Gaussian bounds for propagators perturbed by potentials

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Abstract

We develop the perturbation theory for propagators, with the objective to prove Gaussian bounds. Let U be a strongly continuous propagator, i.e., a family of operators describing the solutions of a non-autonomous evolution equation, on an L_p-space, and assume that U is positive and satisfies Gaussian upper and lower bounds. Let V be a (time-dependent) potential satisfying certain Miyadera conditions with respect to U. We show that then the perturbed propagator enjoys Gaussian upper and lower bounds as well. To prepare the necessary tools, we extend the perturbation theory of strongly continuous propagators and the theory of absorption propagators.

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Introduction

The validity of Gaussian bounds for fundamental solutions of second-order parabolic equations with non-smooth coefficients goes back to the paper of Aronson [4]. Aronson studied the parabolic equation with measurable coefficients that depend on space and time variables. Besides a uniformly elliptic second order part and a first order part, a multiplication term (potential) was allowed that was supposed to satisfy a certain integrability property. Starting from the publication of [4] the topic has a rich history, and numerous extensions and generalisations have been obtained; cf. [11, 13, 7, 8, 37, 33, 20, 6, 27] for (subsets of) $\mathbb{R}^N$ and [18, 28, 9, 38, 14] for manifolds.

For the case of Schrödinger operators $-\Delta + V$, in [1] the Kato class of potentials was shown to be the appropriate class for the L_1-perturbation theory of the corresponding C_0-semigroups, and in [34] it was shown that the fundamental solution of the perturbed heat equation still satisfies upper and lower Gaussian estimates. However, it was only in the last years that parabolic equations with more general time-dependent potentials were studied and the non-autonomous Kato class as the proper extension of the Kato class was introduced; cf. [44], [45], [31], [15], [16].

Essentially following [44] we say that a potential $V \in L_{1,loc}((0,\infty)\times\mathbb{R}^N)$ is in the *non-autonomous Kato class* **NK** if

$$N^-(V) := \lim_{\alpha\to 0} N_\alpha^-(V) = 0 \quad \text{and} \quad N^+(V) := \lim_{\alpha\to 0} N_\alpha^+(V) = 0,$$

where

$$N_\alpha^\pm(V) = \sup_{x,s}\int_0^\alpha \int_{\mathbb{R}^N} k_t(x-y)|V(s\pm t, y)|\,dy\,dt \qquad (0<\alpha\leqslant\infty),$$

with the free heat kernel $k_t(x) = (4\pi t)^{-N/2}e^{-\frac{|x-y|^2}{4t}}$, and $V(\tau,\cdot) := 0$ for $\tau < 0$. We say that V is in the *enlarged* non-autonomous Kato class $\widehat{\mathbf{NK}}$ if $N^-(V) < \infty$ and $N^+(V) < \infty$. (Observe that this differs from [31] where only $N^+(V) < \infty$ was required). Note that $N^+(V) = N^-(V)$ in the case of time-independent V. In fact, **NK** just reduces to the Kato class in this case, and $\widehat{\mathbf{NK}}$ to the extended Kato class (cf. [31]).

In [45], Zhang studied the fundamental solution u_V of the parabolic equation

$$u_t = \nabla\cdot(a(t,x)\nabla u) - V(t,x)u. \tag{0.1}$$

He proved that if a is uniformly elliptic and $V_\varkappa := V(\cdot/\varkappa,\cdot) \in \mathbf{NK}$ for a suitable $\varkappa > 0$ then there are $c_1, c_2 > 1$ such that u_V satisfies the two-sided Gaussian bounds

$$\frac{1}{c_1}(t-s)^{-N/2}e^{-c_2\frac{|x-y|^2}{(t-s)}} \leqslant u_V(x,t;y,s) \leqslant c_1(t-s)^{-N/2}e^{-\frac{|x-y|^2}{c_2(t-s)}} \tag{0.2}$$

for $s < t \leqslant s+1$. (Analogous estimates were shown for parabolic equations with sub-elliptic principal part.) Of course the above bounds can be extended to all $t \geqslant s$ by means of the reproducing kernel property. However, this leads to an additional factor $e^{\omega t}$ in the upper bound, and $e^{-\omega t}$ in the lower bound. The bounds (0.2) are called *global* if they hold for all $t > s$.

As is observed in [31], for the well-posedness of the Cauchy problem for (0.1) in $L_1(\mathbb{R}^n)$ only the condition $N^+(V_\varkappa) < \varkappa/M$ is needed, where $\varkappa, M$ are such that $u_0(x,t;y,s) \leqslant Mk_{\varkappa(t-s)}(x-y)$. This is derived by means of the non-autonomous Miyadera perturbation theorem [25, Thm. 3.4]. In fact, the above condition implies that V is a small Miyadera perturbation of the unperturbed propagator corresponding to the Cauchy problem for (0.1) with $V = 0$. However, a condition controlling $N^+(V_\varkappa)$ (a *forward* Kato condition) is not sufficient for obtaining Gaussian bounds: Zhang's result also requires a *backward* Kato condition, i.e., a condition on $N^-(V_\varkappa)$. The latter condition is responsible for the well-posedness of the adjoint Cauchy problem

$$-u_t = \nabla\cdot(a^T(t,x)\nabla u) - V(t,x)u, \qquad u(T,\cdot) = f$$

in $L_1(\mathbb{R}^n)$ (see Section 3.1).

It was noted in [45] that the Gaussian bounds (0.2) for the fundamental solution of (0.1) remain valid if $V_\varkappa \in \widehat{\mathbf{NK}}$ and the quantities $N^\pm(V_\varkappa)$ are sufficiently small. The results of the present paper imply that the upper bound in (0.2) holds if the negative part V^- of V satisfies $N^\pm(V_\varkappa^-) < \varkappa/M$ (with $\varkappa, M$ as in the previous paragraph); cf. Theorem 3.10. It follows from the general approach that we present in this paper that no restrictions on the positive part V^+ of V are needed. The lower bound in (0.2) is established under the assumption $N^\pm(V^+_\varkappa) < \infty$ and the condition (on V^-) that the Cauchy problem for (0.1) is well-posed in $L_1(\mathbb{R}^n)$; cf. Theorem 3.12. In order to obtain global bounds (0.2) we require the same conditions as above but with $N^\pm_\infty$ in place of $N^\pm$. For the upper bound this essentially is proved in [45]. The global lower bound under these conditions is new. The corresponding result in the autonomous case is due to Semenov; see [32, (6.6)].

The aim of this paper is to develop a perturbation method that enables one to derive the above results in the general framework of positive propagators on $L_p(\mu)$-spaces, thus allowing for a much wider range of applications. A (linear) propagator, or evolution family, describes the time evolution of a system with time-dependent driving force. The (additive time-dependent) perturbation we consider can be thought of as an operator addition to the local infinitesimal generators of the propagator. Since the latter do not exist, in general, the perturbed propagators are constructed by requiring the validity of a Duhamel formula. We refer to the introduction of [25] for a more extensive discussion.

This approach using the Duhamel formula works rather directly in the case of bounded perturbations. In order to include more general perturbations we apply the Miyadera perturbation theorem for C_0-semigroups. The connection between the theory of C_0-semigroups and (non-autonomous) propagators is established by the use of evolution semigroups. This method was first introduced by Howland ([17]) and further developed in [12], [22]; see also [30] and references therein. For the special case of positivity preserving propagators on L_p-spaces and perturbations by potentials, one can still go further by approximating a general potential by bounded potentials and using monotonicity. This approach, known as the theory of absorption semigroups and absorption propagators, is well-established for C_0-semigroups ([42], [43], [2], [21]). So far, in the non-autonomous case only perturbations by positive potentials have been studied; see [24]. In the present paper we develop the theory for sign-changing potentials and provide the link to the evolution semigroup approach; cf. subsection 2.1.

We further develop the perturbation theory for propagators U acting simultaneously on all L_p spaces, $1 \leqslant p \leqslant \infty$. In this case, using Stein interpolation we show that it suffices to require the potential V to be a small Miyadera perturbation of U, with bound γ, in order to obtain that U_V extends to a strongly continuous propagator on L_p, $1 \leqslant p < 1/\gamma$ (see Remark 3.5(b)).

In order to obtain a Gaussian upper bound for the perturbed propagator U_V one first needs to establish an L_1–L_∞ estimate, which is a diagonal upper bound for the integral kernel. Assuming that the unperturbed propagator U on L_1 satisfies an L_1–L_∞ estimate, and V is a small Miyadera perturbation of U, with bound γ, we obtain an L_1–L_κ estimate (with $1 < \kappa < 1/\gamma$) for the perturbed propagator U_V; cf. Proposition 3.3. Under the additional assumption that V satisfies a backward Miyadera condition (see (3.3) below), we also obtain an $L_{\kappa'}$–L_∞ estimate. From these two estimates together we deduce the desired L_1–L_∞ estimate by an iteration argument. The Gaussian upper bound for the perturbed propagator then follows from the upper bound for the unperturbed propagator and the diagonal bound for the perturbed one. We achieve this by means of an interpolation inequality for propagators that is well-known for the case of Schrödinger semigroups (see Proposition 3.8). We provide a purely analytical argument for the proof of this inequality, based on complex interpolation, thus avoiding previously used probabilistic tools or the Trotter product formula (for the latter see [32, (6.2)], [3, Thm. 2.4]).

By the same interpolation inequality we obtain a Gaussian lower bound for the perturbed propagator under the assumption that the unperturbed propagator satisfies a diagonal upper bound and a Gaussian lower bound.

The paper is organised as follows. In Section 1 we recall the notions and further develop the theory of propagators (or evolution families) on Banach spaces and the corresponding evolution semigroups. We provide a short introduction to bounded perturbations of propagators. The main part of Section 1, i.e. subsection 1.3, consists in a review of the Miyadera perturbation theorem for C_0-semigroups and its application to evolution semigroups and propagators. The main result of Section 1, Theorem 1.16, is a slight improvement of [25, Theorem 3.4]. We provide a new and more structured proof of this result, singling out important equivalences between relations occurring in the propagator context and in the semigroup context; cf. Theorems 1.12 and 1.14. The final subsection 1.4 of Section 1 deals with positive perturbations of positive propagators on Banach lattices.

Section 2 deals with absorption propagators. In subsection 2.1 we study the general theory, in subsection 2.2 we apply the results of subsection 1.3 to potentials and show, in particular, that the two approaches are consistent.

Section 3 contains the main results of the paper, namely the stability of Gaussian upper and lower bounds under perturbation by potentials. In subsection 3.1 we show the diagonal upper bound, in subsection 3.2 the Gaussian upper bound, and in subsection 3.3 the Gaussian lower bound.

1 Propagators, evolution semigroups, and perturbation theory

1.1 Propagators and evolution semigroups

Let $J \subseteq \mathbb{R}$ be an interval, $D_J := \{(t,s) \in J \times J;\ t \geqslant s\}$. Let X be a (real or complex) Banach space. A *propagator* on X (with *parameter interval* J) is a function $U\colon D_J \to \mathcal{L}(X)$ satisfying

(P1) $U(t,t) = I,\ U(t,r) = U(t,s)\,U(s,r)$ for all $t \geqslant s \geqslant r$ in J.

If additionally

(P2) $U\colon D_J \to \mathcal{L}(X)$ is strongly continuous

then $U = (U(t,s))_{(t,s)\in D_J}$ is called a *strongly continuous (s.c.) propagator.*

The next lemma is useful in applications for verifying the strong continuity of propagators. For $t \in \mathbb{R}$ we use the notation $J_{\geqslant t} := \{s \in J; s \geqslant t\}$, and analogously $J_{>t}$, $J_{\leqslant t}$, $J_{<t}$.

1.1 Lemma. *([16, Thm. 2.2]) Let U be a propagator on X. Suppose that*

(i) *U is locally bounded;*

(ii) *for any $s \in J$ the mapping $J_{\geqslant s} \ni t \mapsto U(t,s)$ is strongly continuous;*

(iii) *for any $t \in J$ the mapping $J_{\leqslant t} \ni s \mapsto U(t,s)$ is strongly continuous at $s = t$.*

Then U is strongly continuous.

Proof. Fix $(t_0,s_0) \in D_J$. Let $J_0 := J_{<s_0}$ ($J_0 := \{s_0\}$ in case $s_0 = \inf J$). Due to (iii) the set $\bigcup_{r\in J_0} U(s_0,r)X$ is dense in X. Thus, due to (i), it suffices to show that the mapping $D_J \ni (t,s) \mapsto U(t,s)x_0$ is continuous at (t_0,s_0), for $x_0 := U(s_0,r)x$ with $r \in J_0,\ x \in X$.

Let $t_n \to t_0,\ s_n \to s_0$. Without restriction $s_n \geqslant r$ for all $n \in \mathbb{N}$. Denote $x_n := U(s_n, r)x$. Condition (ii) implies that $x_n \to x_0$. Using (i) and (ii) again we conclude that

$$U(t_n, s_n)x_0 = U(t_n, r)x + U(t_n, s_n)(x_0 - x_n) \to U(t_0, r)x = U(t_0, s_0)x_0. \qquad \square$$

The uniform boundedness theorem implies that s.c. propagators are locally bounded. The following example shows that separate strong continuity of a propagator in not sufficient to obtain local boundedness (as was claimed in [5, Def. 2.32]). In particular it shows that condition (i) of Lemma 1.1 is needed for the conclusion.

