligned} \|m^{-1}\rho_\xi T(t)\rho_\xi^{-1}mf\|_{L_1(m^2\lambda^d)} &\leq \int_{\Omega_0} \int_{\Omega_0} x_1^{-1}e^{\xi \cdot x}k_t(x, y)e^{-\xi \cdot y}y_1|f(y)|dy d(m^2\lambda^d)(x) \\ &= \int_{\Omega_0} \int_{\Omega_0} \frac{x_1}{y_1}e^{\xi \cdot (x-y)}k_t(x, y)dx f(y) d(m^2\lambda^d)(y). \end{aligned} \quad (3.4)$$

Consequently, the assertions of (a) follow from Lemma 3.1(b).

(b) Let $f \in L_1(m^2\lambda^d)$ be such that $\rho_\xi^{-1}mf \in L_2(\Omega_0)$. Then we estimate, using (3.2) for the first and (3.3) for the second inequality,

$$\begin{aligned} |m^{-1}\rho_\xi T(t)\rho_\xi^{-1}mf(x)| &\leq \int_{\Omega_0} \frac{1}{x_1y_1} \cdot \frac{x_1y_1}{t}(4\pi t)^{-d/2}e^{\xi \cdot (x-y)}e^{-|x-y|^2/4t}|f(y)|d(m^2\lambda^d)(y) \\ &\leq (4\pi)^{-d/2}t^{-(d/2+1)}e^{|\xi|^2t}\|f\|_{L_1(m^2\lambda^d)} \quad (\text{a.e. } x \in \Omega_0). \quad \square \end{aligned}$$

Remark 3.3. Note that Proposition 3.2(a) with $\xi = 0$ implies that T^m is L_1 -contractive. Actually, it follows from the identity (3.4) and Lemma 3.1(a) that T^m is even stochastic, i.e. $\|T^m(t)f\|_{L_1(m^2\lambda^d)} = \|f\|_{L_1(m^2\lambda^d)}$ for all $f \in L_1 \cap L_2(m^2\lambda^d)_+$.

3.2 Kernel estimates for Schrödinger semigroups with weighted L_1 -bounds on the positive half-space

Throughout this subsection let Ω_0 , m and ρ_ξ be defined as in Subsection 3.1, and let T be the C_0 -semigroup on $L_2(\Omega_0)$ generated by the Dirichlet Laplacian Δ_D on Ω_0 . Let $V: \Omega \rightarrow \mathbb{R}$ be measurable, and assume that V satisfies the form smallness condition (2.8) for some $\alpha \in (0, 1)$, with $\Omega = \Omega_0$. We will now show the kernel estimate (1.5) and then derive the kernel estimate (1.6).

Theorem 3.4. *Let (2.8) be true, and assume that there exists $M \geq 1$ such that*

$$\begin{aligned} \|m^{-1}\rho_\xi T_V(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda^d)\rightarrow L_1(m^2\lambda^d)} &\leq Me^{2|\xi|^2t} & (t \geq 0, \xi \in \mathbb{R}^d, \xi_1 \geq 0), \\ \|m^{-1}\rho_\xi T_V(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda^d)\rightarrow L_1(m^2\lambda^d)} &\leq Me^{|\xi|^2t} & (t \geq 0, \xi \in \mathbb{R}^d, \xi_1 \leq 0). \end{aligned}$$

Then for every $t > 0$ the operator $T_V(t)$ has an integral kernel $k_t^V \in L_\infty(\Omega \times \Omega)$ such that there exists $c > 0$ independent of t with

$$0 \leq k_t^V(x, y) \leq cx_1y_1t^{-(d/2+1)} \left(1 + \frac{|x-y|^2}{4t}\right)^{\alpha(d+2)/4} e^{-|x-y|^2/4t}$$

for a.e. $x, y \in \Omega_0$.

For the proof of Theorem 3.4 we define the unitarily transformed semigroup T_V^m on $L_2(\Omega_0, m^2\lambda^d)$ by $T_V^m(t)f := m^{-1}T_V(t)mf$ ($t \geq 0, f \in L_2(m^2\lambda^d)$). By Lemma 2.1, T_V^m is a C_0 -semigroup and V is T^m -admissible with $(T^m)_V = T_V^m$. (Recall from Proposition 2.2 that V is T -admissible.)

Proof of Theorem 3.4. Without loss of generality let $\mathbb{K} = \mathbb{C}$.

Let $\xi \in \mathbb{R}^d, |\xi| = 1$. By Proposition 3.2(b) we have

$$\|\rho_\xi^{-\beta}T^m(t)\rho_\xi^\beta\|_{L_1(m^2\lambda^d)\rightarrow L_\infty(m^2\lambda^d)} \leq (4\pi)^{-d/2}t^{-(d/2+1)}e^{\beta^2t} \quad (t > 0, \beta \in \mathbb{R}).$$

From (2.10) we deduce that

$$\|\rho_\xi^{-\beta}T^m(t)\rho_\xi^\beta\|_{L_2(m^2\lambda^d)\rightarrow L_2(m^2\lambda^d)} \leq e^{\beta^2t} \quad (t \geq 0, \beta \in \mathbb{R})$$

since $L_2(m^2\lambda) \ni f \mapsto mf \in L_2(0, \infty)$ is an isometric isomorphism. Moreover, the hypotheses imply that

$$\|\rho_\xi^{-\beta}(T^m)_V(t)\rho_\xi^\beta\|_{L_1(m^2\lambda^d)\rightarrow L_1(m^2\lambda^d)} \leq Me^{\beta^2t} \quad (t \geq 0, \beta > 0)$$

if $\xi_1 \geq 0$, and

$$\|\rho_\xi^{-\beta}(T^m)_V(t)\rho_\xi^\beta\|_{L_1(m^2\lambda^d)\rightarrow L_1(m^2\lambda^d)} \leq Me^{2\beta^2t} \quad (t \geq 0, \beta > 0)$$

if $\xi_1 \leq 0$; by duality and self-adjointness of $(T^m)_V$ the latter is equivalent to

$$\|\rho_\xi^{-\beta}(T^m)_V(t)\rho_\xi^\beta\|_{L_\infty(m^2\lambda^d)\rightarrow L_\infty(m^2\lambda^d)} \leq Me^{2\beta^2t} \quad (t \geq 0, \beta > 0)$$

for $\xi_1 \geq 0$.

