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## Weighted norm estimates and $L_p$-spectral independence of linear operators

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# Weighted norm estimates and $L_p$-spectral independence of linear operators

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**Abstract.** We investigate the $L_p$-spectrum of linear operators defined consistently on $L_p(\Omega)$ for $p_0 \le p \le p_1$, where $(\Omega, \mu)$ is an arbitrary $\sigma$-finite measure space and $1 \le p_0 < p_1 \le \infty$. We prove $p$-independence of the $L_p$-spectrum assuming weighted norm estimates or kernel estimates. The assumptions are formulated in terms of a measurable semi-metric $d$ on $(\Omega, \mu)$ which is required to satisfy certain bounds for the volume growth of balls. Roughly speaking, these bounds are of subexponential type but we can also treat certain situations where the volume growth is exponential.

We show how previous results on $L_p$-spectral independence can be treated as special cases of our results and give examples – including strictly elliptic operators in Euclidean space, generators of semigroups that satisfy (generalized) Gaussian bounds, integral and non-integral regularizing operators – to indicate improvements.

Moreover, we study $L_p$-spectral independence in weighted $L_p$-spaces where the condition on the weight depends on the semi-metric $d$ and is again of subexponential type.

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# 1 Introduction and Main Results

Let $(\Omega, \mu)$ be a $\sigma$-finite measure space and $1 \le p < q \le \infty$. Suppose that $A$ is a linear operator which acts consistently in $L_r(\Omega)$ for $r \in [p, q]$, i.e.

$$A\colon L_p(\Omega) \cap L_q(\Omega) \to L_p(\Omega) \cap L_q(\Omega)$$

is linear and extends, for all $r \in [p, q]$, to a bounded operator $A_r\colon L_r(\Omega) \to L_r(\Omega)$ which, in addition, is weak*-continuous if $r = \infty$. The term "consistency" refers to the property

$$A_r f = A_s f \qquad (f \in L_r(\Omega) \cap L_s(\Omega),\ r, s \in [p, q]).$$

It is then a natural question whether the spectrum $\sigma(A_r)$ of $A_r$ in $L_r(\Omega)$ depends on $r \in [p, q]$ or not. This question also makes sense for unbounded operators $A_r$ in $L_r(\Omega)$, $r \in [p, q]$, if consistency is rephrased in terms of resolvents or semigroup operators (for the case that the $A_r$ are generators). We refer to Remark 3(vi) for details.

There are several kinds of assumptions that are sufficient for $r$-independence of $\sigma(A_r)$. We refer to the discussion in [14] and the references there. One method that has been

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widely used in the context of the $L_p$-theory of elliptic operators relies on the exploitation of certain bounds, especially of Gaussian type, that hold for the kernels of associated integral operators, i.e. semigroup or resolvent operators ([9], [1], [20], [7], [10], [13], [14]). If these operators are not integral operators, which is in general the case if $p > 1$ and $q < \infty$, then $r$-independence of the spectrum still holds under the assumption of certain weighted norm estimates ([17], [16], [18], [15]). It is this kind of assumptions, i.e. weighted norm estimates and kernel bounds, that we shall use in this paper.

The case studied in most of the cited papers is that $\Omega$ is an open subset of $\mathbb{R}^N$ and $\mu$ is the Lebesgue measure. We shall deal here with an arbitrary $\sigma$-finite measure space $(\Omega, \mu)$. Our assumptions on the operator $A$, i.e. weighted norm estimates or kernel bounds, will be formulated in terms of a measurable semi-metric $d$ on $\Omega$ which has suitable properties. The underlying philosophy is that, in order to get $r$-independence of the spectrum for a given operator $A$ on $(\Omega, \mu)$ by one of our results, one has *to find* a semi-metric $d$ on $\Omega$ such that $A$ and $d$ satisfy our assumptions. With respect to the problem of $r$-independence of the spectrum as stated above this seems to be the natural procedure. Even if the space $\Omega$ carries a natural metric $d_0$, as open subsets of $\mathbb{R}^N$ do or Riemannian manifolds ([20]), a suitable semi-metric $d$ may be quite different. We want to emphasize that, in the case of weighted norm estimates, the consideration of a general semi-metric space $(\Omega, d)$ instead of (subsets of) $\mathbb{R}^N$ means that we have to use a new approach in the proof which is rather different from the arguments used previously in this context.

In order to make the assumptions in our first main result (which uses weighted norm estimates) less restrictive we now slightly change the setting outlined above.

Let $(\Omega', \mu)$ be a $\sigma$-finite measure space and suppose that $d$ is a measurable semi-metric on $\Omega'$ satisfying

$$\gamma := \operatorname*{ess\,inf}_{z\in\Omega'} \mu(B(z,R)) > 0 \quad \text{for some } R > 0, \tag{1}$$

where $B(z, R)$ denotes the open ball with center $z$ and radius $R$ with respect to $d$. Further assume that

$$\operatorname*{ess\,sup}_{z\in\Omega'} \mu(B(z,r)) \le m(r) \quad (r > 0), \tag{2}$$

where $m\colon [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $m(0) = 0$ and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ satisfying

$$m(r) \le C_\varepsilon e^{\varepsilon r} \quad (r > 0). \tag{3}$$

Note that we do not demand $m(r) \to 0$ $(r \to 0)$, cf. [7, p. 177], and that (2) and (3) mean that the volume growth is uniformly subexponential. We point out that uniformly subexponential volume growth has a more restrictive definition in the context of Riemannian manifolds, see the end of Subsection 2.1.

Before we state our main theorems we introduce some notation. For a measurable function $g$ on $\Omega' \times \Omega'$ and $r \in [1, \infty]$ we define

$$n_r(g) := \max\big(\operatorname*{ess\,sup}_{x\in\Omega'} \|g(x,\cdot)\|_{L_r(\Omega')}, \operatorname*{ess\,sup}_{y\in\Omega'} \|g(\cdot,y)\|_{L_r(\Omega')}\big) \in [0,\infty].$$

If $n_r(g)$ is finite, $g$ defines a bounded integral operator $I_g\colon L_p(\Omega') \to L_q(\Omega')$ for all $1 \le p \le q \le \infty$ satisfying $r^{-1} + p^{-1} = 1 + q^{-1}$. In fact

$$\|I_g\colon L_p(\Omega') \to L_q(\Omega')\| \le n_r(g), \tag{4}$$

which can be proved by an application of Fubini's theorem and Riesz-Thorin interpolation. In the case $r = 1$ the operator $I_g$ is bounded in every space $L_p(\Omega')$ $(1 \le p \le \infty)$.

We define weight functions $\rho_{\varepsilon,z}$ by

$$\rho_{\varepsilon,z}(x) := \rho_\varepsilon(z,x) := e^{-\varepsilon d(z,x)} \quad (x, z \in \Omega',\ \varepsilon \in \mathbb{R}).$$

Then $n_\infty(\rho_\varepsilon) \le 1$ and by assumption (3)

$$n_p(\rho_\varepsilon) \le \left( \int_0^\infty e^{-\varepsilon p r}\, dm(r) \right)^{1/p} < \infty \quad (1 \le p < \infty). \tag{5}$$

Here we have used the fact that, for any non-increasing function $\varphi\colon [0,\infty) \to [0,\infty)$ and almost all $x \in \Omega'$, one can estimate

$$\int_{\Omega'} \varphi(d(x,y))\, d\mu(y) \le \int_0^\infty \varphi(r)\, dm(r)$$

by assumption (2); cf. [7, p. 179].

Now let $\Omega \subset \Omega'$ be a measurable subset. We tacitly assume that functions defined on $\Omega$ are extended by 0 outside $\Omega$ when considered as functions on $\Omega'$. In the following we consider operators on $L_p(\Omega)$, $1 \le p \le \infty$. The reason for introducing the space $\Omega'$ is that condition (1) is not assumed to hold for the balls of $\Omega$ but only for the (possibly larger) balls of $\Omega'$. An important example for this situation is $\Omega' = \mathbb{R}^N$ and a subset $\Omega \subset \mathbb{R}^N$.

We fix an increasing sequence $(\Omega_n)$ of measurable subsets of $\Omega$ that have finite $d$-diameter and finite $\mu$-volume (cf. Remark 3(viii)) such that $\Omega = \bigcup_n \Omega_n$. By $L_{1,loc}(\Omega)$ we denote the set of (equivalence classes of) all measurable functions $f$ on $\Omega$ with $\|f\|_{L_1(\Omega_n)} < \infty$ for all $n \in \mathbb{N}$ and by $L_{\infty,c}(\Omega)$ the space of (equivalence classes of) all measurable functions $f$ on $\Omega$ for which there is an $n \in \mathbb{N}$ such that $f = 0$ a.e. on $\Omega \setminus \Omega_n$ (see also Remark 3(v) for these spaces). Note that the weight functions $\rho_{\varepsilon,z}$ are multiplicators on $L_{1,loc}(\Omega)$ and $L_{\infty,c}(\Omega)$.

For a linear operator $T\colon D(T) \subset L_p(\Omega) \to L_q(\Omega)$, $p, q \in [1,\infty]$, we denote its norm by $\|T\|_{p\to q}$. The following are our main results. The first involves a weighted norm estimate.

**Theorem 1.** *Assume that* (1), (2) *and* (3) *hold. Let* $1 \le p < q \le \infty$ *and* $A\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ *be a linear operator satisfying*

$$\|\rho_{\varepsilon_0,z} A \rho_{\varepsilon_0,z}^{-1}\|_{p\to q} \le C \quad (z \in \Omega') \tag{6}$$

*for some* $C, \varepsilon_0 > 0$. *Then* $A$ *extends to consistent bounded operators* $A_r$ *on* $L_r(\Omega)$ $(r \in [p,q])$, *and the spectrum* $\sigma(A_r)$ *is independent of* $r \in [p,q]$. *In the case* $q = \infty$ *the operator* $A\colon L_{\infty,c}(\Omega) \to L_\infty(\Omega)$ *is* $\sigma(L_{\infty,c}, L_1)$-*weak*-continuous;* $A_\infty$ *denotes the unique weak*-continuous extension.*

Our second result is restricted to integral operators $A$ acting in all $L_p$-spaces, but we are able to relax the lower volume bound (1) and do not need to consider a superset $\Omega' \supset \Omega$. Instead, we require that there exists $R > 0$ such that

$$\mu(B(x,R)) > 0 \quad \text{for almost all } x \in \Omega. \tag{7}$$

This condition is in particular satisfied (even for all $R > 0$) if $(\Omega, d)$ is separable. See also Remark 3(i).

Moreover we are able to adjust our assumptions to the particular volume growth, i.e. to the growth of the function $m$. This is done by considering a modified metric $d_1 := \psi \circ d$ where $\psi\colon [0,\infty) \to [0,\infty)$ is an increasing, continuous and subadditive function satisfying $\psi(0) = 0$. Note that a subadditive function is continuous if and only if it is continuous at 0. We require that

$$\liminf_{r\to\infty}(\psi(\alpha r) - \psi(r)) > 0 \quad \text{for some } \alpha > 1 \tag{8}$$

and

$$n_1\big((x,y) \mapsto e^{-\psi(\varepsilon d(x,y))}\big) < \infty \quad (\varepsilon > 0). \tag{9}$$

The first condition is a lower bound for the growth of $\psi$ at infinity, in particular $\psi$ must grow at least logarithmically. See also Remark 3(ii). The second condition for $\varepsilon = 1$ implies that the volume growth with respect to the metric $d_1$ is at most exponential. Observe that, if $m(r)$ is a bound for the volume of the $d$-balls of radius $r$ in $\Omega$, then $m(\psi^{-1}(r))$ is a bound for the $d_1$-balls of radius $r$ in $\Omega$. This means that $\psi$ may change the volume growth from subexponential to exponential. See Subsection 2.4 for an example where the volume of balls with respect to $d_1$ does grow exponentially.

Under these assumptions we have $L_p$-spectral independence if there is a suitable off-diagonal decay of the kernel of $A$.

**Theorem 2.** *Assume that* (7), (8) *and* (9) *hold. Let $A\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ be a linear operator given by a measurable kernel $k\colon \Omega\times\Omega \to \mathbb{C}$. Let $g(x,y) := |k(x,y)| \exp(d_1(x,y))$ and assume that $n_1(g) < \infty$ and $n_{p_0}(g) < \infty$ for some $p_0 > 1$. Then, for all $p \in [1,\infty]$, $A$ acts as a bounded operator $A_p$ on $L_p(\Omega)$, and $A_\infty$ is weak*-continuous. The $A_p$ are consistent and the spectrum $\sigma(A_p)$ is independent of $p \in [1,\infty]$.*

**Remark 3.** (i) A close inspection of the proof of Theorem 1 below shows that, for $(p,q) = (1,\infty)$, assumption (1) in Theorem 1 may be replaced by (7). But we point out that, for $(p,q) = (1,\infty)$, the weighted norm estimate (6) implies that $A$ has a kernel $k$ satisfying $n_\infty(ke^{\varepsilon_0 d}) < \infty$ and $n_1(ke^{\varepsilon d}) \le n_1(e^{-(\varepsilon_0-\varepsilon)d}) n_\infty(ke^{\varepsilon_0 d}) < \infty$ for $\varepsilon < \varepsilon_0$. Therefore the assumptions of Theorem 2 are fulfilled in this case with $\psi(r) = r$ and $p_0 = \infty$.

We now show that condition (7) is equivalent to the following condition: There exist $R > 0$ and a sequence $(x_n) \subset \Omega$ such that

$$\mu\big(\Omega \setminus \bigcup_n B(x_n,R)\big) = 0. \tag{10}$$

If (10) holds for some $R > 0$ then obviously (7) holds with $2R$ in place of $R$. Assume conversely that (7) holds. Fix $x_0 \in \Omega$. Then, for $r > 0$, the ball $B(x_0,r)$ has finite volume by (viii), and for almost all $x \in B(x_0,r)$ we have $\mu(B(x,R)) > 0$. An application of Zorn's Lemma shows that for all $r > 0$ there exists a sequence $(x_n(r)) \subset \Omega$ such that $\mu(B(x_0,r)\setminus\bigcup_n B(x_n(r),2R)) = 0$. It follows that (10) is valid with $2R$ in place of $R$ since $\Omega = \bigcup_{r\in\mathbb{N}} B(x_0,r)$. Notice that condition (10) holds for all $R > 0$ if $(\Omega,d)$ is separable. Further, note that condition (10) cannot be weakened by assuming it for a superset $\Omega' \supset \Omega$ instead.