1.2 Example. We indicate an example on the Hilbert space ℓ_2.

Let $\big((a_n, b_n)\big)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint subintervals of $[0,1]$, and for each $n \in \mathbb{N}$ let $a_n < s_n < t_n < b_n$. For $n \in \mathbb{N}$ let $\varphi_n\colon [0,1] \to [\frac{1}{n}, 1]$ be a continuous function, $\varphi_n = 1$ on $[0, a_n]$, $\varphi_n(s_n) = \frac{1}{n}$, $\varphi_n(t_n) = 1$, $\varphi_n = \frac{1}{n}$ on $[b_n, 1]$. Define U by

$$U(t,s)\big((x_n)_n\big) := \left(\frac{\varphi_n(t)}{\varphi_n(s)} x_n\right)_n,$$

for $0 \leqslant s \leqslant t \leqslant 1,\ (x_n)_n \in \ell_2$.

Then U is a propagator on ℓ_2 with parameter interval $[0,1]$. Further, $[0,t] \ni s \mapsto U(t,s)$ is strongly continuous for all $t \in [0,1]$, and $[s,1] \ni t \mapsto U(t,s)$ is strongly continuous for all $s \in [0,1]$. However, $\|U(t_n, s_n)\| = n$ for all $n \in \mathbb{N}$, so U is not bounded.

1.3 Remark. Note that, in the previous example, the unboundedness of U occurs close to the diagonal. In fact, it is easy to see that a separately strongly continuous propagator U with parameter interval J is locally bounded on $\big\{(t,s) \in D_J;\ t - s \geqslant \delta\big\}$, for all $\delta > 0$.

For the following definition we refer to [15, Def. 3]. A *backward propagator* on X is a function $W\colon D_J^* \to \mathcal{L}(X)$, where $D_J^* := \{(s,t) \in J \times J;\ s \leqslant t\}$, satisfying

$$W(t,t) = I, \quad W(r,t) = W(r,s)\, W(s,t) \qquad (r \leqslant s \leqslant t \text{ in } J).$$

1.4 Remarks. (a) W is a backward propagator if and only if $\widetilde{W}\colon D_{-J} \to \mathcal{L}(X)$, defined by $\widetilde{W}(t,s) := W(-t,-s)$ $((t,s) \in D_{-J})$ is a propagator.

(b) Let U be a propagator on X. Define $U_*\colon D_J^* \to \mathcal{L}(X^*)$ by $U_*(s,t) := U(t,s)^*$. Then U_* is a backward propagator on X^*. This shows that taking the adjoint of a propagator is connected with time reversal.

Let U be a s.c. propagator on X. If U is *exponentially bounded*, i.e., there exist $M \geqslant 1$, $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leqslant Me^{\omega(t-s)} \qquad \big((t,s) \in D_J\big), \tag{1.1}$$

then with the propagator U one can associate an *evolution semigroup* $\mathcal{T} = (\mathcal{T}(t))_{t\geqslant 0}$ on $L_p(J; X)$ by

$$\mathcal{T}(t)f(s) := \mathbb{1}_J(s-t)\, U(s, s-t) f(s-t).$$

From (1.1) one obtains $\|\mathcal{T}(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. In the present paper we use evolution semigroups on $L_1(J; X)$ only.

We refer to [17], [12], [22], [30] for the concept of evolution semigroups, properties and historical remarks.

1.5 Remarks. (a) It is important to note that the correspondence between s.c. propagators and evolution semigroups is one-to-one: If U and $\tilde{U}$ are s.c. propagators giving rise to the same evolution semigroup $\mathcal{T}$ then $U = \tilde{U}$. Indeed, let $x \in X$. For $\varphi \in C_c^1(\mathbb{R})$, $f := \varphi x \, (\in L_1(J;X))$, $t \geqslant 0$ we have

$$\mathbb{1}_J(s-t)\varphi(s-t)\,\tilde{U}(s,s-t)x = \mathcal{T}(t)f(s) = \mathbb{1}_J(s-t)\varphi(s-t)\,U(s,s-t)x$$

for a.e. $s \in J$. Thus $\tilde{U}(t,s)x = U(t,s)x$ for a.e. $(t,s) \in D_J$ and hence, due to the strong continuity, for all $(t,s) \in D_J$. (See also [22, Thm. 4.9].)

(b) For many purposes, only the time local behaviour of propagators is of interest. In these cases, (exponential) boundedness can be assumed without loss of generality, by replacing the interval J by a compact subinterval.

1.2 Bounded perturbations of propagators

In this subsection we describe the perturbation of propagators by bounded operator functions. As above, let U be a s.c. propagator on X with parameter interval $J \subseteq \mathbb{R}$. Let $B\colon J \to \mathcal{L}(X)$ be strongly measurable and bounded.

The following considerations show that there exists a unique s.c. propagator U_B with parameter interval J satisfying the Duhamel formula

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau, \tag{1.2}$$

for all $(t,s) \in J$, $x \in X$. The standard contraction principle argument shows uniqueness. The solution by iteration leads to the terms

$$U_0(t,s) := U(t,s), \quad U_j(t,s) := \int_s^t U_{j-1}(t,\tau)B(\tau)U(\tau,s)\,d\tau \quad (j = 1,\dots)$$

(strong integrals). Then it is standard to show that

$$U_B(t,s) := \sum_{j=0}^{\infty} U_j(t,s) \tag{1.3}$$

defines a s.c. propagator U_B satisfying (1.2), where the series converges in operator norm, uniformly absolutely on compact subsets of D_J. (With $M := \sup_{s\leqslant\sigma\leqslant\tau\leqslant t}\|U(\tau,\sigma)\|$, $C := \sup_{\tau\in[s,t]}\|B(\tau)\|$ we obtain

$$\|U_j(t,s)\| \leqslant \frac{(t-s)^j}{j\,!}M^{j+1}C^j; \tag{1.4}$$

the propagator property is a consequence of the formula

$$U_j(t,r) = \sum_{k=0}^{j} U_{j-k}(t,s)\,U_k(s,r) \qquad (t \geqslant s \geqslant r \text{ in } J,\ j \in \mathbb{N}),$$

which is proved by induction.) We note that the terms $U_j(t,s)$ also satisfy

$$\begin{aligned} U_j(t,s) &= \int_{s\leqslant\tau_1\leqslant\ldots\leqslant\tau_j\leqslant t} U(t,\tau_j)B(\tau_j)U(\tau_j,\tau_{j-1})\ldots B(\tau_1)U(\tau_1,s)\,d(\tau_1,\ldots,\tau_j) \\ &= \int_s^t U(t,\tau)B(\tau)U_{j-1}(\tau,s)\,d\tau. \end{aligned} \tag{1.5}$$

From this representation one immediately concludes the second Duhamel formula

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\,d\tau \tag{1.6}$$

for all $(t,s) \in J,\ x \in X$.

The existence and uniqueness of U_B was shown in [24, Thm. 2.1] by evolution semigroup methods. The method presented above is used in the proof of the following result and in the proof of Proposition 3.6.

1.6 Proposition. *Let U and B be as above. Then $z \mapsto U_{zB}(t,s)$ is analytic, for all $(t,s) \in D_J$.*

Proof. This is immediate from the series expansion

$$U_{zB}(t,s) = \sum_{j=0}^{\infty} z^j U_j(t,s). \qquad \square$$

1.7 Remark. The following two properties of bounded perturbations can be obtained from uniqueness in Duhamel's formula (or from evolution semigroup considerations).

(a) If $c \in \mathbb{K}$ and B is the constant function $B(\tau) = cI$, then $U_B(t,s) = e^{c(t-s)}U(t,s)$ $((t,s) \in D_J)$.

(b) For a second function B_1 (with the same properties as B) we have

$$(U_B)_{B_1} = U_{B+B_1}\ (= (U_{B_1})_B).$$

This shows that formula (1.6) can also be derived from formula (1.2): consider U as obtained as a perturbation of U_B by $-B$.

The following convergence result for perturbed propagators strengthens [24, Prop. 2.3(b)]. It will be needed in the proof of Theorem 2.7.

1.8 Proposition. *Let U, B be as above, and let $(B_n)_{n\in\mathbb{N}}$ be a sequence of bounded, strongly measurable functions $B_n\colon J \to \mathcal{L}(X)$ with $c := \sup_{t\in J,\, n\in\mathbb{N}} \|B_n(t)\| < \infty$, $B_n(t) \to B(t)$ strongly for a.e. $t \in J$.*

Then $U_{B_n}(t,s) \to U_B(t,s)$ strongly as $n \to \infty$, uniformly for (t,s) in compact subsets of D_J.

Proof. By Remark 1.7(b) we have $U_{B_n} = (U_B)_{B_n - B}$, so without restriction we can assume $B = 0$. Let $(t_0, s_0) \in D_J,\ x \in X$. Then

$$C := \bigl\{U(t,s)x;\ s_0 \leqslant s \leqslant t \leqslant t_0\bigr\}$$

is a compact subset of X. This implies that $B_n(t)y \to B(t)y = 0$ $(n \to \infty)$ uniformly for $y \in C$, for a.e. $t \in J$. Therefore,

$$\int_{s_0}^{t_0} \sup_{y\in C} \|B_n(\tau)y\|\,d\tau \to 0 \qquad (n \to \infty),$$

by the dominated convergence theorem. (Note that there exists a countable dense subset C_0 of C, so

$$\tau \longmapsto \sup_{y\in C} \|B_n(\tau)y\| = \sup_{y\in C_0} \|B_n(\tau)y\|$$

is measurable.) We obtain that

$$\begin{aligned}\|U_{B_n}(t,s)x - U(t,s)x\| &\leqslant \int_s^t \|U_{B_n}(t,\tau)B_n(\tau)U(\tau,s)x\|\,d\tau \\ &\leqslant \sup_{\tau\in[s,t]}\|U_{B_n}(t,\tau)\|\int_{s_0}^{t_0}\sup_{y\in C}\|B_n(\tau)y\|\,d\tau.\end{aligned}$$

With $M := \sup_{s_0\leqslant s\leqslant t\leqslant t_0}\|U(t,s)\|$ we have $\|U_{B_n}(t,\tau)\| \leqslant Me^{Mc(t_0-s_0)}$ by (1.4), so the assertion follows. □

1.3 Unbounded perturbations and the Miyadera perturbation theorem

In this subsection we discuss a non-autonomous version of the Miyadera perturbation theorem and related issues. The main subject is the relation between s.c. propagators and the corresponding evolution semigroups.

For the convenience of the reader, we start with a formulation of the Miyadera perturbation theorem for semigroups (see [41, 42, 25]).

Let T be a C_0-semigroup on a Banach space E, with generator G. Let B be a G-bounded operator. Then B is called a *Miyadera perturbation* of T (*with constants* (α,γ)) if there exist $\alpha\in(0,\infty]$, $\gamma\geqslant 0$ such that

$$\int_0^\alpha \|BT(t)f\|\,dt \leqslant \gamma\|f\| \tag{1.7}$$

for all $f\in D(G)$. The operator B is called a *small* Miyadera perturbation if α, γ can be chosen such that $\gamma<1$, and *infinitesimally small* if the infimum of all possible γ is zero.

For a Miyadera perturbation B of T it is easy to see that for all $t>0$ there exists $c>0$ such that

$$\int_0^t \|BT(\tau)f\|\,d\tau \leqslant c\|f\| \tag{1.8}$$

for all $f\in D(G)$ (cf. [41, p. 168]).

The G-boundedness of B can be replaced by a slightly weaker hypothesis. The fact that B has a Miyadera bounded extension under this weaker hypothesis is in fact part of the Miyadera perturbation theorem as proved in [41]. We single out this part in the following lemma.

1.9 Lemma. *Let E, T, G be as above, and let $D\subseteq D(G)$ be a dense subspace of E with $T(t)D\subseteq D$ $(t\geqslant 0)$. Assume that B is a linear operator with $D(B)=D$ such that*

(i) *$[0,\infty)\ni t\mapsto BT(t)f\in E$ is continuous for all $f\in D$;*

(ii) *there are constants $\alpha\in(0,\infty]$, $\gamma\geqslant 0$ such that* (1.7) *holds for all $f\in D$.*

Then B has a unique G-bounded extension $\hat{B}$ with $D(\hat{B})=D(G)$. Inequality (1.7) *holds for all $f\in D(G)$, with B replaced by $\hat{B}$, i.e., $\hat{B}$ is a Miyadera perturbation of T. If B is closable then $\hat{B}\subseteq\overline{B}$.*

(Note that the hypotheses imply that D is a core for G.)

1.10 Theorem. *Let E, T, G be as above, and let B be a small Miyadera perturbation of T. Then the following assertions hold:*

(a) *$G+B$ generates a C_0-semigroup T_B.*

(b) *For $f \in D(G)$ and $t \geqslant 0$ we have*

$$T_B(t)f = T(t)f + \int_0^t T_B(t-\tau)BT(\tau)f\,d\tau, \tag{1.9}$$

$$T_B(t)f = T(t)f + \int_0^t T(t-\tau)BT_B(\tau)f\,d\tau. \tag{1.10}$$

Moreover, T_B is the only C_0-semigroup satisfying (1.9).

We refer to [41, Thm. 1 and Rem. 2(a)] and [42, Thm. 1.1] for the proof of Lemma 1.9 and Theorem 1.10. (In fact, T_B in Theorem 1.10 is constructed as a solution of the Duhamel formula (1.9); then formula (1.10) is a consequence of part (a).)