Let $p := 1/\alpha (> 1)$. Then pV is T^m -admissible and $(T^m)_{pV}$ is contractive, by Proposition 2.2 and Lemma 2.1. Hence, Theorem 2.7 implies that there exists $c > 0$ such that

$$\|\rho_\xi^{-\beta}T_V^m(t)\rho_\xi^\beta\|_{L_1(m^2\lambda^d)\rightarrow L_\infty(m^2\lambda^d)} \leq ct^{-(d/2+1)}(1 + \beta^2t)^{(d/2+1)/2p}e^{\beta^2t} \quad (t > 0, \beta \in \mathbb{R}),$$

for all $x \in \mathbb{R}^d$, $\xi_1 \geq 0$, $|\xi| = 1$, i.e.,

$$\|\rho_\xi T_V^m(t) \rho_\xi^{-1}\|_{L_1(m^2 \lambda^d) \rightarrow L_\infty(m^2 \lambda^d)} \leq ct^{-(d/2+1)}(1 + |\xi|^2 t)^{\alpha(d+2)/4} e^{|\xi|^2 t} \quad (t > 0, \xi \in \mathbb{R}^d).$$

Thus, by the Dunford-Pettis theorem one concludes that for every $t > 0$ the operator $(T^m)_V(t)$ has an integral kernel $k_t^{V,m} \in L_\infty(\Omega_0 \times \Omega_0)$ such that

$$0 \leq k_t^{V,m}(x, y) \leq ct^{-(d/2+1)}(1 + |\xi|^2 t)^{\alpha(d+2)/4} e^{|\xi|^2 t} \quad (\text{a.e. } x, y \in \Omega_0) \quad (3.5)$$

for all $\xi \in \mathbb{R}^d$. Now one easily shows that

$$k_t^V(x, y) := x_1 k_t^{V,m}(x, y) y_1 \quad (x, y \in \Omega_0)$$

defines an integral kernel k_t^V of $T_V(t)$, and with (3.5) and Davies trick one sees that this integral kernel satisfies the asserted kernel estimate. \square

Remark 3.5. If one could show that

$$\|m^{-1} \rho_\xi T_V(t) \rho_\xi^{-1} m\|_{L_1(m^2 \lambda^d) \rightarrow L_1(m^2 \lambda^d)} \leq M e^{|\xi|^2 t} \quad (t \geq 0) \quad (3.6)$$

also holds for $\xi_1 > 0$, then with the same proof as above one could infer that $k_t^V(x, y) \leq cx_1 y_1 t^{-(d/2+1)} e^{-|x-y|^2/4t}$ for some $c > 0$. Thus one would obtain a kernel estimate better than (1.5), without the polynomial correction factor $(1 + (x-y)^2/4t)^{\alpha(d+2)/4}$. Unfortunately, if $\xi_1 > 0$, then one can show that (3.6) already fails for $V = 0$ (cf. the proof of Lemma 3.1(b)).

As a direct consequence of Theorems 2.11 and 3.4 we now obtain the kernel estimate (1.6).

Corollary 3.6. *Let (2.8) be true, and assume that there exists $M \geq 1$ such that*

$$\begin{aligned} \|\rho_\xi T_V(t) \rho_\xi^{-1}\|_{L_1(\Omega_0) \rightarrow L_1(\Omega_0)} &\leq M e^{|\xi|^2 t} && (t \geq 0, \xi \in \mathbb{R}^d), \\ \|m^{-1} \rho_\xi T_V(t) \rho_\xi^{-1} m\|_{L_1(m^2 \lambda^d) \rightarrow L_1(m^2 \lambda^d)} &\leq M e^{2|\xi|^2 t} && (t \geq 0, \xi \in \mathbb{R}^d, \xi_1 \geq 0), \\ \|m^{-1} \rho_\xi T_V(t) \rho_\xi^{-1} m\|_{L_1(m^2 \lambda^d) \rightarrow L_1(m^2 \lambda^d)} &\leq M e^{|\xi|^2 t} && (t \geq 0, \xi \in \mathbb{R}^d, \xi_1 \leq 0). \end{aligned}$$

Then for every $t > 0$ the operator $T_V(t)$ has an integral kernel $k_t^V \in L_\infty(\Omega_0 \times \Omega_0)$, and there exists $c > 0$ independent of t such that

$$0 \leq k_t^V(x, y) \leq c \left(1 \wedge \left(\frac{x_1 y_1}{t} \left(1 + \frac{|x-y|^2}{4t} \right)^{\alpha(d+2)/4} \right) \right) t^{-d/2} e^{-|x-y|^2/4t}$$

for a.e. $x, y \in \Omega_0$.