(ii) Condition (9) implies that for $\varepsilon \in (0,1]$ and almost all $x \in \Omega$ we have

$$\mu(B(x,r)) \le \int_\Omega e^{\psi(\varepsilon r)-\psi(\varepsilon d(x,y))}\,d\mu(y) \le n_1(e^{-\psi(\varepsilon d)})e^{\psi(\varepsilon r)} \quad (r>0) \tag{11}$$

since $\psi$ is increasing. From this and the subadditivity of $\psi$ it follows that (2) holds for a subexponential function $m\colon [0,\infty) \to [0,\infty)$. If, conversely, (2) holds and for all $\varepsilon > 0$ there exist $C > 0$, $\alpha < 1$ such that $m(r) \le Ce^{\alpha\psi(\varepsilon r)}$, then (9) holds. Therefore, in the particular but important case $\psi(r) = r$ condition (9) is equivalent to the validity of (2) and (3).

If we assume $\psi$ to be concave then it is subadditive since $\psi(0) = 0$. In this case condition (8) is equivalent to the condition $\operatorname{ess\,lim\,inf}_{r\to\infty} r\psi'(r) > 0$, see Section 5. This condition clearly holds for the concave functions $\psi(r) = r^\alpha$ (with $0 < \alpha \le 1$) and $\psi(r) = \alpha\log(1+r)$ (with $\alpha > 0$). The latter function is adjusted to the case of polynomial volume growth: If $m(r) \le c(1+r)^N$ $(r > 0)$ for some $c, N > 0$ (e.g. $\Omega \subset \mathbb{R}^N$ and $\mu$ the Lebesgue measure) and if $\alpha > N$, then condition (9) is fulfilled for $\psi(r) := \alpha\log(1+r)$. Conversely, if (9) is fulfilled for this function $\psi$, then (2) holds with $m(r) = n_1(e^{-d_1})(1+r)^\alpha$ by (11). See Subsection 2.4 for an example.

(iii) In Section 5 we show a variant of Theorem 2 which in particular implies the following: Instead of $n_1(g) < \infty$ assume more restrictively that the volume bound (2) holds and that $g \le h \circ d$ for some non-increasing function $h\colon (0,\infty) \to [0,\infty]$ satisfying $\int_0^\infty h(r)\,dm(r) < \infty$ (note that this implies $h(0)m(0+) < \infty$). Then Theorem 2 is valid without assumption (8).

(iv) The condition $n_1(g) < \infty$ is not needed in the case $\psi(r) = r$, i.e. $d_1 = d$: Let $\tilde\psi(r) := r/2$, then (8) and (9) are obviously fulfilled for $\tilde\psi$ in place of $\psi$. Setting $\rho(x,y) := \exp(-d(x,y)/2)$ and $\tilde g := |k|\exp(\tilde\psi \circ d) = g\cdot\rho$ we further have $n_{p_0}(\tilde g) \le n_{p_0}(g) < \infty$ and $n_1(\tilde g) \le n_{p_0}(g)\cdot n_{p_0'}(\rho) < \infty$ by Hölder's inequality.

(v) Observe that the spaces $L_{1,loc}(\Omega)$ and $L_{\infty,c}(\Omega)$ depend on the choice of the sequence $(\Omega_n)$. The special choice of $(\Omega_n)$ may depend on the problem under consideration. For instance, one may want to choose the sets $\Omega_n$ also bounded with respect to another measurable (semi-)metric $d_0$ on $\Omega'$. To simply consider the space of all functions which are integrable over every $d$-bounded set instead of $L_{1,loc}(\Omega)$ and the space of all bounded functions with support essentially contained in a $d$-bounded set instead of $L_{\infty,c}(\Omega)$ may not always be the most convenient choice. See Subsection 2.3 for an example.

(vi) By the spectral mapping theorem for the resolvent (see, e.g., [5, Lemma 2.11]) we have independence of the spectrum $\sigma(A_r)$ for $r \in [p,q]$ if the $A_r$ are unbounded operators in $L_r(\Omega)$ such that for some $\lambda \in \bigcap_{r\in[p,q]}\rho(A_r)$ the resolvent operators $R := R(\lambda,A_r)$ are consistent and the assumptions of one of our theorems hold for $R\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$.

This is of special interest in the case that the $A_r$ are generators of consistent $C_0$-semigroups $T_r(t)$ on $L_r(\Omega)$. Then the consistency of the resolvents $R(\lambda,A_r)$ holds for

$\lambda$ larger than the maximum of the types of $T_p$ and $T_q$. See Subsections 2.1 and 2.3 for applications.

(vii) If the measure space $(\Omega, \mu)$ is finite, then it is known that in (6) no weights are needed (see [1, Prop. 1.1]) for the assertion of Theorem 1. This follows also from our result if we take $d = 0$ as a semi-metric on $\Omega' = \Omega$. Observe that (1) holds since $B(z, R) = \Omega$ for all $R > 0$.

(viii) The space $(\Omega, \mu)$ must be assumed to be $\sigma$-finite since otherwise condition (2) cannot hold: We show that condition (2) implies an upper bound for the volume of *all* balls in $\Omega'$. Let $x \in \Omega'$. If $\mu(B(x, r)) > 0$ then (2) implies that there exists $z \in B(x, r)$ satisfying $\mu(B(z, 2r)) \leq m(2r)$. It follows that $\mu(B(x, r)) \leq m(2r)$ for all $x \in \Omega'$. A simple example shows that this bound cannot be improved.

This paper is organized as follows. In Section 2 we discuss the relation of our results to previous ones on the problem of $L_p$-spectral independence. In Section 3 we prove Theorem 1, and in Section 4 we prove Theorem 2. In Section 5 we discuss the assumptions of Theorem 2 and present variations of this result. In Section 6 we present results on $p$-independence of the spectrum in weighted $L_p$-spaces, making use of several estimates obtained in Section 3 and Section 4.

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# 2 Comments and examples

In this section we relate our results to the existing literature on $L_p$-spectral independence and we present some examples to indicate the improvements.

The case studied most is $(\Omega, \mu)$ where $\Omega \subset \mathbb{R}^N$ is an open subset and $\mu$ is the Lebesgue measure. Endowed with the Euclidean distance $d$ conditions (2) and (3) hold. Choosing a suitable superset $\Omega'$ of $\Omega$ if necessary, also (1) holds (take, e.g., a $\delta$-neighbourhood of $\Omega$, or simply $\mathbb{R}^N$).

In this setting, $p$-independence of the $L_p$-spectrum has been proved for the generator $A$ of consistent $C_0$-semigroups in $L_p(\Omega, \mu)$ under the assumption that the semigroup operators are given by kernels which satisfy Gaussian bounds. Making use of Remark 3(vi) we will study this case in Subsection 2.1 below in a more general setting. The first proof of $L^p$-spectral independence making use of Gaussian bounds was carried out in [9] for Schrödinger operators with negative part of the potential in the Kato class. In the general case of validity of Gaussian bounds, $L^p$-spectral independence has been proved in [1] under the assumption that $\rho(A_2)$ is connected, for selfadjoint $A_2$ in [7] and under certain commutator estimates in [10]. In [13] those additional assumptions were removed. It is known that Gaussian bounds hold for large classes of uniformly elliptic operators (see, e.g., [2], [3], [4], [6], [19], only to mention a few).

Weighted norm estimates may be used to prove $L_p$-spectral independence if the semigroup does not act on all $L_p$-spaces, $1 \leq p < \infty$. In [17] this was done for Schrödinger operators with form bounded negative part of the potential. In [8] weighted norm estimates were used to prove $L_p$-spectral independence for higher order elliptic operators. The ideas from [17] were put in a more general context in [16], assuming so-called generalized Gaussian bounds which involve weight functions $x \mapsto \exp(\xi \cdot x)$, $\xi \in \mathbb{R}^N$. Those weight functions correspond to the Euclidean metric in $\mathbb{R}^N$. In [18] more general weight functions were used in order to study selfadjoint elliptic operators with unbounded coefficients. There the weights are $x \mapsto \exp(\xi\varphi(x))$ where $\varphi$ is a so-called $L_1$-regular function on $\mathbb{R}^N$. This corresponds to the semi-metric $d(x,y) := |\varphi(x) - \varphi(y)|$ on $\mathbb{R}^N$. In [15], $L_p$-spectral independence was proved for closed (not necessarily selfadjoint) operators assuming a weighted norm estimate for a single resolvent. In Subsection 2.2 we discuss properties of $L_1$-regular functions and show that Theorem 1 of the present paper extends the result of [15, Thm. 1].

We want to emphasize that, until now, all proofs of $L_p$-spectral independence assuming weighted norm estimates relied on the "box method" where $\mathbb{R}^N$ is split up into congruent cubes $Q_j$ and one works in spaces $l_r(L_p(Q_j))$. In contrast, our proof of Theorem 1 below does *not* use the box method but relies on Lemma 12 as a substitute. Indeed, working in a general measure space which carries a semi-metric it is not clear what one should use instead of the partition into cubes of equal size. In Subsection 2.3 below we give an example of a strictly elliptic operator which illustrates the limitations of cube partitions even in Euclidean space.

A modified metric for $(\Omega, \mu)$ was already used in [14] where $L_p$-spectral independence was proved for integral operators whose kernels satisfy a certain estimate. As in our Theorem 2 the metric had the form $d(x,y) = \psi(|x-y|)$ where $\psi: [0,\infty) \to [0,\infty)$ is increasing, subadditive, continuous at 0 and satisfies $\psi(0) = 0$. A slight variant of Theorem 2 is an extension of the main result in [14] (see Section 5). Moreover, Theorem 2 gives an answer to [14, Problem 4.9(iii)]. In Subsection 2.4 we give classes of integral operators in $\mathbb{R}$ where our Theorems 2 or 1 apply but the result of [14] is not applicable.

Finally, in Subsection 2.5 we show how Theorem 1 may be applied to certain non-integral regularizable operators where the result of [15] is not applicable.

## 2.1 Gaussian bounds in metric spaces

In this subsection we discuss a result of Davies [7] which stimulated us to study the problem of $p$-independence in the context of (semi-)metric spaces. The result is as follows:

Let $(\Omega, d)$ be a separable metric space, $\mu$ a Borel measure on $\Omega$ satisfying

$$\mu(B(x,r)) \leq \begin{cases} c_0 r^N & \text{if } 0 \leq r \leq 1 \\ c_0 r^M & \text{if } 1 \leq r < \infty \end{cases} \tag{12}$$

for all $x \in \Omega$ and $r > 0$ for some constants $c_0 > 0$ and $0 < N \leq M < \infty$. Let $H$ be a non-negative selfadjoint operator in $L_2(\Omega)$. Assume that the generated semigroup $e^{-tH}$ on $L_2(\Omega)$ is given by an integral kernel $K(t,x,y)$ satisfying the Gaussian upper bound

$$|K(t,x,y)| \leq c_1 t^{-N/2} \exp(-c_2\, d(x,y)^2/t)$$

for all $t > 0$ and $x, y \in \Omega$ for some positive constants $c_1, c_2$.

Then the semigroup $e^{-tH}$ on $L_1(\Omega) \cap L_\infty(\Omega)$ extends to consistent $C_0$-semigroups $T_p(t)$ on $L_p(\Omega)$ for all $1 \leq p < \infty$. Let $-H_p$ denote the generator of $T_p(t)$. Then the spectrum of $H_p$ is independent of $p \in [1,\infty)$.

Note that in the general context of metric spaces the Gaussian upper bound is much less restrictive than what it seems: Assume that for $d(x,y) \geq 1$ we only have

$$|K(t,x,y)| \leq c_1 t^{-N/2} \exp(-c_2\, d(x,y)^\gamma/t) \quad (t>0)$$

for some $0<\gamma<1$. Then just define another metric $d_1 := \psi \circ d$ on $\Omega$ with the subadditive function

$$\psi(r) := \begin{cases} r & \text{if } r \leq 1 \\ r^{\gamma/2} & \text{if } r \geq 1 \end{cases} .$$

Replacing $d$ by $d_1$, the Gaussian upper bound is trivially fulfilled, and the volume bound (12) holds with $2M/\gamma$ in place of $M$.

