On L^p-spectra and essential spectra of second order elliptic operators

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1 Introduction and Main Results

The aim of this paper is to investigate spectral properties of second order elliptic operators with measurable coefficients. Namely, we study the problems of L^p-independence of the spectrum and stability of the essential spectrum.

The problem of L^p-independence of the spectrum for elliptic operators has a long history going back to B. Simon [30] where the question was posed for Schrödinger operators. The main breakthrough was made by R. Hempel and J. Voigt [14] who answered the question in the affirmative for the case that the negative part of the potential is from the Kato class. This result was a starting point for many extensions in different directions [2, 9, 10, 15, 17, 25, 26, 27] (the list is by no means complete).

Most of the results deal with cases when the operators are selfadjoint in L^2 and can be defined in all L^p, $1 \leqslant p < \infty$. Under these conditions an abstract approach based on a functional calculus was developed by E. B. Davies [9]. In [26] L^p-independence was established for the Schrödinger operator with form bounded negative part of the potential. In this case the operator exists only in L^p for p from a certain interval around $p = 2$. The ideas from [26] were put in a more general context in [25]. Further progress was made by Yu. Semenov [27] who treated selfadjoint elliptic operators with unbounded coefficients, adapting the method from [26]. In the non-symmetric case the best result known so far is proved in [17] under assumption of a Gaussian estimate using ideas from [2].

In the present paper we treat second order non-symmetric divergence form elliptic operators with unbounded coefficients in the main part as well as unbounded lower order terms. Such operators generate C_0-semigroups on L^p for p from an interval. The semigroup does not map L^1 into L^∞ and, as a result, does not enjoy any pointwise Gaussian estimate. All known approaches break down for this case. In order to treat it we develop a new approach (as discussed below and in Section 5) which inherits ideas from [26], [25], [27].

The paper is organized as follows. In this section we describe the framework and state the main results. Their proofs are delegated to the three subsequent sections. In Section 5 we discuss further applications and extensions of the results. Some auxiliary propositions and technicalities are collected in Appendices A, B and C.

In the course of the paper we use the following notation. Ω denotes an open subset of $\mathbb{R}^d$ $(d \geqslant 1)$, $\|\cdot\|_p$ the norm in $L^p := L^p(\Omega)$, L^∞_c the set of bounded measurable functions with compact support. $\langle\cdot,\cdot\rangle$ is the inner product in L^2, $\langle f\rangle := \int_\Omega f(x)dx$, $\|\cdot\|_{p\to q}$ is the norm of a linear operator acting from L^p to L^q. $D(A)$ denotes the domain of an operator A, $Q(A)$ the domain of the associated quadratic form (in L^2), $\restriction$ is the sign of restriction.

Our main abstract result on L^p-independence of the spectrum is contained in Theorem 1. To formulate it we need the following notion (due to Yu. Semenov [27]). We say that a function $\psi : \mathbb{Z}^d \longrightarrow \mathbb{R}^d$ is *L^1-regular*, if

$$\sup_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} e^{-|\psi(k)-\psi(j)|} =: M_\psi < \infty.$$

A function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is called L^1-regular, if it is Lipschitz continuous (i.e. uniformly Lipschitz continuous) and $\psi\restriction_{\mathbb{Z}^d}$ is L^1-regular. Equivalent descriptions are given in Appendix A.

Theorem 1. *Given $1 \leqslant p < q < \infty$ let T_p and T_q be closed operators in $L^p(\Omega)$ and $L^q(\Omega)$, respectively. If there exist $\varepsilon > 0$, $C < \infty$, $\lambda_0 \in \rho(T_p)\cap\rho(T_q)$ and a L^1-regular function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that*

(i) $(\lambda_0 - T_p)^{-1}$, $(\lambda_0 - T_q)^{-1}$ are consistent, i.e. $(\lambda_0 - T_p)^{-1}\restriction_{L^p\cap L^q} = (\lambda_0 - T_q)^{-1}\restriction_{L^p\cap L^q}$,

(ii) $\|e^{\xi\psi}(\lambda_0 - T_p)^{-1}e^{-\xi\psi}\restriction_{L^\infty_c}\|_{p\to q} \leqslant C$ for all $\xi \in \mathbb{R}^d$, $|\xi| \leqslant \varepsilon$,

then $\sigma(T_p) = \sigma(T_q)$.

Remarks. *1. Although the theorem is stated for arbitrary closed operators T_p and T_q, in the following we only consider the case when they are generators of consistent C_0-semigroups. Note that condition (i) is trivially fulfilled in this case for large λ_0.*

2. We call condition (ii) the "weighted" resolvent estimate. It is fulfilled, for instance, in case of the validity of a Gaussian estimate for the corresponding semigroup, with $\psi(x) = x$ (see details in Section 5).

3. It is an obvious but important observation that condition (ii) carries over to generators of semigroups which are dominated by e^{-tT_p}.

Now we pass to our main application of Theorem 1, namely to second order elliptic operators. First we introduce the framework.

Let $a : \Omega \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ be a matrix-valued measurable function, $b : \Omega \longrightarrow \mathbb{R}^d$ be a vector-valued measurable function, $V : \Omega \longrightarrow \mathbb{R}$ be a measurable function.

Consider the second order formal differential expression

$$\hat{T} = -\nabla \cdot a \cdot \nabla + b \cdot \nabla + V := -\sum_{j,k=1}^{d} \partial_j a_{jk} \partial_k + \sum_{j=1}^{d} b_j \partial_j + V.$$

We assume that the following conditions on the coefficients are satisfied:

Assumptions on a.

A1. $a_{jk} \in L^1_{loc}(\Omega)$ and $a_{jk} = a_{kj}$ for $1 \leqslant j,k \leqslant d$.

A2. a is strictly elliptic, i.e. $a(x) \geqslant \sigma I$ in matrix sense, for some $\sigma > 0$ and almost all $x \in \Omega$.

Let $\mathbf{a}$ be the following sesquilinear form in L^2:

$$\mathbf{a}[u,v] := \langle \nabla u, a \cdot \overline{\nabla v} \rangle := \int_\Omega \sum_{j,k=1}^{d} a_{jk}(x) \frac{\partial u(x)}{\partial x_k} \frac{\overline{\partial v}(x)}{\partial x_j} dx, \;\; D(\mathbf{a}) := C_c^\infty(\Omega).$$

It is well-known that the form $\mathbf{a}$ is closable in L^2 and, without additional assumptions on Ω and a_{jk}, may have infinitely many closed extensions. We confine ourselves to the ones which are *Dirichlet forms.* In particular, we define the following extensions of the form $\mathbf{a}$:

$$\tau_D := \bar{\mathbf{a}}, \quad \text{the closure of } \; \mathbf{a},$$

$$\tau_N[u,v] := \langle \nabla u, a \cdot \nabla v \rangle, \;\; D(\tau_N) := \{u \in W^{1,2}(\Omega); \langle \nabla u \cdot a \cdot \overline{\nabla u} \rangle < \infty\},$$

$$\tau_i := \tau_N \restriction_{D(\tau_i)}, \ D(\tau_i) := D(\tau_N) \cap \overset{\circ}{W}{}^{1,2}(\Omega),$$

which represent Dirichlet, Neumann and intermediate boundary conditions.

These forms are Dirichlet forms (cf. [12]), so the associated selfadjoint operators A_D, A_N and A_i, respectively, generate Markov semigroups. Below we denote by τ one of the above forms and by A the associated operator.

Assumptions on b and V.

In order to formulate our assumptions on the first order term we need to introduce the function $b \cdot a^{-1} \cdot b := \sum_{j,k=1}^{d} b_j b_k a^{jk}$ where a^{jk} are the entries of the inverse matrix a^{-1}.

B. $b \cdot a^{-1} \cdot b \in L^1_{loc}(\Omega)$ and there exist $\beta, c(\beta) \geqslant 0$ such that $b \cdot a^{-1} \cdot b \leqslant \beta A + c(\beta)$ in the form sense.

V. $V = V^+ - V^-$, $V^\pm \geqslant 0$, V^+ *admissible* with respect to e^{-tA} (i.e. $D(\tau) \cap Q(V^+)$ is dense in L^2; cf. [31]), $V^- \leqslant \gamma A + c(\gamma)$ in the form sense for some $\gamma, c(\gamma) \geqslant 0$.

FS. $\sqrt{\beta} + \gamma < 1$.

The last condition ensures form smallness of the lower order terms with respect to the main part of the operator. This in turn guarantees that the form

$$\mathbf{t}[u,v] := \tau[u,v] + \langle b \cdot \nabla u, v \rangle + \langle Vu, v \rangle, \quad D(\mathbf{t}) = D(\tau) \cap Q(V^+)$$

is a densely defined closed sectorial form in L^2. By the representation theorem (cf. [16], Ch. VI, Thm. 2.1) $\mathbf{t}$ is associated with an m-sectorial operator T which generates a holomorphic semigroup e^{-tT} on L^2. For the moment let $\eta := 2 - \sqrt{\beta}$. It was shown in [18] (see also [28], Appendix) that

$$e^{-tT} \restriction_{L^\infty_c} \text{ extends to a } C_0\text{-semigroup } e^{-tT_p} \text{ on } L^p$$

for $p \in [p_-, p_+] := [\frac{4}{\eta + \sqrt{\eta^2 - 4\gamma}}, \frac{4}{\eta - \sqrt{\eta^2 - 4\gamma}}]$ (and $p \in [\frac{2}{2-\sqrt{\beta}}, \infty)$ in the case $\gamma = 0$, i.e. V^- bounded). It is positivity preserving, and also L^∞-contractive if $V^- = 0$.