4 L_1 -estimates for weighted Schrödinger semigroups on the positive real axis

In this section we prove Theorem 1.1 by showing that the Schrödinger semigroup T_V on $L_2(0, \infty)$ satisfies the three weighted L_1 -estimates assumed in Corollary 3.6 if the potential V satisfies the integral condition (1.1). The basic tool for the proof of the three L_1 -estimates will be given in Subsection 4.1. In Subsection 4.2 we show a kernel estimate for the resolvents of the Dirichlet Laplacian on $(0, \infty)$. This kernel estimate is used in Subsection 4.3 to show that the operator norm $\|V(\lambda - A)^{-1}\|_{1 \rightarrow 1}$ is small, where A is the Dirichlet Laplacian provided with the weights that are needed for the application of Corollary 3.6; the smallness of the operator norm will enable us to apply the results of Subsection 4.1 to infer the three weighted L_1 -estimates indicated above.

4.1 L_1 -estimates for perturbed C_0 -semigroups

The basis for our L_1 -estimates is the following general result for absorption semigroups on L_1 .

Proposition 4.1. *Let (Ω, μ) be a measure space. Let T be a positive C_0 -semigroup on $L_1(\mu)$ with generator A , and let $M \geq 1$, $\omega \in \mathbb{R}$ be such that*

$$\|T(t)\|_{1 \rightarrow 1} \leq M e^{\omega t} \quad (t \geq 0).$$

Let $V : \Omega \rightarrow \mathbb{R}$ be measurable, and assume that there exist $\lambda > \omega$, $\alpha \in (0, 1)$ such that

$$\|V(\lambda - A)^{-1}\|_{1 \rightarrow 1} \leq \alpha. \quad (4.1)$$

Then V is T -admissible,

$$\|T_V(t)\|_{1 \rightarrow 1} \leq \frac{M}{1 - \alpha} e^{\lambda t} \quad (t \geq 0).$$

Proof. By rescaling we can assume without loss of generality that $\omega = 0$. For $n \in \mathbb{N}$ let $V_n := (V \wedge n) \vee (-n) \in L_\infty(\mu)$.

(i) In the first step we show that

$$\|e^{-\lambda t} T_{V_n}(t)\|_{1 \rightarrow 1} \leq \frac{M}{1 - \alpha} \quad (t \geq 0, n \in \mathbb{N}). \quad (4.2)$$

For this, fix $n \in \mathbb{N}$ and note that $\|V_n(\lambda - A)^{-1}\|_{1 \rightarrow 1} \leq \alpha$ by hypothesis. Since $L_1(\mu) \ni f \mapsto \int_\Omega |V_n| f d\mu \in \mathbb{K}$ is a continuous linear functional, we have

$$\int_0^\infty \|V_n e^{-\lambda t} T(t) f\|_1 dt = \left\| V_n \int_0^\infty e^{-\lambda t} T(t) f dt \right\|_1 = \|V_n(\lambda - A)^{-1} f\|_1 \leq \alpha \|f\|_1$$

for all $0 \leq f \in L_1(\mu)$, so by the positivity of T it follows that

$$\int_0^\infty \|V_n e^{-\lambda t} T(t) f\|_1 dt \leq \alpha \|f\|_1 \quad (f \in L_1(\mu)).$$

Thus, [Voi77, Thm. 1(c)] (applied with $B = -V_n$) implies (4.2).

(ii) Since (4.1) holds with $-V^-$ and $-V^+$ in place of V , we conclude from step (i) and [Voi88, Prop. 2.2] that $-V^-$ and $-V^+$ are T -admissible. By [Voi88, Prop. 3.3(b)] the latter implies that also V^+ is T -admissible, so that V is T -admissible. Now the asserted estimate follows from (4.2) since $T_{V_n}(t) \rightarrow T_V(t)$ strongly for all $t \geq 0$. \square

We will apply the above proposition in the form of the next result on absorption semi-groups on L_2 .

Corollary 4.2. *Let (Ω, μ) be a measure space. Let T be a positive C_0 -semigroup on $L_2(\mu)$ with generator A , and assume that there exist $M \geq 1$, $\omega \in \mathbb{R}$ such that*

$$\|T(t)\|_{1 \rightarrow 1} \leq M e^{\omega t}, \quad \|T(t)\|_{2 \rightarrow 2} \leq M e^{\omega t}.$$

for all $t \geq 0$. Let $V: \Omega \rightarrow \mathbb{R}$ be T -admissible, and assume that there exist a sequence $(\lambda_k)_{k \in \mathbb{N}}$ in (ω, ∞) converging to ω and $\alpha \in (0, 1)$ such that

$$\|V(\lambda_k - A)^{-1}\|_{1 \rightarrow 1} \leq \alpha \quad (k \in \mathbb{N}). \quad (4.3)$$

Then

$$\|T_V(t)\|_{1 \rightarrow 1} \leq \frac{M}{1 - \alpha} e^{\omega t} \quad (t \geq 0).$$

Proof. For every $t \geq 0$ the operator $T(t)|_{L_1 \cap L_2(\mu)}$ extends to a bounded operator $T_1(t)$ on $L_1(\mu)$ with $\|T_1(t)\|_{1 \rightarrow 1} \leq M e^{\omega t}$, and the mapping $T_1: [0, \infty) \rightarrow \mathcal{L}(L_1(\mu))$ thus defined is a positive C_0 -semigroup by [Voi92, Thm. 7]. Since (4.3) holds for $V_n := (V \wedge n) \vee (-n)$ in place of V , it follows from Proposition 4.1 that

$$\|(T_1)_{V_n}(t)\|_{1 \rightarrow 1} \leq \frac{M}{1 - \alpha} e^{\lambda_k t} \quad (t \geq 0, k \in \mathbb{N})$$

for all $n \in \mathbb{N}$. Letting $k \rightarrow \infty$ we obtain

$$\|(T_1)_{V_n}(t)\|_{1 \rightarrow 1} \leq \frac{M}{1 - \alpha} e^{\omega t} \quad (t \geq 0, n \in \mathbb{N}).$$

Since $(T_1)_{V_n}(t)|_{L_1 \cap L_2(\mu)} = T_{V_n}(t)|_{L_1 \cap L_2(\mu)}$ for all $n \in \mathbb{N}$ by [Voi86, Prop. 3.1(a)], the assertion follows by letting $n \rightarrow \infty$ in the last inequality. \square

The final result of this subsection, a version of [Voi86, Proposition 4.6], will be needed for the proof of Theorem 4.10, a kind of converse to Theorem 1.1. It is more or less known; we include a proof for the reader's convenience.