Davies' result is restricted to the symmetric case. In the following we are going to show that without the assumption of symmetry Theorem 2 can be applied to the resolvent $R(\lambda, H)$ for $\lambda$ large enough. This will be a consequence of the following

**Lemma 4.** *Let $m_0(r) := r^N e^r$ $(r \geq 0)$ for some $N>0$ and $dm_0$ be the corresponding measure on $[0,\infty)$. Let $\alpha>0$. Then*

$$t^{-N/2} \| r \mapsto e^{-\alpha r^2/t} \|_{L_p(dm_0)} \leq M t^{-\frac{N}{2p'}} e^{\frac{t}{p\alpha}} \quad (1 \leq p \leq \infty,\ 0<t<\infty)$$

*for some $M>0$.*

*Proof.* For $p=\infty$ the estimate obviously holds with $M=1$. We are going to prove it for $p=1$, then the claim follows by interpolation. We have to show that

$$I := \int_0^\infty t^{-N/2} e^{-\alpha r^2/t} dm_0(r) \leq M e^{t/\alpha} \quad (0<t<\infty).$$

There exists $c>0$ with $dm_0(r) = (1+r/N)e^r dr^N \leq c e^{\sqrt{2}r} dr^N$. Using the elementary inequality $r \leq \frac{t}{\sqrt{2}\alpha} + \frac{\alpha r^2}{2\sqrt{2}t}$ we obtain $I \leq \int_0^\infty t^{-N/2} e^{-(1/2)\alpha r^2/t} c e^{t/\alpha} dr^N$. Substituting $r = \sqrt{t}s$ we deduce

$$I \leq \int_0^\infty e^{-(1/2)\alpha s^2} ds^N \cdot c e^{t/\alpha} \leq M e^{t/\alpha}$$

and conclude the proof. □

Now assume that (2) holds with $m(r) \leq c_0 r^N$ for $r \leq 1$. Let $\psi\colon [0,\infty) \to [0,\infty)$ be an increasing subadditive function with $\psi(r) = r$ $(r \leq 1)$ and $\psi(r) \geq \log \frac{m(r)}{c_0}$ $(r \geq 1)$. Then $m \circ \psi^{-1} \leq c_0 m_0$. As in the introduction $d_1 := \psi \circ d$ is a metric on $\Omega$ and $m(\psi^{-1}(r))$ is a bound for the $d_1$-balls of radius $r$ in $\Omega$. As in (5) we obtain that, for any non-increasing function $\varphi\colon [0,\infty) \to [0,\infty)$ and $p \in [1,\infty]$, we can estimate

$$n_p(\varphi \circ d_1) \leq \|\varphi\|_{L_p(d(m\circ\psi^{-1}))} \leq c_0 \|\varphi\|_{L_p(dm_0)}.$$

Let $T(t)$ be a $C_0$-semigroup on $L_2(\Omega)$ given by an integral kernel $K(t,x,y)$ satisfying the Gaussian upper bound

$$|K(t,x,y)| \leq c_1 t^{-N/2} \exp(-c_2\, d_1(x,y)^2/t) \tag{13}$$

for all $t > 0$ and $x, y \in \Omega$ for some positive constants $c_1, c_2$. Then $\|T(t)\restriction_{L_1(\Omega)\cap L_\infty(\Omega)}\|_{p\to p} \le c_1 n_1(t^{-N/2}\exp(-c_2\, d_1^2/t)) \le Me^{t/c_2}$ by (4) and the above lemma. Hence the semigroup $T(t)$ on $L_1(\Omega)\cap L_\infty(\Omega)$ extends to consistent semigroups $T_p(t)$ on $L_p(\Omega)$ for all $1 \le p \le \infty$. As in [7] one shows that these semigroups are strongly continuous for $p < \infty$ (for $p > 1$ this is clear by interpolation). Let $-A_p$ denote the generator of $T_p(t)$.

By the inequality $r \le \frac{t}{2\alpha} + \frac{\alpha r^2}{2t}$, we have $|K(t,x,y)|e^{d_1(x,y)} \le c_1 t^{-N/2}\exp\big(\frac{t}{2c_2} - \frac{c_2 d_1(x,y)^2}{2t}\big)$. For $\lambda > \lambda_0 := 1/(2c_2)$ define $h_\lambda(r) := c_1\int_0^\infty e^{(\lambda_0-\lambda)t} t^{-N/2}\exp\big(-\frac{c_2 r^2}{2t}\big)\,dt$. By Lemma 4 we conclude

$$\|h_\lambda\|_{L_p(dm_0)} \le c_1\int_0^\infty e^{(\lambda_0-\lambda)t} t^{-N/2}\big\|r\mapsto e^{-\frac{c_2 r^2}{2t}}\big\|_{L_p(dm_0)}\,dt \le c_1\int_0^\infty e^{(\lambda_0-\lambda)t} M t^{-\frac{N}{2p'}} e^{\frac{2t}{pc_2}}\,dt.$$

Note that $h_\lambda$ is a non-increasing function. For $p' > N/2$ and $\lambda > \lambda_p := \lambda_0 + \frac{2}{pc_2}$ it follows that the resolvent $R(\lambda, A_2)$ is given by an integral kernel $k_\lambda$ satisfying $g(x,y) := |k_\lambda(x,y)|e^{d_1(x,y)} \le h_\lambda(d_1(x,y))$ and hence $n_p(g) \le c_0\|h_\lambda\|_{L_p(dm_0)} < \infty$. In the same way $g \le (h_\lambda\circ\psi)\circ d$ and $\int_0^\infty h_\lambda\circ\psi(r)\,dm(r) = \|h_\lambda\|_{L_1(d(m\circ\psi^{-1}))} < \infty$. Now we can apply Theorem 2 in the variant of Remark 3(iii) to obtain

**Proposition 5.** *Let $K$ and $\psi$ be as above, i.e. (13) holds and (2) holds with $m\circ\psi^{-1} \le c_0 m_0$ for some $c_0 > 0$. Then for $\lambda \ge \frac{3}{c_2}$ the resolvent $(\lambda + A_2)^{-1}$ is given by an integral kernel $k_\lambda$ satisfying $n_p(k_\lambda e^{d_1}) < \infty$ for all $p < \frac{N}{N-2}$ ($p < \infty$ if $N = 2$, $p \le \infty$ if $N < 2$). If $\psi$ is continuous and satisfies condition (9), then the spectrum of $A_p$ is independent of $p \in [1, \infty)$.*

This proposition in particular covers the case of usual Gaussian upper bounds (i.e., the case $\psi(r) = r$) if the volume grows uniformly subexponentially. With slight modifications the same can be shown if the semigroup kernel satisfies a Gaussian upper bound of order $m$.

In [12, Sec. 6] it was observed that, for semigroup generators $A$, $p$-independence of the spectrum of $R(\lambda, A_p)^n$ for some $n \in \mathbb{N}$ and all large $\lambda$ implies $p$-independence of $\sigma(A_p)$. If, instead of (13), the kernels $K(t,x,y)$ only satisfy the upper bound

$$|K(t,x,y)| \le c_1 t^{-M/2}\exp(-c_2 d_1(x,y)^2/t) \quad (t > 0, x, y \in \Omega) \tag{14}$$

for some $c_1, c_2 > 0$ and some $M > N$, then $t^{n-1}\exp(-\varepsilon t)K(t,x,y)$ satisfies (13) if $n \ge 1 + (M-N)/2$ and $\varepsilon > 0$. The arguments leading to Proposition 5 show that $R(\lambda, A_2)^n$ has a kernel satisfying the assumptions of Theorem 2 in the variant of Remark 3(iii). Hence, under the assumptions of Proposition 5, we get $p$-independence of the spectrum of $R(\lambda, A_p)^n$ for all large $\lambda$. The observation in [12] then shows that $\sigma(A_p)$ does not depend on $p$.

Another way of proving $L_p$-spectral independence for operators satisfying (14) is to apply Theorem 1 for $\Omega' = \mathbb{R}^N$ and the Euclidean metric: Let $I$ be the interval of all $p \in [1, \infty]$ such that a bound $\|T(t)\|_{p\to p} \le C_p e^{\omega_p t}$ holds. From (14) we obtain

$$\|T(t)\|_{1\to\infty,\varepsilon} \le c_1 t^{-M/2} e^{\mu_1\varepsilon^2 t} \quad (t > 0,\ \varepsilon \ge 0)$$

for some $\mu_1 > 0$. Stein interpolation between this bound and the $L_r$-bound for the semigroup $(T(t))$ in $L_r$ yields

$$\|T(t)\|_{p\to q,\varepsilon} \le Ct^{-1/2} e^{\mu\varepsilon^2 t + \omega_{p,q} t} \quad (t > 0,\ \varepsilon \ge 0)$$

for all $p, q \in I$ satisfying $0 \leq p^{-1} - q^{-1} \leq M^{-1}$. Integration of this bound yields the weighted norm estimate (6) for a suitable resolvent. By Theorem 1 and Remark 3(vi) we obtain $p$-independence of the spectrum of the generator for $p \in I$.

Bound (14) was shown in [4] for semigroups generated by uniformly elliptic operators satisfying Neumann or Robin boundary conditions on certain subsets of $\mathbb{R}^N$. In the same way we could treat situations where a bound

$$|K(t,x,y)| \leq c_1 t^{-N/2} \exp(-c_2 d_1(x,y)^\gamma t^{1-\gamma})$$

with $\gamma \in (1,2)$ holds. This was done in [12] on open subsets of $\mathbb{R}^N$.

In [20], K. T. Sturm showed that the spectrum of uniformly elliptic operators on an $N$-dimensional complete Riemannian manifold with Ricci curvature bounded below is $p$-independent if the volume grows uniformly subexponential in the following sense: Instead of (2) the stronger bound

$$\mu(B(x,r)) \leq m(r)\mu(B(x,1)) \quad (x \in M, r > 0),$$

holds with a function $m$ satisfying (3). This bound is stronger since $\sup_{x\in M} \mu(B(x,1)) < \infty$ by Bishop's comparison principle, but $\inf_{x\in M} \mu(B(x,1)) > 0$ does not hold in general. As a matter of fact, Proposition 5 cannot be used to reprove that result. One of the difficulties is that one only has a weaker Gaussian upper bound for the corresponding semigroup. This problem can be solved by resorting to weighted $L_p$-spaces as described in Section 6. The volume growth bound (2) still holds for the weighted measure, but not with $m(r) \leq c_0 r^N$ $(r \leq 1)$ in general. Nevertheless it is possible to use Theorem 2 directly to prove $p$-independence of the spectrum of elliptic operators on Riemannian manifolds ( cf. [22]).

## 2.2 Weighted norm estimates and $L^1$-regular functions

In this subsection we present a result on $L_p$-spectral independence for closed operators acting in $L_p(\Omega)$ for some open set $\Omega \subset \mathbb{R}^N$ which was proved in [15]. Choosing an appropriate semi-metric $d$ on $\mathbb{R}^N$ we show that this result is also a consequence of Theorem 1 of the present paper. Before stating the result we need to introduce the following notion which is due to Semenov ([18]): We say that a function $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ is *$L^1$-regular*, if it is Lipschitz continuous (i.e. uniformly Lipschitz continuous) and

$$\sup_{k\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} e^{-|\varphi(k)-\varphi(j)|} < \infty.$$

**Theorem ([15], Theorem 1).** *Given $1 \leq p < q < \infty$ let $T_p$ and $T_q$ be closed operators in $L_p(\Omega)$ and $L_q(\Omega)$, respectively. If there exist $\varepsilon > 0$, $C < \infty$, $\lambda_0 \in \rho(T_p) \cap \rho(T_q)$ and an $L^1$-regular function $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ such that*

*(i) $R(\lambda_0, T_p)$, $R(\lambda_0, T_q)$ are consistent,*

*(ii) $\|e^{\xi\varphi} R(\lambda_0, T_p) e^{-\xi\varphi}\restriction_{L_{\infty,c}(\Omega)}\|_{p\to q} \leq C$ for all $\xi \in \mathbb{R}^N$, $|\xi| \leq \varepsilon$,*

*then $\sigma(T_p) = \sigma(T_q)$, and $R(\lambda, T_p)$, $R(\lambda, T_q)$ are consistent for all $\lambda \in \rho(T_p) = \rho(T_q)$.*

In [15], this theorem was applied to (not necessarily selfadjoint) strictly elliptic operators in divergence form with lower order terms. We now show that, for the semi-metric on $\Omega' := \mathbb{R}^N$ defined by $d(x,y) := |\varphi(x) - \varphi(y)|_\infty$ and for $\mu =: |\cdot|$ the Lebesgue measure, assumptions (1) to (3) and the weighted norm estimate (6) hold. The lower volume estimate (1) is a direct consequence of the Lipschitz continuity of $\varphi$: Let $L > 0$ such that $|\varphi(x) - \varphi(y)|_\infty \le L|x-y|_\infty$ for all $x \in \mathbb{R}^N$. Then $B(x,r) \supset x + [-r/L, r/L]^N$ and hence $|B(x,r)| \ge \left(\frac{2}{L}\right)^N r^N$ for all $r > 0$.

The following lemma in particular shows that (2) and (3) hold.

**Lemma 6.** *Let $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz continuous. Define the semi-metric $d$ on $\mathbb{R}^N$ as above. Then the following are equivalent:*

*(i) $\varphi$ is $L_1$-regular,*

*(ii) $n_1(e^{-\varepsilon d}) < \infty$ for all $\varepsilon > 0$,*

*(iii) there exists $C > 0$ such that $|B(x,r)| \le C(1+r)^N$ for all $r > 0$.*

*Proof.* (iii)$\Rightarrow$(ii) is shown in the introduction.

(ii)$\Rightarrow$(i) For $j \in \mathbb{Z}^N$ let $Q_j := j + [-\frac{1}{2}, \frac{1}{2}]^N$. Let $L$ be as above. Then for $k \in \mathbb{Z}^N$ we have

$$\sum_{j\in\mathbb{Z}^N} e^{-|\varphi(k)-\varphi(j)|} \le \sum_{j\in\mathbb{Z}^N} e^{L/2} \int_{Q_j} e^{-|\varphi(k)-\varphi(y)|}\,dy = e^{L/2} \int_{\mathbb{R}^N} e^{-d(k,y)}\,dy.$$

Since $\varphi$ is Lipschitz continuous the function $x \mapsto \int_{\mathbb{R}^N} e^{-d(x,y)}\,dy$ is continuous, and we conclude

$$\sup_{k\in\mathbb{Z}^N} \sum_{j\in\mathbb{Z}^N} e^{-|\varphi(k)-\varphi(j)|} \le e^{L/2} \sup_{k\in\mathbb{R}^N} \int_{\mathbb{R}^N} e^{-d(k,y)}\,dy = e^{L/2} n_1(e^{-d}) < \infty.$$

(i)$\Rightarrow$(iii) (cf. [15, Appendix A]) Let $r > 0$ and let $n \in \mathbb{N}$ with $n - 1 < r \le n$. Then

$$\begin{aligned} B(x,r) &= \left\{y \in \mathbb{R}^N; \varphi(y) \in \varphi(x) + (-r,r)^N\right\} \\ &\subset \bigcup \left\{\varphi^{-1}(\varphi(x) + j + [0,1]^N); j \in \{-n,\dots,n-1\}^N\right\}. \end{aligned}$$

It follows that $|B(x,r)| \le (2n)^N \sup_{z\in\mathbb{R}^N} |\varphi^{-1}(z + [0,1]^N)|$. Since $(2n)^N \le 2^N(1+r)^N$ it remains to show that the supremum is finite.