Theorem 2. *Let A be one of the operators A_D, A_i, A_N defined above. In the case $A = A_N$ assume that Ω satisfies the cone property. Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be L^1-regular. Assume that there exist constants c_0, c_1 such that the inequality*

$$\nabla(\xi\psi) \cdot a \cdot \nabla(\xi\psi) \leqslant c_0(A \dot{+} V^+) + c_1 \tag{1}$$

is fulfilled in the form sense for all $\xi \in \mathbb{R}^d$, $|\xi| \leqslant 1$, where the lefthand side of (1) is to be read as a multiplication operator. Then $\sigma(T_p) = \sigma(T)$ for all $p \in [p_-, p_+]$ (and for $p \in [\frac{2}{2-\sqrt{\beta}}, \infty)$ in the case $\gamma = 0$).

$A\dot{+}V^+$ appearing in condition (1) is the operator associated with the form $\tau_V[u,v] := \tau[u,v] + \langle V^+u, v\rangle$, $D(\tau_V) := D(\tau) \cap Q(V^+)$.

Remarks. *1. For the uniformly elliptic case (a_{jk} bounded), condition (1) is obviously satisfied with $\psi(x) = x$. The a_{jk} are also allowed to grow to infinity as V^+ does. Since ψ can be constant on compact sets, the a_{jk} may have local L^1-singularities on these sets.*

2. The result of Theorem 2 carries over to the case of divergence form elliptic operators containing singular magnetic vector potentials and an additional imaginary scalar potential. This follows from Remark 3 after Theorem 1 (see e.g. [19] for the corresponding constructions).

3. The cone property for Ω in the formulation of the theorem is needed only to ensure the validity of Sobolev's embedding theorem [1].

The next problem we are going to discuss is the stability of the essential spectrum, namely we will show that under quite general conditions on the coefficients of an elliptic operator, its essential spectrum is $[0,\infty]$. This problem was addressed by M. S. Birman [7], M. Reed and B. Simon [23], M. Schechter [24], D. E. Edmunds and W. D. Evans [11], R. Hempel [13], E. M. Ouhabaz [21], and in [22] in a more general setting.

By the essential spectrum we mean here

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C}\,;\, \lambda - A \text{ is not a Fredholm operator}\}.$$

With this notation, if $(\lambda - A)^{-1} - (\lambda - B)^{-1}$ is a compact operator then $\sigma_{ess}(A) = \sigma_{ess}(B)$ (see [16], Ch. IV, §5.6).

In order to formulate our result we need to introduce the spaces $l^p(L^q)$ and $c_0(L^q)$ [8, 29]. We subdivide $\mathbb{R}^d$ into the cubes

$$Q_j := \{x \in \mathbb{R}^d\,;\, |x-j|_\infty \leqslant \frac{1}{2}\},\ j \in \mathbb{Z}^d,$$

so that $\mathbb{R}^d = \bigcup_j Q_j$, $|Q_j| = 1$. Let χ_k denote the characteristic function of Q_k. Define a mapping $J_r : L^r_{loc,unif} \to l^\infty(\mathbb{Z}^d)$ by $J_r f := ((J_r f)(j))_{j\in\mathbb{Z}^d}$, where

$$(J_r f)(j) := \|\chi_j f\|_r.$$

Then we say that $f \in l^p(L^q)$ if $J_q f \in l^p(\mathbb{Z}^d)$, with $\|f\|_{l^p(L^q)} := \|J_q f\|_{l^p}$. Similarly, $c_0(L^q)$ is defined as a subspace of $l^\infty(L^q)$ by: $f \in c_0(L^q)$ if $J_q f \in c_0(\mathbb{Z}^d)$.

Note the following simple inclusions:

$$L^2 = l^2(L^2) \subset c_0(L^2) \subset c_0(L^1),$$
$$c_0(L^1) \cap L^\infty \subset c_0(L^2).$$

Now we consider a more general operator, however acting on the whole space $\mathbb{R}^d$. Namely, let $a_{jk}, b_j, c_j, V \in l^\infty(L^1)$, $1 \leqslant j, k \leqslant d$.

Let $\mathbf{t}_0$ be the form

$$\langle \nabla u, a\nabla v\rangle + \langle \nabla u, bv\rangle - \langle cu, \nabla v\rangle + \langle Vu, v\rangle \tag{2}$$

with $D(\mathbf{t}_0) := C_c^\infty$. We assume that $\mathbf{t}_0$ is sectorial and closable.

Let $\mathbf{t}$ be a closed extension of $\mathbf{t}_0$ such that on $D(\mathbf{t}) \cap W^{1,2}$ it is represented by (2).

Theorem 3. *Let T be the m-sectorial operator associated with $\mathbf{t}$. Assume that the following conditions are satisfied:*

(i) $b, c, V \in c_0(L^2)$,

(ii) *(a)* $D(T), D(T^*) \subset W^{1,2}$,

(b) there exists a positive definite matrix $a^0 \in \mathbb{C}^d \otimes \mathbb{C}^d$ *such that* $a - a^0 \in c_0(L^2)$,

or

(ii′) *(a) a is symmetric and strictly elliptic,*

(b) the form corresponding to the lower order terms b, c, V is relatively bounded with respect to the main part of $\mathbf{t}_0$, with bound less than one,

(c) there exists a positive definite matrix $a^0 \in \mathbb{C}^d \otimes \mathbb{C}^d$ *such that* $a - a^0 \in c_0(L^1)$.

Then $\sigma_{ess}(T) = [0, \infty)$.

Remark. *If a is strictly elliptic (in the sense that $Re \sum_{jk} a_{jk}\xi_j\bar{\xi}_k \geqslant \varepsilon|\xi|^2$ for some $\varepsilon > 0$) and the form corresponding to the lower order perturbations is relatively bounded with respect to the main part of $\mathbf{t}_0$, with bound less than one, then $\mathbf{t}_0$ is closable and sectorial, and condition (ii(a)) of Theorem 3 is fulfilled.*

Theorem 3 is a direct generalization of the main result in [21] and of Example 1 from [13]. In the latter paper only selfadjoint operators are studied while in the former one there is a requirement of ultracontractivity and the conditions on the coefficients of the operator are much more restrictive. In [22] E. M. Ouhabaz and P. Stollmann consider more general operators (with σ_{ess} not necessarily $[0, \infty]$). Applying their results to our operator leads to additional restrictions. In both [21] and [22] unbounded coefficients are treated by an approximation method which requires strict ellipticity of the matrix while the approach described here avoides this. In the case of Schrödinger operators our result is not optimal, but we did not pursue generalizations in order to preserve simplicity.

2 A general criterion of L^p-independence of the spectrum

In this section we prove our abstract criterion of L^p-independence of the spectrum, Theorem 1. In order to prove the inclusion $\rho(T_p) \subset \rho(T_q)$ one has to show that for $\lambda \in \rho(T_p)$ the operator $(\lambda - T_p)^{-1}$ can be extended to a bounded operator on L^q. This is expressed in the following proposition which is stated in a more general context.

Let E, F be Banach spaces. We assume that there exists a vector space G, such that $E \subset G$, $F \subset G$ and $E \cap F$ is dense in both E and F. Let T_E and T_F be closed operators in E and F, respectively.

Proposition 4. *Assume that there exists a $\lambda_0 \in \rho(T_E) \cap \rho(T_F)$ such that $(\lambda_0 - T_E)^{-1}$ and $(\lambda_0 - T_F)^{-1}$ are consistent, i.e. $(\lambda_0 - T_E)^{-1} \restriction_{E\cap F} = (\lambda_0 - T_F)^{-1} \restriction_{E\cap F}$. Let $\lambda \in \rho(T_E)$.*

If $(\lambda - T_E)^{-1} \restriction_{E\cap F}$ extends to a bounded linear operator $R \in \mathcal{L}(F)$, then $\lambda \in \rho(T_F)$ and $(\lambda - T_F)^{-1} = R$.

Proof. For simplicity let $\lambda_0 = 0$. By the assumption $B_E := (\lambda - T_E)T_E^{-1} = \lambda T_E^{-1} - 1$ and $B_F := (\lambda - T_F)T_F^{-1} = \lambda T_F^{-1} - 1$ are consistent bounded operators and B_E is an isomorphism on E, $B_E^{-1} = \lambda(\lambda - T_E)^{-1} - 1$ extends from $E \cap F$ to a bounded operator on F. From this it follows that $B_F := (\lambda - T_F)T_F^{-1} = \lambda T_F^{-1} - 1$ is an isomorphism on F and that B_E^{-1} and B_F^{-1} are consistent, which implies the desired conclusion.

Remark. *In [2] Proposition 4 is proved for generators of C_0-semigroups.*

The main tool needed for the proof of Theorem 1 is the following result

Proposition 5. *Let $1 \leqslant p \leqslant r \leqslant s \leqslant q \leqslant \infty$, $\varepsilon > 0$ and $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be L^1-regular. For a linear operator $A : L_c^\infty(\Omega) \longrightarrow L_{loc}^1(\Omega)$ define*

$$\|A\|_{p\to q,\varepsilon} := \sup\{\|e^{\xi\psi} A e^{-\xi\psi}\|_{p\to q}; \xi \in \mathbb{R}^d, |\xi| \leqslant \varepsilon\} \quad (\in [0,\infty]).$$

(i) There exists $C = C(\varepsilon, \psi) < \infty$ such that

$$\|A\|_{r\to s} \leqslant C \cdot \|A\|_{p\to q,\varepsilon}.$$

(ii) *For* $\varepsilon_0 < \varepsilon$ *we have* $\|A\|_{r\to s,\varepsilon_0} \leqslant C(\varepsilon-\varepsilon_0,\psi)\cdot\|A\|_{p\to q,\varepsilon}$.