Proposition 4.3. *Let (Ω, μ) be a measure space. Let T be a positive C_0 -semigroup on $L_1(\mu)$, and let $V: \Omega \rightarrow (-\infty, 0]$ be measurable. Assume T is stochastic, that V is T -admissible and that $M := \sup_{t \geq 0} \|T_V(t)\|_{1 \rightarrow 1} < \infty$. Then V satisfies the global Kato class condition*

$$\int_0^\infty \|VT(t)f\|_1 dt \leq (M-1)\|f\|_1 \quad (f \in L_1(\mu)).$$

Proof. Since T is a positive semigroup, it suffices to show the estimate for $f \in L_1(\mu)_+$. Let $t > 0$, $n \in \mathbb{N}$ and $V_n := V \vee (-n)$. By Duhamel's formula we have

$$T_V(t)f \geq T_{V_n}(t)f = T(t)f - \int_0^t T(t-s)V_n T_{V_n}(s)f ds.$$

Since $V_n \leq 0$ and T is stochastic, it follows that

$$\|T_V(t)f\|_1 - \|f\|_1 \geq \int_0^t \|V_n T_{V_n}(s)f\|_1 ds \geq \int_0^t \|V_n T(s)f\|_1 ds.$$

Letting $n \rightarrow \infty$ and $t \rightarrow \infty$ we obtain $\int_0^\infty \|VT(s)f\|_1 ds \leq (M-1)\|f\|_1$. \square

The above argument even shows that $\int_0^\infty \|VT_V(t)f\|_1 dt \leq (M-1)\|f\|_1$, but we will not need this stronger estimate.

4.2 Integral kernel for the resolvents of the Dirichlet Laplacian on the positive real axis

In this subsection let Δ_D be the Dirichlet Laplacian on $(0, \infty)$ and T the generated C_0 -semigroup on $L_2(0, \infty)$. We show that for $\lambda > 0$ the integral kernel of the resolvent $(\lambda - \Delta_D)^{-1}$ is given by the Green function $G_\lambda: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$,

$$G_\lambda(x, y) := \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|} \left(1 - e^{-2\sqrt{\lambda}(x \wedge y)}\right).$$

First note that $(\lambda - \Delta_D)^{-1} = \int_0^\infty e^{-\lambda t} T(t) dt$ (strong integral), and recall from (3.1) that for every $t > 0$ the operator $T(t)$ has the integral kernel $k_t: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ defined by

$$k_t(x, y) := (4\pi t)^{-1/2} e^{-|x-y|^2/4t} (1 - e^{-xy/t}) \quad (x, y \in (0, \infty)); \quad (4.4)$$

thus,

$$(\lambda - \Delta_D)^{-1} f(x) = \int_0^\infty \int_0^\infty e^{-\lambda t} k_t(x, y) dt f(y) dy \quad (\text{a.e. } x \in (0, \infty))$$

for all $f \in L_2(0, \infty)$ and all $\lambda > 0$. To conclude that G_λ is the integral kernel of $(\lambda - \Delta_D)^{-1}$, we now show that $(0, \infty) \ni \lambda \mapsto G_\lambda(x, y) \in (0, \infty)$ is the Laplace transform of the function $(0, \infty) \ni t \mapsto k_t(x, y) \in (0, \infty)$, for every $(x, y) \in (0, \infty) \times (0, \infty)$.

Lemma 4.4. *Let $\lambda, x, y \in (0, \infty)$. Then*

$$G_\lambda(x, y) = \int_0^\infty e^{-\lambda t} k_t(x, y) dt.$$

Proof. We observe that

$$\begin{aligned} \int_0^\infty e^{-\lambda t} k_t(x, y) dt &= \int_0^\infty e^{-\lambda t} (4\pi t)^{-1/2} e^{-(x-y)^2/4t} (1 - e^{-xy/t}) dt \\ &= \frac{1}{2\sqrt{\pi}} \left(\int_0^\infty t^{-1/2} e^{-(x-y)^2/4t - \lambda t} dt - \int_0^\infty t^{-1/2} e^{-(x+y)^2/4t - \lambda t} dt \right). \end{aligned} \quad (4.5)$$

To compute the two integrals in the right hand side, let $r > 0$, $G(t) := -\int_t^\infty e^{-s^2} ds$ ($t \in \mathbb{R}$) and

$$F(t) := e^{r\sqrt{\lambda}} G\left(\sqrt{\lambda t} + \frac{r}{2\sqrt{t}}\right) + e^{-r\sqrt{\lambda}} G\left(\sqrt{\lambda t} - \frac{r}{2\sqrt{t}}\right) \quad (t > 0).$$

Using $G'(t) = e^{-t^2}$ ($t \in \mathbb{R}$), one easily verifies that

$$F'(t) = \sqrt{\lambda} t^{-1/2} e^{-r^2/4t - \lambda t} \quad (t > 0).$$

Moreover, since $G(t) \rightarrow 0$ as $t \rightarrow \infty$ and $G(t) \rightarrow -\sqrt{\pi}$ as $t \rightarrow -\infty$, we have

$$F(t) \rightarrow 0 \quad (t \rightarrow \infty), \quad F(t) \rightarrow -\sqrt{\pi} e^{-r\sqrt{\lambda}} \quad (t \rightarrow 0^+).$$