To this end, let $z \in \mathbb{R}^N$ and $Q := z + [0,1]^N$. Let $x_0, y \in \varphi^{-1}(Q)$ and choose $k, j \in \mathbb{Z}^N$ with $x_0 \in Q_k$ and $y \in Q_j$. Then $d(x_0,y) \le 1$ and $d(x_0,k), d(y,j) \le L/2$, hence $d(j,k) \le L+1$. Therefore $\varphi^{-1}(Q) \subset \bigcup\{Q_j; j \in \mathbb{Z}^N, d(j,k) \le L+1\}$ and

$$|\varphi^{-1}(Q)| \le \#\{j \in \mathbb{Z}^N; d(j,k) \le L+1\} \le \sum_{j\in\mathbb{Z}^N} e^{L+1-d(j,k)}.$$

By the $L_1$-regularity of $\varphi$ this show that $|\varphi^{-1}(Q)|$ can be estimated from above independently of the cube $Q$, and the proof is finished. $\square$

To conclude, we show that the weighted norm estimate (ii) of the above theorem with the unbounded weights $e^{\xi\varphi}$ implies estimate (6) with the bounded weights $\rho_{\varepsilon,z} = e^{-\varepsilon d(\cdot,z)}$. To this end let $E := \{\pm\varepsilon e_j; j = 1, \dots, N\}$ where $e_j$ are the standard unit vectors of $\mathbb{R}^N$. Fix $z \in \mathbb{R}^N$ and let $\rho_\xi := e^{\xi(\varphi-\varphi(z))}$ for $\xi \in E$. Then

$$\rho_{\varepsilon,z}^{-1} = e^{\varepsilon|\varphi-\varphi(z)|_\infty} = \max_{\xi\in E} \rho_\xi^{-1}.$$

For $f \in L_{\infty,c}(\Omega)$ and the dual operator $A'$ of $A$ (cf. Lemma 10 of Section 3) we obtain

$$\|\rho_{\varepsilon,z}^{-1} A' \rho_{\varepsilon,z} f\|_{p'} \le \sum_{\xi\in E} \|\rho_\xi^{-1} A' \rho_\xi \rho_\xi^{-1} \rho_{\varepsilon,z} f\|_{p'} \le \sum_{\xi\in E} \|\rho_\xi^{-1} A' \rho_\xi\|_{q'\to p'} \|\rho_\xi^{-1} \rho_{\varepsilon,z} f\|_{q'}.$$

Noting $\rho_\xi^{-1}\rho_{\varepsilon,z} \le 1$, we conclude by Lemma 10(ii) and the definition of $\rho_\xi$:

$$\|\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1}\|_{p\to q} = \|\rho_{\varepsilon,z}^{-1} A' \rho_{\varepsilon,z}\|_{q'\to p'} \le \sum_{\xi\in E} \|\rho_\xi^{-1} A' \rho_\xi\|_{q'\to p'} \le 2N \max_{\xi\in E} \|e^{\xi\varphi} A e^{-\xi\varphi}\|_{p\to q}.$$

Therefore (6) is fulfilled and we can apply Theorem 1, taking into account Remark 3(vi).

## 2.3 Elliptic operators in Euclidean space

We now want to discuss in some detail the fact that even in Euclidean space it is very helpful to have the possibility to resort to a totally different semi-metric and also very helpful not to be restricted to "cube partitions".

Let $\Omega \subset \mathbb{R}^N$ be an open set, $\mu$ the Lebesgue measure on $\Omega$. Let $a_{jk} \in L_{1,loc}(\Omega)$ be real-valued such that $a_{jk} = a_{kj}$ for all $1 \le j,k \le N$ and

$$\sum_{j,k=1}^N a_{jk}\xi_j\xi_k \ge \alpha|\xi|^2 \quad \text{a.e.} \quad (\xi \in \mathbb{R}^N)$$

for some $\alpha > 0$. We define the differential operator $-\nabla\cdot(a\nabla)$ with zero Dirichlet boundary conditions by the form method. Let $\tau$ be the following sesquilinear form in $L^2$:

$$\tau(u,v) := \int_\Omega \sum_{j,k=1}^N a_{jk}(x)\partial_k u(x)\partial_j \bar{v}(x)\,dx, \quad D(\tau) := C_c^\infty(\Omega).$$

It is well-known that the form $\tau$ is closable and that $\bar\tau$ is a Dirichlet form. Let $H$ be the selfadjoint operator in $L_2$ associated with $\bar\tau$. Then $e^{-tH}\restriction_{L_{\infty,c}(\Omega)}$ extends to a $C_0$-semigroup $e^{-tH_p}$ on $L_p$ for all $1 \le p < \infty$. Here $L_{\infty,c}(\Omega)$ means the space corresponding to the sequence $(\Omega_n) := (\{x \in \Omega; |x| \le n, \operatorname{dist}(x,\partial\Omega) \ge 1/n\})$ where dist denotes the Euclidean distance. This is the appropriate choice for later application of [15, Thm. 8].

If $H$ is uniformly elliptic, i.e. the coefficients $a_{jk}$ are bounded, the semigroup $e^{-tH}$ satisfies an upper Gaussian estimate, and $\sigma(H_p)$ is independent of $p \in [1,\infty)$ ([1, Example 5.2]).

In [18], Semenov studied elliptic operators which are not necessarily uniformly elliptic. In this case the semigroup may not satisfy an upper Gaussian estimate (with respect to the Euclidean metric), but under certain conditions $p$-independence of the spectrum still holds. He showed the same if the operator is perturbed by a real-valued potential with form small negative part. Then the semigroup does not necessarily exist in all $L_p$-spaces.

In [15], the conditions from [18] were slighty relaxed, and an additional first order perturbation was included. This required a new method of proof since the resulting operator need no longer be selfadjoint.

We are now going to present an example in dimension $N = 2$ which is not covered by the above mentioned results but is by our Theorem 1: Let $\Omega := \Omega' := \{x \in \mathbb{R}^2; x_1 > 0, 0 < x_2 < x_1^{-2}\} \cup \{x \in \mathbb{R}^2; x_1 \leq 0, 0 < x_2 < 2\}$. Note that $\Omega$ has infinite measure, but the unbounded subset $\Omega_0 := \Omega \cap \{x; x_1 \geq 0, x_2 \leq 2\}$ has finite measure.

Let $a_{jk}$ be as above and assume that $a_{22}(x) \leq c_0 x_2$ for $x_2 > 2$ and $a_{11}(x) \leq c_0$ for $x_1 < 0$. By

$$d(x, y) := |\sqrt{x_2 \vee 2} - \sqrt{y_2 \vee 2}| + |x_1 \wedge 0 - y_1 \wedge 0| \quad (x, y \in \Omega)$$

(where $\vee$ denote the maximum, $\wedge$ the minimum) we define a semi-metric on $\Omega$. We now consider open balls of radius $r$ with respect to this semi-metric. Obviously such a ball contains $\Omega_0$ if and only if it contains a point of $\Omega_0$. A direct calculation shows that $4r \leq \mu(B(x, r)) \leq 4r + 2\sqrt{2}$ for all $x \in \Omega$, $r > 0$. Hence conditions (1) to (3) are fulfilled for the metric $d$, choosing $m(r) := 4r + 2\sqrt{2}$ $(r > 0)$ and $m(0) := 0$.

Observe that in this example the space of all essentially bounded functions with support in a $d$-bounded set is strictly larger than the space $L_{\infty,c}(\Omega)$ we have chosen since it contains the characteristic function of $\Omega_0$ (cf. Remark 3(v)). On the other hand, the space of all functions that are integrable on all $d$-bounded sets is strictly smaller than $L_{1,loc}(\Omega)$.

In order to show the weighted estimate (6) for the resolvents of $H$, we want to apply [15, Thm. 8]. For $0 < \varepsilon \leq 1$, $z \in \Omega$ and $\rho := \rho_{\varepsilon,z}$ we obtain

$$\begin{aligned}\rho^{-2} \sum_{j,k=1}^{2} a_{jk} \partial_k \rho \partial_j \rho &= \sum_{j,k=1}^{2} a_{jk} \varepsilon^2 \partial_k d(\cdot, z) \partial_j d(\cdot, z) \\ &= \varepsilon^2 \big(a_{11} (\partial_1 d(\cdot, z))^2 \chi_{\{x; x_1 < 0\}} + a_{22} (\partial_2 d(\cdot, z))^2 \chi_{\{x; x_2 > 2\}}\big) \leq c_0.\end{aligned}$$

Further $\rho \wedge n$ and $\rho^{-1} \wedge n$ are Lipschitz continuous for all $n \in \mathbb{N}$, so the conditions of [15, Thm. 8] are fulfilled. We conclude that for $1 < p < q < \infty$ there exist $\lambda \in \mathbb{R}$, $C > 0$ such that

$$\|\rho_{\varepsilon,z} (\lambda + H)^{-1} \rho_{\varepsilon,z}^{-1} \restriction_{L_{\infty,c}(\Omega)} \|_{p \to q} \leq C \quad (0 < \varepsilon \leq 1,\ z \in \Omega).$$

By Remark 3(vi), it follows that the spectrum of $H_p$ is independent of $p \in (1, \infty)$. In fact the same is true for $p = 1$, but this is not covered by the above mentioned theorem.

Applying [15, Thm. 2] directly to this situation would lead to stronger conditions on the coefficients $a_{jk}$: On the one hand, the linear growth $a_{22}$ would not be allowed, on the other hand there would be an additional condition on $a_{11}$ on the subset $\Omega_0$. This is clear from the discussion on $L_1$-regular functions in the previous subsection.

## 2.4 Integral operators

The following two propositions present classes of kernel operators which may be treated by the results of this paper but not necessarily by the result in [14] which treated the case $\varphi(y) = y$. As in Section 1 let $\psi\colon [0,\infty) \to [0,\infty)$ be increasing and subadditive with $\psi(0) = 0$. Then $d_1(x,y) := \psi(|x-y|)$ $(x, y \in \mathbb{R})$ defines a metric on $\mathbb{R}$.

**Proposition 7.** *Let $k\colon \mathbb{R}^2 \to \mathbb{C}$ be measurable such that*

$$|k(x,y)| \le h(x - \varphi(y)) \quad (x, y \in \mathbb{R}),$$

*where $h$ and $\varphi$ satisfy the following assumptions: $h\exp(\psi(|\cdot|)) \in L_1 \cap L_{p_0}$ for some $p_0 > 1$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is measurable such that $s := \sup_t |\varphi(t) - t| < \infty$. Let $A$ be the integral operator on $L_{\infty,c}(\mathbb{R})$ with kernel $k$. Then there exists a constant $C > 0$ such that*

$$\|e^{\varepsilon d_1(\cdot,z)} A e^{-\varepsilon d_1(\cdot,z)}\|_{1\to p_0} \le C$$

*for all $\varepsilon \in [0,1]$ and all $z \in \mathbb{R}$.*

*Proof.* For all $\varepsilon \in [0,1]$, $z, y \in \mathbb{R}$ we have

$$\begin{aligned}
\int |e^{\varepsilon d_1(x,z)} k(x,y) e^{-\varepsilon d_1(y,z)}|^{p_0}\, dx &\le \int e^{\varepsilon p_0 d_1(x,y)} h(x - \varphi(y))^{p_0}\, dx \\
&\le e^{\varepsilon p_0 d_1(y,\varphi(y))} \int e^{\varepsilon p_0 d_1(x,\varphi(y))} h(x - \varphi(y))^{p_0}\, dx \\
&\le e^{p_0 \psi(s)} \|e^{\psi(|\cdot|)} h\|_{p_0}^{p_0},
\end{aligned}$$

which yields the claim with $C := \exp(\psi(s)) \|\exp(\psi(|\cdot|))h\|_{p_0}$. □

Assume that $\psi(r)/\log r \to \infty$ $(r \to \infty)$. Then conditions (1) to (3) hold for the balls with respect to the metric $d_1$ and we can apply Theorem 1 to the operator $A$ from Proposition 7. Since $k$ need not decay exponentially as $|x-y| \to \infty$, we may not apply the result of [15], cf. the end of Subsection 2.5.

**Proposition 8.** *Assume in addition to the assumptions of Proposition 7 that $\varphi \in C^1$ is strictly increasing and that $(\varphi^{-1})'$ is bounded. Then we have $n_{p_0}(|k|e^{d_1}) < \infty$.*

*Proof.* We interchange the roles of $x$ and $y$ in the previous proof and use the substitution $y = \varphi^{-1}(\eta)$ to obtain

$$\begin{aligned}
\int |k(x,y) e^{d_1(x,y)}|^{p_0}\, dy &\le \int e^{p_0 d_1(x,y)} h(x - \varphi(y))^{p_0}\, dy \\
&= \int e^{p_0 d_1(x,\varphi^{-1}(\eta))} h(x - \eta)^{p_0} (\varphi^{-1})'(\eta)\, d\eta \\
&\le e^{p_0\psi(s)} \|(\varphi^{-1})'\|_\infty \|e^{\psi(|\cdot|)} h\|_{p_0}^{p_0}
\end{aligned}$$

which together with the previous proof implies $n_{p_0}(|k|e^{d_1}) < \infty$. □

If the conditions of Proposition 8 are fulfilled and $\psi$ is continuous and satisfies conditions (8) and (9), we can apply Theorem 2. Here we in particular have in mind the function $\psi(r) := \alpha\log(1+r)$ for some $\alpha > 1$. Then (9) is fulfilled since $e^{\psi(\varepsilon r)} = (1+\varepsilon r)^\alpha$ and $\int_0^\infty (1+\varepsilon r)^{-\alpha} dr < \infty$ for all $\varepsilon > 0$. The function $g$ in Theorem 2 is $g = |k|(1+d)^\alpha$ in this case. Further observe that the volume of a $d_1$-ball with radius $r$ is $2\psi^{-1}(r) = 2(e^{r/\alpha} - 1)$. Hence $(\mathbb{R}, d_1)$ has exponential volume growth.

We now give an example of a kernel $k$ satisfying the assumptions of Proposition 7 but not of Proposition 8. Let $\alpha \in (0,1)$ and define $h$ by $h(x) := |x|^{-\alpha}$ if $|x| \le 1$ and 0 otherwise. Let $\beta > 1/\alpha$ and define $\varphi$ by $\varphi(y) := y$ if $|y| \ge 1$ and $\varphi(y) := |y|^\beta \operatorname{sgn} y$ if $|y| < 1$. Define the kernel $k$ by $k(x,y) := h(x - \varphi(y))$ and denote the corresponding integral operator by $A$. Let $\psi(r) = r$. Then $A$ satisfies the assumptions of Proposition 7 for all $p_0 < 1/\alpha$ and we can apply Theorem 1.