(iii) *There exists* $c(\xi) = c(\xi;\varepsilon,\psi)$ *with* $c(\xi)\to 0$ $(\xi\to 0)$ *such that*

$$\|e^{\xi\psi}Ae^{-\xi\psi} - A\|_{r\to s} \leqslant c(\xi)\cdot\|A\|_{p\to q,\varepsilon} \quad \text{for all } |\xi| < \varepsilon.$$

Remark. *For* $\psi(x) = x$ *assertions (i) and (ii) are due to G. Schreieck and J. Voigt [26], (iii) is proved in [25]. For the case of* L^1*-regular* ψ *assertion (i) can be found in [27]. Although the proof of the proposition for general* ψ *does not differ substantially from the case* $\psi(x) = x$*, for the reader's convenience we give the detailed proof below.*

Note that for bounded Ω the assertion of Proposition 5 is trivial since then L^r is continuously embedded in L^p, as well as L^q in L^s. In order to prove the proposition for general Ω we subdivide Ω into the bounded subsets

$$Q_j := \{x\in\Omega; |x-j|_\infty \leqslant \frac{1}{2}\},\ j\in\mathbb{Z}^d,$$

so that $\Omega = \bigcup_j Q_j$, $|Q_j|\leqslant 1$. Let χ_k denote the characteristic function of Q_k. For the proof of Proposition 5 we need the following simple lemma which is a modification of Schur's Lemma (cf. [34], Thm. 6.23).

Lemma 6. *Let* $1\leqslant p\leqslant q\leqslant\infty$, $A : L_c^\infty(\Omega)\longrightarrow L_{loc}^1(\Omega)$ *be a linear operator. Let* $K : \mathbb{Z}^d\times\mathbb{Z}^d\to[0,\infty)$ *be symmetric and such that*

$$\sup_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d} K(j,k) =: C_K < \infty.$$

Suppose that

$$\|\chi_k A\chi_j\|_{p\to q} \leqslant K(j,k) \quad \text{for } j,k\in\mathbb{Z}^d.$$

Then $\|A\|_{p\to q}\leqslant C_K$.

Proof. Recall from Section 1 the norm preserving mapping $J_r : L^r(\Omega)\longrightarrow l^r(\mathbb{Z}^d)$ defined for $r\in[1,\infty]$ by $(J_rf)(k) = \|\chi_k f\|_r$. For $f\in L_c^\infty$ we get

$$J_q(Af)(k) \leqslant \sum_j \|\chi_k A\chi_j(\chi_j f)\|_q \leqslant \sum_j K(j,k)\|\chi_j f\|_p = I_K(J_pf)(k).$$

Here I_K denotes the operator on $l^q(\mathbb{Z}^d)$ defined by

$$(I_Kg)(k) := \sum_j K(j,k)g(j).$$

This implies the estimate

$$\|Af\|_q = \|J_q(Af)\|_{l^q} \leqslant \|I_K(J_pf)\|_{l^q} \leqslant \|I_K\|_{l^q\to l^q}\|J_pf\|_{l^q}.$$

Since $\|J_pf\|_q \leqslant \|J_pf\|_p = \|f\|_p$ it remains to show $\|I_K\|_{l^q\to l^q} \leqslant C_K$. For $q=1$ and $q=\infty$ this is readily seen, so the assertion follows from the Riesz-Thorin interpolation theorem.

Proof of Proposition 5. For $x \in Q_j$ one has $|\psi(x)-\psi(j)| \leqslant L|x-j| \leqslant \frac{L}{2}\sqrt{d}$ (with L the Lipschitz constant of ψ), hence $e^{\xi\psi(x)}e^{-\xi\psi(j)} \leqslant e^{|\xi||\psi(x)-\psi(j)|} \leqslant e^{|\xi|\frac{L}{2}\sqrt{d}} =: C_{|\xi|}$. (Notice that $C_{|\xi|} \to 1$ as $|\xi| \to 0$.) Therefore

$$\sup_{x\in Q_j} e^{\xi\psi} \leqslant C_{|\xi|}e^{\xi\psi(j)}. \tag{3}$$

Below we use the shorthand A_ξ for $e^{\xi\psi}Ae^{-\xi\psi}$ and denote $\|f\|_{Q_j,p} := \|\chi_jf\|_p$. Let $j,k \in \mathbb{Z}^d$ and $f \in L_c^\infty$ with $\operatorname{supp} f \subset Q_j$. Then we have

$$\begin{aligned}
\|Af\|_{Q_k,s} &\leqslant \|e^{-\xi\psi}A_\xi e^{\xi\psi}f\|_{Q_k,q} \leqslant C_{|\xi|}e^{-\xi\psi(k)}\|A_\xi e^{\xi\psi}f\|_q \\
&\leqslant C_{|\xi|}e^{-\xi\psi(k)}\|A_\xi\|_{p\to q}\|e^{\xi\psi}f\|_p \leqslant C_{|\xi|}^2e^{-\xi\psi(k)+\xi\psi(j)}\|A_\xi\|_{p\to q}\|f\|_{Q_j,p} \\
&\leqslant C_\varepsilon^2 e^{-\varepsilon|\psi(k)-\psi(j)|}\|A\|_{p\to q,\varepsilon}\|f\|_{Q_j,r},
\end{aligned} \tag{4}$$

where we have chosen $\xi = \varepsilon\frac{\psi(k)-\psi(j)}{|\psi(k)-\psi(j)|}$. Lemma 6 now implies

$$\|A\|_{r\to s} \leqslant C_\varepsilon^2\|A\|_{p\to q,\varepsilon}\sup_k\sum_j e^{-\varepsilon|\psi(k)-\psi(j)|} = C(\varepsilon,\psi)\|A\|_{p\to q,\varepsilon},$$

which completes the proof of (i), taking into account Proposition A1 from Appendix A.

Let $\xi_0 \in \mathbb{R}^d$ and $|\xi_0| \leqslant \varepsilon_0$. If $|\xi| \leqslant \varepsilon-\varepsilon_0$ then $|\xi+\xi_0| \leqslant \varepsilon$, and (i) yields

$$\|A_{\xi_0}\|_{r\to s} \leqslant C(\varepsilon-\varepsilon_0)\|A_{\xi_0}\|_{p\to q,\varepsilon-\varepsilon_0} \leqslant C(\varepsilon-\varepsilon_0)\|A\|_{p\to q,\varepsilon},$$

which proves assertion (ii).

To show (iii) let again $j,k \in \mathbb{Z}^d$, $f \in L_c^\infty$, $\operatorname{supp} f \subset Q_j$. Recall that $\sup_{x\in Q_j} e^{\pm(\xi\psi(j)-\xi\psi(x))} \leqslant C_{|\xi|}$, hence $\sup_{x\in Q_j}|e^{\xi\psi(j)-\xi\psi(x)}-1| \leqslant C_{|\xi|}-1$. Setting $f_{\xi,j} := e^{\xi\psi(j)-\xi\psi}f$ we consequently have

$$\|f_{\xi,j}-f\|_{Q_j,r} \leqslant (C_{|\xi|}-1)\|f\|_{Q_j,r}. \tag{5}$$

Denote $M := C_{|\xi|}e^{|\xi||\psi(k)-\psi(j)|}$. It follows from (3) that

$$\sup_{x\in Q_k} e^{\xi\psi(x)-\xi\psi(j)} \leqslant M \quad \text{and} \quad \sup_{x\in Q_k}|e^{\xi\psi(x)-\xi\psi(j)}-1| \leqslant M-1,$$

so we can estimate

$$\begin{aligned}\|A_\xi f - Af\|_{Q_k,s} &= \|e^{\xi\psi-\xi\psi(j)}Ae^{\xi\psi(j)-\xi\psi}f - Af\|_{Q_k,s} \\ &\leqslant \|e^{\xi\psi-\xi\psi(j)}A(f_{\xi,j}-f)\|_{Q_k,s} + \|(e^{\xi\psi-\xi\psi(j)}-1)Af\|_{Q_k,s} \\ &\leqslant M\|A(f_{\xi,j}-f)\|_{Q_k,s} + (M-1)\|Af\|_{Q_k,s}.\end{aligned}$$

Hence by (4) and (5) we obtain

$$\|A_\xi f - Af\|_{Q_k,s} \leqslant K_\xi(j,k)\|A\|_{p\to q,\varepsilon}\|f\|_{Q_j,r},$$

where

$$K_\xi(j,k) := C_\varepsilon^2 e^{-\varepsilon|\psi(k)-\psi(j)|}(C_{|\xi|}^2 e^{|\xi||\psi(k)-\psi(j)|}-1).$$

By Lemma 6 we conclude

$$\|A_\xi - A\|_{r\to s} \leqslant \sup_k \sum_j K_\xi(j,k)\|A\|_{p\to q,\varepsilon} = c(\xi)\|A\|_{p\to q,\varepsilon},$$

where $c(\xi) := \sup_k \sum_j C_\varepsilon^2 e^{-\varepsilon|\psi(k)-\psi(j)|}(C_{|\xi|}^2 e^{|\xi||\psi(k)-\psi(j)|}-1)$. It remains to prove that $c(\xi)\to 0$ as $\xi \to 0$. (Notice that K_ξ satisfies the conditions of Lemma 6 since $K_\xi(j,k) \leqslant C_\varepsilon^4 e^{-(\varepsilon-|\xi|)|\psi(k)-\psi(j)|}$.)

Let $B_n(k) := \{j; |\psi(k)-\psi(j)| \in [n,n+1)\}$. Then $\#B_n(k) \leqslant cn^d$ by Proposition A1. Using the bound

$$K_\xi(j,k) \leqslant C_\varepsilon^2 e^{-\varepsilon n}(C_{|\xi|}^2 e^{|\xi|(n+1)}-1) \quad \text{for } j \in B_n(k)$$

we obtain

$$\begin{aligned}\sum_j K_\xi(j,k) &= \sum_{n=0}^\infty \sum_{j\in B_n(k)} K_\xi(j,k) \\ &\leqslant \sum_{n=0}^\infty cC_\varepsilon^2 n^d e^{-\varepsilon n}(C_{|\xi|}^2 e^{|\xi|(n+1)}-1) \longrightarrow 0 \quad \text{as } |\xi|\to 0.\end{aligned}$$

Next we prove the following technical lemma which will be used later on.

