Domination of semigroups associated with sectorial forms

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Abstract

Let τ be a closed sectorial form in a Hilbert space H, and let $T = (T(t); t \geqslant 0)$ be the C_0-semigroup associated with τ. We generalize a criterion of Ouhabaz on T-invariance of a closed convex set $C \subseteq H$ in terms of τ and the Hilbert space projection onto C. Using this criterion, we generalize known criteria for domination properties between two C_0-semigroups associated with closed sectorial forms in L_2-spaces. Following recent developments, the dominated semigroup is assumed to act on a Hilbert space valued L_2-space.

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1 Introduction

Let $\mathfrak{a}$, $\mathfrak{b}$ be closed sectorial forms in the Hilbert spaces $L_2(\Omega, \mu; G)$, $L_2(\Omega, \mu)$, respectively, where G is a Hilbert space. Denote by S, T the C_0-semigroups associated with $\mathfrak{a}$, $\mathfrak{b}$, respectively. Our aim is characterizing when S is dominated by T, i.e.,

$$|S(t)u| \leqslant T(t)|u|$$

for all $u \in \overline{D(\mathfrak{a})}$, $t \geqslant 0$ (where $|\cdot|$ denotes the norm in G). The results we are going to present generalize previous results in two ways: On the one hand we will study the case where the forms are not densely defined, and on the other hand we will present core results.

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Our work is motivated by the papers of E. M. Ouhabaz [10], [11]—to which we refer for the history and earlier literature—as well as by the desire to find manageable criteria for applications. Considering domination for semigroups acting in Hilbert space valued L_2-spaces was initiated by Shigekawa [12]. A treatment of invariance and domination properties in the non-linear setting has been given by Barthélemy [1].

The application we have in mind is the theory of Schrödinger operators with magnetic fields. In this case, Ω is an open subset of $\mathbb{R}^d$, $\mu = \sigma\, dx$ with a (singular) weight $\sigma\colon \Omega \to [0,\infty)$, and $\mathfrak{a}$, $\mathfrak{b}$ are associated with the formal differential operators

$$\sigma^{-1}(\nabla - ib)^* a(\nabla - ib) \ \Big(= -\sigma^{-1}\sum_{j,k=1}^{d} (\partial_j - ib_j)a_{jk}(\partial_k - ib_k)\Big),$$

$\sigma^{-1}\nabla^* a\nabla$, respectively. Here, $b\colon \Omega \to \mathbb{R}^d$ is a (singular) vector potential and $a(x) = (a_{jk}(x))_{j,k=1,\ldots,d}$ is a nonnegative real symmetric matrix for all $x \in \Omega$. It was this situation in which the first named author was faced with the two problems mentioned above: In general, the forms $\mathfrak{a}$ and $\mathfrak{b}$ are not densely defined, and they are explicitly given only on cores (cf. [6; Kap. 14]).

In order to illustrate the type of characterization we obtain, we mention here the result that S is dominated by T if and only if $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$, and

$$\operatorname{Re}\mathfrak{a}(u,v) \geqslant \mathfrak{b}(|u|,|v|)$$

for all $u, v \in D(\mathfrak{a})$ satisfying $(u\,|\,v)_G = |u|\,|v|$; cf. Theorem 4.1.

Our results on domination are derived from a result characterizing semigroups leaving invariant a closed convex set in a Hilbert space (Theorem 2.1). This is parallel to the treatment in [10], [11], the difference being that we push this analysis further. As a consequence, our proof of the characterization of domination is in all parts based on the result of invariance. It is by this method that we are able to treat the case where $D(\mathfrak{a})$ and $D(\mathfrak{b})$ are not densely defined and where inequalities are assumed only on suitable cores of $D(\mathfrak{a})$ and $D(\mathfrak{b})$.

As far as possible, we treat the real case and the complex case simultaneously. Throughout the paper, we assume (Ω,μ) to be a measure space, and by $M(\mu;G)$ we denote the set of measurable G-valued functions, where G is the Hilbert space mentioned initially. For $G = \mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, the space $M(\mu;\mathbb{K})$ in fact is a lattice.

The main result of Section 2 is our result presenting various characterizations for invariance of a closed convex set in a Hilbert space under a semigroup which is associated with a sectorial form. This result is applied to characterizing when a semigroup is real, positive, or sub-Markovian, respectively.

Section 3 contains preparations for the application of the main result of Section 2 to domination. We point out the importance of the convex cone

$$C := \big\{(u,v) \in M(\mu;G) \times M(\mu;\mathbb{R});\ |u| \leqslant v\big\}$$

for domination of operators, we compute a distinguished projection P from $M(\mu;G)\times M(\mu;\mathbb{R})$ onto this cone, and we characterize subspaces $U\subseteq M(\mu;G)$, $V\subseteq M(\mu;\mathbb{R})$ for which $U\times V$ is invariant under P. It is in this context that generalized ideals appear as a natural concept.

In Section 4, the main result of Section 2 is applied to characterizing domination of a semigroup S by a positive semigroup T. As special situations we study the two cases that both S and T are symmetric, and that S is a positive semigroup, too.

In Section 5 we supply some additional information concerning generalized ideals.

2 The invariance criterion

The aim of this section is to generalize Ouhabaz' criterion on invariance of closed convex sets ([10; Thm. 2.1], [11; Thm. 3]) in three directions. Firstly, we want to state the criterion in the setting of not necessarily densely defined forms; this is in fact an easy task. Secondly, we want to present a core version which is useful in applications. Thirdly, we want to remove the (standard) assumption that the form is non-negative. The latter generalization originated in the paper [4]; see also the comments at the end of this section.

Throughout this section let τ be a closed sectorial form in a (real or complex) Hilbert space H. For the theory of sectorial forms we refer to [3; Ch. VI], [5; Ch. I]. We do not assume τ to be densely defined or non-negative. Let $H_\tau := \overline{D(\tau)}^H$, A the m-sectorial operator in H_τ associated with τ and $T=(T(t);t\geqslant 0)$ the C_0-semigroup on H_τ generated by $-A$.

Let $-\omega_0(\tau)$ be the lower bound of τ, i.e.,

$$\omega_0(\tau) := \inf\bigl\{\omega\in\mathbb{R};\ \operatorname{Re}\tau(u)\geqslant -\omega\|u\|^2 \text{ for all } u\in D(\tau)\bigr\}.$$

Recall that $\omega_0(\tau)$ coincides with the type of the semigroup T. In particular, T is contractive if and only if τ is non-negative, i.e., $\operatorname{Re}\tau(u)\geqslant 0$ for all $u\in D(\tau)$.

Before stating the main theorem of this section we introduce the following notation. If $\varnothing\neq C$ is a closed convex subset of H, we denote by $P_C^H\colon H\to C$ the projection (= Hilbert space projection) from H onto C, i.e., for $u\in H$ the element $P_C^H u$ is the unique element of C satisfying

$$\|u-P_C^H u\| = \min_{v\in C}\|u-v\|.$$

It is known that $P_C^H u$ is characterized by

$$\operatorname{Re}\,(v-P_C^H u\,|\,u-P_C^H u)\leqslant 0 \quad \text{for all } v\in C. \tag{2.1}$$

We say that a set $M\subseteq H$ is invariant under a semigroup S on H if $S(t)M\subseteq M$ for all $t\geqslant 0$.

In the case that the form τ is densely defined and non-negative, the equivalences '(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv)' of the following theorem are due to Ouhabaz ([10; Thm. 2.1], [11; Thm. 3]).