But $A$ does not satisfy the assumptions of Theorem 2 (in particular it does not satisfy the assumptions of the result in [14]) since $n_1(k) = \infty$: We have $\int_{-1}^1 |y|^{-\alpha\beta}\, dy = \infty$ by the assumption $\beta > 1/\alpha$. By the monotone convergence theorem we conclude

$$\|k(x,\cdot)\|_1 = \int_{\mathbb{R}} h(x - \varphi(y))\, dy \ge \int_{-1/2}^{1/2} \left|x - |y|^\beta \operatorname{sgn} y\right|^{-\alpha} dy \ge \int_{-1/2}^{1/2} \left(|x| + |y|^\beta\right)^{-\alpha} dy \to \infty$$

as $x \to 0$. This implies $n_1(k) = \infty$.

In the following we show that $A$ is even not $L_r$-bounded for $r > \frac{\beta-1}{\alpha\beta-1}$. To this aim let $1 - \alpha\beta < \gamma < 1$ and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(y) := |y|^{-\gamma}$ if $|y| \le 1$ and 0 otherwise. Then $(\gamma-1)/\beta > -\alpha$ and hence $\delta := \alpha + (\gamma-1)/\beta > 0$. Since $\alpha, \gamma < 1$ we get $\delta - \alpha + 1 \in (0,1)$. Using this and the substitution $z = |y|^\beta \operatorname{sgn} y$ it is elementary to show that $Af(x) \sim |x|^{-\delta}$ as $x \to 0$. Hence $Af \notin L_r(\mathbb{R})$ for $r^{-1} \le \delta$, but $f \in L_r(\mathbb{R})$ for $\gamma < r^{-1}$. It remains to show that for $r > \frac{\beta-1}{\alpha\beta-1}$ there exists $1 - \alpha\beta < \gamma < 1$ such that $\gamma < r^{-1} \le \alpha + (\gamma-1)/\beta$. A direct calculation yields that $\gamma := r^{-1}\beta + 1 - \alpha\beta$ fulfils these inequalities. If, e.g., $\alpha = 3/4$ and $\beta = 3$, then taking $\gamma = 1/4$ shows that $A$ is not $L_2$-bounded.

## 2.5 Non-integral regularizing operators

According to [21], for all $1 < p < q < \infty$ with $p^{-1} - q^{-1} < 1/2$, there exists a linear non-integral operator $A\colon L_p(\mathbb{R}) \to L_q(\mathbb{R})$, given by convolution with a certain finite positive Borel measure $\nu$ with $\operatorname{supp}\nu \subset [0,1]$. Let $\psi$ be a subadditive increasing function satisfying $\psi(0) = 0$ and $\psi(r)/\log r \to \infty$ as $r \to \infty$. By $d(x,y) := \psi(|x-y|)$ we define a metric on $\mathbb{R}$. Then conditions (1) to (3) hold for the balls with respect to the metric $d$ if we endow $\mathbb{R}$ with the Lebesgue measure.

For $\varepsilon_0 = 1$ and $C := e^{\psi(1)}\|A\|_{p\to q}$, estimate (6) holds, since

$$\begin{aligned} |e^{-d(x,z)}(A\, e^{d(\cdot,z)} f)(x)| &\le e^{-d(x,z)} \int_0^1 e^{d(z,x-y)} |f(x-y)|\, d\nu(y) \\ &\le \int_0^1 e^{\psi(|y|)} |f(x-y)|\, d\nu(y) \le e^{\psi(1)} (A|f|)(x) \quad (x,z \in \mathbb{R}) \end{aligned}$$

for all $f \in L_{\infty,c}(\mathbb{R})$. Now let $k$ be a measurable function on $\mathbb{R}^2$ satisfying $|k(x,y)| \le \exp\bigl(-2\psi(|x-y|)\bigr)$ $(x,y \in \mathbb{R})$ and define $S$ to be the integral operator with kernel $k$.

Then we can apply Theorem 1 to $A+S$ but none of the theorems requiring a pointwise defined kernel; in particular, we cannot apply the result of [14].

Now assume that $\exp\bigl(-c\psi(|x|)-c\psi(|y|)\bigr) \le |k(x,y)| \le \exp\bigl(-2\psi(|x-y|)\bigr)$ for some $c \ge 2$. Using Lemma 6 from Subsection 2.2, we are going to show that we cannot apply the result of [15] to $A+S$ if $\psi(r)/r \to 0$ as $r \to \infty$, e.g. $\psi(r)=\sqrt{r}$. To this end suppose there exist $\varepsilon > 0$ and an $L_1$-regular function $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\|e^{\xi\varphi}(A+S)e^{-\xi\varphi}\|_{p\to q} < \infty \quad \text{for all } \xi \in [-\varepsilon,\varepsilon].$$

Without loss of generality assume that $\varphi(0)=0$. Let $0 \le f \in L_{\infty,c}(\mathbb{R})$, $f \ne 0$. Then

$$e^{\xi\varphi}(A+S)e^{-\xi\varphi}f(x) \ge e^{\xi\varphi}Ae^{-\xi\varphi}f(x) + e^{\xi\varphi(x)-c\psi(|x|)}\int_{\mathbb{R}} e^{-c\psi(|y|)-\xi\varphi(y)}f(y)\,dy.$$

for all $\xi \in [-\varepsilon,\varepsilon]$. The first summand on the right hand side is in $L_{\infty,c}(\mathbb{R})$. Suppose the second is in $L_q(\mathbb{R})$ for some $q \in [1,\infty]$ and $\xi = \pm\varepsilon$. Then in particular $\sup_{x\in\mathbb{R}}\bigl(\varepsilon|\varphi(x)| - c\psi(|x|)\bigr) < \infty$, i.e., $|\varphi(x)| \le c\psi(|x|)/\varepsilon + k$ for some $k > 0$. Since $\varphi(0)=0$, Lemma 6 implies that

$$\bigl|\bigl\{y\in\mathbb{R};\psi(|y|) \le \tfrac{\varepsilon}{c}(r-k)\bigr\}\bigr| \le \bigl|\bigl\{y\in\mathbb{R};|\varphi(0)-\varphi(y)| \le r\bigr\}\bigr| \le C(1+r) \quad (r>k).$$

This contradicts the assumption $\psi(r)/r \to 0$ as $r \to \infty$.

# 3 Proof of Theorem 1

We start with a series of preparatory results. For this whole section we fix $1 \le p \le q \le \infty$.

In order to prove the inclusion $\rho(A_r) \subset \rho(A_s)$ for $r,s \in [p,q]$ one has to show that for $\lambda \in \rho(A_r)$ the operator $R(\lambda,A_r)\restriction_{L_{\infty,c}(\Omega)}$ extends to a bounded operator on $L_s(\Omega)$. This is expressed in the following elementary lemma which is stated in the general context of topological spaces (cf. [15, Prop. 4] or [1, Prop. 2.3]).

Let $E,F,G$ be Hausdorff spaces with $E,F \hookrightarrow G$ such that $E\cap F$ is dense in both $E$ and $F$. Let $D \subset E\cap F$ be a subset which is dense with respect to the initial topology coming from the embeddings $E\cap F \hookrightarrow E$ and $E\cap F \hookrightarrow F$.

**Lemma 9.** *Let $A_E\colon E\to E$ and $A_F\colon F\to F$ be continuous mappings satisfying $A_E\restriction_D = A_F\restriction_D$. Assume that $A_E$ is continuously invertible and $A_E^{-1}\restriction_D$ extends to a continuous mapping $R\colon F\to F$. Then $A_F$ is continuously invertible, and $A_F^{-1}=R$.*

*Proof.* Since $D$ is dense in $E\cap F$ and $E,F \hookrightarrow G$, we have $A_E\restriction_{E\cap F} = A_F\restriction_{E\cap F}$ and $A_E^{-1}\restriction_{E\cap F} = R\restriction_{E\cap F}$. Hence $RA_F = A_FR = I$ on $E\cap F$. The density of $E\cap F$ in $F$ yields the claim. □

To apply this lemma we further need the following. Consider the dual system $\langle L_{\infty,c}(\Omega), L_{1,loc}(\Omega)\rangle$ and endow $L_{\infty,c}(\Omega)$ and $L_{1,loc}(\Omega)$ with the corresponding weak topologies. If $A\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ is a weakly continuous operator, then $A$ has a dual operator $A'\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$, i.e. $\int Af\cdot g = \int f\cdot A'g$ for all $f,g \in L_{\infty,c}(\Omega)$.

**Lemma 10.** *Let $A\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ be a linear operator, $\|A\|_{p\to q} < \infty$.*

*(i)* *If $p < \infty$, then $A$ is weakly continuous.*

*(ii)* *If $A$ is weakly continuous, then for the dual operator $A'$ one has $\|A'\|_{q'\to p'} = \|A\|_{p\to q}$.*

*(iii)* *If $A$ is weakly continuous and $p = q = \infty$, then $A$ has a unique weak*-continuous extension $A_\infty$: $L_\infty(\Omega) \to L_\infty(\Omega)$.*

*Proof.* (i) Let $A_p$: $L_p(\Omega) \to L_q(\Omega)$ be the continuous extension of $A$. Then $A_p$ is weakly continuous. Since $L_p(\Omega)$ is reflexive, $A_p$ is weak*-continuous. This implies (i).

(ii) follows from $\|A\|_{p\to q} = \sup\big\{ |\int Af \cdot g|; f, g \in L_{\infty,c}(\Omega), \|f\|_p = \|g\|_{q'} = 1\big\}$.

(iii) By (ii) we have $\|A'\|_{1\to 1} < \infty$. Let $\tilde{A}$: $L_1(\Omega) \to L_1(\Omega)$ denote the continuous extension of $A'$. Then $A_\infty := \tilde{A}'$ is the unique weak*-continuous extension of $A$. □

**Corollary 11.** *Let $A_r$: $L_r(\Omega) \to L_r(\Omega)$ be consistent bounded operators ($p \le r \le q$), $A_\infty$ weak*-continuous if $q = \infty$. If $\lambda \in \rho(A_r)$ and $\|R(\lambda, A_r)\lceil_{L_{\infty,c}(\Omega)}\|_{s\to s} < \infty$ for some $r, s \in [p, q]$, then $\lambda \in \rho(A_s)$.*

*Proof.* For $r, s < \infty$ the result directly follows from Lemma 9 with $G = L_{1,loc}(\Omega)$ and $D = L_{\infty,c}(\Omega)$.

We endow $L_\infty(\Omega)$ with the weak*-topology. Then, for $t < \infty$, $L_{\infty,c}(\Omega)$ is dense in $L_t(\Omega) \cap L_\infty(\Omega)$, and $L_t(\Omega) \cap L_\infty(\Omega)$ is dense in $L_t(\Omega)$ and $L_\infty(\Omega)$.

For the case $r = \infty$ it remains to note that $\lambda - A_\infty$ has a bounded inverse if and only if it has a weak*-continuous inverse. In the case $r < s = \infty$ the assumptions imply that $R(\lambda, A_r)\lceil_{L_{\infty,c}(\Omega)}$ is $L_r$- and $L_\infty$-bounded. Hence, by Lemma 10(i) and (iii), $R(\lambda, A_r)\lceil_{L_{\infty,c}(\Omega)}$ has a weak*-continuous extension $R$: $L_\infty(\Omega) \to L_\infty(\Omega)$, and again we are done by Lemma 9. □

Now we construct a norm equivalent to the $L_q$-norm using the weights $\rho_{\varepsilon,z}$. Here is the only point where assumption (1) is needed.

**Lemma 12.** *For all $f \in L_q(\Omega)$ and $\varepsilon > 0$ we have*

$$\|f\|_{q,\varepsilon} := \big\| z \mapsto \|\rho_{\varepsilon,z} f\|_{L_q(\Omega)} \big\|_{L_q(\Omega')} \ge c_{q,\varepsilon}^{-1} \|f\|_q,$$

*where $c_{q,\varepsilon} := \gamma^{-1/q} e^{\varepsilon R}$.*

*Proof.* By Fubini's theorem we obtain $\|f\|_{q,\varepsilon} = \big\| x \mapsto \|\rho_{\varepsilon,x}\|_{L_q(\Omega')} f(x) \big\|_{L_q(\Omega)}$. Assumption (1) implies $\|\rho_{\varepsilon,x}\|_{L_q(\Omega')} \ge \|e^{-\varepsilon R}\chi_{B(x,R)}\|_{L_q(\Omega')} \ge e^{-\varepsilon R}\gamma^{1/q}$ for almost all $x \in \Omega$. This gives the desired conclusion. □

The following consequence of (4) will be used throughout:

$$\big\| \Omega' \ni z \mapsto \|g(z,\cdot) f\|_p \big\|_q \le n_t(g) \|f\|_s \quad (p \le s, t \le q,\ p^{-1} + q^{-1} = s^{-1} + t^{-1}), \tag{15}$$

where $g$ is a measurable function on $\Omega' \times \Omega'$, $f$ a measurable function on $\Omega$. In particular, for $p = q$, $g = \rho_\varepsilon$ we have $\|f\|_{q,\varepsilon} \le n_q(\rho_\varepsilon)\|f\|_q$, so $\|\cdot\|_{q,\varepsilon}$ is indeed equivalent to $\|\cdot\|_q$.

For a linear operator $A$: $L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ and $\varepsilon \in \mathbb{R}$ we will use the following notation:

$$\begin{aligned}\|A\|_{p\to q,\varepsilon} &:= \sup_{z\in\Omega'} \|\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1}\|_{p\to q} \\ &= \inf\big\{c > 0; \forall f \in L_{\infty,c}(\Omega), z \in \Omega' : \|\rho_{\varepsilon,z} A f\|_q \le c \|\rho_{\varepsilon,z} f\|_p\big\} \in [0, \infty].\end{aligned}$$

With this notation assumption (6) reads $\|A\|_{p\to q,\varepsilon_0} \le C$.