Hence,

$$\int_0^\infty t^{-1/2} e^{-r^2/4t - \lambda t} dt = \frac{1}{\sqrt{\lambda}} \left(\lim_{t \rightarrow \infty} F(t) - \lim_{t \rightarrow 0^+} F(t) \right) = \sqrt{\frac{\pi}{\lambda}} e^{-r\sqrt{\lambda}}.$$

Note that this identity also holds for $r = 0$. Using (4.5), we conclude that

$$\begin{aligned} \int_0^\infty e^{-\lambda t} k_t(x, y) dt &= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{\lambda}} \left(e^{-\sqrt{\lambda}|x-y|} - e^{-\sqrt{\lambda}(x+y)} \right) \\ &= \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|} \left(1 - e^{-\sqrt{\lambda}(x+y-|x-y|)} \right) \\ &= \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|} \left(1 - e^{-2\sqrt{\lambda}(x \wedge y)} \right) = G_\lambda(x, y). \quad \square \end{aligned}$$

Using the elementary inequality $1 - e^{-r} \leq r$ ($r \geq 0$) we obtain the following estimate for the Green function G_λ that will be crucial in the next subsection:

$$G_\lambda(x, y) \leq (x \wedge y) e^{-\sqrt{\lambda}|x-y|} \quad (\lambda > 0, x, y \in (0, \infty)). \quad (4.6)$$

4.3 L_1 -estimates for weighted Schrödinger semigroups on $(0, \infty)$

Let again Δ_D be the Dirichlet Laplacian on $(0, \infty)$ and T the generated C_0 -semigroup on $L_2(0, \infty)$. Let $V: (0, \infty) \rightarrow \mathbb{R}$ be measurable and assume that there exists $\alpha \in (0, 1)$ such that

$$\int_0^\infty x|V(x)| dx \leq \alpha. \quad (4.7)$$

Our first goal is to show L_1 -estimates for the Schrödinger semigroup T_V with exponential weights $\rho_\xi: (0, \infty) \rightarrow (0, \infty)$ defined by $\rho_\xi(x) := e^{\xi \cdot x}$ ($x > 0$), for all $\xi \in \mathbb{R}$. We achieve this by first showing that (4.7) implies an estimate of the form (4.3) (see Proposition 4.5(a) below) and then applying Corollary 4.2 to the semigroup T^ξ defined as follows, for $\xi \in \mathbb{R}$: due to (2.10), $\rho_\xi T(t) \rho_\xi^{-1}$ extends to a bounded operator $T^\xi(t)$ on $L_2(0, \infty)$ for all $t \geq 0$, and by Proposition 2.10 the family $(T^\xi(t))_{t \geq 0}$ thus defined is a (positive) C_0 -semigroup on $L_2(0, \infty)$. We denote by $\Delta_{D, \xi}$ the generator of T^ξ .

The second goal is to prove L_1 -estimates for T_V with exponential weight ρ_ξ and weight $m: (0, \infty) \rightarrow (0, \infty)$ defined by $m(x) := x$ ($x > 0$) on the measure space $((0, \infty), m^2 \lambda)$. We will show this in a manner similar to the first goal from above, working with the unitarily transformed semigroup $T^{\xi, m}$ on $L_2(\Omega_0, m^2 \lambda)$ defined by

$$T^{\xi, m}(t)f := m^{-1}T^\xi(t)mf \quad (t \geq 0, f \in L_2(m^2 \lambda)).$$

Note that $T^{\xi, m}$ is a positive C_0 -semigroup satisfying

$$\|T^{\xi, m}(t)\|_{L_2(m^2 \lambda) \rightarrow L_2(m^2 \lambda)} \leq e^{\xi^2 t} \quad (t \geq 0) \quad (4.8)$$

since $L_2(m^2 \lambda) \ni f \mapsto mf \in L_2(0, \infty)$ is an isometric lattice isomorphism and T^ξ is positive and satisfies (2.10). We denote by $\Delta_{D, \xi}^m$ the generator of $T^{\xi, m}$.

Proposition 4.5. *Let (4.7) be true, and let $\xi \in \mathbb{R}$. Then*

- (a) $\|V(\mu - \Delta_{D, \xi})^{-1}\|_{1 \rightarrow 1} \leq \alpha$ for all $\mu > \xi^2$, and
- (b) $\|V(\mu - \Delta_{D, \xi}^m)^{-1}\|_{L_1(m \lambda^2) \rightarrow L_1(m \lambda^2)} \leq \alpha$ for all $\mu > \xi^2$.

Proof. We assume without loss of generality that $V \in L_\infty(0, \infty)$. (This can be done since $V_n := |V| \wedge n \in L_\infty(0, \infty)$ satisfies (4.7) for every $n \in \mathbb{N}$ and, if the assertions of (a) and (b) hold with V_n in place of V , then they also hold for V by monotone convergence.)

