A weak Gordon type condition for absence of eigenvalues of one-dimensional Schrödinger operators

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Abstract

We study one-dimensional Schrödinger operators with complex measures as potentials and present an improved criterion for absence of eigenvalues which involves a weak local periodicity condition. The criterion leads to sharp quantitative bounds on the eigenvalues. We apply our result to quasiperiodic measures as potentials.

MSC2010: 34L15, 34L40, 81Q10, 81Q12

Key words: Schrödinger operators, eigenvalue problem, quasiperiodic potential

1 Introduction

In this paper we study Schrödinger operators of the form

$$H_\mu = -\Delta + \mu$$

in $L_2(\mathbb{R})$, where the potential μ is locally a complex Radon measure. We are going to study absence of eigenvalues of H_μ under the condition that μ can be approximated by periodic measures in a suitable sense. Roughly speaking, we require that there are arbitrarily large periods $p > 0$ such that the three "pieces" $\mathbb{1}_{[-p,0]}\mu$, $\mathbb{1}_{[0,p]}\mu$ and $\mathbb{1}_{[p,2p]}\mu$ look very similar.

The above so-called *Gordon condition* first appeared in [5]. Gordon treated the case of bounded potentials and measured the distance of the "pieces" in the L_∞-norm. In [2] this was generalised to uniformly locally integrable potentials and the $L_{1,\mathrm{loc,unif}}$-norm. Recently, a similar result was shown in [7] for real measures as potentials, and the distance was measured in the total variation metric. We will extend these results in two directions.

The most important new aspect is that we work with a weaker Wasserstein type metric in the Gordon condition, thus allowing a wider range of applications. For linear combinations of Dirac measures this means that the positions of the Dirac deltas are allowed to vary, not only their coefficients as in [7].

The second extension concerns sharp quantitative results. We show that the operator H_μ has no eigenvalues z with $|z| < E_\mu$, where E_μ is a constant given by an explicit expression in terms of μ. Moreover, we give an example that $-E_\mu$ can occur as an eigenvalue. We note that we can also deal with complex measures, thus obtaining non-selfadjoint operators.

Our result can be applied to quasiperiodic measures where the ratio of the periods satisfies a strong Liouville condition. Such potentials were also considered in [2, 7]; however, absence of eigenvalues was only shown under some additional hypotheses. Working with our weaker metric we can prove the result without any further assumptions.

The question of the absence of eigenvalues appears in the study of one-dimensional quasicrystals. Here, it is crucial to obtain a good criterion when working with a weak metric to measure the distances of potentials (i.e., measures). The results in [8, Chapters 4 and 6] suggest that the topology of vague convergence of measures is appropriate in the study of continuum quasicrystals. Our Wasserstein type metric is precisely of this quality, see Lemma 2.4.

We now describe in more detail the Schrödinger operator H_μ we are going to study. We say that

$$\mu\colon \{B \subseteq \mathbb{R};\ B \text{ is a bounded Borel set}\} \to \mathbb{C}$$

is a *local measure* if $\mathbb{1}_K\mu := \mu(\cdot \cap K)$ is a complex Radon measure for any compact set $K \subseteq \mathbb{R}$. Then there exist a (unique) nonnegative Radon measure ν on $\mathbb{R}$ and a measurable function $\sigma\colon \mathbb{R} \to \mathbb{C}$ such that $|\sigma| = 1$ ν-a.e. and $\mathbb{1}_K\mu = \mathbb{1}_K\sigma\nu$ for all compact sets $K \subseteq \mathbb{R}$. The *total variation* of μ is defined by $|\mu| := \nu$. Let $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ be the space of all local measures on $\mathbb{R}$.

A local measure $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ is called *uniformly locally bounded* if

$$\|\mu\|_{\mathrm{unif}} := \sup_{a\in\mathbb{R}} |\mu|((a, a+1]) < \infty.$$

Let $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ denote the space of all uniformly locally bounded local measures. The space $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ naturally extends $L_{1,\mathrm{loc,unif}}(\mathbb{R})$ to measures.

Given $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, we now define the operator H_μ in $L_2(\mathbb{R})$ as the maximal operator associated with $-\Delta+\mu$ (in the distributional sense). More precisely,

$$\begin{aligned} D(H_\mu) &:= \{u \in L_2(\mathbb{R}) \cap C(\mathbb{R});\ -u'' + u\mu \in L_2(\mathbb{R})\}, \\ H_\mu u &:= -u'' + u\mu, \end{aligned}$$

i.e., for $u, f \in L_2(\mathbb{R})$ one has $u \in D(H_\mu)$, $H_\mu u = f$ if and only if

$$\int_{\mathbb{R}} f\varphi = -\int_{\mathbb{R}} u\varphi'' + \int_{\mathbb{R}} u\varphi\, d\mu \qquad \big(\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R})\big).$$

The operator H_μ can also be defined via the form method [6] or along the lines of Sturm-Liouville differential operators [1]; see Remark 3.5 for details. In Theorem 3.6 we will show that the three approaches yield the same operator (cf. [8, Corollary 1.3.7] for the self-adjoint case). This result may be of independent interest.