For the remainder of this subsection we fix the following assumptions. Let X be a Banach space, U an exponentially bounded s.c. propagator on X with parameter interval J, $\mathcal{T}$ the corresponding evolution semigroup on $E := L_1(J;\, X)$, and $\mathcal{G}$ the generator of $\mathcal{T}$. Further, let $X_s \subseteq X$ $(s \in J)$ be dense subspaces such that $U(t,s)X_s \subseteq X_t$ for all $(t,s) \in D_J$. Let

$$\mathcal{D} := \operatorname{lin}\{J \ni s \mapsto \varphi(s)U(s,r)x;\ r \in J,\ x \in X_r,\ \varphi \in C_c^1(\mathbb{R}) \text{ with } \operatorname{spt}\varphi \subseteq [r,\infty)\} \subseteq E.$$

We emphasise that the space $\mathcal{D}$ depends on the choice of the family $(X_s)_{s\in J}$. Let $(B(t))_{t\in J}$ be a family of linear operators in X. We define a linear operator $\mathcal{B}$ in E by

$$D(\mathcal{B}) := \bigl\{f \in E;\ f(s) \in D(B(s)) \text{ for a.e. } s \in J,\ \mathcal{B}f := B(\cdot)f(\cdot) \in E\bigr\}.$$

1.11 Remarks. (a) Due to the strong continuity of U, the space $\mathcal{D}$ is dense in $C_c(\mathring{J};\, X)$ and hence in E. (Use a suitable partition of unity; cf. [29, proof of Prop. 1.13]). Let $r \in J$, $x \in X_r$, $\varphi \in C_c^1(\mathbb{R})$ with $\operatorname{spt}\varphi \subseteq [r,\infty)$, so that $f_0 := \varphi(\cdot)U(\cdot,r)x \in \mathcal{D}$. Then for all $t \geqslant 0$ we have

$$\mathcal{T}(t)f_0 = \mathbf{1}_J(\cdot - t)U(\cdot,\cdot - t)\varphi(\cdot - t)U(\cdot - t, r)x = \varphi(\cdot - t)U(\cdot,r)x \in \mathcal{D},$$

so $\mathcal{T}(t)\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D} \subseteq D(\mathcal{G})$. (As a consequence, $\mathcal{D}$ is a core for $\mathcal{G}$.)

(b) Assume that $\mathcal{D} \subseteq D(\mathcal{B})$. Then $\mathcal{B}f_0 = \varphi(\cdot)B(\cdot)U(\cdot,r)x \in E$, with f_0 as in (a). We infer that $B(\cdot)U(\cdot,r)x \in L_{1,loc}(J_{>r})$ for all $r \in J$, $x \in X_r$.

Moreover, $\mathcal{B}\mathcal{T}(t)f_0 = \varphi(\cdot - t)B(\cdot)U(\cdot,r)x$ for all $t \geqslant 0$. Thus, for all $f \in \mathcal{D}$ the function $[0,\infty) \ni t \mapsto \mathcal{B}\mathcal{T}(t)f \in E$ is continuous (even differentiable).

1.12 Theorem. *Let $\alpha \in (0,\infty]$, $\gamma \geqslant 0$. Then the following are equivalent.*

(i) *For all $s \in J$, $x \in X_s$ the function $J_{\geqslant s} \ni t \mapsto B(t)U(t,s)x$ is defined almost everywhere and measurable, and*

$$\int_s^{s+\alpha} \mathbf{1}_J(t)\|B(t)U(t,s)x\|\,dt \leqslant \gamma\|x\|. \tag{1.11}$$

If this condition is satisfied, we will say that $(B(t))_{t\in J}$ is a Miyadera perturbation *of U (*with constants (α,γ)*), and that γ is a* Miyadera bound.

(ii) *$\mathcal{D} \subseteq D(\mathcal{B})$ and, for all $f \in \mathcal{D}$,*

$$\int_0^\alpha \|\mathcal{B}\mathcal{T}(t)f\|_E\,dt \leqslant \gamma\|f\|_E.$$

(Under this condition, the operator $\widehat{\mathcal{B}|_{\mathcal{D}}}$ from Lemma 1.9 is a Miyadera perturbation of $\mathcal{T}$; recall from Remark 1.11(b) that $t \mapsto \mathcal{B}\mathcal{T}(t)f$ is continuous.)

Proof. '(i)⇒(ii)' is shown in [25, proof of Thm. 3.4(a)].

'(ii)⇒(i)'. Let $s \in J,\ x \in X_s$. For $\varepsilon \in (0,\alpha)$ let $0 \leqslant \varphi_\varepsilon \in C_c^1(\mathbb{R})$ such that $\operatorname{spt}\varphi_\varepsilon \subseteq [s, s+\varepsilon]$, $\int \varphi_\varepsilon = 1$, and let $f_\varepsilon := \varphi_\varepsilon(\cdot)U(\cdot, s)x\,(\in \mathcal{D})$.

From Remark 1.11(b) we obtain that the function

$$J \times [0,\alpha] \ni (t,\tau) \mapsto \mathcal{B}\mathcal{T}(\tau)f_\varepsilon(t) = \varphi_\varepsilon(t-\tau)B(t)U(t,s)x$$

is measurable, so

$$\begin{aligned}\int_J \int_0^\alpha \varphi_\varepsilon(t-\tau)\,d\tau\,\|B(t)U(t,s)x\|\,dt &= \int_0^\alpha \int_J \|\varphi_\varepsilon(t-\tau)B(t)U(t,s)x\|\,dt\,d\tau \\ &= \int_0^\alpha \|\mathcal{B}\mathcal{T}(\tau)f_\varepsilon\|_E\,d\tau \leqslant \gamma\|f_\varepsilon\|_E.\end{aligned}$$

The assumptions on φ_ε imply that $\int_0^\alpha \varphi_\varepsilon(t-\tau)\,d\tau = 1$ for $t \in [s+\varepsilon, s+\alpha]$. Therefore,

$$\int_J \mathbf{1}_{[s+\varepsilon,s+\alpha]}(t)\|B(t)U(t,s)x\|\,dt \leqslant \gamma \int_J \varphi_\varepsilon(t)\|U(t,s)x\|\,dt.$$

Letting $\varepsilon \to 0$ we obtain (1.11). □

1.13 Remark. It is interesting to note that $\widehat{\mathcal{B}|_\mathcal{D}}$ is a Miyadera perturbation of $\mathcal{T}$ if and only if $\widehat{\mathcal{B}|_\mathcal{D}}$ is $\mathcal{G}$-bounded. Indeed, necessity is clear, so assume that $\widehat{\mathcal{B}|_\mathcal{D}}$ is $\mathcal{G}$-bounded. Then there are $\beta, c \geqslant 0$ such that $\|\mathcal{B}f\|_1 \leqslant \beta\|\mathcal{G}f\|_1 + c\|f\|_1$ for all $f \in \mathcal{D}$. Let $\alpha > 0$. For $s \in J,\ x \in X_s$, $\varphi \in C_c^1(\mathbb{R}),\ 0 \leqslant \varphi \leqslant \mathbf{1}_{[s,s+\alpha]},\ \|\varphi'\|_1 \leqslant 2,\ f := \varphi(\cdot)U(\cdot,s)x$ we have $\mathcal{G}f = -\varphi'(\cdot)U(\cdot,s)x$ by Remark 1.11(a) and hence

$$\begin{aligned}\int_J \varphi(t)\|B(t)U(t,s)x\|\,dt = \|\mathcal{B}f\|_1 &\leqslant \beta \int_J |\varphi'(t)|\|U(t,s)x\|\,dt + c\int_J \varphi(t)\|U(t,s)x\|\,dt \\ &\leqslant (2\beta + c\alpha) \sup_{s\leqslant t\leqslant s+\alpha} \|U(t,s)x\|.\end{aligned}$$

Now choose $(\varphi_n) \subseteq C_c^1(\mathbb{R})$ such that $0 \leqslant \varphi_n \uparrow \mathbf{1}_{[s,s+\alpha]},\ \|\varphi_n'\|_1 \leqslant 2$. Then we obtain (1.11) with $\gamma = (2\beta + c\alpha)\sup_{0\leqslant t-s\leqslant\alpha} \|U(t,s)\|$.

For $t \in J,\ t > \inf J$ we define

$$X_{t-} := \bigcup_{s\in J_{<t}} U(t,s)X_s.$$

Note that $X_{t-} \subseteq X_t,\ U(t,s)X_{s-} \subseteq X_{t-}$, and that X_{t-} is a dense subspace of X, for all $(t,s) \in D_J,\ s > \inf J$.

1.14 Theorem. *Let W be an exponentially bounded s.c. propagator on X, $\mathcal{S}$ the corresponding evolution semigroup on E. Let U, $\mathcal{T}$, $\mathcal{B}$ be as above and assume that $\mathcal{D} \subseteq D(\mathcal{B})$. Then the following are equivalent.*

(i) *For all $(t,s) \in D_J,\ s > \inf J,\ x \in X_{s-}$:*

$$W(t,s)x = U(t,s)x + \int_s^t W(t,\tau)B(\tau)U(\tau,s)x\,d\tau. \tag{1.12}$$

(ii) *For all* $t \geqslant 0,\ f \in \mathcal{D}$*:*

$$\mathcal{S}(t)f = \mathcal{T}(t)f + \int_0^t \mathcal{S}(t-\tau)\mathcal{B}\mathcal{T}(\tau)f\, d\tau.$$

(The assumption $\mathcal{D} \subseteq D(\mathcal{B})$ *implies that the integrals in* (i) *and* (ii) *are defined; see Remark 1.11(b).)*

If (ii) *holds and, in addition,* $B(\cdot)U(\cdot,s)x \in L_{1,loc}(J_{\geqslant s})$ *for all* $s \in J,\ x \in X_s$*, then* (1.12) *holds for all* $(t,s) \in D_J,\ x \in X_s$*. This additional assumption is satisfied if* $(B(t))_{t\geqslant 0}$ *is a Miyadera perturbation of* U*.*

Proof. We first show that (ii) is equivalent to

(ii') For all $t \geqslant 0,\ r \in J,\ x \in X_r$, and a.e. $s > r$ with $s+t \in J$:

$$W(s+t,s)U(s,r)x = U(s+t,r)x + \int_s^{s+t} W(s+t,\tau)B(\tau)U(\tau,r)x\, d\tau. \tag{1.13}$$

Let $t \geqslant 0,\ r \in J,\ x \in X_r$. By Remark 1.11(b), $\mathbb{1}_{J_{>r}}(\cdot)B(\cdot)U(\cdot,r)x$ is measurable. Since W is strongly continuous we obtain that

$$v\colon D_J \to X, \quad v(s,\tau) := \mathbb{1}_{J_{>r}}(\tau)W(s,\tau)B(\tau)U(\tau,r)x$$

is measurable.

Let now $\varphi \in C_c^1(\mathbb{R})$ with $\operatorname{spt}\varphi \subseteq [r,\infty),\ f := \varphi(\cdot)U(\cdot,r)x\,(\in\mathcal{D})$. Let $0 \leqslant \tau \leqslant t$. Then $\mathcal{B}\mathcal{T}(\tau)f = \varphi(\cdot-\tau)B(\cdot)U(\cdot,r)x$, so

$$\begin{aligned}\mathcal{S}(t-\tau)\mathcal{B}\mathcal{T}(\tau)f &= \mathbb{1}_J(\cdot-t+\tau)W(\cdot,\cdot-t+\tau)\varphi(\cdot-t)B(\cdot-t+\tau)U(\cdot-t+\tau,r)x\\ &= \varphi(\cdot-t)v(\cdot,\cdot-t+\tau) =: u(\tau,\cdot),\end{aligned}$$

with $u\colon [0,t]\times J \to X$ measurable, $u(\tau,\cdot) \in E$ for all $\tau \in [0,t]$. Moreover,

$$\mathcal{S}(t)f = \mathbb{1}_J(\cdot-t)W(\cdot,\cdot-t)\varphi(\cdot-t)U(\cdot-t,r)x = \varphi(\cdot-t)W(\cdot,\cdot-t)U(\cdot-t,r)x.$$

By the above computations, (ii) is equivalent to

$$\varphi(\cdot-t)W(\cdot,\cdot-t)U(\cdot-t,r)x = \varphi(\cdot-t)U(\cdot,r)x + \int_0^t u(\tau,\cdot)\, d\tau \tag{1.14}$$

for all $t \geqslant 0,\ r \in J,\ x \in X_r,\ \varphi \in C_c^1(\mathbb{R})$ with $\operatorname{spt}\varphi \subseteq [r,\infty)$. Since u is measurable,

$$\Big(\int_0^t u(\tau,\cdot)\, d\tau\Big)(s) = \int_0^t u(\tau,s)\, d\tau = \varphi(s-t)\int_{s-t}^s v(s,\tau)\, d\tau$$

for a.e. $s \in J$. We conclude that (1.14) holds for all $\varphi \in C_c^1(\mathbb{R})$ with $\operatorname{spt}\varphi \subseteq [r,\infty)$ if and only if

$$W(s,s-t)U(s-t,r)x = U(s,r)x + \int_{s-t}^s v(s,\tau)\, d\tau$$

for a.e. $s \in J$ with $s-t > r$, i.e., if (1.13) holds for a.e. $s > r$ with $s+t \in J$.

We now prove the equivalence of (i) and (ii'). Obviously, (i) implies (ii'), so assume that (ii') holds. It is easy to see that the terms in (1.13) depend continuously on s, so (1.13) holds for all $s > r$ with $s + t \in J$. For $r \in J,\ y \in X_r,\ (t,s) \in D_J$ with $s > r$ we thus obtain

$$W(t,s)U(s,r)y = U(t,s)U(s,r)x + \int_s^t W(t,\tau)B(\tau)U(\tau,s)U(s,r)y\,d\tau,$$

i.e., (i) holds.