2.1 Theorem. *Let $\varnothing \neq C \subseteq H$ be a closed convex set, $P := P_C^H$. Then the following are equivalent:*

(i) *H_τ is invariant under P, and $C \cap H_\tau$ is invariant under T.*

(ii) *$D(\tau)$ is invariant under P, and*

$$\operatorname{Re} \tau(Pu, u - Pu) \geqslant 0 \tag{2.2}$$

for all $u \in D(\tau)$.

(iii) *There exists a dense subset D of $D(\tau)$ (i.e., D is a core for τ) such that $P(D) \subseteq D(\tau)$ and (2.2) holds for all $u \in D$.*

(iv) *$D(\tau)$ is invariant under P, and*

$$\operatorname{Re} \tau(u, u - Pu) \geqslant -\omega \|u - Pu\|^2 \tag{2.3}$$

for all $u \in D(\tau)$, with $\omega = \omega_0(\tau)$.

(v) *There exist $\omega \in \mathbb{R}$ and $D \subseteq D(\tau)$ dense, convex and invariant under P such that (2.3) holds for all $u \in D$.*

Proof. Since P is continuous, either of the conditions implies that $H_\tau = \overline{D(\tau)} = \overline{D}$ is invariant under P. This shows $P\restriction_{H_\tau} = P_{C\cap H_\tau}^{H_\tau}$. We may therefore assume $H_\tau = H$, i.e., τ is densely defined.

(i) $\Rightarrow$ (ii). The first part of the proof of [10; Thm. 2.1] shows that $D(\tau)$ is invariant under P. Let $u \in D(\tau)$. Then $T(t)Pu \in C$, so [10; Lemma 1.1] and characterization (2.1) imply

$$\operatorname{Re} \tau(Pu, u - Pu) = \lim_{t\to 0+} \frac{1}{t} \operatorname{Re} (Pu - T(t)Pu \,|\, u - Pu) \geqslant 0.$$

(ii) $\Rightarrow$ (iv) follows from the equality

$$\tau(u, u - Pu) = \tau(Pu, u - Pu) + \tau(u - Pu)$$

and the definition of $\omega_0(\tau)$.

(iv) $\Rightarrow$ (i). We only have to prove the second statement. The following is an adaptation of the proof of [10; Thm. 2.1].

It is sufficient to show that $\lambda(\lambda + A)^{-1}(C) \subseteq C$, for all $\lambda > \max(\omega_0(\tau), 0)$. Let $u \in C$, $v := \lambda(\lambda + A)^{-1}u$. Then $Av = \lambda(u - v)$. Characterization (2.1) yields $\operatorname{Re}(u \,|\, v - Pv) \leqslant \operatorname{Re}(Pv \,|\, v - Pv)$. Inequality (2.3) now implies

$$\begin{aligned} -\omega_0(\tau)\|v - Pv\|^2 &\leqslant \operatorname{Re} \tau(v, v - Pv) = \operatorname{Re}(Av \,|\, v - Pv) = \lambda \operatorname{Re}(u - v \,|\, v - Pv) \\ &\leqslant \lambda \operatorname{Re}(Pv - v \,|\, v - Pv) = -\lambda\|v - Pv\|^2. \end{aligned}$$

Because of $\lambda > \omega_0(\tau)$ we obtain $v = Pv \in C$.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (iii). (cf. [10; proof of Cor. 2.5]) Let $u \in D$. Then $u_\varepsilon := Pu + \varepsilon(u - Pu) \in D$ for all $\varepsilon \in (0,1)$, and $u_\varepsilon \to Pu$ in $D(\tau)$ as $\varepsilon \to 0$. Moreover, $Pu_\varepsilon = Pu$ by characterization (2.1). Thus (iii) follows by letting $\varepsilon \to 0$ in

$$\operatorname{Re}\tau(u_\varepsilon, u - Pu) = \frac{1}{\varepsilon}\operatorname{Re}\tau(u_\varepsilon, u_\varepsilon - Pu_\varepsilon) \geqslant -\frac{1}{\varepsilon}\omega\|u_\varepsilon - Pu_\varepsilon\|^2 = -\varepsilon\omega\|u - Pu\|^2.$$

(iii) $\Rightarrow$ (ii). Let $u \in D(\tau)$. Let $(u_n) \subseteq D$ such that $u_n \to u$ in $D(\tau)$ as $n \to \infty$. By (iii) we have

$$\operatorname{Re}\tau(Pu_n) \leqslant \operatorname{Re}\tau(Pu_n, u_n) \quad (n \in \mathbb{N}).$$

As in [10; proof of Thm. 2.1] one deduces that there exists $c \geqslant 1$ such that

$$\operatorname{Re}\tau(Pu_n) \leqslant c\big(\operatorname{Re}\tau(u_n) + \|u_n\|^2\big) + \|Pu_n\|^2 \quad (n \in \mathbb{N}).$$

Therefore, (Pu_n) is a bounded sequence in $D(\tau)$. Since $Pu_n \to Pu$ in H we obtain that $Pu_n \to Pu$ weakly in $D(\tau)$ (see, e.g., [5; Lemma I.2.12]). Thus, the lower semicontinuity of $\operatorname{Re}\tau$ implies

$$\operatorname{Re}\tau(Pu) \leqslant \liminf_{n\to\infty}\operatorname{Re}\tau(Pu_n) \leqslant \liminf_{n\to\infty}\operatorname{Re}\tau(Pu_n, u_n) = \operatorname{Re}\tau(Pu, u),$$

so (ii) holds. □

The next result states an equivalent reformulation of property (i) of the above theorem, which holds in a more general context.

2.2 Proposition. *Let S be a C_0-semigroup on a closed subspace H_S of H. Let S_0 be the semigroup on H defined by $S_0(t) := S(t)P^H_{H_S}$ for all $t \geqslant 0$. Let $\varnothing \neq C \subseteq H$ be a closed convex set, $P := P^H_C$. Then the following are equivalent:*

(i) *C is invariant under S_0.*

(ii) *H_S is invariant under P, and $C \cap H_S$ is invariant under S.*

Proof. It is immediate that one only has to show that C is invariant under $S_0(0) = P^H_{H_S}$ if and only if H_S is invariant under $P = P^H_C$. This equivalence is a special case of the following lemma. □

2.3 Lemma. *([1; footnote on p. 9]) Let $C_1, C_2 \subseteq H$ be closed convex sets, $P_j := P^H_{C_j}$ $(j = 1, 2)$. Then C_1 is invariant under P_2 if and only if C_2 is invariant under P_1.*

Proof. We only need to show necessity. Let C_1 be invariant under P_2, $x \in C_2$. We have to show that $y := P_1x \in C_2$. Since $x \in C_2$, i.e. $P_2x = x$, we have

$$\|x - P_2y\| = \|P_2x - P_2y\| \leqslant \|x - y\| = \|x - P_1x\|.$$

Note that $P_2y \in C_1$ since $y \in C_1$. Since P_1x is the best approximation of x in C_1, we conclude that $y = P_1x = P_2y \in C_2$. □

In Corollary 2.6 below we will need the following application of Lemma 2.3. Recall that a complex Hilbert lattice H is the complexification of a real Hilbert lattice H_r. If H is a real or complex Hilbert lattice then it is easy to show that the projection from H onto H_+ $(= (H_r)_+)$ is given by $u \mapsto (\operatorname{Re} u)^+$.