Lemma 7. *Let X be a Banach space, B an isomorphism of X and A a closed densely defined operator in X. Suppose that*

(i) A is injective, $\rho(A) \neq \varnothing$,

(ii) there exists a core D for A with $BA : D \longrightarrow R(A)$,

(iii) $A^{-1}BA\restriction_D$ is a bounded operator in X, and its extension $B_A \in \mathcal{L}(X)$ satisfies the inequality

$$\|\mu(B_A - B)(\mu - A)^{-1}\| < \|B^{-1}\|^{-1}$$

for some $\mu \in \rho(A)$.

Then $A^{-1}BA$ is an isomorphism of $D(A)$, in particular B^{-1} is a bijection from $R(A)$ to $R(A)$, and $B_A^{-1}x = A^{-1}B^{-1}Ax$ for $x \in D(A)$.

Proof. Since A is closed, it is easy to conclude from (ii) that

$$BAx \in R(A),\ A^{-1}BAx = B_A x \quad \text{for all } x \in D(A).$$

According to (iii) the operator

$$\begin{aligned}(\mu - A)A^{-1}BA(\mu - A)^{-1} &= \big[\mu A^{-1}BA - \mu B + B(\mu - A)\big](\mu - A)^{-1} \\ &= \mu(B_A - B)(\mu - A)^{-1} + B =: \hat{B}\end{aligned}$$

is an isomorphism of X. Hence $A^{-1}BA$ can be represented as the composition of the isomorphisms $\mu - A : D(A) \longrightarrow X$, $\hat{B} : X \longrightarrow X$ and $(\mu - A)^{-1} : X \longrightarrow D(A)$.

Proof of Theorem 1. We only show that $\rho(T_p) \subset \rho(T_q)$. The proof of the other inclusion is almost the same. Let $\lambda \in \rho(T_p)$, set $B := (\lambda - T_p)(\lambda_0 - T_p)^{-1} = 1 + (\lambda - \lambda_0)(\lambda_0 - T_p)^{-1}$. According to Proposition 5(ii) there exists $C_1 < \infty$ such that

$$\|e^{\xi\psi}(\lambda_0 - T_p)^{-1}e^{-\xi\psi}\|_{p\to p} \leqslant C_1 \quad \text{for } |\xi| \leqslant \varepsilon/2.$$

This yields $\|e^{\xi\psi}Be^{-\xi\psi}\|_{p\to p} \leqslant 1 + |\lambda - \lambda_0|C_1$ for $|\xi| \leqslant \varepsilon/2$. Applying Proposition 5(iii) we obtain

$$\|e^{\xi\psi}Be^{-\xi\psi} - B\|_{p\to p} \to 0 \quad \text{as } |\xi| \to 0.$$

Let B_ξ be the continuous extension of $e^{\xi\psi}Be^{-\xi\psi}$ to the whole of L^p. There exists $\varepsilon_0 \in (0, \varepsilon/2)$ such that $\|B_\xi - B\|_{p\to p} \leqslant \frac{1}{2}\|B^{-1}\|^{-1}$ for all $|\xi| \leqslant \varepsilon_0$. Since B is an isomorphism of L^p, it follows that B_ξ is also an isomorphism, and

$$\|B_\xi^{-1}\|_{p\to p} \leqslant 2\|B^{-1}\|_{p\to p} \quad \text{for } |\xi| \leqslant \varepsilon_0. \tag{6}$$

Next we apply Lemma 7 to show $B_\xi^{-1}\restriction_{D(e^{-\xi\psi})}= e^{\xi\psi}B^{-1}e^{-\xi\psi}$: In the notation of Lemma 7 we set $A = e^{-\xi\psi}$, $D = L_c^\infty$, $\mu = -1$ and note that $\|(-1-e^{-\xi\psi})^{-1}\| \leqslant 1$.

Let $f \in L_c^\infty$, $|\xi| \leqslant \varepsilon_0$. Using (6) we obtain

$$\begin{aligned}\|e^{\xi\psi}(\lambda - T_p)^{-1}e^{-\xi\psi}f\|_q &= \|e^{\xi\psi}(\lambda_0 - T_p)^{-1}e^{-\xi\psi}e^{\xi\psi}(\lambda_0 - T_p)(\lambda - T_p)^{-1}e^{-\xi\psi}f\|_q \\ &\leqslant \|e^{\xi\psi}(\lambda_0 - T_p)^{-1}e^{-\xi\psi}\|_{p\to q}\|B_\xi^{-1}\|_{p\to p}\|f\|_p \leqslant 2C\|B^{-1}\|_{p\to p}\|f\|_p.\end{aligned}$$

Applying Proposition 5(i) we finally obtain $\|(\lambda - T_p)^{-1}\restriction_{L_c^\infty}\|_{q\to q} < \infty$. According to Proposition 4 this implies $\lambda \in \rho(T_q)$ and the consistency of the resolvents, since L_c^∞ is dense in $L^p \cap L^q$ in the sum norm.

3 L^p-independence of the spectrum. Proof of Theorem 2

We apply Theorem 1 in order to prove Theorem 2. Since the condition (i) of Theorem 1 is evidently satisfied for generators of consistent C_0-semigroups, for sufficiently large λ_0, all we have to prove is the appropriate "weighted" resolvent estimate. The next theorem serves this purpose.

Theorem 8. *Let A be one of the operators A_D, A_i, A_N defined above. In the case $A = A_N$ assume that Ω satisfies the cone property. Let $p, q \in [p_-, p_+]$ ($p, q \in [\frac{2}{2-\sqrt{\beta}}, \infty)$ in the case $\gamma = 0$) be such that $0 < \frac{1}{p} - \frac{1}{q} \leqslant \frac{2}{d}$. Let $\rho : \Omega \to (0,\infty)$ be such that $\rho \wedge n$ and $\rho^{-1} \wedge n$ are Lipschitz for all $n \in \mathbb{N}$. Then there exist $\delta > 0$ and $C < \infty$ such that if an inequality of the type*

$$\rho^{-2}\nabla\rho \cdot a \cdot \nabla\rho \leqslant \delta(A\dot{+}V^+) + c_\delta \tag{7}$$

holds in the form sense then there exists a $\lambda_0 \in \mathbb{R}$ satisfying

$$\|\rho(\lambda + T)^{-1}\rho^{-1}\|_{p\to q} \leqslant C \quad \text{for all } \lambda \geqslant \lambda_0.$$

For the proof of Theorem 8 we need the following proposition. Recall that A is associated with the form τ, $A\dot{+}V^+$ with $\tau_V = \tau + V^+$.

Proposition 9. *Let $\rho \in W^{1,\infty}$, i.e. ρ is bounded and Lipschitz. Assume that there exist constants $c_0, c_1 \in \mathbb{R}$ such that*

$$\nabla\rho \cdot a \cdot \nabla\rho \leqslant c_0(A\dot{+}V^+) + c_1 \tag{8}$$

in the form sense. Then ρ is a bounded multiplication operator on $(\tau_V, D(\tau_V))$.

Proof. First, observe that ρ is a bounded multiplicator on $Q(V^+)$. For $\tau = \tau_N$ or $\tau = \tau_i$ the assertion follows from the inequality

$$\begin{aligned}\tau[\rho u] &\leqslant 2\|\rho\|_\infty^2 \tau[u] + 2\langle \nabla\rho \cdot a \cdot \nabla\rho, |u|^2\rangle \\ &\leqslant 2(\|\rho\|_\infty^2 + c_0)\tau_V[u] + 2c_1\|u\|_2^2.\end{aligned}$$

To finish the proof for $\tau = \tau_D$, we apply Lemma B4 from Appendix B.

Proof of Theorem 8. In order to prove the theorem it suffices to prove the estimate

$$\|\rho(\lambda+T)^{-1}\rho^{-1}f\|_q \leqslant C\|f\|_p \quad \text{for } 0 \leqslant f \in L_c^\infty, \lambda \geqslant \lambda_0 \tag{9}$$

for some $C \leqslant \infty$, $\lambda_0 \in \mathbb{R}$.

In the following let $\lambda > \frac{c(\beta)}{\sqrt{\beta}} + c(\gamma)$. Then $\lambda \in \rho(-T_p)$ for $p \in [p_-, p_+]$ ($p \in [\frac{2}{2-\sqrt{\beta}}, \infty)$ in the case $\gamma = 0$) by [18], Thm. 5.

First, notice that we can reduce the proof to the case $\rho, \rho^{-1} \in W^{1,\infty}$, Indeed, let (9) hold for $\rho_n := (\rho \wedge n) \vee \frac{1}{n}$. Then $\rho_n = \rho$ on supp f for sufficiently large n. ρ_n satisfies (7) with the same constants since $\nabla\rho_n = (\nabla\rho)\chi_{[1/n \leqslant \rho \leqslant n]}$. Clearly, $u_n := \rho_n(\lambda+T)^{-1}\rho_n^{-1}f \to u := \rho(\lambda+T)^{-1}\rho^{-1}f$ pointwise. Therefore by Fatou's lemma $\|u\|_q \leqslant \liminf_n \|u_n\|_q \leqslant C\|f\|_p$. So from now on we assume without loss of generality $\rho^{\pm 1} \in W^{1,\infty}$.