Finally, assume that (ii) holds, and that $B(\cdot)U(\cdot,s)x \in L_{1,loc}(J_{\geqslant s})$ for all $s \in J,\ x \in X_s$. If $r \in J,\ x \in X_r,\ t \geqslant 0$ with $r + t \in \mathring{J}$ then, again by continuity in s, (1.13) holds for $s = r$. This proves (1.12) for all $(t,s) \in D_J$ with $t \in \mathring{J},\ x \in X_s$. To conclude the proof, observe that the terms in (1.12) are continuous in t. □

In the following lemma we need the hypothesis that J is left half open. This assumption will be avoided further on.

1.15 Lemma. *Assume that J is left open and right closed, that $\mathcal{B}|_{\mathcal{D}}$ has a $\mathcal{G}$-bounded extension $\widehat{\mathcal{B}|_{\mathcal{D}}}$, and that $\mathcal{G}+\widehat{\mathcal{B}|_{\mathcal{D}}}$ generates a C_0-semigroup $\mathcal{T}_{\mathcal{B}}$ on E. Then $\mathcal{T}_{\mathcal{B}}$ is an evolution semigroup.*

Proof. The assertion follows from [25, Thm. 2.4, (a)⇔(c)] (see also [22, Thm. 4.12] for the case of bounded J). For details we refer the reader to [25, middle of p. 352]; note that, by the closed graph theorem, condition (R) ([25, p. 345]) is equivalent to the requirement that $D(G)$ is a dense subset of $C_0(J)$. □

The essentials of the following result were shown in [25]. In fact, equation (1.15) was shown there only for $x \in X_{s-}$, and J was assumed to be right closed.

We give a new proof of this result, where in Theorems 1.12 and 1.14 we have singled out important equivalences between relations occuring in the propagator context and in the semigroup context.

1.16 Theorem. *([25, Thm. 3.4, Cor. 3.5]) Assume that $(B(t))_{t\in J}$ is a small Miyadera perturbation of U, i.e., property* (i) *of Theorem 1.12 holds for some $\alpha > 0,\ 0 \leqslant \gamma < 1$.*

(a) *Then there is a unique exponentially bounded s.c. propagator U_B on X satisfying*

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau \tag{1.15}$$

for all $(t,s) \in D_J$ and $x \in X_s$.

(b) *Assume in addition that $B(t)$ is closed for a.e. $t \in J$. Then property* (i) *of Theorem 1.12 holds with $X_t = X\ (t \in J)$; in particular,* (1.15) *holds for all $x \in X,\ (t,s) \in D_J$.*

Moreover, for $x \in X,\ s \in J$, the function $B(\cdot)U_B(\cdot,s)x$ is defined almost everywhere and locally integrable on $J_{\geqslant s}$, and

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\,d\tau \tag{1.16}$$

for all $x \in X,\ (t,s) \in D_J$.

The evolution semigroup associated with U_B has generator $\mathcal{G} + \mathcal{B}$.

Proof. (a) We first show that any s.c. propagator U_B satisfying (1.15) is exponentially bounded. Let $c := \sup\{\|U(t,s)\|;\ (t,s) \in D_J,\ t-s \leqslant \alpha\}$. Let $(t,s) \in D_J$ with $t-s \leqslant \alpha$, $M := \sup_{s\leqslant\tau\leqslant t} \|U_B(t,\tau)\|$. Then, for all $s \leqslant \sigma \leqslant t,\ x \in X_\sigma$,

$$\|U_B(t,\sigma)x\| \leqslant \|U(t,\sigma)x\| + M\int_\sigma^t \|B(\tau)U(\tau,\sigma)x\|\,d\tau \leqslant c\|x\| + M\gamma\|x\|,$$

so $\|U_B(t,\sigma)\| \leqslant c + M\gamma$ for all $s \leqslant \sigma \leqslant t$. Thus $M \leqslant \frac{c}{1-\gamma}$, and we conclude that U_B is exponentially bounded.

The uniqueness assertion is obtained by an analogous argument.

By Remark 1.11(a) and Theorem 1.12, the conditions of Lemma 1.9 are satisfied for $\mathcal{T}, \mathcal{G}, \mathcal{D}, \mathcal{B}|_\mathcal{D}$ as above. Hence, $\mathcal{B}|_\mathcal{D}$ has a $\mathcal{G}$-bounded extension $\widehat{\mathcal{B}|_\mathcal{D}}$, which is a small Miyadera perturbation of $\mathcal{T}$. Thus, by Theorem 1.10, $\mathcal{G} + \widehat{\mathcal{B}|_\mathcal{D}}$ generates a C_0-semigroup $\mathcal{T}_\mathcal{B}$ on E satisfying

$$\mathcal{T}_\mathcal{B}(t)f = \mathcal{T}(t)f + \int_0^t \mathcal{T}_\mathcal{B}(t-\tau)\mathcal{B}\mathcal{T}(\tau)f\,d\tau \tag{1.17}$$

for all $t \geqslant 0,\ f \in \mathcal{D}$.

For the case that J is left open and right closed, the assertion now follows from Lemma 1.15 and Theorem 1.14, (ii)$\Rightarrow$(i).

Next we study the case that J is right closed but $a := \inf J \in J$, i.e., J is not left open. Let $\tilde{J} := J \setminus \{a\}$. By the above we obtain a unique exponentially bounded s.c. propagator U_B with parameter interval $\tilde{J}$ satisfying (1.15) for all $(t,s) \in D_{\tilde{J}}$ and $x \in X_s$. We now extend this propagator to a propagator with parameter interval J, defining operators $U_B(s,a) \in \mathcal{L}(X)$ by

$$U_B(s,a)x := U(s,a)x + \int_a^s U_B(s,\tau)B(\tau)U(\tau,a)x\,d\tau \qquad (s \in J,\ x \in X_a). \tag{1.18}$$

Applying $U_B(t,s)$ to both sides of (1.18) we obtain that

$$U_B(t,s)U_B(s,a)x = U_B(t,s)U(s,a)x + \int_a^s U_B(t,\tau)B(\tau)U(\tau,a)x\,d\tau.$$

Now the propagator property $U_B(t,s)U_B(s,a) = U_B(t,a)$ easily follows from (1.15). By the dominated convergence theorem, $U_B(s,a)x \to x$ strongly as $s \to a$, for all $x \in X_a$. By Lemma 1.1 we conclude that U_B is strongly continuous on D_J.

Finally, assume that J is not right closed. Then there exists a sequence (J_n) of right closed subintervals of J with $\bigcup J_n = J$. By the above, for each n we obtain a unique perturbed propagator satisfying (1.15) for all $(t,s) \in D_{J_n}$ and $x \in X_s$. The uniqueness implies that two such perturbed propagators coincide on their common domain.

(b) Observe that $\mathcal{B}$ is closed due to our additional assumption. By Lemma 1.9 we obtain that $\mathcal{B}\,(\supseteq \widehat{\mathcal{B}|_\mathcal{D}})$ is a small Miyadera perturbation of $\mathcal{T}$ (in particular, $\mathcal{G} + \mathcal{B} = \mathcal{G} + \widehat{\mathcal{B}|_\mathcal{D}}$ is the generator of $\mathcal{T}_\mathcal{B}$). This means that the inequality in Theorem 1.12(ii) is satisfied for all $f \in D(\mathcal{G})$. Let now $\mathcal{D}$ be as above, but with $X_t = X$ $(t \in J)$. Then $\mathcal{D} \subseteq D(\mathcal{G})$, so we obtain that property (i) of Theorem 1.12 holds with $X_t = X$. Thus, by part (a), (1.15) holds for all $x \in X,\ (t,s) \in D_J$.

Let $\widetilde{\mathcal{T}}_\mathcal{B}$ be the evolution semigroup associated with U_B. By Theorem 1.14 we infer from the above that (1.17) also holds with $\mathcal{T}_\mathcal{B}$ replaced by $\widetilde{\mathcal{T}}_\mathcal{B}$, so the uniqueness assertion of Theorem 1.10 implies $\widetilde{\mathcal{T}}_\mathcal{B} = \mathcal{T}_\mathcal{B}$. We conclude that $\mathcal{G} + \mathcal{B}$ is the generator of $\widetilde{\mathcal{T}}_\mathcal{B}$.

By [23, Lemma 1.1], $\mathcal{B}$ is also a (not necessarily small) Miyadera perturbation of $\mathcal{T}_{\mathcal{B}}$, i.e., there exist $\alpha_1 > 0,\ c \geqslant 0$ such that

$$\int_0^{\alpha_1} \|\mathcal{B}\mathcal{T}_{\mathcal{B}}(\tau)f\|_E \, d\tau \leqslant c\|f\|_E$$

for all $f \in D(\mathcal{G})$. Moreover, by Theorem 1.10(b) we obtain

$$\mathcal{T}_{\mathcal{B}}(t)f = \mathcal{T}(t)f + \int_0^t \mathcal{T}(t-\tau)\mathcal{B}\mathcal{T}_{\mathcal{B}}(\tau)f \, d\tau$$

for all $t \geqslant 0,\ f \in D(\mathcal{G})$. The remaining assertions of (b) now follow from Theorems 1.12 and 1.14, applied with $\mathcal{D}\,(\subseteq D(\mathcal{G}))$ constructed from U_B instead of U, and $X_t = X$. □

1.17 Remark. (a) If in Theorem 1.16 the operators $B(t)$ are not closable then the perturbed propagator U_B may depend on the choice of the spaces X_s.

(b) We note that one can also prove Theorem 1.16 without using evolution semigroups; cf. [40].

We conclude this subsection with a result that is used to establish strong continuity in cases when both Duhamel formulas hold. In this result we assume X_s to be a dense subspace of X, for all $s \in J$, but we do not require $U(t,s)X_s \subseteq X_t$.

1.18 Proposition. *Let W be a locally bounded propagator on X. Assume that, for all $(t,s) \in D_J,\ x \in X_s$, the functions $B(\cdot)W(\cdot,s)x,\ W(t,\cdot)B(\cdot)U(\cdot,s)x$ are integrable on $[s,t]$, and*

$$W(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)W(\tau,s)x \, d\tau, \tag{1.19}$$

$$W(t,s)x = U(t,s)x + \int_s^t W(t,\tau)B(\tau)U(\tau,s)x \, d\tau. \tag{1.20}$$

Then W is strongly continuous.

Proof. This is an immediate consequence of Lemma 1.1 and the following lemma. □

1.19 Lemma. *Let W be a locally bounded propagator on X.*

(a) *Assume that, for all $(t,s) \in D_J,\ x \in X_s$, the function $B(\cdot)W(\cdot,s)x$ is integrable on $[s,t]$, and* (1.19) *holds. Then $W(\cdot,s)$ is strongly continuous on $J_{\geqslant s}$ for all $s \in J$.*

(b) *Assume that, for all $(t,s) \in D_J,\ x \in X_s$, the function $W(t,\cdot)B(\cdot)U(\cdot,s)x$ is integrable on $[s,t]$, and* (1.20) *holds. Then $W(t,\cdot)$ is strongly continuous on $J_{\leqslant t}$ for all $t \in J$.*

Proof. (a) Let $s \in J,\ x \in X_s$. Since

$$W(t,s)x = U(t,s)x + \int_J \mathbf{1}_{[s,t]}(\tau)U(t,\tau)B(\tau)W(\tau,s)x \, d\tau \quad (t \in J_{\geqslant s})$$

and U is strongly continuous (and hence locally bounded), $W(\cdot,s)x$ is continuous on $J_{\geqslant s}$ by dominated convergence. This shows the assertion since X_s is dense in X and W is locally bounded.

(b) Fix $t \in J,\ s_0 \in J_{\leqslant t}$. Let $r \in J_{<s_0}$ ($r = s_0$ in case $s_0 = \inf J$), $x \in X_r,\ x_s := U(s,r)x$ for $s \in [r,t]$. By (1.20) we have

$$x_s = W(s,r)x - \int_r^s W(s,\tau)B(\tau)U(\tau,r)x\,d\tau.$$

Applying $W(t,s)$ to this equation and using (1.20) again, we obtain

$$\begin{aligned} W(t,s)x_s &= W(t,r)x - \int_r^s W(t,\tau)B(\tau)U(\tau,r)x\,d\tau \\ &= U(t,r)x + \int_s^t W(t,\tau)B(\tau)U(\tau,r)x\,d\tau. \end{aligned}$$

(Note that we cannot obtain this by setting $x = x_s$ in (1.20).) Thus, by dominated convergence,

$$W(t,s)x_s \to U(t,r)x + \int_{s_0}^t W(t,\tau)B(\tau)U(\tau,r)x\,d\tau = W(t,s_0)x_{s_0}$$

as $[r,t] \ni s \to s_0$. Since $x_s \to U(s_0,r)x = x_{s_0}$, we obtain

$$W(t,s)x_{s_0} = W(t,s)x_s + W(t,s)(x_{s_0} - x_s) \to W(t,s_0)x_{s_0} \quad \text{as } [r,t] \ni s \to s_0.$$

The set of elements x_{s_0} under consideration is dense in X, so the assertion follows. □

1.4 Domination for propagators and evolution semigroups

In this subsection we assume that X is a Banach lattice. Let U, $\tilde U$ be exponentially bounded s.c. propagators on X, and let $\mathcal{T}$, $\tilde{\mathcal{T}}$ be the evolution semigroups on $L_1(J;X)$, associated with U, $\tilde U$, respectively.