2.4 Lemma. *Let H be a (real or complex) Hilbert lattice, and let V be a closed subspace of H. Then P_V^H is positive if and only if V is a sublattice of H.*

Proof. By Lemma 2.3, applied with $C_1 = V$, $C_2 = H_+$, we obtain: $P_V := P_V^H$ is positive if and only if $P_{H_+}^H v = (\operatorname{Re} v)^+ \in V$ for all $v \in V$. It remains to observe that, if P_V is positive then $|v| \in V$ for all $v \in V$. The latter holds because the inequalities $\|P_V|v|\| \leqslant \|v\|$, $|v| = |P_V v| \leqslant P_V|v|$ imply $|v| = P_V|v| \in V$. ☐

In the remainder of this section we apply Theorem 2.1 and Proposition 2.2 to real, positive, and sub-Markovian semigroups, respectively. Let τ, H_τ, T be as above and define T_0 like S_0 in Proposition 2.2.

If H is the complexification of a real Hilbert space H_r and X is a subspace of H, we define $X_r := H_r \cap X$. We say that X is *real* if $X = \operatorname{lin}(X_r)$. If X is real then a semigroup on X is called *real* if it leaves X_r invariant.

2.5 Corollary. *(cf. [9; Prop. 2.2]) Assume that H is the complexification of a real Hilbert space H_r. Then the following are equivalent:*

(i) *T_0 is real.*

(ii) *H_τ is real and T is real.*

(iii) *$D(\tau)$ is real, and $\tau(u,v) \in \mathbb{R}$ for all $u, v \in D(\tau)_r$.*

(iv) *There exists a dense subset D of $D(\tau)$ such that $u \in D$ implies $\operatorname{Re} u \in D(\tau)$ and $\tau(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}$.*

Proof. Note that $P\colon x \mapsto \operatorname{Re} x$ is the projection from H onto $C := H_r$, and that a subspace of H is real if and only if it is invariant under P. Thus, (i) $\Leftrightarrow$ (ii) is immediate from Proposition 2.2. Moreover, $\operatorname{Re}\tau(Pu, u - Pu) = \operatorname{Im}\tau(\operatorname{Re} u, \operatorname{Im} u)$ for all $u \in D(\tau)$, so (iv) $\Rightarrow$ (ii) follows from Theorem 2.1, (iii)$\Rightarrow$(i).

(iii) $\Rightarrow$ (iv) is trivial, so it remains to show (ii) $\Rightarrow$ (iii). Let $u, v \in D(\tau)_r$, $w := u + iv$. By Theorem 2.1, (i)$\Rightarrow$(ii), $\operatorname{Im}\tau(u,v) = \operatorname{Re}\tau(Pw, w - Pw) \geqslant 0$ as well as $-\operatorname{Im}\tau(u,v) = \operatorname{Re}\tau(P\overline{w}, \overline{w} - P\overline{w}) \geqslant 0$, so (iii) holds. ☐

2.6 Corollary. *(cf. [9; Thm. 2.4], [10; Cor. 2.5]) Let H be a (real or complex) Hilbert lattice. Then the following are equivalent:*

(i) *T_0 is positive.*

(ii) *H_τ is a sublattice of H, and T is positive.*

(iii) *T is real, $D(\tau)$ is a sublattice of H, and $\tau(u^+, u^-) \leqslant 0$ for all $u \in D(\tau)_r$.*

(iv) *T is real, and there exists a dense subset D of $D(\tau)_r$ such that $u \in D$ implies $u^+ \in D(\tau)$ and $\tau(u^+, u^-) \leqslant 0$.*

Moreover, if (ii) *holds then $\tau(|u|) \leqslant \operatorname{Re}\tau(u)$ for all $u \in D(\tau)$.*

Proof. In the case $\mathbb{K} = \mathbb{R}$, the assertion is a consequence of Proposition 2.2 and Theorem 2.1, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii), applied with $C = H_+$.

Assume now $\mathbb{K} = \mathbb{C}$. By Lemma 2.4, (i) implies that H_τ is a sublattice of H. It remains to show that (ii) implies that $|u| \in D(\tau)$, $\tau(|u|) \leqslant \operatorname{Re}\tau(u)$ for all $u \in D(\tau)$.

If τ is symmetric, this is proved in the same way as the corresponding assertion in the first Beurling-Deny criterion; see, e.g., [2; Theorem 1.3.2].

If τ is not symmetric, we use the symmetric part τ_s of τ, which is defined by $\tau_s(u,v) := \frac{1}{2}(\tau(u,v) + \overline{\tau(v,u)})$, for $u, v \in D(\tau_s) := D(\tau)$. Then $\tau(u^+, u^-) \leqslant 0$, $\tau(u^-, u^+) = \tau((-u)^+, (-u)^-) \leqslant 0$ for all $u \in D(\tau)_r$, and hence $\tau_s(u^+, u^-) \leqslant 0$ for all $u \in D(\tau_s)_r$. Moreover, $\tau_s(u,v) \in \mathbb{R}$ for all $u, v \in D(\tau_s)_r$, so by Corollaries 2.5 and 2.6 (real) we obtain that the semigroup associated with τ_s is real, and positive on H_r. Consequently Corollary 2.6 (complex, form symmetric) implies $|u| \in D(\tau)$, $\tau(|u|) = \tau_s(|u|) \leqslant \tau_s(u) = \operatorname{Re}\tau(u)$ for all $u \in D(\tau)$. $\square$

2.7 Remark. In the case that τ is symmetric, the inequality in (iii) is in fact equivalent to '$\tau(|u|) \leqslant \operatorname{Re}\tau(u)$ for all $u \in D(\tau)$' (if T is real and $D(\tau)$ is a sublattice of H). The latter inequality is one of the equivalences in the first Beurling-Deny criterion.

Finally, let H be a *Stonian* Hilbert sublattice of $M(\mu;\mathbb{R})$ (i.e., $f \wedge 1 \in H$ for all $f \in H$). A bounded operator S on H is called *sub-Markovian* if S is positive and $|Sf| \leqslant 1$ for all $f \in H$ such that $|f| \leqslant 1$. Note that S is sub-Markovian if and only if $Sf \leqslant 1$ for all $f \in H$ such that $f \leqslant 1$. A semigroup T is called *sub-Markovian* if $T(t)$ is sub-Markovian for all $t \geqslant 0$.

Observe that the projection from H onto $C := \{f \in H;\, f \leqslant 1\}$ is given by $f \mapsto f \wedge 1$. Applying Proposition 2.2, Theorem 2.1, and Corollary 2.6, we obtain

2.8 Corollary. *(cf. [4; Cor. of Thm. 2], [10; Cor. 2.7]) Let H be a Stonian Hilbert sublattice of $M(\mu;\mathbb{R})$. Then the following are equivalent:*

(i) T_0 *is sub-Markovian.*

(ii) H_τ *is a Stonian sublattice of* H*, and* T *is sub-Markovian.*

(iii) $D(\tau)$ *is a Stonian sublattice of* H*, and* $\tau(u \wedge 1, (u-1)^+) \geqslant 0$ *for all* $u \in D(\tau)$.

(iv) *There exists a dense subset* D *of* $D(\tau)$ *such that* $u \in D$ *implies* $u \wedge 1 \in D(\tau)$ *and* $\tau(u \wedge 1, (u-1)^+) \geqslant 0$.