Using Lemma 12 and the estimate (15) only, we see that assumption (6) implies that the operator $A$ is $L_q$-bounded: Let $f \in L_{\infty,c}(\Omega)$, then

$$\|Af\|_q \le c_{q,\varepsilon_0}\|Af\|_{q,\varepsilon_0} \le c_{q,\varepsilon_0}\big\|z \mapsto \|A\|_{p\to q,\varepsilon_0}\|\rho_{\varepsilon_0,z}f\|_p\big\|_q \le c_{q,\varepsilon_0}n_p(\rho_{\varepsilon_0})\|A\|_{p\to q,\varepsilon_0}\|f\|_q.$$

But we are going to establish much more general estimates. For $\varepsilon > 0$, $M \ge 1$ we define a class of weight functions

$$P(\varepsilon, M) := \big\{\rho\colon \Omega' \to (0,\infty) \text{ measurable}; \forall u, x \in \Omega' : \rho(x)/\rho(u) \le M\rho_{-\varepsilon,u}(x)\big\}.$$

For $\rho \in P(\varepsilon, M)$, $\varepsilon' \in \mathbb{R}$ and $u \in \Omega'$ we have

$$\rho_{\varepsilon',u}\rho\rho(u)^{-1} \le M\rho_{\varepsilon'-\varepsilon,u} \quad \text{and} \quad \rho_{\varepsilon',u}\rho(u)\rho^{-1} \le M\rho_{\varepsilon'-\varepsilon,u}. \tag{16}$$

Note that due to the triangle inequality $\rho_{\varepsilon,z} \in P(|\varepsilon|, 1)$ for $\varepsilon \in \mathbb{R}$ and $z \in \Omega'$.

We now estimate several operator norms in terms of $\|A\|_{p\to q,\varepsilon_0}$ (cf. [17, Prop. 3.2]).

**Proposition 13.** *(i) Let $p \le s \le q$, $t^{-1} := p^{-1} + q^{-1} - s^{-1}$. For $0 \le \varepsilon < \varepsilon_0$, $M \ge 1$ and $\rho \in P(\varepsilon, M)$ we have*

$$\|\rho A\rho^{-1}\|_{s\to q} \le M^2 c_{q,\varepsilon_0+\varepsilon}n_t(\rho_{\varepsilon_0-\varepsilon})\|A\|_{p\to q,\varepsilon_0}.$$

*(ii) There exist constants $C_{\varepsilon_0,\varepsilon} < \infty$, bounded as $\varepsilon \to 0$, such that*

$$\|A\|_{s\to t,\varepsilon} \le C_{\varepsilon_0,\varepsilon}\|A\|_{p\to q,\varepsilon_0}$$

*for all $p \le s \le t \le q$ and $|\varepsilon| < \varepsilon_0$.*

*Proof.* We begin with (i). Let $\varepsilon' := \varepsilon_0 + \varepsilon$ and $f \in L_{\infty,c}(\Omega)$. By Lemma 12 we have

$$\|\rho A\rho^{-1}f\|_q \le c_{q,\varepsilon'}\big\|u \mapsto \|\rho_{\varepsilon',u}\rho\rho(u)^{-1}A\rho(u)\rho^{-1}f\|_q\big\|_q.$$

Using (16) and the definition of $\|A\|_{p\to q,\varepsilon_0}$ we get

$$\begin{aligned}\|\rho_{\varepsilon',u}\rho\rho(u)^{-1}A\rho(u)\rho^{-1}f\|_q &\le M\|\rho_{\varepsilon_0,u}A\rho(u)\rho^{-1}f\|_q \le M\|A\|_{p\to q,\varepsilon_0}\|\rho_{\varepsilon_0,u}\rho(u)\rho^{-1}f\|_p \\ &\le M^2\|A\|_{p\to q,\varepsilon_0}\|\rho_{\varepsilon_0-\varepsilon,u}f\|_p.\end{aligned}$$

By (15) we conclude

$$\|\rho A\rho^{-1}f\|_q \le c_{q,\varepsilon'}M^2\|A\|_{p\to q,\varepsilon_0}\big\|u \mapsto \|\rho_{\varepsilon_0-\varepsilon,u}f\|_p\big\|_q \le M^2 c_{q,\varepsilon'}\|A\|_{p\to q,\varepsilon_0}n_t(\rho_{\varepsilon_0-\varepsilon})\|f\|_s.$$

This proves (i).

For $t = q$, part (ii) is just a special case of (i) since $\rho_{\varepsilon,z} \in P(|\varepsilon|, 1)$ for $z \in \Omega'$. Note that the constant $C_{\varepsilon_0,\varepsilon}$ does not depend on $p, q, s, t$ since $c_{q,\varepsilon'} \le c_{1,\varepsilon'} \vee c_{\infty,\varepsilon'}$ and $n_q(\rho_{\varepsilon_0-\varepsilon}) \le n_1(\rho_{\varepsilon_0-\varepsilon}) \vee n_\infty(\rho_{\varepsilon_0-\varepsilon})$. Further note that the constants stay bounded as $\varepsilon \to 0$.

We now prove (ii) for $s = t = p$. Note that the case $p = \infty$ is treated above, so we assume $p < \infty$. By Lemma 10(i) we know that $\rho_{\varepsilon_0,z}A\rho_{\varepsilon_0,z}^{-1}\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ is weakly continuous for all $z \in \Omega'$. It easily follows that $A$ is weakly continuous and that $(\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1})' = \rho_{\varepsilon,z}^{-1} A' \rho_{\varepsilon,z}$ for all $\varepsilon \in \mathbb{R}$, $z \in \Omega'$. Hence by Lemma 10(ii) and by (i) we obtain

$$\|A'\|_{q' \to p', \varepsilon_1} = \|A\|_{p \to q, -\varepsilon_1} \le c_{q, \varepsilon_0 + \varepsilon_1} n_q(\rho_{\varepsilon_0 - \varepsilon_1}) \|A\|_{p \to q, \varepsilon_0}$$

for $\varepsilon_0 > \varepsilon_1 > 0$. For $|\varepsilon| < \varepsilon_0$, choosing $|\varepsilon| < \varepsilon_1 < \varepsilon_0$ and using (i) again for the dual situation we conclude

$$\begin{aligned} \|A\|_{p \to p, \varepsilon} &= \|A'\|_{p' \to p', -\varepsilon} \\ &\le c_{p', \varepsilon_1 + |\varepsilon|} n_{q'}(\rho_{\varepsilon_1 - |\varepsilon|}) \|A'\|_{q' \to p', \varepsilon_1} \\ &\le c_{p', \varepsilon_1 + |\varepsilon|} n_{q'}(\rho_{\varepsilon_1 - |\varepsilon|}) c_{q, \varepsilon_0 + \varepsilon_1} n_q(\rho_{\varepsilon_0 - \varepsilon_1}) \|A\|_{p \to q, \varepsilon_0}. \end{aligned}$$

Now Riesz-Thorin interpolation between the cases treated above proves the remaining cases. □

**Remark 14.** Let $A$: $L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ be weakly continuous with $\|A\|_{p \to q, \varepsilon_0} < \infty$ for some $\varepsilon_0 > 0$. By Lemma 10(i) the latter implies weak continuity if — as in the assumptions of Theorem 1 — $p < \infty$.

(i) Proposition 13(ii) yields that $A$ extends to a bounded operator $A_r$ on $L_r(\Omega)$ for $r \in [p, q]$, $r < \infty$. In the case $q = \infty$ Lemma 10(iii) implies that $A$ has a unique weak*-continuous extension $A_\infty$: $L_\infty(\Omega) \to L_\infty(\Omega)$. The operators $A_r$ are consistent since $L_{\infty,c}(\Omega)$ is dense in $L_r(\Omega) \cap L_s(\Omega)$ if $L_\infty(\Omega)$ is endowed with the weak*-topology.

(ii) If $\rho \in P(\varepsilon, M)$ for some $\varepsilon < \varepsilon_0$, $M \ge 1$, then $\rho A \rho^{-1}$ extends to a bounded operator $A_\rho$: $L_q(\Omega) \to L_q(\Omega)$ by Proposition 13(i). In the case $q = \infty$, again $A_\rho$: $L_\infty(\Omega) \to L_\infty(\Omega)$ is the unique weak*-continuous extension. In case $\rho = \rho_{\varepsilon,z}$ for some $|\varepsilon| < \varepsilon_0, z \in \Omega'$ we write $A_{\varepsilon,z}$ for $A_{\rho_{\varepsilon,z}}$.

(iii) If an operator $A$ is defined on $D(A) \supset L_{\infty,c}(\Omega)$, we also write $\|A\|_{p \to q, \varepsilon}$ instead of $\|A\restriction_{L_{\infty,c}(\Omega)}\|_{p \to q, \varepsilon}$. If $A$: $L_p(\Omega) \to L_q(\Omega)$ is bounded (weak*-continuous in case $p = \infty$), then $\|\rho_{\varepsilon,z} A f\|_q \le \|A\|_{p \to q, \varepsilon} \|\rho_{\varepsilon,z} f\|_p$ for all $f \in L_p(\Omega)$. Therefore, if $B$: $L_q(\Omega) \to L_r(\Omega)$ is a bounded operator (weak*-continuous in case $q = \infty$), then $\|BA\|_{p \to r, \varepsilon} \le \|A\|_{p \to q, \varepsilon} \|B\|_{q \to r, \varepsilon}$.

The crucial part in the proof of Theorem 1 is the following estimate which implies convergence of weighted operators (cf. [16, Lemma 3.2.3] or [15, Prop. 5(iii)]).

**Proposition 15.** *There exist $\delta_{\varepsilon_0,\varepsilon} > 0$ with $\delta_{\varepsilon_0,\varepsilon} \to 0$ $(\varepsilon \to 0)$ such that*

$$\|\rho A \rho^{-1} - A\|_{s \to t} \le \delta_{\varepsilon_0, \varepsilon} \|A\|_{p \to q, \varepsilon_0}$$

*for all $p \le s \le t \le q$, $0 \le \varepsilon < \varepsilon_0$ and $\rho \in P(\varepsilon, 1)$.*

*Proof.* First note that $\|A\|_{s \to t, \varepsilon_1} \le C_{\varepsilon_0, \varepsilon_1} \|A\|_{p \to q, \varepsilon_0}$ for all $0 < \varepsilon_1 < \varepsilon_0$ by Proposition 13(ii). Hence it suffices to treat the case $s = p$, $t = q$. Using Lemma 12, for $f \in L_{\infty,c}(\Omega)$ we have

$$\|(\rho A \rho^{-1} - A) f\|_q \le c_{q, \varepsilon_0 + \varepsilon} \big\| u \mapsto \|\rho_{\varepsilon_0 + \varepsilon, u} (\rho A \rho^{-1} - A) f\|_q \big\|_q. \tag{17}$$

We now write

$$\rho A \rho^{-1} - A = \rho \rho(u)^{-1} A (\rho(u) \rho^{-1} - 1) + (\rho \rho(u)^{-1} - 1) A,$$

insert this into (17), use the triangle inequality and estimate the two resulting terms separately. For the second we have, using (16) and (15),

$$\big\|u \mapsto \|\rho_{\varepsilon_0+\varepsilon,u}(\rho\rho(u)^{-1}-1)Af\|_q\big\|_q \le n_q(\rho_{\varepsilon_0+\varepsilon}(\rho_{-\varepsilon}-1))\|A\|_{p\to q}\|f\|_p.$$

Using (16) and (15) again, the first term can be estimated by

$$\begin{aligned}\big\|u \mapsto \|\rho_{\varepsilon_0+\varepsilon,u}\rho\rho(u)^{-1}A(\rho(u)\rho^{-1}-1)f\|_q\big\|_q &\le \big\|u \mapsto \|\rho_{\varepsilon_0,u}A(\rho(u)\rho^{-1}-1)f\|_q\big\|_q \\ &\le \|A\|_{p\to q,\varepsilon_0}\big\|u \mapsto \|\rho_{\varepsilon_0,u}(\rho_{-\varepsilon,u}-1)f\|_p\big\|_q \\ &\le \|A\|_{p\to q,\varepsilon_0} n_q(\rho_{\varepsilon_0}(\rho_{-\varepsilon}-1))\|f\|_p.\end{aligned}$$

The claim follows after noting that $\|A\|_{p\to q} \le C_{\varepsilon_0,0}\|A\|_{p\to q,\varepsilon_0}$ and

$$n_q(\rho_{\varepsilon_0}(\rho_{-\varepsilon}-1)) \le n_q(\rho_{\varepsilon_0/2}) \sup_{r\ge 0} e^{-\varepsilon_0 r/2}(e^{\varepsilon r}-1) \to 0 \quad (\varepsilon \to 0),$$

where we used $\rho_{\varepsilon_0} = {\rho_{\varepsilon_0/2}}^2$. □

The following consequence of Proposition 15 will be used in the proof of Theorem 1.

**Corollary 16.** *Let $A$ be an operator on $L_q(\Omega)$ satisfying $\|A\|_{q\to q,\varepsilon_0} < \infty$ for some $\varepsilon_0 > 0$. In case $q = \infty$ assume that $A$ is weak*-continuous. Then for all $\lambda \in \rho(A)$ there exists $\varepsilon > 0$ such that $\lambda \in \rho(A_{\varepsilon,z})$ for all $z \in \Omega'$, and $\|R(\lambda, A)\|_{q\to q,\varepsilon} < \infty$.*

*Proof.* Let $\lambda \in \rho(A)$. By assumption we have $\|\lambda - A\|_{q\to q,\varepsilon_0} < \infty$. Applying Proposition 15 and recalling that inversion is continuous in the open set of invertible elements in $\mathcal{L}(L_q(\Omega))$, we have for some $\varepsilon > 0$ that $\lambda - A_{\varepsilon,z}$ is invertible for all $z \in \Omega'$ (that is, the first claim), and that

$$\sup_{z\in\Omega'} \|R(\lambda, A_{\varepsilon,z})\|_{q\to q} < \infty. \tag{18}$$

To prove $\|R(\lambda, A)\|_{q\to q,\varepsilon} < \infty$ observe that for all $z \in \Omega'$ the function $\rho_{\varepsilon,z}^{-1}: D(\rho_{\varepsilon,z}^{-1}, L_q) \to L_q$ is surjective since $\rho_{\varepsilon,z}$ is bounded. Hence $R(\lambda, A_{\varepsilon,z})\restriction_{D(\rho_{\varepsilon,z}^{-1},L_q)} = \rho_{\varepsilon,z}R(\lambda, A)\rho_{\varepsilon,z}^{-1}$ and we are done by (18). □

**Proof of Theorem 1.** In Remark 14(i) it is shown that $A$ extends to consistent bounded operators $A_r$ on $L_r(\Omega)$ ($p \le r \le q$, $A_\infty$ weak*-continuous if $q = \infty$). So we need to prove the inclusion $\rho(A_r) \subset \rho(A_s)$ for all $r, s \in [p, q]$. Let $\lambda \in \rho(A_r)$.