(a) Let $D := \{L_1 \cap L_2(0, \infty); \rho_\xi^{-1} f \in L_2(0, \infty)\}$, and let $f \in D$. Then $(\mu - \Delta_{D, \xi})^{-1} f = \rho_\xi (\mu - \Delta_D)^{-1} \rho_\xi^{-1} f$ since $(\mu - \Delta_{D, \xi})^{-1}$ is the Laplace transform of T^ξ . Thus, using the estimate (4.6) for the integral kernel G_μ of $(\mu - \Delta_D)^{-1}$ and the assumption (4.7), we

estimate

$$\begin{aligned}
\|V(\mu - \Delta_{D,\xi})^{-1}f\|_1 &= \int_0^\infty |V\rho_\xi(\mu - \Delta_D)^{-1}\rho_\xi^{-1}f|(x) dx \\
&= \int_0^\infty |V(x)|e^{\xi x} \int_0^\infty G_\mu(x,y)e^{-\xi y}|f(y)| dy dx \\
&\leq \int_0^\infty |f(y)| \int_0^\infty e^{\xi(x-y)}e^{-\sqrt{\mu}|x-y|}(x \wedge y)|V(x)| dx dy \\
&\leq \int_0^\infty |f(y)| \int_0^\infty x|V(x)| dx dy \leq \alpha\|f\|_1.
\end{aligned}$$

Now the assertion of (a) follows since D is dense in $L_1 \cap L_2(0, \infty)$.

(b) Let $D := \{L_1 \cap L_2(m^2\lambda); \rho_\xi^{-1}f \in L_2(m^2\lambda)\}$, and let $f \in D$. Then $(\mu - \Delta_{D,\xi}^m)^{-1}f = m^{-1}\rho_\xi(\mu - \Delta_D)^{-1}\rho_\xi^{-1}mf$, and similarly as in the proof of (a) we estimate

$$\begin{aligned}
\|V(\mu - \Delta_{D,\xi}^m)^{-1}f\|_{L_1(m^2\lambda)} &= \int_0^\infty |Vm^{-1}\rho_\xi(\mu - \Delta_D)^{-1}\rho_\xi^{-1}mf|(x) d(m^2\lambda)(x) \\
&= \int_0^\infty |V(x)|\frac{1}{x}e^{\xi x} \int_0^\infty G_\mu(x,y)e^{-\xi y}y|f(y)| dy d(m^2\lambda)(x) \\
&\leq \int_0^\infty |f(y)| \int_0^\infty \frac{x}{y}e^{\xi(x-y)}e^{-\sqrt{\mu}|x-y|}(x \wedge y)|V(x)| dx d(m^2\lambda)(y) \\
&\leq \int_0^\infty |f(y)| \int_0^\infty x|V(x)| dx d(m^2\lambda)(y) \leq \alpha\|f\|_{L_1(m^2\lambda)}.
\end{aligned}$$

Now the assertion of (b) follows since D is dense in $L_1 \cap L_2(m\lambda^2)$. \square

Remark 4.6. The assertions of Proposition 4.5 have a kind of converse: below we will show that

$$\int_0^\infty x|V(x)| dx = \lim_{\mu \rightarrow 0^+} \|V(\mu - \Delta_D)^{-1}\|_{1 \rightarrow 1} = \lim_{\mu \rightarrow 0^+} \|V(\mu - \Delta_D^m)^{-1}\|_{L_1(m^2\lambda) \rightarrow L_1(m^2\lambda)};$$

since $\lim_{\mu \rightarrow 0^+} \|V(\mu - \Delta_D)^{-1}f\|_1 = \int_0^\infty \|VT(t)f\|_1 dt$ for all $f \in L_1 \cap L_2(0, \infty)_+$ (cf. the proof of Proposition 4.1), it then follows that (4.7) is equivalent to the global Kato class condition

$$\int_0^\infty \|VT(t)f\|_1 dt \leq \alpha\|f\|_1 \quad (f \in L_1 \cap L_2(0, \infty)).$$

In the same way one sees that (4.7) is true if and only if

$$\int_0^\infty \|VT^m(t)f\|_{L_1(m^2\lambda)} dt \leq \alpha\|f\|_{L_1(m^2\lambda)} \quad (f \in L_1 \cap L_2(m^2\lambda)).$$

Remarkably, (4.7) is thus equivalent to both an unweighted and a weighted global Kato class condition on V .

The inequality $\lim_{\mu \rightarrow 0^+} \|V(\mu - \Delta_D)^{-1}\|_{1 \rightarrow 1} \leq \int_0^\infty x|V(x)| dx$ holds by Proposition 4.5(a). For the converse inequality first observe that $G_\mu(x, y) \uparrow x \wedge y$ as $\mu \downarrow 0$, for all $x, y > 0$. Thus, by the monotone convergence theorem and a computation as in the proof of Proposition 4.5 one sees that

$$\lim_{\mu \rightarrow 0^+} \|V(\mu - \Delta_D)^{-1}f\|_1 = \int_0^\infty f(y) \int_0^\infty (x \wedge y)|V(x)| dx dy \geq \|f\|_1 \int_0^\infty (x \wedge n)|V(x)| dx$$

for all $f \in L_1 \cap L_2(0, \infty)_+$ with $\text{spt } f \subseteq (n, \infty)$. Letting $n \rightarrow \infty$ one obtains the desired converse inequality.

Noting that $\frac{x}{y}G_\mu(x, y) \uparrow \frac{x^2}{y} \wedge x$ as $\mu \downarrow 0$ one shows the second equality $\int_0^\infty x|V(x)| dx = \lim_{\mu \rightarrow 0^+} \|V(\mu - \Delta_D^m)^{-1}\|_{L_1(m\lambda^2) \rightarrow L_1(m\lambda^2)}$ in a similar way, now using f with $\text{spt } f \subseteq (0, 1/n)$.

Using Proposition 4.5(a) with $\xi = 0$, we now show that (4.7) implies the form smallness condition (2.8).

Proposition 4.7. *Let (4.7) be true. Then*

$$\int_0^\infty |V||u|^2 dx \leq \alpha \langle -\Delta_D u, u \rangle \quad (u \in \text{dom}(\Delta_D)). \quad (4.9)$$

Proof. As in the proof of Proposition 4.5 we assume without loss of generality that $V \in L_\infty(0, \infty)$.