It is this application of Theorem 2.1 that shows why it is important not to assume that τ is non-negative in that theorem. The other applications could also be proven by passing to the non-negative form $\tau + \omega_0(\tau)$.

It seems that (variants of) the above corollary are well-known only in the case that τ is non-negative. Nevertheless, it was already observed by H. Kunita [4] that this assumption is not needed. We point out that Kunita also did not assume the form to be symmetric—his result deals with an arbitrary densely defined closed sectorial form. We did not find a reference to Kunita's paper [4] in the recent literature on non-symmetric Dirichlet forms.

We note that in [11; Theorems 1 and 2], the result that is generalized by our Theorem 2.1 has been applied to characterize L_∞-contractivity of semigroups acting on Hilbert space valued L_2-spaces.

3 Generalized ideals

In this section we provide the tools needed to formulate and prove the domination criterion in Section 4.

As announced in the introduction, G will be a (real or complex) Hilbert space. The norm in G is denoted by $|\cdot|$. By $M(\mu;G)$ we denote the measurable functions from the measure space (Ω,μ) to G.

We define the convex cone

$$C := \big\{(u,v) \in M(\mu;G) \times M(\mu;\mathbb{R});\ |u| \leqslant v\big\} \tag{3.1}$$

(cf. [11; proof of Thm. 4]). The following elementary observation establishes the relationship between C and domination properties of operators.

3.1 Remark. Let V be a sublattice of $M(\mu;\mathbb{R})$, and let U be a subspace of $M(\mu;G)$ with $|u| \in V$ for all $u \in U$. Let $T\colon V \to V$, $S\colon U \to U$ be linear operators. Then T is positive and S is dominated by T (i.e., $|Su| \leqslant T|u|$ for all $u \in U$) if and only if $C \cap (U \times V)$ is invariant under $S \times T$.

In view of this remark one can expect that Theorem 2.1 is applicable to domination of semigroups. In order to apply Theorem 2.1, we have to study the projection from $H := L_2(\mu;G) \times L_2(\mu;\mathbb{R})$ (considered as a real Hilbert space) onto $C \cap H$ and the invariant subspaces for this projection.

Let $P\colon M(\mu;G) \times M(\mu;\mathbb{R}) \to C$ be defined by

$$P(u,v) := \frac{1}{2}\big((|u| + |u| \wedge v)^+ \operatorname{sgn} u, (|u| \vee v + v)^+\big). \tag{3.2}$$

Here we use the notation

$$\operatorname{sgn} x := \begin{cases} \frac{1}{|x|}x & \text{if } x \neq 0,\\ 0 & \text{if } x = 0,\end{cases}$$

for $x \in G$, and correspondingly for a function $u\colon \Omega \to G$.

3.2 Lemma. (a) *Let $\widetilde{C} := \big\{(x,y) \in G \times \mathbb{R};\ |x| \leqslant y\big\}$. Then $\widetilde{C}$ is a closed convex subset of $G \times \mathbb{R}$, and for $(x,y) \in G \times \mathbb{R}$ one has*

$$P_{\widetilde{C}}^{G\times\mathbb{R}}(x,y) = \frac{1}{2}\big((|x| + |x| \wedge y)^+ \operatorname{sgn} x, (|x| \vee y + y)^+\big)$$

(= $P(x,y)$ if G and $\mathbb{R}$ are interpreted as $M(\mu;G)$ and $M(\mu;\mathbb{R})$, respectively, with a one-point measure space).

(b) *The restriction of P to $H = L_2(\mu;G) \times L_2(\mu;\mathbb{R})$ is the projection from H onto $C \cap H$.*

Proof. (a) Let $x \in G,\ |x| = 1$. Obviously, $\widetilde{C}$ is invariant under the projection $P^{G\times\mathbb{R}}_{\mathbb{R}x\times\mathbb{R}}$. By Lemma 2.3 this implies $P^{G\times\mathbb{R}}_{\widetilde{C}}(\alpha x, y) \in \mathbb{R}x \times \mathbb{R}$ for all $\alpha, y \in \mathbb{R}$.

A simple geometric consideration shows that on $[0,\infty)x \times \mathbb{R}$ the projection $P^{G\times\mathbb{R}}_{\widetilde{C}}$ is given by

$$\begin{aligned} P^{G\times\mathbb{R}}_{\widetilde{C}}(\alpha x, y) &= \begin{cases} (\alpha x, y) & \text{for } y \geqslant \alpha, \\ \frac{1}{2}((\alpha+y)x, \alpha+y) & \text{for } |y| < \alpha, \\ (0,0) & \text{for } y \leqslant -\alpha \end{cases} \\ &= \frac{1}{2}\big((\alpha + \alpha\wedge y)^+ x, (\alpha \vee y + y)^+\big). \end{aligned}$$

For arbitrary x this transforms into

$$P^{G\times\mathbb{R}}_{\widetilde{C}}(x, y) = \frac{1}{2}\big((|x| + |x|\wedge y)^+ \operatorname{sgn} x, (|x| \vee y + y)^+\big).$$

(b) is a direct consequence of (a). □

In order to describe P-invariant subspaces of $M(\mu;G) \times M(\mu;\mathbb{R})$, we need the following notion.

3.3 Definition. Let U, V be subspaces of $M(\mu;G)$, $M(\mu;\mathbb{K})$, respectively. We call U a *generalized ideal* of V if

(i) $u \in U$ implies $|u| \in V$,

(ii) $u \in U,\ v \in V,\ |v| \leqslant |u|$ implies $v \operatorname{sgn} u \in U$.

If V is a generalized ideal of V, we say that V is a *generalized ideal of itself.*

3.4 Remark. (a) If V is a real subspace of $M(\mu;\mathbb{K})$ then it is sufficient to require (ii) for $v \in V_r$; in particular, U is a generalized ideal of V if and only if U is a generalized ideal of V_r.

(b) Ouhabaz [10], [11] introduced the object defined in Definition 3.3 under the name 'ideal'. We reserve the term 'ideal' for lattice ideals.

(c) The example given in Remark 4.2(d) illustrates that, in the case $G = \mathbb{K}$, it may happen that $U \neq \{0\}$ is a generalized ideal of V, and $U \cap V = \{0\}$.

The following proposition establishes the relationship between the projection P and the notion of generalized ideal.

3.5 Proposition. *Let U, V be subspaces of $M(\mu;G)$, $M(\mu;\mathbb{R})$, respectively. Then the following are equivalent:*

(i) *$U \times V$ is invariant under P.*

(ii) *V is a sublattice of $M(\mu;\mathbb{R})$, and U is a generalized ideal of V.*

Proof. (i) ⇒ (ii). Let $v \in V$. Then $(0, v^+) = P(0,v) \in U \times V$, which proves the first assertion of (ii).

Let $u \in U$. Then $\frac{1}{2}(u, |u|) = P(u, 0) \in U \times V$, so $|u| \in V$. Further, let $v \in V$, $|v| \leqslant |u|$. Then $\frac{1}{2}(u + v \operatorname{sgn} u, |u| + v) = P(u, v) \in U \times V$, and therefore $v \operatorname{sgn} u = (u + v \operatorname{sgn} u) - u \in U$. This shows the second assertion.