First we treat the case $\lambda \neq 0$. Then we can rewrite the resolvent $R(\lambda)$ of $A_r$ as follows

$$R(\lambda) = \lambda^{-1}I + \lambda^{-2}A_r + \lambda^{-2}A_r R(\lambda)A_r.$$

We have to show that $\lambda \in \rho(A_s)$ which by Corollary 11 amounts to showing $L_s$-boundedness of $R(\lambda)\restriction_{L_{\infty,c}(\Omega)}$.

It is clear that $\lambda^{-1}I + \lambda^{-2}A$ is $L_s$-bounded; we will show that $A_r R(\lambda)A$ is $L_s$-bounded. According to Corollary 16 we have $\|R(\lambda)\|_{r\to r,\varepsilon} < \infty$ for some $0 < \varepsilon < \varepsilon_0$. By Proposition 13(ii) $\|A\|_{p\to r,\varepsilon} + \|A\|_{r\to q,\varepsilon} < \infty$. Remark 14(iii) implies $\|A_r R(\lambda)A\|_{p\to q,\varepsilon} < \infty$. Another application of Proposition 13(ii) gives $L_s$-boundedness of $A_r R(\lambda)A$.

In the case $\lambda = 0$ we simply write $R(\lambda) = A_r R(\lambda)^3 A_r$, hence by the above we have $\|R(\lambda)\|_{p\to q,\varepsilon} < \infty$. Again $\|R(\lambda)\|_{s\to s} < \infty$ by Proposition 13(ii). □

**Remark 17.** Let $A_p$ be a bounded operator on $L_p(\Omega)$ with $0 \in \rho(A_p)$ and $\|A_p\|_{p\to q} < \infty$. Then $\|I\colon L_p(\Omega) \to L_q(\Omega)\| \le \|A_p^{-1}\|_{p\to p}\|A_p\|_{p\to q} < \infty$. Therefore $(\Omega,\mu)$ cannot contain a sequence $(M_n)$ of subsets satisfying $M_n \supset M_{n+1}$ $(n \in \mathbb{N})$ and $0 < \mu(M_n) \to 0$ $(n \to \infty)$.

Then $\Omega = \bigcup_{n=0}^{\infty} M_n$ where $\mu(M_0) = 0$, $M_n$ are pairwise disjoint atoms of $(\Omega,\mu)$ $(n \ge 1)$ and $\inf_{n\ge 1}\mu(M_n) > 0$. Therefore, for all $s < \infty$, the space $L_s(\Omega)$ is isometrically isomorphic to the weighted space of sequences $\{(x_n); \sum_n |x_n|^s \mu(M_n) < \infty\}$, and $L_\infty(\Omega)$ is isometrically isomorphic to $l_\infty$. In this case we have $\|A_p{\restriction}_{L_1(\Omega)\cap L_p(\Omega)}\|_{1\to\infty} < \infty$.

# 4 Proof of Theorem 2

For the proof of Theorem 2 we use a different type of weight functions $\rho_{\varepsilon,z}$ which we define for $\varepsilon \in [0,1]$ and $z \in \Omega$ by $\rho_{\varepsilon,z}(x) := \exp(-d_\varepsilon(x,z))$ where $d_\varepsilon(x,z) := \psi(\varepsilon d(x,z))$. Note that this is consistent with the definition of $d_1$ in the introduction and that each $d_\varepsilon$ is a semi-metric (and $d_0 \equiv 0$), since $\psi$ is increasing and subadditive, $\psi \ge 0$ and $\psi(0) = 0$.

Let $A\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ be a linear operator given by a measurable kernel $k\colon\Omega\times\Omega \to \mathbb{C}$. Let $g(x,y) := |k(x,y)|\exp(d_1(x,y))$ and assume that $n_1(g) < \infty$ and $n_{p_0}(g) < \infty$ for some $p_0 > 1$. We begin with some preparatory results.

**Proposition 18.** *If $1 \le p \le q \le \infty$ and $1 + q^{-1} - p^{-1} \ge p_0^{-1}$, we have for all $\varepsilon \in [0,1]$*

$$\|A\|_{p\to q,\varepsilon} := \sup_{z\in\Omega} \|\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1}\|_{p\to q} < \infty.$$

*Proof.* The operator $\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1}$ is given by the kernel

$$k_{\varepsilon,z}(x,y) := \exp(-d_\varepsilon(x,z))k(x,y)\exp(d_\varepsilon(y,z)), \tag{19}$$

so by (4) we only have to show $n_r(k_{\varepsilon,z}) < \infty$ for $r^{-1} := 1 + q^{-1} - p^{-1}$. For $\varepsilon \in [0,1]$ we have the estimate

$$|k_{\varepsilon,z}(x,y)| \le \exp(d_\varepsilon(x,y))|k(x,y)| \le g(x,y)$$

since $\psi$ is increasing. Taking into account that $1 \le r \le p_0$, the assumption $n_1(g) + n_{p_0}(g) < \infty$ implies the assertion. ☐

**Proposition 19.** *For all $p \in [1,\infty]$ we have*

$$\sup_{z\in\Omega} \|\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1} - A\|_{p\to p} \to 0 \quad (\varepsilon \to 0).$$

*Proof.* The operators $\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1} - A$ are given by kernels $m_{\varepsilon,z} = k_{\varepsilon,z} - k$ with $k_{\varepsilon,z}$ from (19). For $x,y,z \in \Omega$ and $\varepsilon \in (0,1)$ we have

$$|m_{\varepsilon,z}(x,y)| \le (e^{-d_\varepsilon(x,z)+d_\varepsilon(y,z)} - 1)|k(x,y)| \le e^{-d_1(x,y)}(e^{d_\varepsilon(x,y)} - 1)g(x,y). \tag{20}$$

Hence $|m_{\varepsilon,z}| \le \sup_r e^{-\psi(r)}(e^{\psi(\varepsilon r)} - 1)\cdot g$ which implies $n_1(m_{\varepsilon,z}) \le \sup_r e^{-\psi(r)}(e^{\psi(\varepsilon r)} - 1)\cdot n_1(g)$. The proof is complete after we prove the following lemma. ☐

**Lemma 20.** *Let* $\psi\colon [0,\infty) \to [0,\infty)$ *be increasing with* $\psi(r) \to 0$ *as* $r \to 0$. *Then* (8) *is equivalent to*

$$\sup_{r\geq 0} e^{-\psi(r)}(e^{\psi(\varepsilon r)} - 1) \to 0 \quad (\varepsilon \to 0).$$

*Proof.* We have $\sup_{r\geq 0} e^{-\psi(r)}(e^{\psi(\varepsilon r)}-1) = \sup_{r\geq 0} e^{-\psi(r/\varepsilon)}(e^{\psi(r)}-1)$. First we show that (8) implies $\sup_{0\leq r\leq r_0} e^{-\psi(r/\varepsilon)}(e^{\psi(r)} - 1) \to 0$ as $\varepsilon \to 0$ for all $r_0 > 0$. This follows from the estimate

$$e^{-\psi(r/\varepsilon)}(e^{\psi(r)} - 1) \leq \begin{cases} 1\cdot(e^{\psi(\sqrt{\varepsilon})} - 1) & , 0 \leq r \leq \sqrt{\varepsilon} \\ e^{-\psi(1/\sqrt{\varepsilon})}(e^{\psi(r_0)} - 1) & , \sqrt{\varepsilon} \leq r \leq r_0 \end{cases}$$

since $\psi(r) \to 0$ as $r \to 0$ and $\psi(r) \to \infty$ as $r \to \infty$. Now $e^{\psi(r)} \geq e^{\psi(r_0)} > 1$ for all $r \geq r_0 > 0$, so there is a constant $c > 0$ such that $ce^{\psi(r)} \leq e^{\psi(r)} - 1 \leq e^{\psi(r)}$ for all $r \geq r_0$. Hence $\sup_{r\geq r_0} e^{-\psi(r/\varepsilon)}(e^{\psi(r)} - 1) \to 0$ as $\varepsilon \to 0$ is equivalent to

$$c(\varepsilon, r_0) := \inf_{r\geq r_0} (\psi(r/\varepsilon) - \psi(r)) \to \infty \quad (\varepsilon \to 0).$$

This in turn holds for some $r_0 > 0$ if and only if (8) holds. □

**Proof of Theorem 2.** The proof differs only slightly from the proof of Theorem 1. By Proposition 18 (with $\varepsilon = 0$) $A$ acts as a bounded operator $A_p$ on $L_p(\Omega)$ $(1 \leq p \leq \infty)$. The operators $A_p$ are consistent since they are all defined by the same integral kernel. Let $r \in [1,\infty]$ and $\lambda \in \rho(A_r)$. First we assume that $\lambda \neq 0$.

Fix $n \in \mathbb{N}$ such that $1-p_0^{-1} \geq n^{-1}$. Proposition 18 then gives $\|A_r^n\|_{1\to r,\varepsilon}+\|A_r^n\|_{r\to\infty,\varepsilon} < \infty$. For the resolvent $R(\lambda)$ of $A_r$ we have

$$R(\lambda) = \sum_{j=1}^{2n} \lambda^{-j} A_r^{j-1} + \lambda^{-2n} A_r^n R(\lambda) A_r^n. \tag{21}$$

As in Section 3 we have to show $L_s$-boundedness of $R(\lambda)\restriction_{L_{\infty,c}(\Omega)}$ for all $s \in [1,\infty]$.

This is clear for $\sum_{j=1}^{2n}$ in (21); we will show it for $A_r^n R(\lambda) A_r^n$. As in the proof of Theorem 1 let $A_{\varepsilon,z}$ denote the (weak*-)continuous extension of $\rho_{\varepsilon,z} A \rho_{\varepsilon,z}^{-1}$ to $L_r(\Omega)$. Applying Proposition 19 and recalling that inversion is continuous in the open set of invertible elements in $\mathcal{L}(L_r(\Omega))$ we have for some $\varepsilon > 0$ that $\lambda \in \rho(A_{\varepsilon,z}, L_r)$ for all $z \in \Omega$, and that

$$\sup_{z\in\Omega'} \|R(\lambda, A_{\varepsilon,z})\|_{r\to r} < \infty.$$

Since again $R(\lambda, A_{\varepsilon,z}) = \rho_{\varepsilon,z} R(\lambda) \rho_{\varepsilon,z}^{-1}$ this implies $c := \|A_r^n R(\lambda) A_r^n\|_{1\to\infty,\varepsilon} < \infty$. By the Dunford-Pettis theorem it follows that $A_r^n R(\lambda) A_r^n$ is given by a kernel $h$ and that, for all $z \in \Omega$,

$$|h(x,y)| \leq c\exp(d_\varepsilon(x,z) - d_\varepsilon(y,z)) \quad \text{(almost all } x,y \in \Omega).$$

Choose $R > 0$ and a sequence $(x_n)$ according to (10). Then, for almost all $(x,y) \in B(x_n,R)\times\Omega$, we have

$$|h(x,y)| \leq c\exp(d_\varepsilon(x,x_n) - d_\varepsilon(y,x_n)) \leq c\exp(2\psi(\varepsilon R) - d_\varepsilon(x,y)). \tag{22}$$

By (9) we get $n_1(h) \le c\exp(2\psi(\varepsilon R))n_1(e^{-d_\varepsilon}) < \infty$. Hence $A_r^n R(\lambda) A_r^n\restriction_{L_{\infty,c}(\Omega)}$ is bounded on $L_s(\Omega)$ for all $s \in [1,\infty]$.

In the case $\lambda = 0$ we write $R(0) = A_r^n R(0)^{2n+1} A_r^n$ and obtain as above that $R(\lambda)\restriction_{L_{\infty,c}(\Omega)}$ is bounded on $L_s(\Omega)$ for all $s \in [1,\infty]$. □

# 5 Variants of Theorem 2

In this section we discuss variants of the assumptions in Theorem 2.

**Remark 21.** Let $\psi\colon [0,\infty) \to [0,\infty)$ be increasing and continuous with $\psi(0) = 0$. If $\psi$ is subadditive and a.e. differentiable with $\operatorname{ess\,lim\,inf}_{r\to\infty} r\psi'(r) > 0$, then condition (8) holds. If $\psi$ is concave, then the converse is also true.

For the proof of the first claim take $\gamma, r_0 > 0$ such that $\psi'(r) \ge \gamma r^{-1}$ for almost all $r \ge r_0$. Then (8) holds for all $\alpha > 1$ by the estimate

$$\psi(\alpha r) - \psi(r) \ge \int_r^{\alpha r} \psi'(\eta)\,d\eta \ge \gamma \int_r^{\alpha r} \eta^{-1}\,d\eta = \gamma(\log(\alpha r) - \log r) = \gamma \log\alpha \quad (r \ge r_0).$$

If $\psi$ is concave then $\psi'(r) \ge \frac{\psi(\alpha r)-\psi(r)}{\alpha r - r}$ for all $\alpha > 1$ and almost all $r > 0$. If (8) holds we obtain that $\operatorname{ess\,lim\,inf}_{r\to\infty} r\psi'(r) \ge \frac{1}{\alpha-1}\liminf_{r\to\infty}(\psi(\alpha r) - \psi(r)) > 0$. Finally notice that if $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ is concave and continuous then it is subadditive and absolutely continuous (but the converse is not true).