Let $p \in (1, 1/\alpha)$. Then $\| -p|V|(\lambda - \Delta_D)^{-1} \|_{1 \rightarrow 1} \leq p\alpha$ for all $\lambda > 0$ by Proposition 4.5(a). Therefore, since T is contractive and L_1 -contractive,

$$\|T_{-p|V|}(t)\|_{1 \rightarrow 1} \leq \frac{1}{1 - p\alpha} \quad (t \geq 0)$$

by Corollary 4.2. Using duality and the self-adjointness of $T_{-p|V|}$, we also get the estimate

$$\|T_{-p|V|}(t)\|_{\infty \rightarrow \infty} \leq \frac{1}{1 - p\alpha} \quad (t \geq 0).$$

The Riesz-Thorin interpolation theorem now implies that $\|T_{-p|V|}(t)\|_{2 \rightarrow 2} \leq 1/(1 - p\alpha)$ for all $t \geq 0$. Thus $-(\Delta_D + p|V|)$ is accretive, i.e.,

$$\int_0^\infty p|V||u|^2 dx \leq \langle -\Delta_D u, u \rangle \quad (u \in \text{dom}(\Delta_D)).$$

Letting $p \rightarrow 1/\alpha$ we obtain the asserted estimate (4.9). \square

Now we are ready to prove the L_1 -estimates mentioned in the beginning of this section. We start with the exponentially weighted L_1 -estimate for Schrödinger semigroups on $(0, \infty)$. Note that V is T -admissible by Proposition 4.7 and Proposition 2.2.

Proposition 4.8. *Let (4.7) be true. Let $\xi \in \mathbb{R}$. Then*

$$\|\rho_\xi T_V(t) \rho_\xi^{-1}\|_{L_1(0, \infty) \rightarrow L_1(0, \infty)} \leq \frac{1}{1 - \alpha} e^{\xi^2 t} \quad (t \geq 0).$$

Proof. Recall the definition of T^ξ from the first paragraph of the present subsection. By (2.10) we have $\|T^\xi(t)\|_{1 \rightarrow 1} \leq e^{\xi^2 t}$ and $\|T^\xi(t)\|_{2 \rightarrow 2} \leq e^{\xi^2 t}$ for all $t \geq 0$. Now fix $n \in \mathbb{N}$ and let $V_n := (V \wedge n) \vee (-n) \in L_\infty(0, \infty)$. Then

$$(T^\xi)_{V_n}(t)f = \rho_\xi T_{V_n}(t)\rho_\xi^{-1}f \quad (t \geq 0, f \in \text{dom}(\rho_\xi^{-1})) \quad (4.10)$$

by Proposition 2.10. Moreover, we have $\|V_n(\lambda - \Delta_{D,\xi})^{-1}\|_{1 \rightarrow 1} \leq \alpha$ for all $\lambda > \xi^2$, by Proposition 4.5(a). Hence, Corollary 4.2 implies that

$$\|(T^\xi)_{V_n}(t)\|_{1 \rightarrow 1} \leq \frac{1}{1 - \alpha} e^{\xi^2 t} \quad (t \geq 0).$$

Now the assertion follows because of (4.10) and the fact that $T_{V_n}(t) \rightarrow T_V(t)$ strongly for all $t \geq 0$. \square

The next proposition deals with the exponentially weighted L_1 -estimates for Schrödinger semigroups on $((0, \infty), m^2\lambda)$.

Proposition 4.9. *Let (4.7) be true. Let $\xi \in \mathbb{R}$. Then*

$$\|m^{-1}\rho_\xi T_V(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda) \rightarrow L_1(m^2\lambda)} \leq \frac{2\sqrt{2}}{1 - \alpha} e^{2\xi^2 t} \quad (t \geq 0).$$

Moreover, if $\xi \leq 0$, then

$$\|m^{-1}\rho_\xi T_V(t)\rho_\xi^{-1}m\|_{L_1(m^2\lambda) \rightarrow L_1(m^2\lambda)} \leq \frac{1}{1 - \alpha} e^{\xi^2 t} \quad (t \geq 0).$$

Proof. Let $T^{\xi,m}$ be defined as in the paragraph preceding Proposition 4.5, and recall (4.8). Moreover, note that Proposition 3.2 yields

$$\|T^{\xi,m}(t)\|_{L_1(m^2\lambda) \rightarrow L_1(m^2\lambda)} \leq c_\xi^{3/2} e^{c_\xi \xi^2 t} \quad (t \geq 0),$$

where $c_\xi := 2$ if $\xi > 0$ and $c_\xi := 1$ if $\xi \leq 0$. Now one completes the proof by the same argumentation as in the proof of Proposition 4.8, using part (b) of Proposition 4.5 instead of part (a). \square

With the above weighted L_1 -estimates at hand we can finally prove our main result.

Proof of Theorem 1.1. It follows from Propositions 4.7 and 2.2 that V satisfies the form smallness condition (2.8) and hence is T -admissible. Now Propositions 4.8 and 4.9 show that Corollary 3.6 is applicable, and this gives the desired kernel estimate. \square

To conclude, we prove a kind of converse to Theorem 1.1, for negative potentials.