1.20 Lemma. (a) *$\tilde{\mathcal{T}}(t) \geqslant 0$ for all $t \geqslant 0$ if and only if $\tilde U(t,s) \geqslant 0$ for all $(t,s) \in D_J$.*

(b) *Let $\tilde{\mathcal{T}}(t) \geqslant 0$ for all $t \geqslant 0$. Then $|\mathcal{T}(t)f| \leqslant \tilde{\mathcal{T}}(t)|f|$ for all $t \geqslant 0,\ f \in L_1(J;X)$ if and only if $|U(t,s)x| \leqslant \tilde U(t,s)|x|$ for all $(t,s) \in D_J,\ x \in X$.*

Proof. (a) The "if" part is obvious. Suppose that $\tilde{\mathcal{T}}(t) \geqslant 0$ for all $t \geqslant 0$. Let $0 \leqslant x \in X$, $t \geqslant 0$. Then $\mathbf{1}_J(s-t)\tilde U(s,s-t)x \geqslant 0$ for a.e. $s \in J$. Therefore $(t,s) \mapsto \tilde U(t,s)x$ is positive a.e. and continuous, hence $\tilde U(t,s)x \geqslant 0$ for all $(t,s) \in D_J$.

(b) is proved in the same way as (a). □

1.21 Proposition. *Assume that $U(t,s) \geqslant 0$ for all $(t,s) \in D_J$. Let $(\tilde B(t))_{t\in J}$ be a small Miyadera perturbation of U, and assume that the spaces X_t occuring in the definition (see Theorem 1.12(i)) are sublattices of X. Let $B(t)$ be operators in X with $D(\tilde B(t)) \subseteq D(B(t))$ and*

$$\tilde B(t)x \geqslant B(t)x \geqslant 0 \qquad \big(0 \leqslant x \in D(\tilde B(t)) \cap X_{t-}\big)$$

for all $t \in J,\ t > \inf J$. Then $(B(t))_{t\in J}$ is a small Miyadera perturbation of U, and

$$U_{\tilde B}(t,s) \geqslant U_B(t,s) \geqslant 0 \qquad \big((t,s) \in D_J\big).$$

Proof. Let $s \in J$. From the assumptions it is clear that (1.11) holds for all $0 \leqslant x \in X_s$ (with α, γ from the Miyadera condition for $(\tilde{B}(t))_{t\in J})$. Thus we can define an operator A_s: $X_s \to L_1\big((s, s+\alpha) \cap J; X\big)$ by

$$A_s x(t) := B(t)U(t,s)x.$$

The assumptions imply that A_s is positive. Therefore, $|A_s x| \leqslant A_s|x|$ for all $x \in X_s$, and we obtain (1.11) for all $x \in X_s$, i.e., $(B(t))_{t\in J}$ is a small Miyadera perturbation of U.

Let now $\mathcal{B}$, $\tilde{\mathcal{B}}$ be defined as in the previous subsection. By Theorem 1.12 we have $\mathcal{D} \subseteq D(\mathcal{B}) \cap D(\tilde{\mathcal{B}})$, with $\mathcal{D} \subseteq E = L_1(J;X)$ constructed from the spaces X_t.

By the definition of $\mathcal{D}$, $f(t) \in X_{t-}$ for all $f \in \mathcal{D}$, $t \in J$. For $0 \leqslant f \in \mathcal{D}$ we thus obtain $\tilde{\mathcal{B}}f \geqslant \mathcal{B}f \geqslant 0$. From the Dyson-Phillips expansions for $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\tilde{\mathcal{B}}}$ (cf. [26]) we conclude that $\mathcal{T}_{\tilde{\mathcal{B}}}(t)f \geqslant \mathcal{T}_{\mathcal{B}}(t)f \geqslant 0$ for all $t \geqslant 0$, $0 \leqslant f \in \mathcal{D}$. As in Remark 1.11(a) we see that $\{f \in \mathcal{D};\ f \geqslant 0\}$ is dense in $\{f \in E;\ f \geqslant 0\}$. Thus the assertion follows from Lemma 1.20(b). □

2 Absorption propagators

In this section we study the perturbation of propagators on $L_p(\mu)$ by real-valued potentials.

2.1 Abstract theory of positive absorption propagators

Let (Ω, μ) be a measure space, $1 \leqslant p < \infty$, U a positive s.c. propagator on $L_p(\mu)$ with parameter interval $J \subseteq \mathbb{R}$. (A positive operator-valued function is, by definition, a function taking its values in the positive operators on $L_p(\mu)$.) Let V: $J \times \Omega \to \mathbb{R}$ be measurable. Let V^+, V^- denote the positive and negative parts of V, respectively, i.e. $V^+ = V \vee 0$, $V^- = (-V)^+$. We are going to apply the perturbation results of subsection 1.2 to bounded potentials and to extend the definition of a perturbed propagator to suitable unbounded real-valued potentials.

For real-valued $V \in L_\infty(J \times \Omega)$ we define a strongly measurable function B: $J \to \mathcal{L}(L_p(\mu))$ by

$$B(\tau) := -V(\tau, \cdot) \qquad (\tau \in J).$$

(In order to make this a valid definition we choose a bounded representative of V with measurable $V(\tau, \cdot)$ for all $\tau \in J$.) With this B we define

$$U_V := U_B,$$

in the sense of Section 1.2. We choose the negative sign in order to stay compatible with the notation usually chosen for Schrödinger operators; positive V corresponds to absorption, negative V to excitation.

The next proposition, which is a complete analogue of the corresponding statement for semigroups [43, Prop. 1.3], is the corner stone to the whole approach.

2.1 Proposition. *Let U, U_1, U_2 be positive s.c. propagators on $L_p(\mu)$, $V, V_1, V_2 \in L_\infty(J \times \Omega)$ real-valued. Then*

(a) $V_1 \leqslant V_2$ *implies* $U_{V_1} \geqslant U_{V_2} \geqslant 0$*;*

(b) $V \geqslant 0,\ U_1 \leqslant U_2$ *implies* $0 \leqslant (U_2)_V - (U_1)_V \leqslant U_2 - U_1$.

Proof. (a) is proved in [24, Lemma 3.1].

(b) Choose $c \in \mathbb{R}$ with $V - c \leqslant 0$ a.e. Formula (1.3) implies

$$(U_1)_{V-c} \leqslant (U_2)_{V-c}.$$

By Remark 1.7 we have

$$(U_1)_V(t,s) = e^{-c(t-s)}(U_1)_{V-c}(t,s) \quad \big((t,s) \in D_J\big),$$

and analogously for U_2. This proves the first of the asserted inequalities. The second, in the form $U_1 - (U_1)_V \leqslant U_2 - (U_2)_V$, then follows from the Duhamel formula (1.2). □

As in the theory of absorption semigroups ([42], [43]), we approximate V by means of cut-offs $V^{(n)} := (V \wedge n) \vee (-n)$ $(n \in \mathbb{N})$ and define

$$U_V(t,s) := \operatorname*{s-lim}_{n\to\infty} U_{V^{(n)}}(t,s) \quad \big((t,s) \in D_J\big), \tag{2.1}$$

if the limits exist.

Obviously, U_V is a propagator in this case, and the inequalities stated in Proposition 2.1 for bounded potentials V, V_1, V_2 are valid whenever the corresponding limits in (2.1) exist.

In the cases $V \geqslant 0$ and $V \leqslant 0$, the sequence $(U_{V^{(n)}}(t,s))_{n\in\mathbb{N}}$ is monotone due to Proposition 2.1. This leads to the following definition.

2.2 Definition. Let $V\colon J \times \Omega \to \mathbb{R}$ be measurable.

(a) $V \geqslant 0$ is called *U-admissible* if (2.1) defines a s.c. propagator.

(b) $V \leqslant 0$ is called *U-admissible* if the limit in (2.1) exists and defines a s.c. propagator.

(c) V is called *U-admissible* if V^+ and $-V^-$ are U-admissible.

The crucial fact concerning the notion of admissibility with respect to s.c. propagators is in analogy to the corresponding fact for semigroups [43, Thm. 2.6], though the proof requires an additional argument involving Dini's theorem. An important feature of the result is that the potential V is written as a difference of two positive potentials in a more general way than $V = V^+ - V^-$.

2.3 Proposition. *Let $V_\pm \geqslant 0$, assume that $-V_-$ is U-admissible, and denote $V := V_+ - V_-$.*

(a) *Then*

$$(U_{-V_-})_{V_+}(t,s) = \operatorname*{s-lim}_{n\to\infty} U_{V^{(n)}}(t,s) \,\big(= U_V(t,s)\big)$$

for all $(t,s) \in D_J$.

(b) *Assume additionally that V_+ is U-admissible. Then V_+ is U_{-V_-}-admissible, V_- is U_{V_+}-admissible, and*

$$U_V = (U_{-V_-})_{V_+} = (U_{V_+})_{-V_-};$$

in particular, U_V is strongly continuous.

Proof. The arguments of the proof are taken from [39, proof of Prop. 1.6], except for the last argument in which Dini's theorem is used.

(a) Let $f \in L_p(\mu)$, $f \geqslant 0$, $(t,s) \in D_J$. For $m,n \in \mathbb{N}$ we have, by Proposition 2.1,

$$0 \leqslant (U_{-V_-})_{V_+^{(n)}}(t,s)f - (U_{-V_-^{(m)}})_{V_+^{(n)}}(t,s)f \leqslant U_{-V_-}(t,s)f - U_{-V_-^{(m)}}(t,s)f. \tag{2.2}$$

In order to proceed, we note the inequalities

$$V_+^{(n)} - V_-^{(2n)} \leqslant (V_+ - V_-)^{(n)} \leqslant V_+^{(2n)} - V_-^{(n)},$$

which imply

$$U_{V_+^{(n)}-V_-^{(2n)}}(t,s)f \geqslant U_{V^{(n)}}(t,s)f \geqslant U_{V_+^{(2n)}-V_-^{(n)}}(t,s)f. \tag{2.3}$$

Recall that $(U_{-V_-^{(m)}})_{V_+^{(n)}} = U_{V_+^{(n)}-V_-^{(m)}}$ by Remark 1.7(b). Thus, using (2.2) first with $m = 2n$, and then with n replaced by $2n$ and $m = n$, we obtain that both the left hand side and the right hand side of (2.3) tend to $(U_{-V_-})_{V_+}(t,s)f$, and hence also $U_{V^{(n)}}(t,s)f \to (U_{-V_-})_{V_+}(t,s)f$ as $n \to \infty$.

(b) Let f and (t,s) be as in the proof of (a). For $m,n \in \mathbb{N}$ we have, by Proposition 2.1,

$$0 \leqslant (U_{V_+^{(n)}})_{m-V_-^{(m)}}(t,s)f - (U_{V_+})_{m-V_-^{(m)}}(t,s)f \leqslant U_{V_+^{(n)}}(t,s)f - U_{V_+}(t,s)f.$$

Hence, by Remark 1.7,

$$\begin{aligned}(U_{-V_-^{(m)}})_{V_+^{(n)}}(t,s)f &= e^{m(t-s)}(U_{V_+^{(n)}})_{m-V_-^{(m)}}(t,s)f \\ &\to e^{m(t-s)}(U_{V_+})_{m-V_-^{(m)}}(t,s)f = (U_{V_+})_{-V_-^{(m)}}(t,s)f\end{aligned}$$

as $n \to \infty$. Therefore, letting $n \to \infty$ in (2.2) we obtain

$$0 \leqslant (U_{-V_-})_{V_+}(t,s)f - (U_{V_+})_{-V_-^{(m)}}(t,s)f \leqslant U_{-V_-}(t,s)f - U_{-V_-^{(m)}}(t,s)f \to 0 \quad (m \to \infty),$$

and due to Dini's theorem the latter convergence is uniform with respect to (t,s) on compact subsets of D_J. We conclude that $(U_{-V_-})_{V_+} = (U_{V_+})_{-V_-}$ and that this propagator is strongly continuous. ☐

2.4 Remark. Let V be U-admissible and assume that U_{-V^-} is exponentially bounded. Let $\mathcal{T}_V$, $\mathcal{T}_{V^{(n)}}$ denote the evolution semigroups on $E := L_1(J; L_p(\mu))$ corresponding to U_V, $U_{V^{(n)}}$, respectively. Then $\mathcal{T}_{V^{(n)}}(t) \to \mathcal{T}_V(t)$ strongly for all $t \geqslant 0$. Indeed, by Proposition 2.3(a), $U_V(t,s) = \text{s-lim}_{n\to\infty} U_{V^{(n)}}(t,s)$ exists for all $(t,s) \in D_J$, and for $f \in E$, $n \in \mathbb{N}$ we have

$$|\mathcal{T}_{V^{(n)}}(t)f| \leqslant \mathbb{1}_J(\cdot - t)U_{-V^-}(\cdot, \cdot - t)|f(\cdot - t)|.$$

Thus the asserted convergence follows from the dominated convergence theorem.

Let $\mathcal{T}$ be the evolution semigroup on E corresponding to U, $\mathcal{G}$ the generator of $\mathcal{T}$. Recall from Theorem 1.16(b) that $\mathcal{G} - V^{(n)}$ is the generator of $\mathcal{T}_{V^{(n)}}$. In the same way as in [42, Cor. 2.7] we now obtain that the generator $\mathcal{G}_V$ of $\mathcal{T}_V$ is an extension of $\mathcal{G} - V$. As a consequence, if $\mathcal{G} - V$ is a generator then $\mathcal{G}_V = \mathcal{G} - V$.