Let T_n be the operator associated with the form $\mathbf{t}_n[u,v] := \tau[u,v] + \langle \nabla u, bv\rangle + \langle V_n u, v\rangle$ ($n \in \mathbb{N}_0$), with $V_n := V \vee (-n)$, $D(\mathbf{t}_n) := D(\tau_V)$. Let $f \in L_c^\infty$. Then $u_n := \rho(\lambda+T_n)^{-1}\rho^{-1}f \to u := \rho(\lambda+T)^{-1}\rho^{-1}f$ in L^2 (see e.g. [18], [31]), so passing to a subsequence we can assume that $u_n \to u$ a.e. Therefore by Fatou's lemma

$$\|\rho(\lambda+T)^{-1}\rho^{-1}f\|_q \leqslant \liminf_n \|\rho(\lambda+T_n)^{-1}\rho^{-1}f\|_q, \tag{10}$$

and the desired estimate reduces to the case $V^- \in L^\infty$.

In the rest of the proof we distinguish two cases.

Case 1. $\frac{1}{p} + \frac{1}{q} \leqslant 1$.

Let $f \in L_c^\infty$, $\rho \in W^{1,\infty}$, $\lambda > \frac{c(\beta)}{\sqrt{\beta}} + c(\gamma)$, $V^- \in L^\infty$ and set $u := \rho(\lambda+T)^{-1}\rho^{-1}f$. Then $\rho^{-1}u \in D(\tau_V)$. We also claim that $\rho^{-1}u \in L^\infty$. Indeed, $(\lambda+T)^{-1}$ maps L_c^∞ to L^∞: Take $g \in L_c^\infty$, then by the second resolvent identity we can write the representation

$$(\lambda+T)^{-1}g = (\lambda+T_0)^{-1}g + (\lambda+T_0)^{-1}V^-(\lambda+T)^{-1}g. \tag{11}$$

The semigroup e^{-tT_0} is L^∞-contractive and $(\lambda+T_{0,p})^{-1}$ maps L^p into L^q for $0\leqslant 1/p-1/q<2/d$ (cf. [18], Thms. 1 and 4, [28], Appendix). This implies $(\lambda+T_0)^{-1}g\in L^\infty$. Moreover, if $(\lambda+T)^{-1}g\in L^p$, then from (11) it follows that $(\lambda+T)^{-1}g\in L^q$ for $1/p-1/q<2/d$. Repeating this argument $[\frac{d}{4}]+1$ times we conclude that $(\lambda+T)^{-1}g\in L^\infty$.

By Proposition 9 we now have $0\leqslant u\in L^\infty\cap D(\tau_V)$. Let $\nu:=1+\frac{q}{p'}$, hence $(\nu-1)p'=q$ and $\nu\geqslant 2$. Then $u^{\nu/2},u^{\nu-1}\in D(\tau_V)$ since they are multiples of normal contractions of u, and also $\rho u^{\nu-1}\in D(\tau_V)$ by Proposition 9. Multiplying the equality $\rho(\lambda+T)\rho^{-1}u=f$ scalarly in L^2 by $u^{\nu-1}$ we obtain

$$\lambda\|u\|_\nu^\nu+\mathbf{t}[\rho^{-1}u,\rho u^{\nu-1}]=\langle f,u^{\nu-1}\rangle. \tag{12}$$

Claim 1. There exist $\varepsilon>0$ and c such that

$$\mathbf{t}[\rho^{-1}u,\rho u^{\nu-1}]\geqslant\varepsilon\tau[u^{\nu/2}]-c\|u\|_\nu^\nu. \tag{13}$$

Under the stated conditions inequality (13) is a standard quadratic estimate. The details of its proof are delegated to Appendix C.

Using (13) we estimate the LHS of (12) as follows

$$\begin{aligned}\lambda\|u\|_\nu^\nu+\mathbf{t}[\rho^{-1}u,\rho u^{\nu-1}]&\geqslant\varepsilon\tau[u^{\nu/2}]+(\lambda-c)\|u\|_\nu^\nu\\&\geqslant\varepsilon\sigma\|\nabla u^{\nu/2}\|_2^2+\varepsilon\sigma\|u\|_\nu^\nu+(\lambda-c-\varepsilon\sigma)\|u\|_\nu^\nu\\&\geqslant\varepsilon\sigma\|u^{\nu/2}\|_{W^{1,2}}^2\geqslant\varepsilon\sigma C_d\|u^{\nu/2}\|_{2q/\nu}^2,\end{aligned}$$

provided $\lambda\geqslant c+\varepsilon\sigma=:\lambda_0$. In the last step we used Sobolev's inequality with the constant C_d, taking into account $\frac{1}{p}-\frac{1}{q}\leqslant\frac{2}{d}$. Applying Hölder's inequality to the RHS of (12) we get

$$\varepsilon\sigma C_d\|u\|_q^\nu\leqslant\|f\|_p\|u^{\nu-1}\|_{p'}=\|f\|_p\|u\|_q^{\nu-1}.$$

This completes the proof of (9) with $C:=(\varepsilon\sigma C_d)^{-1}$.

Case 2. $\frac{1}{p}+\frac{1}{q}>1$.

We have to treat this case separately since in the above proof ν becomes less than 2, and one cannot conclude that $u^{\nu-1}$ belongs to the form domain. Instead of (9) we will prove the equivalent estimate

$$\|\rho^{-1}(\lambda+T^*)^{-1}\rho f\|_{p'}\leqslant C\|f\|_{q'}. \tag{14}$$

As was shown above we can assume without loss of generality that $\rho^{-1}\in W^{1,\infty}$ and $V^-\in L^\infty$. Let $0\leqslant f\in L_c^\infty$, $\lambda>\frac{c(\beta)}{\sqrt{\beta}}+c(\gamma)$, set

$u := \rho^{-1}(\lambda + T^*)^{-1}\rho f$. Then $\rho u = (\lambda + T^*)^{-1}\rho f \in D(\tau_V)$, and Proposition 9 implies $u \in D(\tau_V)$. Let $(\nu - 1)q = p'$, then $\nu \in [2, p')$. By Proposition C1 (see Appendix C) and Proposition 9 we have $u^{\nu/2} = \rho^{-\nu/2}(\rho u)^{\nu/2} \in D(\tau_V)$.

Note that

$$\lambda u + \rho^{-1}T^*\rho u = f.$$

Multiplying this equality by $u^{\nu-1}$ and integrating over Ω we obtain

$$\lambda\|u\|_\nu^\nu + \langle \rho^{-1}T^*\rho u, u^{\nu-1}\rangle = \langle f, u^{\nu-1}\rangle.$$

Claim 2. There exist $\varepsilon_1 > 0$ and c_1 such that

$$\langle \rho^{-1}T^*\rho u, u^{\nu-1}\rangle \geqslant \varepsilon_1\tau[u^{\nu/2}] - c_1\|u\|_\nu^\nu. \tag{15}$$

The proof is delegated to Appendix C. The final part of the proof is the same as in Case 1.

Proof of Theorem 2. Set $\rho := e^{\xi\psi}$, then $\nabla\rho = \rho\nabla(\xi\psi)$. It follows that

$$\begin{aligned}\rho^{-2}\nabla\rho\cdot a\cdot\nabla\rho &= \sum_{j,k=1}^{d}\xi_j\xi_k\nabla\psi_j\cdot a\cdot\nabla\psi_k\\ &\leqslant c|\xi|^2\sum_{j=1}^{d}\nabla\psi_j\cdot a\cdot\nabla\psi_j,\end{aligned}$$

with a constant c depending only on the dimension d. Therefore by the assumption of the theorem there exists $\varepsilon > 0$ such that for $|\xi| \leqslant \varepsilon$ the inequality

$$\rho^{-2}\nabla\rho\cdot a\cdot\nabla\rho \leqslant \delta(A\dot{+}V^+) + 1$$

holds in the form sense. Notice also that $\rho\wedge n = e^{\xi\psi\wedge\ln n}$ and $\rho^{-1}\wedge n = e^{(-\xi\psi)\wedge\ln n}$ are Lipschitz continuous, so that all assumptions of Theorem 8 are fulfilled. Applying now Theorem 1 we obtain the result.

4 Comparison of essential spectra

In this section we present the proof of Theorem 3. For this we need some auxiliary results and simple properties of $l^p(L^q)$ spaces.

First, note that the following Hölder's inequality holds:

$$\|fg\|_{l^{r_1}(L^{r_2})} \leqslant \|f\|_{l^{p_1}(L^{p_2})}\|g\|_{l^{q_1}(L^{q_2})},$$

where $\frac{1}{r_j} = \frac{1}{p_j} + \frac{1}{q_j}, j = 1, 2$. We will use it for the case $r = r_1 = r_2$, then $l^r(L^r) = L^r$. By Sobolev's embedding theorem we also have

$$W^{m,2} = l^2(W^{m,2}) \subset l^2(L^\infty)$$

for $m > \frac{d}{2}$, where $l^2(W^{m,2})$ is defined analogously to $l^2(L^q)$.

Lemma 10. *Let $h \in c_0(L^2), m \in \mathbb{N}, m > d/2$. Then the multiplicator $h : W^{m,2} \to L^\infty$ is a compact operator.*

Proof. First, let $\operatorname{supp} h$ be a compact $K \subset \mathbb{R}^d$, then $h = h\chi_K \in L^2$. The embedding $W^{m,2} \subset L^\infty$ is compact for bounded domains (see [1], Thm. 6.2). Therefore $\chi_K : W^{m,2} \to L^\infty$ is compact. The operator $h : L^\infty \to L^2$ is bounded, so $h : W^{m,2} \to L^2$ is compact.