Theorem 4.10. *Let V be T -admissible, and assume that $V \leq 0$. Further, assume that for every $t > 0$ the operator $T_V(t)$ has an integral kernel $k_t^V \in L_\infty((0, \infty) \times (0, \infty))$ such that (1.2) holds for some $c > 0$ independent of t . Then*

$$\int_0^\infty x|V(x)| dx < \infty. \quad (4.11)$$

Proof. For $t > 0$ let $k_t: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ be the integral kernel of $T(t)$; see (4.4). One easily sees that

$$\left(1 + \frac{(x-y)^2}{4t}\right)^{3\alpha/4} \leq M e^{(x-y)^2/8t} \quad (t, x, y > 0)$$

for some $M \geq 1$, so it follows from (1.2) and (3.2) that

$$k_t^V(x, y) \leq cM \left(1 \wedge \frac{xy}{t}\right) t^{-1/2} e^{-(x-y)^2/8t} \leq 4\sqrt{2} cM k_{2t}(x, y) \quad (\text{a.e. } x, y \in (0, \infty))$$

for all $t > 0$. With $C := (4\sqrt{2} cM) \vee 1$ we deduce that $T_V(t) \leq CT(2t)$ and hence $(T^m)_V(t) = (T_V)^m(t) \leq CT^m(2t)$ for all $t \geq 0$. This together with the L_1 -stochasticity of T^m (see Remark 3.3) implies that

$$\sup_{t \geq 0} \|(T^m)_V(t)\|_{L_1(m^2\lambda) \rightarrow L_1(m^2\lambda)} \leq C. \quad (4.12)$$

Now $T^m(t)|_{L_1 \cap L_2(m^2\lambda)}$ extends to a stochastic operator $T_1^m(t)$ on $L_1(m^2\lambda)$, and the mapping $T_1^m: [0, \infty) \rightarrow \mathcal{L}(L_1(m^2\lambda))$ thus defined is a positive C_0 -semigroup by [Voi92, Thm. 7]. Further observe that, due to (4.12), we can apply [Voi86, Prop. 3.1(a)] and obtain the T_1^m -admissibility of V and the identity $(T_1^m)_V(t)f = (T^m)_V(t)f$ for all $f \in L_1 \cap L_2(m^2\lambda)$. Therefore, Proposition 4.3 implies that V satisfies the global Kato class condition

$$\int_0^\infty \|VT_1^m(t)f\|_1 dt \leq (C-1)\|f\|_1 \quad (f \in L_1(m^2\lambda)).$$

The asserted inequality (4.11) now follows from Remark 4.6. □

Acknowledgement.

This paper is a contribution to the project M1 of the Collaborative Research Centre TRR 181 ‘‘Energy Transfers in Atmosphere and Ocean’’ funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under project number 274762653.

References

- [ArBa93] W. ARENDT AND C.J.K. BATTY: Absorption semigroups and Dirichlet boundary conditions, *Math. Ann.* **295** (1993), no. 3, 427–448.
- [Cou90] TH. COULHON: Dimension a l’infini d’un semi-groupe analytique, *Bull. Sci. Math.* **114** (1990), no. 4, 485–500.
- [DaPa89] E.B. DAVIES AND M.M.H. PANG: Sharp heat kernel bounds for some Laplace operators, *Quart. J. Math. Oxford Ser. (2)* **40** (1989), no. 159, 281–290.

- [Dav87] E.B. DAVIES: Explicit constants for Gaussian upper bounds on heat kernels, *Amer. J. Math.* **109** (1987), no. 2, 319–333.
- [EnNa00] K.-J. ENGEL AND R. NAGEL: *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. **194**, Springer Verlag, New York, 2000.
- [Haa07] M. HAASE, Convexity inequalities for positive operators, *Positivity* **11** (2007), no. 1, 57–68.
- [LiSe98] V. LISKEVICH AND Y. SEMENOV: Two-sided estimates of the heat kernel of the Schrödinger operator, *Bull. London Math. Soc.* **30** (1998), no. 6, 596–602.
- [Kat95] T. KATO: *Perturbation Theory for Linear Operators*, corrected printing of 2nd edition, Springer Verlag, Berlin-Heidelberg, 1995.
- [Mol75] S.A. MOLCHANOV: Diffusion processes, and Riemannian geometry, *Uspehi Mat. Nauk* **30** (1975), no. 1, 3–59.
- [Ouh05] E.M. OUHABAZ: *Analysis of heat equations on domains*, London Mathematical Society Monographs Series **31**, Princeton University Press, Princeton, NJ, 2005.
- [Ouh06] E.M. OUHABAZ: Sharp Gaussian bounds and L^p -growth of semigroups associated with elliptic and Schrödinger operators, *Proc. Amer. Math. Soc.* **134** (2006), no. 12, 3567–3575.
- [Sik04] A. SIKORA: Riesz transform, Gaussian bounds and the method of wave equation, *Math. Z.* **207** (2004), no. 3, 643–662.
- [StVo96] P. STOLLMANN AND J. VOIGT: Perturbation of Dirichlet forms by measures. *Potential Anal.* **5** (1996), no. 2, 109–138.
- [Voi77] J. VOIGT: On the perturbation theory for strongly continuous semigroups, *Math. Ann.* **229** (1977), no. 2, 163–171.
- [Voi86] J. VOIGT: Absorption semigroups, their generators, and Schrödinger semigroups, *J. Funct. Anal* **67** (1986), no. 2, 167–205.
- [Voi88] J. VOIGT: Absorption semigroups, *J. Operator theory* **20** (1988), no. 1, 117–131.
- [Voi92] J. VOIGT: One-parameter semigroups acting simultaneously on different L_p -spaces, *Bull. Soc. Roy. Sci. Liège* **61** (1992), no. 6, 465–470.
- [Vog10] H. Vogt: *Perturbation theory for parabolic equations*. Habilitationsschrift, TU Dresden, 2010.
- [Vog19] H. VOGT: L_∞ -estimates for the torsion function and L_∞ -growth of semigroups satisfying Gaussian bounds, *Potential Anal.* **51** (2019), no. 1, 37–47.