(ii) $\Rightarrow$ (i). Let $(u, v) \in U \times V$. Then $|u| \in V$ since U is a generalized ideal of V. Since V is a sublattice of $M(\mu; \mathbb{R})$, we obtain

$$\frac{1}{2}(|u| + |u| \wedge v)^+, \ \frac{1}{2}(|u| \vee v + v)^+ \in V.$$

The inequality $0 \leqslant \frac{1}{2}(|u| + |u| \wedge v)^+ \leqslant |u|$ and U being a generalized ideal of V then imply $\frac{1}{2}(|u| + |u| \wedge v)^+ \operatorname{sgn} u \in U$. Hence $P(u, v) \in U \times V$. □

The following proposition establishes the relationship between the notions of ideal and generalized ideal.

3.6 Proposition. *(cf. [10; Prop. 3.6]) Let $U \subseteq V$ be subspaces of $M(\mu; \mathbb{K})$.*

(a) *Assume that V is a real subspace, and that U is a generalized ideal of V. If $u \in U$, $v \in V$, $|v| \leqslant |u|$, then $v \in U$.*

(b) *Assume that V is a generalized ideal of itself and a sublattice of $M(\mu; \mathbb{K})$. Then U is an ideal of V if and only if U is a generalized ideal of V.*

Proof. (a) If $u \in U$, $v \in V$, $|v| \leqslant |u|$ then $\overline{u} \in V$, $|\overline{u}| \leqslant |u|$, therefore $|u| = \overline{u} \operatorname{sgn} u \in U$, $v = v \operatorname{sgn} |u| \in U$.

(b) By (a) we only have to show: If U is an ideal of V then U is a generalized ideal of V. For $u \in U \subseteq V$ we have $|u| \in V$, by Definition 3.3(i). Let moreover $v \in V$, $|v| \leqslant |u|$. Then $v \operatorname{sgn} u \in V$ by Definition 3.3(ii). But also $|v \operatorname{sgn} u| \leqslant |u|$, so U being an ideal of V implies $v \operatorname{sgn} u \in U$. □

3.7 Corollary. *Let U, V be sublattices of $M(\mu; \mathbb{K})$, V a generalized ideal of itself. Then U is an ideal of V if and only if U is a generalized ideal of V.*

Proof. By the above proposition, we only have to show that $U \subseteq V$. This is clear if U is an ideal of V. If U is a generalized ideal of V then $u \in U$ implies $(\operatorname{Re} u)^+ \in U$ and hence $(\operatorname{Re} u)^+ = |(\operatorname{Re} u)^+| \in V$. This shows $U \subseteq V$. □

4 The domination criterion

In this section we assume $\mathfrak{a}$, $\mathfrak{b}$ to be two closed sectorial forms in $L_2(\mu; G)$, $L_2(\mu; \mathbb{K})$, respectively, where G is a Hilbert space. Let $H_{\mathfrak{a}} := \overline{D(\mathfrak{a})}^{L_2(\mu;G)}$. Denote by A the m-sectorial operator in $H_{\mathfrak{a}}$ associated with $\mathfrak{a}$, by S the C_0-semigroup on $H_{\mathfrak{a}}$ generated by $-A$, and define the semigroup S_0 on $L_2(\mu; G)$ as in Proposition 2.2. Similarly for $\mathfrak{b}$, with the operator B, the C_0-semigroup T on $H_{\mathfrak{b}} := \overline{D(\mathfrak{b})}^{L_2(\mu;\mathbb{K})}$, and the semigroup T_0 on $L_2(\mu; \mathbb{K})$.

The following theorem is the generalization of Ouhabaz' criterion announced in the introduction.

4.1 Theorem. *Assume that $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu;\mathbb{K})$ and that T is positive. Then the following statements* (i)–(iv) *are equivalent:*

(i) *S_0 is dominated by T_0, i.e., $|S_0(t)u| \leqslant T_0(t)|u|$ for all $u \in L_2(\mu;G)$, $t \geqslant 0$.*

(ii) *$H_{\mathfrak{a}}$ is a generalized ideal of $H_{\mathfrak{b}}$, and S is dominated by T.*

(iii) *$D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$, and*

$$\operatorname{Re}\mathfrak{a}(u,v) \geqslant \mathfrak{b}(|u|,|v|) \quad \bigl(u,v \in D(\mathfrak{a}),\ (u\,|\,v)_G = |u|\,|v|\bigr). \tag{4.1}$$

(iv) *There exist dense subspaces $D_{\mathfrak{a}} \subseteq D(\mathfrak{a})$, $D_{\mathfrak{b}} \subseteq D(\mathfrak{b})_r$ such that $D_{\mathfrak{b}}$ is a sublattice of $L_2(\mu;\mathbb{R})$, $D_{\mathfrak{a}}$ is a generalized ideal of $D_{\mathfrak{b}}$, and*

$$\operatorname{Re}\mathfrak{a}(u, v\operatorname{sgn} u) \geqslant \mathfrak{b}(|u|, v) \quad \bigl(u \in D_{\mathfrak{a}},\ v \in D_{\mathfrak{b}},\ 0 \leqslant v \leqslant |u|\bigr).$$

Moreover, if (ii) *holds then*

(v) *$D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$, and*

$$\mathfrak{b}(|u|) - \mathfrak{b}(v) \leqslant \operatorname{Re}\bigl(\mathfrak{a}(u) - \mathfrak{a}(v\operatorname{sgn} u)\bigr) \quad \bigl(u \in D(\mathfrak{a}),\ v \in D(\mathfrak{b})_r,\ |v| \leqslant |u|\bigr).$$

Conversely, if $\mathfrak{a}$ and $\mathfrak{b}$ are symmetric and

(vi) *there exist dense subspaces $D_{\mathfrak{a}} \subseteq D(\mathfrak{a})$, $D_{\mathfrak{b}} \subseteq D(\mathfrak{b})_r$ such that $D_{\mathfrak{b}}$ is a sublattice of $L_2(\mu;\mathbb{R})$, $D_{\mathfrak{a}}$ is a generalized ideal of $D_{\mathfrak{b}}$, and*

$$\mathfrak{b}(|u|) - \mathfrak{b}(v) \leqslant \mathfrak{a}(u) - \mathfrak{a}(v\operatorname{sgn} u) \quad \bigl(u \in D_{\mathfrak{a}},\ v \in D_{\mathfrak{b}},\ 0 \leqslant v \leqslant |u|\bigr),$$

then (i)–(iv) *hold.*

4.2 Remarks. Assume that $G = \mathbb{K}$.

(a) Assume that $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu;\mathbb{K})$ and that T is positive. Then, by Corollary 2.6, T_0 is positive and hence dominated by itself. By Theorem 4.1, (i)⇒(ii),(iii), we obtain that $H_{\mathfrak{b}}$ is a generalized ideal of itself, and so is $D(\mathfrak{b})$.

(b) Since $(u\,|\,v)_{\mathbb{K}} = u\overline{v}$, the condition '$(u\,|\,v)_G = |u|\,|v|$' in (4.1) in statement (iii) can be replaced by '$u\overline{v} \geqslant 0$'.

(c) In the case $H_{\mathfrak{a}} \subseteq H_{\mathfrak{b}}$, the word 'generalized' can be dropped in statement (ii). This is immediate from Proposition 3.6(b) and part (a) above.