Note that $L_1(J; L_1(\mu))$ is isomorphic to $L_1(J \times \Omega, \lambda^1 \otimes \mu)$, where λ^1 is one-dimensional Lebesgue measure. Thus, in the case $p = 1$, $\mathcal{T}_V$ is an absorption semigroup on $L_1(J \times \Omega, \lambda^1 \otimes \mu)$, in the sense of [42]. (In the case $p > 1$, $\mathcal{T}_V$ is an absorption semigroup on an appropriate Banach function space; cf. [21].)

The remainder of this subsection is devoted to giving some more information concerning approximation and convergence of absorption propagators. These results will not be needed in the remaining part of the present paper.

2.5 Lemma. *Let $V_1 \leqslant V_2 \leqslant 0$, V_1 U-admissible. Then V_2 is U-admissible.*

Proof. First note that $\operatorname{s-lim}_{n\to\infty} U_{V_2^{(n)}}(t,s)$ exists for all $(t,s) \in D_J$ since the sequence is dominated by $U_{V_1}(t,s)$. The inequality

$$V_1 + V_2^{(n)} - V_1^{(n)} \leqslant V_1 + V_2 - V_1 = V_2$$

implies $U_{V_1+V_2^{(n)}-V_1^{(n)}} \geqslant U_{V_2}$. By Proposition 2.3(a) and Proposition 2.1 we obtain

$$0 \leqslant U_{V_2} - U_{V_2^{(n)}} \leqslant U_{V_1+V_2^{(n)}-V_1^{(n)}} - U_{V_2^{(n)}} = (U_{V_1})_{V_2^{(n)}-V_1^{(n)}} - (U_{V_1^{(n)}})_{V_2^{(n)}-V_1^{(n)}} \leqslant U_{V_1} - U_{V_1^{(n)}}.$$

By Dini's theorem, the right hand side tends to zero strongly, uniformly on compact subsets of D_J, and the estimate shows that the same holds for $U_{V_2} - U_{V_2^{(n)}}$. □

2.6 Remark. One would expect an analogous result for positive potentials, namely that $V_1 \geqslant V_2 \geqslant 0$, V_1 U-admissible implies that V_2 is U-admissible. However, we can only show this under the additional assumption $\limsup_{s\to t-} \|U(t,s)\| \leqslant 1$ $(t \in J)$.

We now prove an analogue of the non-monotone convergence result for semigroups (cf. [43, Thm. 2.6], [19, Prop. 1.11]). This generalises [24, Lemma 3.1(c)].

2.7 Theorem. *Let $V_\pm \geqslant 0$, $-V_-$ U-admissible, $V := V_+ - V_-$. Let $(V_n)_{n\in\mathbb{N}}$ be such that $-V_- \leqslant V_n \leqslant V_+$ for all $n \in \mathbb{N}$, $V_n \to V$ a.e.*

(a) *Then $U_{V_n}(t,s) \to U_V(t,s)$ strongly as $n \to \infty$, for all $(t,s) \in D_J$.*

(b) *If additionally V_+ is U-admissible then the strong convergence in (a) is uniform for (t,s) in compact subsets of D_J.*

(Note that $-V_n^-$ is U-admissible by Lemma 2.5 since $V_n^- \leqslant V_-$, and that therefore $U_{V_n}(t,s) = \operatorname{s-lim}_{k\to\infty} U_{V_n^{(k)}}(t,s)$ exists for all $(t,s) \in D_J$, by Proposition 2.3(a).)

Proof. As a first preliminary step, we prove the theorem for the case $V_+ = 0$.

Let $0 \leqslant f \in L_p(\mu)$, $(t_0,s_0) \in D_J$, $\varepsilon > 0$. From Dini's theorem we obtain $m \in \mathbb{N}$ such that

$$\sup_{s_0\leqslant s\leqslant t\leqslant t_0} \|U_V(t,s)f - U_{V^{(m)}}(t,s)f\| \leqslant \varepsilon.$$

Observe that $V_n^{(m)}(t,\cdot) \to V^{(m)}(t,\cdot)$ strongly as $n \to \infty$, for all $t \in J$, where we consider $V_n^{(m)}(t,\cdot)$, $V^{(m)}(t,\cdot)$ as multiplication operators in $\mathcal{L}(L_p(\mu))$. Thus, applying Proposition 1.8 we obtain $N \in \mathbb{N}$ such that

$$\sup_{s_0\leqslant s\leqslant t\leqslant t_0} \|U_{V^{(m)}}(t,s)f - U_{V_n^{(m)}}(t,s)f\| \leqslant \varepsilon.$$

for all $n \geqslant N$. From $U_V(t,s)f \geqslant U_{V_n}(t,s)f \geqslant U_{V_n^{(m)}}(t,s)f$ we now conclude that

$$\sup_{s_0\leqslant s\leqslant t\leqslant t_0} \|U_V(t,s)f - U_{V_n}(t,s)f\| \leqslant \sup_{s_0\leqslant s\leqslant t\leqslant t_0} \|U_V(t,s)f - U_{V_n^{(m)}}(t,s)f\| \leqslant 2\varepsilon$$

for all $n \geqslant N$. This proves part (b) in this special case.

As a second preliminary step, we prove the theorem for the case $V_- = 0$.

Part (a) is proved in [24, Lemma 3.1(c)] (cf. also [42, Prop. A1]). The proof of part (b) is the same as the proof given above, only that now $U_V(t,s)f \leqslant U_{V_n}(t,s)f \leqslant U_{V_n^{(m)}}(t,s)f$. (Incidentally, in order to prove part (a), one just has to leave away the sup's.)

Now we give the proof for the general case.

(a) Following [43, proof of Thm. 2.6] we define

$$V_{n,+} := (V_n + V_-) \wedge V_+, \quad V_{n,-} := V_{n,+} - V_n \big(= (-V_n + V_+) \wedge V_- \big).$$

Then

$$0 \leqslant V_{n,\pm} \leqslant V_\pm, \qquad V_{n,\pm} \to V_\pm \quad \text{a.e.}$$

By Proposition 2.3(a) we have $U_{V_n} = (U_{-V_{n,-}})_{V_{n,+}}$, so Proposition 2.1 implies

$$0 \leqslant (U_{-V_-})_{V_{n,+}} - U_{V_n} \leqslant U_{-V_-} - U_{-V_{n,-}}. \tag{2.4}$$

From this inequality, the assertion of (a) follows since $U_{-V_{n,-}} \to U_{-V_-}$ and $(U_{-V_-})_{V_{n,+}} \to (U_{-V_-})_{V_+} = U_V$ as $n \to \infty$, as a consequence of the special cases $V_+ = 0,\ V_- = 0$ studied initially.

(b) Under the additional hypothesis, V_+ is U_{-V_-}-admissible by Proposition 2.3(b). Thus, inequality (2.4) together with the preliminary steps shows the assertion. □

2.2 Miyadera class for potentials

Let (Ω, μ) be a measure space, $1 \leqslant p < \infty$. Let U be an exponentially bounded positive s.c. propagator on $L_p(\mu)$ with parameter interval $J \subseteq \mathbb{R}$. Let $V\colon J \times \Omega \to \mathbb{R}$ be measurable. Without restriction, $V(t,\cdot)$ is measurable for all $t \in J$ and hence defines a (closed) multiplication operator in $L_p(\mu)$. Thus, by Theorem 1.16(b), V is a Miyadera perturbation of U if and only if there exist $\alpha > 0,\ \gamma \geqslant 0$ such that

$$\int_s^{s+\alpha} \mathbb{1}_J(t) \|V(t)U(t,s)f\|_p \, dt \leqslant \gamma \|f\|_p \tag{2.5}$$

for all $s \in J,\ f \in L_p(\mu)$, where we use the notation $V(t) = V(t,\cdot)$.

In the following two theorems we formulate conditions under which the potential V is U-admissible. The first of these results is a generalisation of the strong continuity assertion in [24, Thm. 3.3].

2.8 Theorem. *Let $V \geqslant 0$. Assume that for $s \in J$ there exists a dense sublattice X_s of $L_p(\mu)$ such that $\|V(\cdot)U(\cdot,s)f\|_p$ is locally integrable on $J_{\geqslant s}$, for all $f \in X_s$. (This assumption is satisfied with $X_s = X$ if V is a Miydera perturbation of U.) Then V is U-admissible, and for all $(t,s) \in D_J,\ f \in X_s$ we have*

$$\begin{aligned} U_V(t,s)f &= U(t,s)f - \int_s^t U_V(t,\tau)V(\tau)U(\tau,s)f \, d\tau \qquad (2.6)\\ &= U(t,s)f - \int_s^t U(t,\tau)V(\tau)U_V(\tau,s)f \, d\tau. \qquad (2.7)\end{aligned}$$

Proof. Let $(t,s) \in D_J,\ f \in X_s$. For $\tau \in [s,t]$ we have, since $U_{V^{(n)}} \leqslant U$,

$$\begin{aligned} \big|U_{V^{(n)}}(t,\tau)V^{(n)}(\tau)U(\tau,s)f\big| &\leqslant U(t,\tau)V(\tau)U(\tau,s)|f|,\\ \big|U(t,\tau)V^{(n)}(\tau)U_{V^{(n)}}(\tau,s)f\big| &\leqslant U(t,\tau)V(\tau)U(\tau,s)|f|.\end{aligned}$$

Thus, by dominated convergence we can pass to the limit in

$$\begin{aligned}U_{V^{(n)}}(t,s)f &= U(t,s)f - \int_s^t U_{V^{(n)}}(t,\tau)V^{(n)}(\tau)U(\tau,s)f\,d\tau \\ &= U(t,s)f - \int_s^t U(t,\tau)V^{(n)}(\tau)U_{V^{(n)}}(\tau,s)f\,d\tau,\end{aligned}$$

and obtain (2.6), (2.7). Now from Proposition 1.18 we conclude that U_V is strongly continuous. ☐

2.9 Theorem. *Let $V \leqslant 0$ be a small Miyadera perturbation of U. Then V is U-admissible, and (2.6), (2.7) hold for all $(t,s) \in D_J$, $f \in L_p(\mu)$.*

Proof. Let $\mathcal{T}$ be the evolution semigroup associated with U, with generator $\mathcal{G}$. By Theorem 1.16 there exists a s.c. propagator $\widetilde{U}_V$ on $L_p(\mu)$ such that the evolution semigroup $\widetilde{\mathcal{T}}_V$ associated with $\widetilde{U}_V$ has generator $\mathcal{G} - V$. Also by Theorem 1.16(b), $\mathcal{G} - V^{(n)}$ is the generator of the evolution semigroup $\mathcal{T}_{V^{(n)}}$ associated with $U_{V^{(n)}}$. Since $-V^{(n)} \leqslant -V$ we obtain $U_{V^{(n)}} \leqslant \widetilde{U}_V$ by Proposition 1.21.

Let now $(t,s) \in D_J$, $f \in L_p(\mu)$. From Theorem 1.16 we know that $|V|(\cdot)U(\cdot,s)f$, $|V|(\cdot)\widetilde{U}_V(\cdot,s)f$ are locally integrable on $J_{\geqslant s}$. Thus, using

$$\begin{aligned}\left|U_{V^{(n)}}(t,\tau)V^{(n)}(\tau)U(\tau,s)f\right| &\leqslant \widetilde{U}_V(t,\tau)|V|(\tau)U(\tau,s)|f|, \\ \left|U(t,\tau)V^{(n)}(\tau)U_{V^{(n)}}(\tau,s)f\right| &\leqslant U(t,\tau)|V|(\tau)\widetilde{U}_V(\tau,s)|f|,\end{aligned}$$

we obtain (2.6), (2.7) and hence the strong continuity of U_V as in the proof of Theorem 2.8. ☐

For the case of a small Miyadera perturbation V of U, a perturbed propagator can be defined either by Theorem 1.16 or by using absorption propagators. The following result shows that these two approaches lead to the same object.

2.10 Corollary. *Let $\mathcal{T}$ be the evolution semigroup associated with U, $\mathcal{G}$ the generator of $\mathcal{T}$. Let V be a Miyadera perturbation of U, with V^- Miyadera small. Then V is U-admissible, and $\mathcal{G} - V$ is the generator of the evolution semigroup $\mathcal{T}_V$ associated with U_V.*

Proof. We start with the following simple observation. If $\tilde{U}$ is a positive s.c. propagator on $L_p(\mu)$ with $\tilde{U} \leqslant U$ then a potential that is a small Miyadera perturbation of U is also Miyadera small with respect to $\tilde{U}$.

The U-admissibility of V follows from Theorems 2.8 and 2.9. There exists $n \in \mathbb{N}$ such that $\frac{1}{n}V^+$ is Miyadera small with respect to U. By Theorems 1.12 and 1.10, $\mathcal{G} - \frac{1}{n}V^+$ is a generator, so by Remark 2.4, $\mathcal{G} - \frac{1}{n}V^+$ is the generator of the evolution semigroup associated with $U_{\frac{1}{n}V^+}$. Proposition 2.1(a) implies that $U_{\frac{1}{n}V^+} \leqslant U$. By the initial observation, $\frac{1}{n}V^+$ is also Miyadera small with respect to $U_{\frac{1}{n}V^+}$, so we can repeat the argument and obtain that $\mathcal{G} - V^+$ is the generator of the evolution semigroup associated with U_{V^+}.