Now let $h_R := \chi_{B(0;R)}h$. Then $h_R \to h$ in the operator norm as operators from $W^{m,2}$ to L^2. Indeed, let $f \in W^{m,2}$. Then

$$\|(h_R - h)f\|_2 \leqslant \|h_R - h\|_{l^\infty(L^2)}\|f\|_{l^2(L^\infty)} \leqslant \|\chi_{B(0;R)^c}h\|_{l^\infty(L^2)}c\|f\|_{W^{m,2}},$$

and $\|\chi_{B(0;R)^c}h\|_{l^\infty(L^2)} \to 0$ by the definition of $c_0(L^2)$.

This shows that $h : W^{m,2} \to L^\infty$ is a compact operator.

We use the notation $|a(x)|$ for the norm of the operator $a(x) : \mathbb{C}^d \to \mathbb{C}^d$.

Lemma 11. *Let $\mathbf{t}$ be the form introduced in Section 1, $m > d/2 + 1$. Then $W^{m,2} \subset D(\mathbf{t})$.*

Proof. We only have to show that $(C_c^\infty, \|\cdot\|_{m,2})$ is continuously embedded in $(\mathbf{t}, D(\mathbf{t}))$. Let $u \in C_c^\infty$. Then $\|u\|_{l^2(L^\infty)} \leqslant c_0\|u\|_{m,2}$ and $\|\nabla u\|_{l^2(L^\infty)} \leqslant c_0\|\nabla u\|_{m-1,2} \leqslant c_0\|u\|_{m,2}$. Therefore

$$\begin{aligned}|\langle \nabla u, a\nabla u\rangle| &\leqslant \||a| \cdot |\nabla u|^2\|_1 \\ &\leqslant \|a\|_{l^\infty(L^1)}\|\nabla u\|^2_{l^2(L^\infty)} \leqslant c_0^2\|a\|_{l^\infty(L^1)}\|u\|^2_{m,2}.\end{aligned}$$

Estimating the lower order terms similarly, we get

$$|\mathbf{t}[u]| \leqslant c_0^2(\|a\|_{l^\infty(L^1)} + \|b\|_{l^\infty(L^1)} + \|c\|_{l^\infty(L^1)} + \|V\|_{l^\infty(L^1)})\|u\|^2_{m,2},$$

which leads to the desired conclusion.

Proof of Theorem 3. Let τ^0 be the form $\langle \nabla u, a^0 \nabla v\rangle$ defined on $W^{1,2}$, and A^0 be the associated operator. It is well-known that the spectrum of A^0 is $[0,\infty)$ and coincides with the essential spectrum (see [24], Ch. 3, Cor. 3.3). Therefore it suffices to show that $(\lambda+T)^{-1} - (\lambda+A^0)^{-1}$ is a compact operator (see [16], Ch. IV, §5.6). According to a result by E. M. Ouhabaz and P. Stollmann ([22], Prop. 2.2) we only have to show that both

$$[(\lambda+T)^{-1} - (\lambda+A^0)^{-1}](\lambda+A^0)^{-k} \tag{16}$$

and

$$(\lambda+A^0)^{-k}[(\lambda+T)^{-1} - (\lambda+A^0)^{-1}] \tag{17}$$

are compact for some $k \in \mathbb{N}$ since both T and A^0 are accretive and $D(T) \subset D(A^{0\frac{1}{2}})$. Observe that T and T^* have the same form and (17) with T^* in place of T is the adjoint operator to (16). So we confine ourselves to showing the compactness of (17).

Next we use the quadratic form method in order to obtain a representation of (17) which will lead to compactness. For $u, v \in L^2$ we have (with the shorthand $R = (\lambda+A^0)^{-k-1}$)

$$\begin{aligned}
&\langle (\lambda+A^0)^{-k}[(\lambda+A^0)^{-1} - (\lambda+T)^{-1}]u, v\rangle \\
&\quad = \langle u, (\lambda+A^0)^{-k-1}v\rangle - \langle (\lambda+T)^{-1}u, (\lambda+A^0)^{-k}v\rangle \\
&\quad = (\lambda+\mathbf{t})[(\lambda+T)^{-1}u, Rv] - (\lambda+\tau^0)[(\lambda+T)^{-1}u, Rv] \\
&\quad = \langle \nabla(\lambda+T)^{-1}u, ((a-a^0)\nabla + b)Rv\rangle + \langle (\lambda+T)^{-1}u, (\bar{c}\nabla + V)Rv\rangle \\
&\quad = \langle [(a-a^0)\nabla R]^* \nabla(\lambda+T)^{-1}u, v\rangle + \langle [bR]^*\nabla(\lambda+T)^{-1}u, v\rangle \\
&\qquad + \langle [\bar{c}\nabla R]^*(\lambda+T)^{-1}u, v\rangle + \langle [VR]^*(\lambda+T)^{-1}u, v\rangle,
\end{aligned}$$

taking into account Lemma 11. This implies

$$\begin{aligned}
&(\lambda+A^0)^{-k}[(\lambda+T)^{-1} - (\lambda+A^0)^{-1}] \\
&\qquad = [(a-a^0)\nabla R]^*\nabla(\lambda+T)^{-1} + [bR]^*\nabla(\lambda+T)^{-1} \\
&\qquad\quad + [\bar{c}\nabla R]^*(\lambda+T)^{-1} + [VR]^*(\lambda+T)^{-1}.
\end{aligned} \tag{18}$$

By Lemma 10 the factors in the square brackets in the RHS of (18) are compact for $k > d/4$. Using the assumption $D(T) \subset W^{1,2}$ and the closed graph theorem one concludes that $\nabla(\lambda+T)^{-1}$ is bounded, so the first assertion of the theorem follows.

Now let a be symmetric and strictly elliptic, that is $a(x) \geqslant \varepsilon I$ for some $\varepsilon > 0$. Then we can rewrite the first term of the RHS of (18) in the form

$$[a^{-\frac{1}{2}}(a - a^0)\nabla R]^*[a^{\frac{1}{2}}\nabla(\lambda + T)^{-1}]. \tag{19}$$

Note that the second factor of (19) is bounded, so for the second assertion of the theorem it remains to show that $a^{-\frac{1}{2}}(a - a^0) \in c_0(L^2)$.

By the next lemma, for $x \in \mathbb{R}^d$ we have

$$|a(x)^{-\frac{1}{2}}(a(x) - a^0)| \leqslant c_1|a(x) - a^0|^{\frac{1}{2}} + c_2(|a(x) - a^0| \wedge |a^0|).$$

But $|a-a^0|^{\frac{1}{2}} \in c_0(L^2)$ and $|a-a^0| \wedge |a^0| \in c_0(L^1) \cap L^\infty \subset c_0(L^2)$, so we arrive at the desired conclusion.

Lemma 12. *Let $A, B \in \mathbb{C}^d \otimes \mathbb{C}^d$ be positive symmetric matrices, $A \geqslant \varepsilon$ in the matrix sense. Then*

$$|A^{-\frac{1}{2}}(A - B)| \leqslant c_1|A - B|^{\frac{1}{2}} + c_2(|A - B| \wedge |B|), \tag{20}$$

where $c_1 = \sqrt{\frac{|B|}{\varepsilon}} \vee 1$, $c_2 = \frac{d^2-d}{\sqrt{\varepsilon}}$.

Proof. Since A is symmetric and the assertion is invariant under unitary transformations, we assume that A is diagonal. Let C denote the matrix on the LHS of (20), then $c_{jk} = a_{jj}^{-1/2}(a_{jj}\delta_{jk} - b_{jk})$. The off-diagonal entries of C are easily estimated by

$$|c_{jk}| = a_{jj}^{-1/2}|b_{jk}| \leqslant \varepsilon^{-1/2}(|A - B| \wedge |B|),$$

since b_{jk} is an entry of $A - B$ as well as of B.

The matrix norm of the diagonal part C_{diag} of C is the greatest of the absolute values of the diagonal entries, so without loss of generality $|C_{diag}| = a_{11}^{-1/2}|a_{11} - b_{11}|$. By distinguishing the cases $a_{11} \leqslant b_{11}$ and $a_{11} > b_{11}$ (note that b_{11} must be positive) one gets

$$|C_{diag}| \leqslant c_1|a_{11} - b_{11}|^{\frac{1}{2}} \leqslant c_1|A - B|^{\frac{1}{2}}.$$

Putting these estimates together yields inequality (20).

5 Concluding remarks

1. Let e^{-tT} be a C_0-semigroup on $L^{p_0}(\Omega)$, $\Omega \subset \mathbb{R}^d$, for some $p_0 \in [1,\infty)$. We say that e^{-tT} satisfies a generalized Gaussian estimate of order $2m$ ($m > 0$) if there exist $q_0 \in (p_0, \infty]$ and constants $\varepsilon_0 > 0$, $C < \infty$, $\omega \in \mathbb{R}$ such that

$$\|e^{\xi x} e^{-tT} e^{-\xi x}\|_{p\to q} \leqslant C t^{-\frac{d}{2m}(\frac{1}{p}-\frac{1}{q})} e^{\omega t} \tag{21}$$

for all $|\xi| \leqslant \varepsilon_0, t > 0$ and $p_0 \leqslant p \leqslant q \leqslant q_0$. (We adopt the notion from [25] where the case $m = 1$ was studied.) Then by [33] the semigroup extends to a C_0-semigroup on L^p, for $p_0 \leqslant p \leqslant q_0$ ($p_0 \leqslant p < \infty$ if $q_0 = \infty$). Now representing the resolvent via the semigroup, by (21) we show that the condition (ii) of Theorem 1 is fulfilled for $\frac{1}{p} - \frac{1}{q} < \frac{2m}{d}$, $p_0 \leqslant p < q \leqslant q_0$. Therefore by Theorem 1 we conclude that the spectrum of T_p is p-independent.