**Remark 22.** If $\psi$ satisfies all the assumptions in Theorem 2 except (8), we still obtain $p$-independence of the spectrum if the function $g = e^{d_1}|k|$ in Theorem 2 satisfies as before $n_1(g) + n_{p_0}(g) < \infty$ for some $p_0 > 1$ and the additional assumption

$$\lim_{n\to\infty} n_1(g_n) = 0 \quad \text{where } g_n(x,y) := 1_{\{d(x,y)\ge n\}} g(x,y) \text{ for } n \in \mathbb{N}. \tag{23}$$

The only change in the proof is that in the proof of Proposition 19 the last expression in (20) should now be estimated as follows: Letting $h_\varepsilon(r) := \exp(-\psi(r))(\exp(\psi(\varepsilon r)) - 1)$ we have

$$|m_{\varepsilon,z}(x,y)| \le h_\varepsilon(d(x,y))g(x,y) \quad (x,y \in \Omega),$$

and $h_\varepsilon(r) \le 1$ gives the estimate

$$n_1(m_{\varepsilon,z}) \le n_1(g_n) + \sup_{r\in[0,n]} h_\varepsilon(r)\cdot n_1(g)$$

for all $n \in \mathbb{N}$. We have $\sup_{r\in[0,n]} h_\varepsilon(r) \to 0$ as $\varepsilon \to 0$ which implies $\limsup_{\varepsilon\to 0} n_1(m_{\varepsilon,z}) \le n_1(g_n)$ for all $n \in \mathbb{N}$. The assumption $n_1(g_n) \to 0$ now gives the assertion of Proposition 19.

Assumption (23) is in particular fulfilled if $g(x,y) \le h(d(x,y))$ for some non-increasing function $h\colon (0,\infty) \to [0,\infty)$ satisfying $\int_0^\infty h(r)\,dm(r) < \infty$, with $m$ from (2). To show this observe that $h(r) \to 0$ as $r \to \infty$ since $m(r) \to \infty$ as $r \to \infty$. As in (5) we obtain

$$n_1(g_n) \le \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega h(d(x,y)) \wedge h(n)\,d\mu(y) \le \int_0^\infty h(r) \wedge h(n)\,dm(r) \to 0 \quad (n \to \infty)$$

by dominated convergence. This proves Remark 3(iii).

Now we can reprove the main result of [14]: Let $\Omega \subset \mathbb{R}^N$ be open and let $\tilde{\Omega} := \Omega - \Omega$. Let $\psi\colon [0,\infty) \to [0,\infty)$ be increasing, continuous and subadditive such that $e^{-\psi(\varepsilon|\cdot|)} \in L_1(\tilde{\Omega})$ for all $\varepsilon \in (0,1]$. Let $k\colon \Omega \times \Omega \to \mathbb{C}$ be a measurable kernel satisfying

$$|k(x,y)| \le h(x-y) \quad (x,y \in \Omega) \tag{24}$$

for a function $h$ that satisfies $he^{\psi(|\cdot|)} \in L_1(\tilde{\Omega}) \cap L_{p_0}(\tilde{\Omega})$. Then the assertion of Theorem 2 holds.

Defining $g$ and $g_n$ as above we have for $n \in \mathbb{N}_0$ (note $g = g_0$!)

$$g_n(x,y) \le 1_{\{|x-y|\ge n\}} h(x-y) e^{\psi(|x-y|)} \quad (x,y \in \Omega).$$

For $n = 0$, this implies $n_1(g) + n_{p_0}(g) < \infty$ by the assumption on $h$. Moreover, for $n \in \mathbb{N}$,

$$n_1(g_n) \le \|1_{\{|\cdot|\ge n\}} h e^{\psi(|\cdot|)}\|_{L_1(\tilde{\Omega})}$$

and the right hand side clearly tends to 0 as $n \to \infty$ by dominated convergence. Hence the assumptions of the variant of Theorem 2 as stated in Remark 22 hold, and the proof is complete.

# 6 Weighted $L_p$-spaces

In this section we give sufficient conditions for operators $A$ to have the same spectrum in different weighted $L_p$-spaces. For previous results in this direction see [7] and [13].

A weight is a measurable function $\varphi\colon \Omega \to (0,\infty)$ such that $\varphi, \varphi^{-1} \in L_{\infty,loc}(\Omega)$, and we define $L_p(\varphi) := \{f; \varphi f \in L_p(\Omega)\}$ for all $1 \le p \le \infty$. If $p < \infty$ then $L_p(\varphi) = L_p(\Omega, \varphi^p \mu)$. In the sequel we shall denote $L_p(\Omega)$ simply by $L_p$. The multiplication operator $\varphi\colon L_p(\varphi) \to L_p$ is an isometry.

Our first result uses the weighted norm estimate (6) for $p = q$. Recall from Section 3 the definition of the classes $P(\varepsilon, M)$, which contain weight functions $\varphi$ defined on $\Omega'$. We call a weight function $\varphi\colon \Omega' \to (0,\infty)$ *subexponential* if, for all $\varepsilon > 0$, $\varphi \in P(\varepsilon, M_\varepsilon)$ for some $M_\varepsilon > 0$.

**Theorem 23.** *Assume that* (1), (2) *and* (3) *hold. Let* $1 \le p < \infty$ *and* $A$ *be a bounded operator on* $L_p(\Omega)$ *satisfying*

$$\|A\|_{p\to p,\varepsilon_0} < \infty \tag{25}$$

*for some* $\varepsilon_0 > 0$. *Let* $\varphi$ *be a subexponential weight function. Then* $A\restriction_{L_p \cap L_p(\varphi)}$ *extends to a bounded operator* $A_\varphi$ *on* $L_p(\varphi)$ *and* $\sigma(A_\varphi) = \sigma(A)$.

Here $\|\cdot\|_{p\to p,\varepsilon}$ is defined as in Section 3.

**Remark 24.** (i) The above theorem still holds if $p = \infty$ and $A$ is weak*-continuous. For the proof just consider the dual situation in $L_1$.

(ii) We point out that in the above theorem only the weighted $L_p$-estimate (25) but no $p-q$-smoothing is responsible for the equality of the spectra.

**Proof of Theorem 23.** By Proposition 13(i), $\varphi A\varphi^{-1}\restriction_{L_{\infty,c}}$ is $L_p$-bounded, hence $A\restriction_{L_{\infty,c}}$ is $L_p(\varphi)$-bounded. The first claim follows from the density of $L_{\infty,c}$ in $L_p \cap L_p(\varphi)$.

We now show the inclusion $\rho(A, L_p) \subset \rho(A_\varphi, L_p(\varphi))$. Let $\lambda \in \rho(A, L_p)$. By Corollary 16 we have $\|R(\lambda)\|_{p\to p,\varepsilon} < \infty$ for some $0 < \varepsilon < \varepsilon_0$. Proposition 13(i) yields

$$\|R(\lambda, A)\restriction_{L_{\infty,c}(\Omega)}\|_{L_p(\varphi)\to L_p(\varphi)} = \|\varphi R(\lambda, A)\varphi^{-1}\restriction_{L_{\infty,c}(\Omega)}\|_{L_p\to L_p} < \infty.$$

Lemma 9 implies $\lambda \in \rho(A_\varphi, L_p(\varphi))$.

For the other inclusion we observe that the operator $\varphi A_\varphi \varphi^{-1}$ on $L_p$ satisfies (25) for $\varepsilon_1 := \varepsilon_0/3$ in place of $\varepsilon_0$: Since $\varphi$ is subexponential, we have $\rho := \rho_{\varepsilon_1,z}\varphi \in P(\varepsilon_0/2, M)$ for all $z \in \Omega'$. Applying Proposition 13(i) we get $\|\varphi A\varphi^{-1}\|_{p\to p,\varepsilon_1} < \infty$. Since $\varphi^{-1}$ is subexponential and $\varphi A_\varphi\varphi^{-1}$ on $L_p \cap L_p(\varphi^{-1})$ extends to $\varphi A \varphi^{-1}$ on $L_p(\varphi^{-1})$, the first part of the proof shows

$$\rho(A_\varphi, L_p(\varphi)) = \rho(\varphi A_\varphi\varphi^{-1}, L_p) \subset \rho(\varphi A\varphi^{-1}, L_p(\varphi^{-1})) = \rho(A, L_p).$$

This completes the proof of the theorem. □

Our second result deals with the situation of Theorem 2. Since we now use the semi-metrics $d_\varepsilon = \psi \circ (\varepsilon d)$ from Section 4, we introduce modified classes of weight functions. Namely, for $\varepsilon \in (0, 1)$ and $M \geq 1$ let

$$P_\psi(\varepsilon, M) := \bigl\{\varphi\colon \Omega \to (0,\infty) \text{ measurable}; \forall x, y \in \Omega : \varphi(x)/\varphi(y) \leq M e^{d_1(x,y) - d_{1-\varepsilon}(x,y)}\bigr\}.$$

**Remark 25.** (i) In case $\psi(r) = r$ this definition coincides with the previous one, i.e. $P_\psi(\varepsilon, M) = P(\varepsilon, M)$.

(ii) In case $\psi(r) = \alpha\log(1 + r)$ for some $\alpha > 0$ it is easy to see that the class $P_\psi(\varepsilon, M)$ only contains bounded weight functions.

(iii) There exist examples of $(\Omega, \mu)$, $d$ and functions $\psi$ satisfying (8) and (9) such that $\psi(r)/\log r \to \infty$ $(r \to \infty)$, but the volume growth of $d_1$-balls is exponential ($\Omega = \mathbb{R}$, $\mu$ the Lebesgue measure, $\psi(r) := \log^2(1 + r)$ and $d(x, y) := m^{-1}(2|x - y|)$ where $m(r) := e^{\psi(r)/2} - 1$). This means that it is not always possible to reduce the situation to the case $\psi(r) = r$.

We say that $\varphi$ is *$\psi$-subordinated* if, for all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that $\varphi \in P_\psi(\varepsilon, M)$. In case $\psi(r) = r$ the two notions $\psi$-subordinated and subexponential coincide by Remark 25(i). In the following we shall assume that $\psi$ is concave, not only subadditive. Then for all $0 < \varepsilon' < \varepsilon < 1$ we have

$$d_1 - d_\varepsilon \leq d_{1-\varepsilon'} - d_{\varepsilon-\varepsilon'}. \tag{26}$$

This follows from

$$\psi(r) - \psi(\varepsilon r) = \int_{\varepsilon r}^{r} \psi'(s)\,ds \leq \int_{(\varepsilon-\varepsilon')r}^{(1-\varepsilon')r} \psi'(s)\,ds = \psi((1-\varepsilon')r) - \psi((\varepsilon-\varepsilon')r) \quad (r > 0),$$

where we have used that $\psi'$ is non-increasing. Conversely, $\psi(r) - \psi(\varepsilon r) \leq \psi((1 - \varepsilon')r) - \psi((\varepsilon - \varepsilon')r)$ for all $r > 0$, $0 < \varepsilon' < \varepsilon < 1$ implies that $\psi$ is concave.

**Theorem 26.** *Assume that $\psi$ is concave and that (7), (8) and (9) hold. Let $A$ be a bounded operator on $L_p$ given by a measurable kernel $k\colon \Omega\times\Omega\to\mathbb{C}$. Let $g := |k|\exp(d_1)$ and assume that $n_1(g)<\infty$ and $n_{p_0}(g)<\infty$ for some $p_0>1$. Let $\varphi$ be a $\psi$-subordinated weight function. Then $A$ acts as a bounded operator $A_\varphi$ on $L_p(\varphi)$ and $\sigma(A_\varphi)=\sigma(A)$.*

*Proof.* The assumptions on $\psi$ and on the kernel $k$ of $A$ imply that $\varphi A\varphi^{-1}$ is $L_p$-bounded; cf. Proposition 18. Hence $A\restriction_{L_{\infty,c}(\Omega)}$ is $L_p(\varphi)$-bounded and the first claim is proved.

We now show the inclusion $\rho(A,L_p)\subset\rho(A_\varphi,L_p(\varphi))$. Let $\lambda\in\rho(A,L_p)$. We use (21) for the resolvent $R(\lambda)$ and — as before — we show $L_p(\varphi)$-boundedness of $R(\lambda)\restriction_{L_{\infty,c}(\Omega)}$.

It is clear that $\sum_{j=1}^{2n}$ in (21) is $L_p(\varphi)$-bounded. The operator $\varphi A^n R(\lambda)A^n\varphi^{-1}$ has the kernel $h_1(x,y):=\varphi(x)h(x,y)\varphi(y)^{-1}$ with $h$ as in the proof of Theorem 2. By (22), the assumption on $\varphi$ and (26) we have the estimate

$$|h_1|\le C_{\varepsilon/2}e^{d_1-d_{1-\varepsilon/2}}c_1e^{-d_\varepsilon}\le c_1C_{\varepsilon/2}e^{-d_{\varepsilon/2}}.$$

By assumption (9) this implies $L_p$-boundedness of $\varphi A^n R(\lambda)A^n\varphi^{-1}$. Hence $A^nR(\lambda)A^n$ is $L_p(\varphi)$-bounded, and we have shown $L_p(\varphi)$-boundedness of $R(\lambda)$. By Lemma 9 we obtain $\lambda\in\rho(A_\varphi,L_p(\varphi))$.

For the proof of the other inclusion we first show that $\varphi A_\varphi\varphi^{-1}$ satisfies the assumptions of Theorem 26 for $\tilde\psi(r):=\psi(r/2)$ in place of $\psi$. Let $\tilde d_\varepsilon:=\tilde\psi\circ(\varepsilon d)$. The kernel of $\varphi A_\varphi\varphi^{-1}$ is $\tilde k(x,y):=\varphi(x)k(x,y)\varphi^{-1}(y)$, and we have the estimate

$$|\tilde k|e^{\tilde d_1}\le Ce^{d_1-d_{1/2}}|k|e^{d_{1/2}}=C|k|e^{d_1}.$$

The weight function $\varphi^{-1}$ is $\tilde\psi$-subordinated since, for all $\varepsilon\in(0,1)$, we can estimate by (26):

$$\varphi^{-1}(x)/\varphi^{-1}(y)\le M_{\varepsilon/2}e^{d_1(x,y)-d_{1-\varepsilon/2}(x,y)}\le M_{\varepsilon/2}e^{d_{1/2}(x,y)-d_{1/2-\varepsilon/2}(x,y)}=M_{\varepsilon/2}e^{\tilde d_1(x,y)-\tilde d_{1-\varepsilon}(x,y)}.$$

Since $\tilde\psi$ is concave and satisfies (8) and (9), the first part of the proof shows

$$\rho(A_\varphi,L_p(\varphi))=\rho(\varphi A_\varphi\varphi^{-1},L_p)\subset\rho(\varphi A\varphi^{-1},L_p(\varphi^{-1}))=\rho(A,L_p),$$

as in the proof of Theorem 23. ☐

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