Again by the initial observation, $-V^-$ is Miyadera small with respect to U_{V^+}. Repeating the above argument once more, we conclude that $(\mathcal{G} - V^+) + V^- = \mathcal{G} - V$ is the generator of $\mathcal{T}_V$. ☐

3 Bounds for the integral kernels of propagators

In this section we assume that (Ω, μ) is a σ-finite measure space. For $p, q \in [1, \infty]$ and a linear operator $B\colon D(B) \subseteq L_p(\mu) \to L_q(\mu)$, we denote its norm by $\|B\|_{p\to q}$.

Throughout this section we assume that

(A1) U is a positive s.c. propagator on $L_1(\mu)$ with parameter interval $J \subseteq \mathbb{R}$, and there is a constant $L \geqslant 1$ such that

$$\|U(t,s)\|_{1\to 1} \leqslant L \qquad \big((t,s) \in D_J\big).$$

3.1 Remark. If U is only exponentially bounded then U can be rescaled to a propagator satisfying the bound in (A1); cf. Remark 1.7(a). Observe that a small Miyadera perturbation of U is also a small Miyadera perturbation of the rescaled propagator. (Make α in the definition smaller if necessary.) Thus, all the qualitative assertions of the subsequent results remain true in the more general case of exponentially bounded U.

We will also use the assumption

(A2) $U(t,s)$ is bounded from L_1 to L_∞, and there are constants $K, \nu > 0,\ A \in \mathbb{R}$ such that

$$\|U(t,s)\|_{1\to\infty} \leqslant K(t-s)^{-\nu} e^{A(t-s)} \qquad \big((t,s) \in D'_J\big),$$

where $D'_J := \big\{(t,s) \in J \times J;\, t > s\big\}$.

3.2 Remarks. (a) In assumption (A2), only the case $A \leqslant 0$ is of interest. In fact, if (A1) holds and $\|U(t,s)\|_{1\to\infty} \leqslant K(t-s)^{-\nu}$ for all $(t,s) \in D'_J,\ t-s \leqslant 1$ then for $(t,s) \in D'_J$ with $t-s > 1$ we have

$$\|U(t,s)\|_{1\to\infty} \leqslant \|U(t,t-1)\|_{1\to\infty}\|U(t-1,s)\|_{1\to 1} \leqslant KL.$$

With $C := KL,\ f(t) := t^{-\nu} \vee 1$ we thus obtain

(A2') $\|U(t,s)\|_{1\to\infty} \leqslant Cf(t-s)$ for all $(t,s) \in D'_J$

(which is slightly weaker than (A2) with $A = 0$, but much stronger than (A2) with $A > 0$).

More generally, let us briefly discuss the assumption (A2') with

$$f(t) := t^{-\nu_0}\ (0 < t \leqslant 1), \quad f(t) := t^{-\nu_1}\ (t > 1), \tag{3.1}$$

where $\nu_0, \nu_1 \geqslant 0$. (Above we had $\nu_0 = \nu,\ \nu_1 = 0$.) For heat propagators on manifolds, the different t-exponents for $t \leqslant 1$ and $t > 1$ are important; e.g., for a compact complete Riemannian manifold one has $\nu_1 = 0$. With the appropriate changes, our subsequent results are valid under assumption (A2') with f as defined in (3.1), too, but we confine ourselves to assumption (A2) for the sake of simplicity.

(b) By the Dunford-Pettis theorem, assumptions (A1) and (A2) imply that the operator $U(t,s)$ is an integral operator, with a positive kernel bounded by $K(t-s)^{-\nu}e^{A(t-s)}$.

3.1 Diagonal upper bound

3.3 Proposition. *Assume (A1) and (A2). Let V be a small Miyadera perturbation of U, i.e., condition* (2.5) *is satisfied with some $\alpha \in (0, \infty]$ and $\gamma \in [0, 1)$. Then for $1 \leqslant \kappa < \frac{1}{\gamma}$ there exists $c > 0$ such that*

$$\|U_V(t,s)\|_{1\to\kappa} \leqslant c(t-s)^{-\nu(1-\frac{1}{\kappa})} e^{\omega(t-s)} \qquad \big((t,s) \in D'_J\big), \tag{3.2}$$

where $\omega = \frac{1}{\kappa}\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma\kappa}\big) + (1-\frac{1}{\kappa})A$. In particular, the operators $U_V(t,s)$ are integral operators.

Proof. We will prove (3.2) for bounded V, with a constant c depending on V only via the Miyadera bound γ. The case of unbounded V then is immediate from the definition (2.1) of U_V.

Let $\mathcal{T}$ be the evolution semigroup on $L_1(J; L_1(\mu))$ corresponding to U, $\mathcal{G}$ the generator of $\mathcal{T}$. By Theorem 1.12, κV is a small Miyadera perturbation of $\mathcal{T}$ with constants $(\alpha, \gamma\kappa)$, so by [41, Thm. 1c)] we obtain

$$\|\mathcal{T}_{\kappa V}(t)\| \leqslant \tfrac{L}{1-\gamma\kappa} e^{\left(\frac{1}{\alpha}\log\frac{L}{1-\gamma\kappa}\right)t} \qquad (t \geqslant 0).$$

Recall from Theorem 1.16(b) that $\mathcal{T}_{\kappa V}$ is the evolution semigroup corresponding to $U_{\kappa V}$. The definition of evolution semigroups thus implies

$$\|U_{\kappa V}(t,s)\| \leqslant \tfrac{L}{1-\gamma\kappa} e^{\left(\frac{1}{\alpha}\log\frac{L}{1-\gamma\kappa}\right)(t-s)} \qquad \big((t,s) \in D_J\big).$$

Now we consider the family of propagators $U_{\kappa z V}$, with z from the strip $0 \leqslant \operatorname{Re} z \leqslant 1$. The corresponding evolution semigroups $\mathcal{T}_{\kappa z V}$ have generators $\mathcal{G} - \kappa z V$. Thus, by the Trotter product formula, $|\mathcal{T}_{\kappa z V}(t))| \leqslant \mathcal{T}_{\kappa(\operatorname{Re} z)V}(t)$ for all $t \geqslant 0$. By Lemma 1.20(b) we obtain, for all $(t,s) \in D'_J$, that

$$|U_{\kappa i b V}(t,s)| \leqslant U(t,s), \quad |U_{\kappa(1+ib)V}(t,s)| \leqslant U_{\kappa V}(t,s) \qquad (b \in \mathbb{R}),$$

and that $z \mapsto U_{\kappa z V}(t,s)$ is bounded. Moreover, $z \mapsto U_{\kappa z V}(t,s)$ is analytic by Proposition 1.6. Thus from the Stein interpolation theorem [35, p. 69] we obtain

$$\|U_V(t,s)\|_{1\to\kappa} \leqslant \|U_{\kappa V}(t,s)\|_{1\to 1}^{1/\kappa} \|U(t,s)\|_{1\to\infty}^{1-1/\kappa}$$

and hence (3.2).

By [10, p. 63, Thm. 5] we obtain from (3.2) that $U_V(t,s)$ is an integral operator. □

3.4 Remark. Let V be such that V^- is a small Miyadera perturbation of U. Then Proposition 3.3 can be applied to U_{-V^-}, and the inequality $U_V \leqslant U_{-V^-}$ implies that (3.2) also holds under the above more general assumption about V.

Similar observations apply to most of the subsequent results.

In order to obtain L_1–L_∞-estimates, we will need two more assumptions about the propagator U. The first of these is boundedness in L_∞,

(A3) $\|U(t,s)\|_{\infty\to\infty} \leqslant L$ for all $(t,s) \in D_J$.

3.5 Remarks. (a) Let assumptions (A1) and (A3) be satisfied. Then $\|U(t,s)\|_{p\to p} \leqslant L$ for all $1 \leqslant p \leqslant \infty,\ (t,s) \in D_J$ by Riesz-Thorin interpolation. Thus, U extends to a consistent family of s.c. propagators U_p on $L_p(\mu),\ 1 \leqslant p < \infty$ (for $f \in L_1 \cap L_\infty(\mu)$, the continuity of $(t,s) \mapsto U_p(t,s)f$ can be obtained from the inequality $\|g\|_p \leqslant \|g\|_1^{1/p}\|g\|_\infty^{1-1/p}$, valid for all $g \in L_1 \cap L_\infty(\mu)$).

(b) In addition, let V be a small Miyadera perturbation of U, with bound γ. Let $1 \leqslant p < \frac{1}{\gamma}$. For $(t,s) \in D_J$ we then obtain as in the proof of Proposition 3.3 that

$$\|U_V(t,s)\|_{p\to p} \leqslant \|U_{pV}(t,s)\|_{1\to 1}^{1/p}\|U(t,s)\|_{\infty\to\infty}^{1/p'}.$$

Thus, U_V extends to a consistent family of exponentially bounded s.c. propagators $U_{p,V}$ on $L_p(\mu),\ 1 \leqslant p < \frac{1}{\gamma}$. If additionally (A2) holds then by Riesz-Thorin interpolation the above inequality together with (3.2) implies the estimates

$$\|U_{p,V}(t,s)\|_{p\to q} \leqslant C(t-s)^{-\nu(\frac{1}{p}-\frac{1}{q})}\mathrm{e}^{\omega(t-s)} \qquad (1 \leqslant p \leqslant q < \tfrac{1}{\gamma}),$$

where $C > 0,\ \omega \in \mathbb{R}$ are constants depending on q.

In particular, if V^- is infinitesimally Miyadera small then the above holds with $\frac{1}{\gamma}$ replaced by ∞. This observation generalises [15, Thm. 2].

If (A3) holds then $\|U(t,s)^*f\|_1 \leqslant L\|f\|_1$ for all $(t,s) \in D_J,\ f \in L_1 \cap L_\infty(\mu)$. We thus obtain a bounded backward propagator U_* on $L_1(\mu)$ (we do not consider U_* as a backward propagator on $L_\infty(\mu)$ as in Remark 1.4(b)). The following assumption on U is needed for technical reasons. (Our results are in fact true without this assumption; cf. [40]).

(A4) U_* is strongly continuous on $L_1(\mu)$.

The definition of a Miyadera perturbation for a s.c. backward propagator is reduced to the case of s.c. propagators by means of Remark 1.4(a). With the obvious modifications, Proposition 3.3 also holds for s.c. backward propagators. We note that V is a Miyadera perturbation of the s.c. backward propagator U_* on $L_1(\mu)$ if there exist $\alpha \in (0,\infty],\ \gamma \geqslant 0$ such that

$$\int_{s-\alpha}^{s} \mathbf{1}_J(\tau)\|V(\tau)U(s,\tau)^*f\|_1\,d\tau \leqslant \gamma\|f\|_1 \qquad (s \in J,\ f \in L_1 \cap L_\infty(\mu)). \tag{3.3}$$

In the case of the heat equation the above coincides with the backward Kato condition mentioned in the introduction.

3.6 Proposition. *Assume (A1) to (A4). Let V be a small Miyadera perturbation of U and U_*, with constants $(\alpha,\gamma),\ \gamma < 1$ (valid for both U and U_*). Then there exist $c > 0,\ \omega \in \mathbb{R}$ such that*

$$\|U_V(t,s)\|_{1\to\infty} \leqslant c(t-s)^{-\nu}e^{\omega(t-s)} \qquad \big((t,s) \in D_J'\big).$$

More precisely, for $\theta \in (\gamma,1)$ there exists $c > 0$ such that the above holds with $\omega = \theta\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma/\theta}\big) + (1-\theta)A$.

Proof. (The following arguments are similar to those in [36, Thm. 5.1]. In [3, Thm. 3.3], a similar result is proved by an extrapolation argument.) Let $\theta \in (\gamma,1)$. There exists $k \in \mathbb{N}$ such that $\beta := \theta k' \in (\gamma,1)$, where $\frac{1}{k} + \frac{1}{k'} = 1$. Then $V_1 := V/\beta$ is a small Miyadera perturbation of U and U_*, with constants (α, γ_1), $\gamma_1 = \frac{\gamma}{\beta}$. Thus, by Proposition 3.3 there exists $c_1 > 0$ such that

$$\|U_{V_1}(t,s)\|_{1\to k'} \leqslant c_1(t-s)^{-\frac{\nu}{k}} e^{\omega_1(t-s)} =: \varphi(t-s) \quad \text{and} \quad \|(U_*)_{V_1}(s,t)\|_{1\to k'} \leqslant \varphi(t-s)$$

for all $(t,s) \in D'_J$, with $\omega_1 := \frac{1}{k'}\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma/\theta}\big) + \frac{1}{k}A$ (note that $\gamma_1 k' = \gamma/\theta$).