(d) The following elementary example illustrates that, in the situation of Theorem 4.1, the domination property is possible without $H_{\mathfrak{a}}$ being a subset of $H_{\mathfrak{b}}$, even if $G = \mathbb{K}$. Let Ω consist of two points, $L_2(\mu;\mathbb{K}) = \mathbb{K}^2$, $\mathfrak{b} = 0$ on $D(\mathfrak{b}) := \{(x,x);\, x \in \mathbb{K}\}$. Let $\gamma \in \mathbb{K}$, $|\gamma| = 1$, $\gamma \neq 1$, $\mathfrak{a} = 0$ on $D(\mathfrak{a}) := \{(x,\gamma x);\, x \in \mathbb{K}\}$. Then $H_{\mathfrak{a}} = D(\mathfrak{a})$ is a generalized ideal of $H_{\mathfrak{b}} = D(\mathfrak{b})$, and $S = (\mathrm{id}_{H_{\mathfrak{a}}};\, t \geqslant 0)$ is dominated by $T = (\mathrm{id}_{H_{\mathfrak{b}}};\, t \geqslant 0)$.

Proof of Theorem 4.1. First observe that the complex case can be reduced to the real case: Consider G as a real Hilbert space $G_{\mathbb{R}}$ with scalar product defined by $\operatorname{Re}(x\,|\,y)$. In the same way, replace $\mathfrak{a}$ by the form $\mathfrak{a}_{\mathbb{R}}$ defined by $\mathfrak{a}_{\mathbb{R}}(u,v) := \operatorname{Re}\bigl(\mathfrak{a}(u,v)\bigr)$. Then $\mathfrak{a}_{\mathbb{R}}$ is a closed sectorial form in $L_2(\mu;G_{\mathbb{R}})$, and $\mathfrak{a}_{\mathbb{R}}$

is associated with S, considered as a semigroup on $(H_{\mathfrak{a}})_{\mathbb{R}}$. Finally, restrict the (complex) domains of $\mathfrak{b}$, T, and T_0 to the respective real spaces, and take into account Remark 3.4(a).

For the remainder of the proof we assume $\mathbb{K} = \mathbb{R}$. Following the proof of [11; Thm. 4, (i)⇒(ii)], we define the sesquilinear form $\tau =: \mathfrak{a} \times \mathfrak{b}$ by

$$D(\tau) := D(\mathfrak{a}) \times D(\mathfrak{b}), \quad \tau\big((u_1, v_1), (u_2, v_2)\big) := \mathfrak{a}(u_1, u_2) + \mathfrak{b}(v_1, v_2).$$

Then τ is a closed sectorial form in $H := L_2(\mu; G) \times L_2(\mu; \mathbb{R})$. Moreover, the C_0-semigroup $W = S \times T$ on $H_\tau = H_{\mathfrak{a}} \times H_{\mathfrak{b}}$ associated with τ is given by

$$W(t)(f, g) = \big(S(t)f, T(t)g\big) \quad \big((f, g) \in H_\tau,\ t \geqslant 0\big),$$

and we have $W_0 = S_0 \times T_0$.

Let C and P be as in Section 3, (3.1) and (3.2). Recall from Lemma 3.2(b) that $P{\restriction_H} = P^H_{C \cap H}$. By Remark 3.1, statement (i) of the theorem holds if and only if $C \cap H$ is invariant under W_0.

Since $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu; \mathbb{R})$, Proposition 3.5 implies that $H_{\mathfrak{a}}$ is a generalized ideal of $H_{\mathfrak{b}}$ if and only if $H_\tau = H_{\mathfrak{a}} \times H_{\mathfrak{b}}$ is invariant under P. We can thus reformulate statement (ii) of the theorem as follows.

(ii') H_τ is invariant under P, and $C \cap H_\tau$ is invariant under W.

After these preparations we proceed to the proof of the asserted implications.

(i) ⇔ (ii') is a direct consequence of Proposition 2.2.

(ii') ⇒ (iii), (v). Theorem 2.1, (i)⇒(ii), implies that $D(\tau)$ is invariant under P, and hence, by Proposition 3.5, that $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$. Moreover,

$$\tau\big(P(u, v), (1 - P)(u, v)\big) \geqslant 0 \quad \big((u, v) \in D(\tau)\big).$$

Let $u \in D(\mathfrak{a})$, $v \in D(\mathfrak{b})$, $|v| \leqslant |u|$. Then $P(u, v) = \frac{1}{2}(u + v \operatorname{sgn} u, |u| + v)$, $(1 - P)(u, v) = \frac{1}{2}(u - v \operatorname{sgn} u, v - |u|)$, and we obtain

$$\mathfrak{a}(u + v \operatorname{sgn} u, u - v \operatorname{sgn} u) \geqslant \mathfrak{b}(|u| + v, |u| - v). \tag{4.2}$$

Replacing v by $-v$ in (4.2) and adding the resulting inequality to (4.2) yields (v).

In order to show (iii), let $u, v \in D(\mathfrak{a})$ with $(u\,|\,v)_G = |u| \cdot |v|$. Let $\tilde{u} := \frac{1}{2}(u + v)$, $\tilde{v} := \frac{1}{2}(|u| - |v|)$. Then $\tilde{u} \in D(\mathfrak{a})$, $\tilde{v} \in D(\mathfrak{b})$, $|\tilde{v}| \leqslant |\tilde{u}|$. Moreover,

$$u = \tilde{u} + \tilde{v} \operatorname{sgn} \tilde{u}, \quad v = \tilde{u} - \tilde{v} \operatorname{sgn} \tilde{u}, \quad |u| = |\tilde{u}| + \tilde{v}, \quad |v| = |\tilde{u}| - \tilde{v}.$$

Hence (4.2) applied to $\tilde{u}$, $\tilde{v}$ yields (iii).

(iii) ⇒ (iv). Take $D_{\mathfrak{a}} := D(\mathfrak{a})$, $D_{\mathfrak{b}} := D(\mathfrak{b})$, and recall from Corollary 2.6 that $D(\mathfrak{b})$ is a sublattice of $L_2(\mu; \mathbb{R})$.

(iv) ⇒ (ii'). Without restriction $\mathfrak{a}, \mathfrak{b}$ are non-negative, so τ is non-negative. By Proposition 3.5, the assumptions imply that $D_{\mathfrak{a}} \times D_{\mathfrak{b}}$ is invariant under P. Let $u \in D_{\mathfrak{a}}$, $v \in D_{\mathfrak{b}}$. By Theorem 2.1, (v)⇒(i), we only have to show

$$\alpha := \tau\big((u, v), (1 - P)(u, v)\big) \geqslant 0.$$

Let $P =: (P_1, P_2)$. Then $P_1(u,v) \in D_{\mathfrak{a}}$. Further, note that $|P_1(u,v)| = P_1(|u|,v) \leqslant |u|$ and $P_1(u,v) = P_1(|u|,v)\operatorname{sgn} u$. Hence $\tilde{v} := |u| - P_1(|u|,v) \in D_{\mathfrak{b}}$, $0 \leqslant \tilde{v} \leqslant |u|$, and $\tilde{v}\operatorname{sgn} u = u - P_1(u,v)$. Therefore,

$$\begin{aligned}\alpha &= \mathfrak{a}(u, u - P_1(u,v)) + \mathfrak{b}(v, v - P_2(u,v)) \\ &= \mathfrak{a}(u, \tilde{v}\operatorname{sgn} u) + \mathfrak{b}(v, v - P_2(|u|,v)) \\ &\geqslant \mathfrak{b}(|u|, \tilde{v}) + \mathfrak{b}(v, v - P_2(|u|,v)) = (\mathfrak{b} \times \mathfrak{b})\big((|u|,v), (1-P)(|u|,v)\big).\end{aligned}$$

To see that the latter quantity is non-negative, recall that T_0 is dominated by itself, so $C \cap H$ is invariant under $T_0 \times T_0$; then apply Proposition 2.2 and Theorem 2.1, (i)⇒(iv).