Note that (21) is valid for all $1 \leqslant p \leqslant q \leqslant \infty$ if the semigroup satisfies the Gaussian estimate pointwise, that is, it has an integral kernel $K_t(x,y)$ satisfying

$$|K_t(x,y)| \leqslant C_1 t^{-\frac{d}{2m}} \exp(-C_2 |x-y|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}} + \omega t) \tag{22}$$

with some constants $C_1 < \infty$, $C_2 > 0$ and $\omega \in \mathbb{R}$.

Estimates of this kind hold, for example, in the following cases

(a) second order uniformly elliptic operators in divergence form

- with real coefficients [3],
- with complex coefficients in dimensions 1 and 2 [5],
- with uniformly continuous complex coefficients in higher dimensions [4];

(b) superelliptic operators of order $2m$ in dimensions $d < 2m$ [10].

(For more detailed discussions of examples for which (22) is valid, we refer to [15], [17].)

Moreover, (21) holds in a certain interval around 2, for higher order superelliptic operators for which the L^p-theory is developed in [10], Sec. 7, in absence of pointwise Gaussian estimates.

2. In Theorem 2 we considered the operator associated with the formal expression $-\nabla \cdot a \cdot \nabla + b \cdot \nabla + V$, where $b \cdot a^{-1} \cdot b$ and V^- are form bounded with

respect to A with bounds β and γ, respectively, and $\sqrt{\beta}+\gamma<1$. For $V^-=0$ it is even possible to treat the case $1\leqslant\beta<4$ if $b\cdot a^{-1}\cdot b\in L^1+L^\infty$. The operator cannot be constructed via the form method but by approximation with bounded drifts, for $p\in[\frac{2}{2-\sqrt{\beta}},\infty)$ (see [18]). In the same way as it is done in the proof of Theorem 8 (see (10)) the estimate for the "weighted" resolvent can be reduced to the case of bounded drifts, so the result on L^p-independence of the spectrum also holds in this case.

3. Under the conditions of Theorem 3 the second order elliptic operator T has essential spectrum $\sigma_{ess}(T)=[0,\infty)$. Following the argument due to E. M. Ouhabaz [21], Cor. 3, one concludes from this that $\sigma(T)\setminus\sigma_{ess}(T)$ consists of isolated eigenvalues of finite algebraic multiplicity (see also [16], Ch. IV, §5).

4. Theorem 3 can be easily reformulated for domains in $\mathbb{R}^d$. The conclusion of the theorem then will read as follows: the essential spectra of the operators with variable and with constant coefficients coincide.

Appendix A

We prove here a proposition which gives a characterization for a function ψ to be L^1-regular.

Proposition A 1. *For $\psi:\mathbb{Z}^d\longrightarrow\mathbb{R}^d$ the following are equivalent:*

(i) *ψ is L_1-regular,*

(ii) $\sup_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}e^{-\varepsilon|\psi(k)-\psi(j)|}<\infty$ *for all $\varepsilon>0$,*

(iii) *there exists $C<\infty$ such that for all unit cubes $Q=x+[0,1]^d\subset\mathbb{R}^d$ we have $\#\psi^{-1}(Q)\leqslant C$,*

(iv) *there exists $C<\infty$ such that for all $x\in\mathbb{R}^d$, $n\in\mathbb{N}$ we have*

$$\#\psi^{-1}(x+[0,n]^d)\leqslant Cn^d.$$

Remark. *In [27], L^1-regularity was defined by property (ii) above. It is an easy consequence of part (iv) of the proposition that there exists $K\in\mathbb{N}$ such that*

$$\textit{diam}\ \psi(j+[0,Kn]^d)\geqslant n \quad \textit{for all } j\in\mathbb{Z}^d,\ n\in\mathbb{N}.$$

This means that ψ cannot be more than linearly contractive.

Proof of Proposition A1. (i) $\Longrightarrow$ (iii) Let $Q \subset \mathbb{R}^d$ be a unit cube. Fix $k \in \psi^{-1}(Q)$. Then for $j \in \psi^{-1}(Q)$ we have $\psi(j), \psi(k) \in Q$, so $|\psi(k)-\psi(j)| \leqslant \sqrt{d}$. Therefore

$$\#\psi^{-1}(Q) = \sum_{j\in\psi^{-1}(Q)} 1 \leqslant \sum_{j\in\mathbb{Z}^d} e^{\sqrt{d}-|\psi(k)-\psi(j)|} \leqslant e^{\sqrt{d}} M_\psi =: C.$$

(iv) $\Longrightarrow$ (ii) Let $k \in \mathbb{Z}^d$. For $n \in \mathbb{N}_0$ define $B_n := \psi^{-1}(\psi(k) + (-n,n)^d)$. Then $|\psi(k) - \psi(j)| \geqslant n$ for $j \notin B_n$. By (iv) we have $\#B_n \leqslant C(2n)^d$, hence

$$\sum_{j\in\mathbb{Z}^d} e^{-\varepsilon|\psi(k)-\psi(j)|} \leqslant \sum_{n=0}^{\infty} \sum_{j\in B_{n+1}\setminus B_n} e^{-\varepsilon n} \leqslant \sum_{n=0}^{\infty} C(2n+2)^d e^{-\varepsilon n} < \infty.$$

The implications (iii) $\Longrightarrow$ (iv) and (ii) $\Longrightarrow$ (i) are trivial.

Appendix B

Here we collect some auxiliary facts on Dirichlet forms. We refer to [12], [20] for main definitions and results of the theory of Dirichlet forms. Recall that $\phi : \mathbb{C} \to \mathbb{C}$ is called a *normal contraction* if $\phi(0) = 0$ and $|\phi(x)-\phi(y)| \leqslant |x-y|$ for all $x, y \in \mathbb{C}$.

Below we constantly use the following simple proposition (see [16], Ch. VI, Thm 1.16, [20], Lemma 2.12).

Proposition B 1. *Let τ be a closed symmetric form in a Hilbert space $\mathcal{H}$. Let $u_n \in D(\tau)$ ($n \in \mathbb{N}$), $u_n \to u$ in $\mathcal{H}$. Suppose that* $\sup_n \tau[u_n] < \infty$. *Then $u \in D(\tau)$ and* $\tau[u] \leqslant \liminf_n \tau[u_n]$.

The next two propositions will be needed to regularize approximating sequences in the domain of a Dirichlet form.

Proposition B 2. *Let $(M, \mathcal{M}, \mu)$ be a measure space. Let $(\tau, D(\tau))$ be a symmetric Dirichlet form on $L^2(M, \mu)$, $u \in D(\tau)$ and ϕ a normal contraction. If $(u_n) \subset D(\tau)$ and $u_n \to u$ in $(\tau, D(\tau))$, then $\phi(u_n) \to \phi(u)$ weakly in $(\tau, D(\tau))$. If in addition* $\liminf \tau[\phi(u_n)] \leqslant \tau[\phi(u)]$ *then the convergence is strong. In particular, if $\phi(u) = u$ a.e. then $\phi(u_n) \to u$ in $(\tau, D(\tau))$.*

For the proof we refer to [6]. We use the above proposition for two cases: a) if $u \geqslant 0$ then $\phi(z) := |z|$; b) if $0 \leqslant u \leqslant M$ then $\phi(z) := |z| \wedge M$.

Proposition B 3. *Let $(\tau, D(\tau))$ be a symmetric Dirichlet form on $L^2(M,\mu)$. Suppose that $u, u_n \in D(\tau)$, $u \geqslant 0$ and $u_n \to u$ in $(\tau, D(\tau))$. Then $|u_n| \wedge u \to u$ in $(\tau, D(\tau))$.*

Proof. By Proposition B2 we have $|u_n| \to u$ in $(\tau, D(\tau))$, so without loss of generality we assume that $u_n \geqslant 0$. Now $\tau[|u_n - u|] \to 0$ since $\tau[|u_n - u|] \leqslant \tau[u_n - u]$. Therefore $u_n \wedge u = \frac{1}{2}(u_n + u - |u_n - u|) \to u$ in $(\tau, D(\tau))$.

Now we study the form τ_D introduced in Section 1.

Lemma B 4. *(i) Let $\rho \in W^{1,\infty}$. If $u \in D(\tau_D) \cap L_c^\infty$, then $\rho u \in D(\tau_D)$.*

(ii) $D(\tau_V) \cap L_c^\infty$ is a core of the form $\tau_V = \tau_D + V^+$.

Proof. (i) Let $\tau := \tau_D$. *Claim.* $W_c^{1,\infty} \subset D(\tau)$.

In order to prove the Claim take $u \in W_c^{1,\infty}$. Let γ_n be the standard mollifier. Then $v_n := u * \gamma_n \in C_c^\infty(\Omega)$ for sufficiently large n, and $v_n \to u$ in L^2. Choose a compact set $K \subset \Omega$ such that $\operatorname{supp} v_n \subset K$ for $n \geqslant n_0$. Since $\|\nabla v_n\|_\infty \leqslant \|\nabla u\|_\infty$, we have

$$|\nabla v_n \cdot a \cdot \overline{\nabla v_n}| \leqslant \|\nabla u\|_\infty^2 \max_i \sum_j |a_{ij}| \chi_K \in L^1.$$

Therefore $\sup_n \tau(v_n) < \infty$, and by Proposition B1 we obtain $u \in D(\tau)$.

To prove the assertion it suffices to consider $0 \leqslant u \in D(\tau) \cap L_c^\infty$. Let $\tilde{u}_n \in C_c^\infty$ be such that $\tilde{u}_n \to u$ in $(\tau, D(\tau))$ as $n \to \infty$. Then $u_n := |\tilde{u}_n| \wedge \|u\|_\infty \in W_c^{1,\infty} \subset D(\tau)$ and $u_n \to u$ in $(\tau, D(\tau))$ by Proposition B2. In particular $\sup_n \tau[u_n] < \infty$.