At the end of the proof we will show that $(U_*)_{V_1}(s,t) = U_{V_1}(t,s)^*$ on $L_1 \cap L_\infty(\mu)$. By duality we then obtain

$$\|U_{V_1}(t,s)\|_{k\to\infty} \leqslant \varphi(t-s).$$

The Riesz-Thorin convexity theorem implies that

$$\|U_{V_1}(t,s)\|_{p\to q} \leqslant \varphi(t-s) \quad \text{for } 1 \leqslant p < q \leqslant \infty,\ \tfrac{1}{p} - \tfrac{1}{q} = \tfrac{1}{k}.$$

Now choosing $p_0 = 1 < p_1 < \cdots < p_k = \infty$ such that $\frac{1}{p_{j-1}} - \frac{1}{p_j} = \frac{1}{k}$ and $t_0 = s < t_1 < \cdots < t_k = t$ such that $t_j - t_{j-1} = \frac{t-s}{k}$ $(j = 1, \ldots, k)$, we obtain

$$\|U_{V_1}(t,s)\|_{1\to\infty} \leqslant \prod_{j=1}^{k} \|U_{V_1}(t_j, t_{j-1})\|_{p_{j-1}\to p_j} \leqslant \varphi\big(\tfrac{t-s}{k}\big)^k = c_1^k k^\nu (t-s)^{-\nu} e^{\omega_1(t-s)}.$$

This implies, as in the proof of Proposition 3.3, that

$$\|U_V(t,s)\|_{1\to\infty} \leqslant \|U_{V_1}(t,s)\|_{1\to\infty}^{\beta} \|U(t,s)\|_{1\to\infty}^{1-\beta} \leqslant c(t-s)^{-\nu} e^{\omega(t-s)},$$

with $\omega = \beta\omega_1 + (1-\beta)A = \theta\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma/\theta}\big) + \big(1 - \beta(1-\frac{1}{k})\big)A$, and the assertion follows.

Finally, we show that $\langle (U_*)_{V_1}(s,t)f, g\rangle = \langle f, U_{V_1}(t,s)g\rangle$ for all $f, g \in L_1 \cap L_\infty(\mu)$. Taking into account Remark 1.4(a), we obtain from (1.3) that

$$(U_*)_{V_1}(s,t) = \sum_{j=0}^{\infty} U_{*,j}(s,t),$$

with

$$U_{*,0}(s,t) := U_*(s,t), \quad U_{*,j}(s,t) := \int_s^t U_{*,j-1}(s,\tau)\big(-V_1(\tau)\big)U_*(\tau,t)\,d\tau \quad (j = 1, \ldots)$$

(strong integrals). From (1.5) we infer by induction that

$$\langle U_{*,j}(s,t)f, g\rangle = \langle f, U_j(t,s)g\rangle \qquad (j = 0, \ldots)$$

where U_j are the constituents of the series for U_{V_1}. Summing the last identity over j we obtain the assertion. □

3.2 Off-diagonal upper bound

More strongly than in (A2) we now assume that the operators $U(t,s)$ are integral operators with kernels $u_0(t,\cdot;s,\cdot)$ satisfying the Gaussian type upper bound

$$u_0(t,x;s,y) \leqslant K(t-s)^{-\nu} e^{A(t-s)-\psi(t,x;s,y)} \qquad \big((t,s) \in D'_J,\ x, y \in \Omega\big), \tag{3.4}$$

with constants $K, \nu > 0$, $A \in \mathbb{R}$ and a measurable function $\psi \geqslant 0$.

3.7 Remark. If $\Omega = \mathbb{R}^N$, $\nu = \frac{N}{2}$, $A = 0$ and $\psi(t,x;s,y) = a\frac{d(x,y)^2}{t-s}$ for some $a > 0$ then (3.4) becomes the classical Gaussian bound. By the results of [4], this bound holds for the fundamental solution of (0.1) (with $V = 0$).

Let now Ω be a complete N-dimensional Riemannian manifold with Ricci curvature bounded below, μ the Riemannian volume and d the Riemannian distance. Assume that $\inf_{x\in\Omega}\mu(B(x,1)) > 0$. Then it follows from [9, Thm. 3 and bottom of p. 3] that a modified form of the above Gaussian bound also holds for the heat kernel on Ω: The term $(t-s)^{-\frac{N}{2}}$ has to be replaced by $(t-s)^{-\frac{N}{2}} \vee 1$ (cf. Remark 3.2(a)), and one can choose $A = -\lambda_0$ and any $0 < a < \frac{1}{4}$, where $\lambda_0 \geqslant 0$ is the bottom of the spectrum of the Laplace-Beltrami operator in $L_2(\Omega,\mu)$. In particular, for a negatively curved manifold one can have $A < 0$. However, without a uniform lower bound on the volume of unit balls, not even a bound of the type (A2) is valid (so our results do not apply to this case).

The next result is the main tool to obtain a off-diagonal upper bound for the kernel of the perturbed propagator. It will also be crucial for the lower bound.

3.8 Proposition. *Assume (A1) and (A2). Let V_0, V_1 be measurable, with V_0^-, V_1^- small Miyadera perturbations of U. Let $\theta \in (0,1)$, $V_\theta := (1-\theta)V_0 + \theta V_1$. Then V_θ^- is a small Miyadera perturbation of U, and for all $f \in L_1(\mu)$, $g \in L_\infty(\mu)$ we have*

$$|\langle U_{V_\theta}(t,s)f, g\rangle| \leqslant \langle U_{V_0}(t,s)|f|,|g|\rangle^{1-\theta}\,\langle U_{V_1}(t,s)|f|,|g|\rangle^{\theta} \qquad \big((t,s)\in D_J\big).$$

Moreover, $U_{V_\theta}(t,s)$ is an integral operator for $(t,s) \in D'_J$, and for the kernel we have

$$u_{V_\theta} \leqslant u_{V_0}^{1-\theta} u_{V_1}^{\theta}. \tag{3.5}$$

Proof. The first assertion is clear. By the definition of absorption propagators it suffices to prove the second assertion in the case of bounded V_0, V_1. We now proceed as in [43, proof of Prop. 1.4]. Fix $(t,s) \in D_J$, $f \in L_1(\mu)$, $g \in L_\infty(\mu)$. Define the function

$$F(z) := \big\langle U_{(1-z)V_0+zV_1}(t,s)f, g\big\rangle = \big\langle (U_{V_0})_{z(V_1-V_0)}(t,s)f, g\big\rangle$$

on the strip $0 \leqslant \operatorname{Re} z \leqslant 1$. As in the proof of Proposition 3.3, we obtain that F is bounded and analytic, and

$$|F(0+ib)| \leqslant \langle U_{V_0}(t,s)|f|,|g|\rangle, \quad |F(1+ib)| \leqslant \langle U_{V_1}(t,s)|f|,|g|\rangle \qquad (b\in\mathbb{R}).$$

By the Hadamard three lines lemma we conclude that

$$|\langle U_{V_\theta}(t,s)f,g\rangle| = |F(\theta)| \leqslant \langle U_{V_0}(t,s)|f|,|g|\rangle^{1-\theta}\,\langle U_{V_1}(t,s)|f|,|g|\rangle^{\theta}.$$

From Proposition 3.3 we obtain that $U_{V_\theta}(t,s)$ is an integral operator. The last assertion now follows from Lemma 3.9 below. ☐

3.9 Lemma. *Let A, B, C be positive integral operators on $L_p(\mu)$. Let a, b, c be the corresponding integral kernels. Suppose that there exists $\theta \in (0,1)$ such that for $0 \leqslant f, g \in L_1 \cap L_\infty(\mu)$ one has*

$$\langle Af, g\rangle \leqslant \langle Bf, g\rangle^{\theta}\langle Cf, g\rangle^{1-\theta}.$$

Then

$$a \leqslant b^{\theta}c^{1-\theta} \ a.e.$$

Proof. It follows from Young's inequality that, for any $\delta > 0$,

$$\langle Af, g\rangle \leqslant \theta\delta\langle Bf, g\rangle + (1-\theta)\delta^{-\frac{\theta}{1-\theta}}\langle Cf, g\rangle.$$

This implies

$$a \leqslant \theta\delta b + (1-\theta)\delta^{-\frac{\theta}{1-\theta}}c \quad \text{a.e.}$$

Optimizing the right hand side over rational δ's we obtain the assertion. □

3.10 Theorem. *Let U be a propagator satisfying (A1) to (A4) and the Gaussian type upper bound (3.4). Let V be a Miyadera perturbation of both U and U_*, V^- Miyadera small with constants (α, γ), $\gamma < 1$. Then the propagator U_V consists of integral operators $U_V(t,s)$, and for $\gamma < \beta < 1$ there exist $c > 0$, $\omega \in \mathbb{R}$ such that the kernel u_V satisfies*

$$u_V(t,x;s,y) \leqslant c(t-s)^{-\nu}e^{\omega(t-s)-(1-\beta)\psi(t,x;s,y)} \qquad \big((t,s) \in D'_J,\ x,y \in \Omega\big).$$

More precisely, for $\theta \in (\gamma, \beta)$ there exists $c > 0$ such that the above holds with $\omega = \theta\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma/\theta}\big) + (1-\theta)A$.

Proof. Let $V_1 := V/\beta,\ \theta_1 := \theta/\beta$. By Proposition 3.6 there exists $c_1 > 0$ such that

$$\|U_{V_1}(t,s)\|_{1\to\infty} \leqslant c_1(t-s)^{-\nu}e^{\omega_1(t-s)} \qquad \big((t,s) \in D'_J\big),$$

with $\omega_1 = \theta_1\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma/\theta}\big) + (1-\theta_1)A$. From Proposition 3.8 we obtain $u_V \leqslant u_{V_1}^{\beta}u_0^{1-\beta}$, so the assertion follows with $\omega = \beta\omega_1 + (1-\beta)A$. □

3.11 Remark. If V^- is an infinitesimally small Miyadera perturbation of U and U^* then one can choose any $\beta > 0$ in Theorem 3.10, and we obtain sharp upper bounds for $t - s \leqslant 1$. In applications to second order parabolic equations this observation shows the difference between the non-autonomous Kato class **NK** and the enlarged non-autonomous Kato class $\widehat{\mathbf{NK}}$.

The validity of the upper bound (3.4) with $A = 0$, i.e., a bound global in time, is of special interest for questions of long time behaviour. By Theorem 3.10 the global upper bound (3.4) is stable under perturbation by the potential V if V^- is a Miyadera perturbation of U and U_* with constants (∞, γ), $\gamma < 1$.

3.3 Lower bound

With the methods from the previous subsections we obtain the following result about the lower bound of the integral kernel.

3.12 Theorem. *Let U be a propagator satisfying (A1) to (A4) and the Gaussian type lower bound*

$$u_0(t,x;s,y) \geqslant K_1(t-s)^{-\nu}e^{-A_1(t-s)-\psi(t,x;s,y)} \qquad \big((t,s) \in D'_J,\ x,y \in \Omega\big), \tag{3.6}$$

with some $K_1, \nu_1 > 0$, $A_1 \in \mathbb{R}$, and a measurable function $\psi \geqslant 0$. Let V be a Miyadera perturbation of U, with V^- Miyadera small. Assume that V^+ is a Miyadera perturbation of U_*, and let (α, γ) be Miyadera constants of V^+ with respect to both U and U_*. Then for $\beta > \gamma$ there are constants $c > 0$, $\omega \in \mathbb{R}$ such that the kernel u_V of U_V satisfies

$$u_V(t,x;s,y) \geqslant c(t-s)^{-\frac{N}{2}}e^{-\omega(t-s)-(1+\beta)\psi(t,x;s,y)} \qquad \big((t,s) \in D'_J,\ x,y \in \Omega\big).$$

More precisely, for $\theta \in (\gamma, \beta)$ there exists $c > 0$ such that the above holds with $\omega = \theta\big(\frac{1}{\alpha}\log\frac{L}{1-\gamma/\theta}\big) + (\beta-\theta)A + (1+\beta)A_1$.

Proof. Note that $U_V \geqslant U_{V^+}$. Thus, U_V has a kernel by Proposition 3.3, and it suffices to prove the lower estimate for $V = V^+$. By the assumption, $-V/\beta$ is a small Miyadera perturbation of U and U_*. Therefore, by Proposition 3.8,

$$u_0 \leqslant u_{-V/\beta}^{\frac{\beta}{1+\beta}} u_V^{\frac{1}{1+\beta}}, \quad \text{so} \quad u_V \geqslant (u_{-V/\beta})^{-\beta} u_0^{1+\beta}.$$

By Proposition 3.6 there exists $c_1 > 0$ such that $\|U_{-V/\beta}(t,s)\|_{1\to\infty} \leqslant c_1(t-s)^{-\nu}e^{\omega_1(t-s)}$ for all $(t,s) \in D'_J$, with ω_1 as in the proof of Theorem 3.10. Thus the assertion follows with $\omega = \beta\omega_1 + (1+\beta)A_1$. □

3.13 Remark. Observations similar to those in Remark 3.11 concerning sharp and global upper bounds, respectively, also apply to the lower bounds. Assume that V^- is a small Miyadera perturbation of U. If V^+ is an infinitesimally small Miyadera perturbation of U and U^* we obtain sharp lower bounds for $t - s \leqslant 1$.

If $A = 0$ in the diagonal upper bound and $A_1 = 0$ in (3.6) then the global lower bound is stable under perturbation by the potential V if V^+ is a Miyadera perturbation of U and U_* with constants (∞, γ), with any $\gamma \geqslant 0$. Up to our knowledge, the latter result is new even in the case of the heat equation with a time-dependent potential.

3.14 Remark. Assume that U and U^* are L_1-norm preserving and that U satisfies classical Gaussian upper and lower bounds (cf. Remark 3.7). This is the case, e.g., if U corresponds to the Cauchy problem for (0.1) with $V = 0$. If $V \leqslant 0$ then the conditions of Theorem 3.10 are close to necessary. Indeed, if U_V satisfies a Gaussian upper bound then U_V and U_V^* are exponentially bounded on $L_1(\mathbb{R}^N)$. Then one can follow the argument in the proof of [42, Prop. 4.6] in order to show that V is a (not necessarily small) Miyadera perturbation of U and U^*. If $V \geqslant 0$ then the conditions of Theorem 3.12 are in fact necessary. This follows by a similar argument.

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