(vi) ⇒ (iv). If $\mathfrak{a}$ and $\mathfrak{b}$ are symmetric then (vi) is equivalent to the validity of (4.2) for all $u \in D_{\mathfrak{a}}$, $v \in D_{\mathfrak{b}}$, $0 \leqslant v \leqslant |u|$. Defining $\tilde{u} := \frac{1}{2}(u + v\operatorname{sgn} u)$, $\tilde{v} := \frac{1}{2}(|u| - v)$, we deduce (iv) as in the proof of (ii') ⇒ (iii). ☐

In the following result we study the special case that $G = \mathbb{K}$ and that both semigroups S_0 and T_0 are positive.

4.3 Corollary. *Assume that $H_{\mathfrak{a}}$, $H_{\mathfrak{b}}$ are sublattices of $L_2(\mu;\mathbb{K})$ and that S, T are positive. Then the following are equivalent:*

(i) *$H_{\mathfrak{a}}$ is an ideal of $H_{\mathfrak{b}}$, and S is dominated by T.*

(ii) *$D(\mathfrak{a})$ is an ideal of $D(\mathfrak{b})$, and*

$$\mathfrak{a}(u,v) \geqslant \mathfrak{b}(u,v) \quad \big(0 \leqslant u, v \in D(\mathfrak{a})\big). \tag{4.3}$$

Proof. By Remark 4.2(a), S_0 and T_0 are positive, $H_{\mathfrak{b}}$ is a generalized ideal of itself, and so is $D(\mathfrak{b})$. By Corollary 2.6, $D(\mathfrak{a})$ and $D(\mathfrak{b})$ are sublattices of $L_2(\mu;\mathbb{K})$. Moreover, since S_0 is dominated by S_0, we obtain by Theorem 4.1, (i)⇒(iii), that (4.3) implies (4.1):

$$\operatorname{Re}\mathfrak{a}(u,v) \geqslant \mathfrak{a}(|u|,|v|) \geqslant \mathfrak{b}(|u|,|v|) \quad \big(u, v \in D(\mathfrak{a}),\ u\overline{v} \geqslant 0\big).$$

Now the assertion follows from Theorem 4.1, (ii)⇔(iii), and Corollary 3.7. ☐

We conclude this section by showing the connection between the domination results of Theorem 4.1 and the well-established 'generalized Kato's inequality for operators'.

4.4 Theorem. *Assume that $H_{\mathfrak{b}}$ is a sublattice of $L_2(\mu;\mathbb{K})$, that T is positive, and that $H_{\mathfrak{a}}$ is a generalized ideal of $H_{\mathfrak{b}}$. Then the following are equivalent:*

(i) *S is dominated by T.*

(ii) *$u \in D(A)$ implies $|u| \in D(\mathfrak{b})$, and*

$$\operatorname{Re}\,(Au \,|\, v\operatorname{sgn} u) \geqslant \mathfrak{b}(|u|, v) \quad \big(u \in D(A),\ 0 \leqslant v \in D(\mathfrak{b})\big).$$

(iii) *$\operatorname{Re}\,(Au \,|\, v\operatorname{sgn} u) \geqslant \big(|u| \,\big|\, B^*v\big)$ for all $u \in D(A)$, $0 \leqslant v \in D(B^*)$, where B^* is the adjoint of B in the Hilbert space $H_{\mathfrak{b}}$.*

Proof. (i) $\Rightarrow$ (ii). The first assertion of (ii) follows from Theorem 4.1, (ii)$\Rightarrow$(iii). Let $u \in D(A),\ v \in D(\mathfrak{b})_+$. The domination hypothesis implies

$$\begin{aligned}\operatorname{Re}\frac{1}{t}\big(u - S(t)u \,\big|\, v \operatorname{sgn} u\big) &= \frac{1}{t}\big(|u| \,\big|\, v\big) - \frac{1}{t}\operatorname{Re}\big(S(t)u \,\big|\, v \operatorname{sgn} u\big) \\ &\geqslant \frac{1}{t}\big(|u| - T(t)|u| \,\big|\, v\big) \quad (t > 0).\end{aligned}$$

By [10; Lemma 1.1] we obtain

$$\frac{1}{t}\big(|u| - T(t)|u| \,\big|\, v\big) \longrightarrow \mathfrak{b}(|u|, v) \quad (t \to 0),$$

and thus

$$\operatorname{Re}\,(Au \,|\, v \operatorname{sgn} u) = \lim_{t\to 0}\operatorname{Re}\frac{1}{t}\big(u - S(t)u \,\big|\, v \operatorname{sgn} u\big) \geqslant \mathfrak{b}(|u|, v).$$

(ii) $\Rightarrow$ (iii). Recall that $D(B^*) \subseteq D(\mathfrak{b})$, and $\mathfrak{b}(w, v) = (w|B^*v)$ for all $w \in D(\mathfrak{b}),\ v \in D(B^*)$; cf. [3; Thm. VI.2.5]. This shows that (iii) is a special case of (ii).

(iii) $\Rightarrow$ (i) is shown as in [8; C-II, proof of Thm. 4.2, (ii)$\Rightarrow$(i)]. $\square$

4.5 Remarks. (a) Corollary 4.3, (ii)$\Rightarrow$(i), was proved in [14; Cor. B.3] for symmetric forms. In [10; Thm. 3.7], Corollary 4.3 was proved for the case $H_{\mathfrak{a}} = H_{\mathfrak{b}}$.

(b) The equivalence '(i)$\Leftrightarrow$(ii)' in Theorem 4.4 is due to Simon [13; Thm. 1] for the special case of densely defined symmetric forms. In the case that both semigroups act on the same space, equivalence '(i)$\Leftrightarrow$(iii)' is true in the more general framework of countably order complete Banach lattices (see [8; C-II, Thm. 4.2]).

5 Supplement on generalized ideals

We first study the relationship between the notions of 'sublattice of $M(\mu;\mathbb{K})$' and of 'generalized ideal of itself' (recall Definition 3.3).

5.1 Proposition. *Let $V \subseteq M(\mu;\mathbb{K})$ be a generalized ideal of itself. Then V is a sublattice of $M(\mu;\mathbb{K})$.*

Proof. For $\mathbb{K} = \mathbb{R}$ there is nothing to show. In the case $\mathbb{K} = \mathbb{C}$ we only have to show that $v \in V$ implies $(\operatorname{Re} v)^+ \in V$. So, let $v \in V$. Applying alternatingly conditions (i) and (ii) of Definition 3.3, we obtain $\overline{v} \operatorname{sgn} v = |v| \in V,\ v \operatorname{sgn} v \in V$, thus $(\operatorname{Re} v) \operatorname{sgn} v \in V,\ |\operatorname{Re} v| = |(\operatorname{Re} v) \operatorname{sgn} v| \in V,\ |\operatorname{Re} v| \operatorname{sgn} v \in V$, and finally $(\operatorname{Re} v)^+ \operatorname{sgn} v = \frac{1}{2}\big((\operatorname{Re} v) \operatorname{sgn} v + |\operatorname{Re} v| \operatorname{sgn} v\big) \in V,\ (\operatorname{Re} v)^+ = |(\operatorname{Re} v)^+ \operatorname{sgn} v| \in V$. $\square$

The following result shows that the converse of the above proposition holds for $\mathbb{K} = \mathbb{R}$, whereas for $\mathbb{K} = \mathbb{C}$ one needs an additional hypothesis.