Choose $\varphi \in C_c^\infty$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi\restriction_{\operatorname{supp} u} = 1$, then $\varphi u = u$. For $v_n := \rho\varphi u_n \in W_c^{1,\infty}$ we have $v_n \to \rho\varphi u = \rho u$ in L^2. Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \tau[v_n] &\leqslant 2\|\rho\varphi\|_\infty^2 \tau[u_n] + 2\|u_n\|_\infty^2 \tau(\rho\varphi) \\ &\leqslant 2\|\rho\|_\infty^2 \tau[u_n] + 2\|u\|_\infty^2 \tau(\rho\varphi), \end{aligned}$$

which implies that $\sup_n \tau[v_n] < \infty$ (note that $\rho\varphi \in W_c^{1,\infty} \subset D(\tau)$). Hence by Proposition B1 we conclude that $\rho u \in D(\tau)$.

(ii) Let $0 \leqslant u \in D(\tau_V)$ and $\tilde{u}_n \in C_c^\infty$ be such that $\tilde{u}_n \to u$ in $(\tau, D(\tau))$ as $n \to \infty$. Then $u_n := |\tilde{u}_n| \wedge u \in D(\tau_V) \cap L_c^\infty$ and $u_n \to u$ in $(\tau, D(\tau))$ by Proposition B3. Also $V^+ u_n^2 \leqslant V^+ u^2 \in L^1$, therefore $(V^+)^{1/2} u_n \to (V^+)^{1/2} u$ in L_2 which proves the assertion.

Appendix C

Proof of Claim 1. Denote $\Phi := \rho^{-1}\nabla\rho$ and $\psi := u^{\nu/2}$, then $\psi \in D(\tau) \cap Q(V^+)$. A straightforward computation shows that

$$\mathbf{t}[\rho^{-1}u, \rho u^{\nu-1}] = 4\frac{\nu-1}{\nu^2}\tau[\psi] + \frac{2}{\nu}\langle\nabla\psi, b\psi\rangle - 2(1-\frac{2}{\nu})\langle\nabla\psi \cdot a \cdot \Phi\psi\rangle$$
$$- \langle\Phi \cdot a \cdot \Phi\psi, \psi\rangle - \langle\Phi\psi, b\psi\rangle + \langle V^+\psi, \psi\rangle - \langle V^-\psi, \psi\rangle. \quad (23)$$

Using the Cauchy-Schwarz inequality and condition B on b one obtains

$$|\langle\nabla\psi, b\psi\rangle| \leqslant \sqrt{\beta}\tau[\psi] + \frac{c(\beta)}{2\sqrt{\beta}}\|\psi\|_2^2. \quad (24)$$

By condition (7) on ρ one has

$$\langle\Phi \cdot a \cdot \Phi\psi, \psi\rangle \leqslant \delta\tau[\psi] + \delta\langle V^+\psi, \psi\rangle + c_\delta\|\psi\|_2^2.$$

So similarly to (24) we can estimate

$$|\langle\nabla\psi \cdot a \cdot \Phi\psi\rangle| \leqslant \sqrt{\delta}\tau[\psi] + \frac{1}{2}\sqrt{\delta}\langle V^+\psi, \psi\rangle + \frac{c_\delta}{2\sqrt{\delta}}\|\psi\|_2^2,$$

$$|\langle\Phi\psi, b\psi\rangle| \leqslant \sqrt{\beta\delta}\tau[\psi] + \frac{1}{2}\sqrt{\beta\delta}\langle V^+\psi, \psi\rangle + \left(\frac{c(\beta)\sqrt{\delta}}{2\sqrt{\beta}} + \frac{c_\delta\sqrt{\beta}}{2\sqrt{\delta}}\right)\|\psi\|_2^2.$$

Using the above inequalities and condition V on V^- for the terms in the RHS of (23) we obtain

$$\mathbf{t}[\rho^{-1}u, \rho u^{\nu-1}] \geqslant \left(4\frac{\nu-1}{\nu^2} - \frac{2}{\nu}\sqrt{\beta} - \gamma - 4\sqrt{\delta}\right)\tau[\psi]$$
$$+ (1 - \frac{5}{2}\sqrt{\delta})\langle V^+\psi, \psi\rangle - c\|\psi\|_2^2,$$

where we have taken into account that $\beta, \delta < 1$. The constant c depends upon $\beta, c(\beta), c(\gamma), \delta, c_\delta$. Let $\varepsilon := \frac{1}{2}\left(4\frac{\nu-1}{\nu^2} - \frac{2}{\nu}\sqrt{\beta} - \gamma\right)$. Then $\varepsilon > 0$ by the conditions of Theorem 8 since $\nu \in (p, q)$. To complete the proof of Claim 1 one has to choose $\delta = \min\{\frac{\varepsilon^2}{16}, \frac{4}{25}\}$.

Proposition C 1. *Let T be the operator defined in Section 1. Let $0 \leqslant f \in L_c^\infty$, $\lambda > \frac{c(\beta)}{\sqrt{\beta}} + c(\gamma)$, $v := (\lambda + T^*)^{-1}f$, $p \in [2, p'_-)$. Then $v^{p/2} \in D(\tau) \cap Q(V^+)$.*

Proof. It is easy to see that $v \in D(\tau) \cap Q(V^+) \cap L^p$. Therefore $v_n := v \wedge n \in D(\tau) \cap Q(V^+) \cap L^\infty$. Since $p \geqslant 2$ the functions $v_n^{p/2}$ and v_n^{p-1} are multiples of normal contractions of u, so $v_n^{p/2}, v_n^{p-1} \in D(\tau) \cap Q(V^+)$. Multiplying the equality $(\lambda + T^*)v = f$ scalarly in L^2 by v_n^{p-1} we have

$$\lambda\langle v, v_n^{p-1}\rangle + \mathbf{t}^*[v, v_n^{p-1}] = \langle f, v_n^{p-1}\rangle$$

or

$$\lambda\langle v, v_n^{p-1}\rangle + 4\frac{p-1}{p^2}\tau[v_n^{p/2}] + \frac{2(p-1)}{p}\langle bv_n^{p/2}, \nabla v_n^{p/2}\rangle + \langle V^+ v, v_n^{p-1}\rangle$$
$$= \langle V^- v, v_n^{p-1}\rangle + \langle f, v_n^{p-1}\rangle.$$

Using (24), Hölder's inequality for $\langle f, v_n^{p-1}\rangle$ and the fact that $0 \leqslant v_n \leqslant v \in L^p$ we obtain

$$(\lambda - \frac{c(\beta)}{p'\sqrt{\beta}})\|v_n^{p/2}\|_2^2 + \left(4\frac{p-1}{p^2} - \frac{2(p-1)}{p}\sqrt{\beta}\right)\tau[v_n^{p/2}] + \langle V^+ v_n^{p/2}, v_n^{p/2}\rangle$$
$$\leqslant \langle V^- v^p\rangle + \|f\|_p\|v\|_p^{p-1}.$$

Recall that $V^- \in L^\infty$, so that the RHS of the last inequality is finite. Note also that $v_n^{p/2} \to v^{p/2}$ in L^2. Therefore by Proposition B1 we conclude that $v^{p/2} \in D(\tau) \cap Q(V^+)$.

Proof of Claim 2. First notice that $\rho^{-1}T^*\rho u = f - \lambda u \in L^\nu$, $u^{\nu-1} \in L^{\nu'}$. Therefore

$$\langle \rho^{-1}T^*\rho u, u^{\nu-1}\rangle = \lim_{n\to\infty}\langle \rho^{-1}T^*\rho u, u_n^{\nu-1}\rangle = \lim_{n\to\infty}\mathbf{t}^*[\rho u, \rho^{-1}u_n^{\nu-1}],$$

where $u_n := u \wedge n \in D(\tau) \cap Q(V^+) \cap L^\infty$, so $\rho^{-1}u_n^{\nu-1} \in D(\mathbf{t}^*)$. As in the proof of Claim 1 we use the notation: $\Phi := \rho^{-1}\nabla\rho$, $\psi := u^{\nu/2}$, $\psi_n := u_n^{\nu/2}$. A straightforward computation gives

$$\mathbf{t}^*[\rho u, \rho^{-1}u_n^{\nu-1}] = 4\frac{\nu-1}{\nu^2}\tau[\psi_n] + 2\frac{\nu-1}{\nu}\langle\nabla\psi_n \cdot a \cdot \Phi\psi\rangle - \langle\nabla u \cdot a \cdot \Phi, u_n^{\nu-1}\rangle$$
$$- \langle\Phi \cdot a \cdot \Phi u, u_n^{\nu-1}\rangle + 2\frac{\nu-1}{\nu}\langle\nabla\psi_n, b\psi_n\rangle + \langle\Phi u, bu_n^{\nu-1}\rangle$$
$$+ \langle V^+ u, u_n^{\nu-1}\rangle - \langle V^- u, u_n^{\nu-1}\rangle.$$

From this point proceeding along the same lines as in the proof of Claim 1 we obtain the inequality

$$\mathbf{t}^*[\rho u, \rho^{-1}u_n^{\nu-1}] \geqslant \left(4\frac{\nu-1}{\nu^2} - 2\frac{\nu-1}{\nu}\sqrt{\beta} - 2\sqrt{\delta}\right)\tau[\psi_n] + \langle V^+\psi_n, \psi_n\rangle$$
$$- (3\sqrt{\delta} + \gamma)\tau[\psi] - 3\sqrt{\delta}\langle V^+\psi, \psi\rangle - c\|\psi\|_2^2,$$

where the constant c depends upon $\beta, c(\beta), c(\gamma), \delta, c_\delta$. Taking into account that $\tau[\psi_n] \to \tau[\psi]$ and $\langle V^+\psi_n, \psi_n\rangle \to \langle V^+\psi, \psi\rangle$ as $n \to \infty$, one can pass to the limit and complete the proof in the same manner as in Claim 1.

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