5.2 Proposition. *Let V be a sublattice of $M(\mu;\mathbb{K})$.*

(a) *If $\mathbb{K}=\mathbb{R}$ then V is a generalized ideal of itself.*

(b) *If $\mathbb{K}=\mathbb{C}$ and V is uniformly complete then V is a generalized ideal of itself. (We recall that for an Archimedean vector lattice E,* uniform completeness *means: For all $x\in E_+$, the (complex) principal ideal E_x, endowed with the norm*

$$\|y\|_x := \inf\bigl\{c\geqslant 0;\ |y|\leqslant cx\bigr\},$$

is complete; cf. [7; Def. 1.1.7 and Prop. 1.2.13].)

Proof. (a) For all $u,v\in V,\ |v|\leqslant|u|$, one has $v\operatorname{sgn}u=|u+v|-|u|\in V$.

(b) Observe that, for all $w,z\in\mathbb{C},\ |w|\leqslant|z|,\ 0<\lambda\leqslant 1$, one has

$$\Bigl|\frac{1}{\lambda}\bigl(|\overline{z}+\lambda w|-|\overline{z}|\bigr)-\operatorname{Re}\,(w\operatorname{sgn}z)\Bigr|\leqslant\lambda|w|.$$

(It is easy to see that it is sufficient to prove the inequality for the special case $w=1$. For the proof of this case one estimates $\frac{1}{\lambda}\int_0^\lambda\bigl|\frac{d}{dt}|\overline{z}+t|-\operatorname{Re}\,(\operatorname{sgn}z)\bigr|\,dt$ suitably.)

Let $u,v\in V$ with $|v|\leqslant|u|$. Then $\bigl||\overline{u}+\lambda v|-|\overline{u}|\bigr|\leqslant\lambda|v|$ and hence $\frac{1}{\lambda}\bigl(|\overline{u}+\lambda v|-|\overline{u}|\bigr)\in V_{|v|}$ for all $\lambda>0$, where $V_{|v|}$ is the principal ideal in V generated by $|v|$. The above inequality shows that $\frac{1}{\lambda}\bigl(|\overline{u}+\lambda v|-|\overline{u}|\bigr)\to\operatorname{Re}\,(v\operatorname{sgn}u)$ as $\lambda\to 0$, in the $|v|$-norm on $M(\mu;\mathbb{C})_{|v|}$. The uniform completeness of V implies that $V_{|v|}$ is closed in $M(\mu;\mathbb{C})_{|v|}$ and therefore $\operatorname{Re}\,(v\operatorname{sgn}u)\in V$. Applying this result to $-iv$ in place of v, we obtain $\operatorname{Im}(v\operatorname{sgn}u)=\operatorname{Re}(-iv\operatorname{sgn}u)\in V,\ v\operatorname{sgn}u\in V$. This shows property (ii) of Definition 3.3. ☐

If $V\subseteq M(\mu;\mathbb{K})$ is a generalized ideal of itself one may ask whether $v\operatorname{sgn}u\in V$ for all $u,v\in V$. This is evidently true if V is an ideal of $M(\mu;\mathbb{K})$. With $(\Omega,\mathcal{A},\mu)=(\mathbb{N},2^{\mathbb{N}},\text{counting measure})$ and V the set of all convergent sequences one obtains a generalized ideal of itself, but there exist $u,v\in V$ such that $v\operatorname{sgn}u\notin V$.

5.3 Proposition. *Let V be a sublattice of $M(\mu;\mathbb{K})$. Assume that, for each countable order bounded set $A\subseteq V_r$, the supremum of A taken in $M(\mu;\mathbb{R})$ belongs to V. Then $v\operatorname{sgn}u\in V$ for all $u,v\in V$.*

Proof. The assumption implies that V is countably order complete and hence uniformly complete (cf. [7; Prop. 1.1.8]). Thus we obtain from Proposition 5.2 that V is a generalized ideal of itself.

Since V is a lattice it is sufficient to show the assertion for $v\geqslant 0$. Note that $|ku|\wedge v\in V$ for all $k\in\mathbb{N}$. Thus property (ii) of Definition 3.3 implies $(|ku|\wedge v)\operatorname{sgn}u\in V$ $(k\in\mathbb{N})$. Since $(|ku|\wedge v)\operatorname{sgn}u\to v\operatorname{sgn}u$ $(k\to\infty)$ pointwise and boundedly (by v), the assumption on V implies $v\operatorname{sgn}u\in V$. ☐

5.4 Remarks. (a) The assumptions of Proposition 5.3 are fulfilled, in particular, if V is a closed sublattice of $L_p(\mu;\mathbb{K})$, with $1 \leqslant p < \infty$ (or more generally, of a Banach function space with order continuous norm).

(b) Proposition 5.3 can be interpreted as the statement that in a sublattice V of $M(\mu;\mathbb{K})$ satisfying the assumption of Proposition 5.3, the signum operator S_u with respect to an element $u \in V$ is explicitly given by

$$S_u v = v \operatorname{sgn} u \quad (v \in V)$$

(cf. [8; C-I, 8.] for the signum operator).

(c) Incidentally, Proposition 5.3 also supplies a detail which seems to have been neglected in [10; proof of Prop. 3.2]. Indeed, if H is a closed sublattice of $L_2(\Omega,\mathcal{A},\mu)$, then $H \cong L_2(\widetilde{\Omega},\widetilde{\mathcal{A}},\tilde{\mu})$ (as a Hilbert space and a lattice) for a suitable measure space; cf. [7; Cor. 2.7.5]. Now, if $u \in L_2(\mu)$ belongs to H and $\tilde{u}$ is the corresponding element of $L_2(\tilde{\mu})$, then part (a) above shows that the signum operator S_u is given by $S_u v = v \operatorname{sgn} u$, but evidently in the representation $L_2(\tilde{\mu})$ one has $S_{\tilde{u}}\tilde{v} = \tilde{v} \operatorname{sgn} \tilde{u}$. Since the signum operator is unique this implies that $v \operatorname{sgn} u$ is represented by the function $\tilde{v} \operatorname{sgn} \tilde{u}$ in $L_2(\tilde{\mu})$, for arbitrary $u, v \in H$.

(d) In Proposition 5.3, the hypothesis on V may not simply be replaced by assuming V_r countably order complete. Indeed, let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$, and define a sublattice V of $M(\mathbb{N};\mathbb{R})$ ($\mathbb{N}$ with counting measure) by

$$V := \big\{u \in \ell_\infty(\mathbb{N});\, u(1) = \lim u(\mathcal{F})\big\}.$$

Then V is order complete, but, defining $u, v \in V$ by $u(1) := 0$, $u(n) := 1/n$ $(n \geqslant 2)$, $v(n) := 1$ $(n \in \mathbb{N})$, we obtain $v \operatorname{sgn} u \notin V$. We remark that, for $u \in V$, the signum operator S_u in V is given by multiplication with

$$\tilde{u}(n) := \begin{cases} \lim(\operatorname{sgn} u)(\mathcal{F}) & \text{for } n = 1, \\ \operatorname{sgn} u(n) & \text{for } n \geqslant 2. \end{cases}$$

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