Notation. For $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $s, t \in \mathbb{R}$ we write

$$\int_s^t \ldots\, d\mu = \begin{cases} \int_{(s,t]} \ldots\, d\mu & \text{if } t \geqslant s, \\ -\int_{(t,s]} \ldots\, d\mu & \text{if } t < s. \end{cases}$$

The paper is organized as follows. In Section 2 we introduce the seminorms on the space of measures we will work with. Estimates on solutions of the eigenvalue equation are provided in Section 3. These will then be used to prove the equivalence of various possibilities to define the Schrödinger operator. In Section 4 we estimate the difference of solutions to different potentials. Our main theorem is stated and proved in Section 5. We also give an application to quasiperiodic potentials there. Finally, in Section 6 we construct an example to show that our results are sharp.

2 Seminorms on measures

In this section we define suitable seminorms on $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. The Lipschitz continuous functions on $\mathbb{R}$ will be denoted by $W_\infty^1(\mathbb{R})$.

Definition. For $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and a set $I \subseteq \mathbb{R}$ (which will usually be an interval) we define

$$\|\mu\|_I := \sup\Big\{ \Big| \int u\, d\mu \Big|;\ u \in W_\infty^1(\mathbb{R}),\ \operatorname{spt} u \subseteq I,\ \operatorname{diam} \operatorname{spt} u \leqslant 2,\ \|u'\|_\infty \leqslant 1 \Big\}.$$

Choosing supports of length at most 2 yields nice values for "typical" measures.

Example 2.1. (a) For the Lebesgue measure λ on $\mathbb{R}$ we have $\|\lambda\|_{\mathbb{R}} = 1$.

(b) For the Dirac measure δ_t at $t \in \mathbb{R}$ we have $\|\delta_t\|_{\mathbb{R}} = 1$. Moreover, $\left\|\sum_{n\in\mathbb{Z}} \delta_n\right\|_{\mathbb{R}} = 1$.

(c) Let $\varepsilon \in (0,1)$ and $\mu := \delta_0 - \delta_\varepsilon + \delta_2 - \delta_{2+\varepsilon}$. Then $\|\mu\|_{\mathbb{R}} = \varepsilon$, but there exists $u \in W_\infty^1(\mathbb{R})$ with $\operatorname{diam}\operatorname{spt} u = 2 + \varepsilon$ and $\|u'\|_\infty \leqslant 1$ such that $\int u\,d\mu = 2\varepsilon$. Thus, increasing the diameter of $\operatorname{spt} u$ in the definition of $\|\mu\|_I$ only slightly can change the norm by a factor of 2.

Next we explain what happens if one further increases $\operatorname{diam}\operatorname{spt} u$.

Remark 2.2. Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $n \in \mathbb{N}$, and let $u \in W_\infty^1(\mathbb{R})$ such that $\operatorname{spt} u \subseteq [-n, n]$, $\|u'\|_\infty \leqslant 1$. Then

$$\left|\int u\,d\mu\right| \leqslant n^2 \|\mu\|_{[-n,n]}.$$

Indeed, there exist $u_k \in W_\infty^1(\mathbb{R})$ with $\operatorname{diam}\operatorname{spt} u_k \leqslant 2$ $(k = 1-n, \ldots, n-1)$ such that $u = \sum u_k$ and $\sum \|u_k'\|_\infty \leqslant n^2$: let

$$u_k(t) = \begin{cases} u(t) - u(k-1)\cdot(k-t) & \text{if } k-1 \leqslant t \leqslant k, \\ u(k)\cdot(k+1-t) & \text{if } k < t \leqslant k+1 \end{cases}$$

for $k < 0$,

$$u_0(t) = \begin{cases} u(t) - u(-1)\cdot(-t) & \text{if } -1 \leqslant t \leqslant 0, \\ u(t) - u(1)\cdot t & \text{if } 0 < t \leqslant 1, \end{cases}$$

and u_k for $k > 0$ defined similarly as for $k < 0$. Then $\|u_k'\|_\infty \leqslant n - |k|$ for all k since $|u(t)| \leqslant n - |t|$ for all $|t| \leqslant n$, and $\sum_{k=1-n}^{n-1}(n - |k|) = n^2$.

Remark 2.3. (a) It is easy to see that $\|\cdot\|_{\mathbb{R}}$ defines a norm on $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ satisfying $\|\mu\|_{\mathbb{R}} \leqslant \|\mu\|_{\mathrm{unif}}$: for $u \in W_\infty^1(\mathbb{R})$ with $\operatorname{spt} u \subseteq [a-1, a+1]$ and $\|u'\|_\infty \leqslant 1$ (with some $a \in \mathbb{R}$) one has

$$|u(s)| \leqslant \max\{1 - |s-a|, 0\} = \int_{a-1}^{a} 1_{(t,t+1]}(s)\,dt \qquad (s \in \mathbb{R})$$

and hence

$$\left|\int u\,d\mu\right| \leqslant \int_{a-1}^{a} \int 1_{(t,t+1]}\,d|\mu|\,dt \leqslant \|\mu\|_{\mathrm{unif}}.$$

(b) For $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, the translates $\mu(\cdot+\varepsilon)$ converge to μ in the $\|\cdot\|_{\mathbb{R}}$-norm as $\varepsilon \to 0$, so $\|\cdot\|_{\mathbb{R}}$ is weaker than $\|\cdot\|_{\mathrm{unif}}$: for $\varepsilon>0$ one has

$$\|\mu-\mu(\cdot+\varepsilon)\|_{\mathbb{R}} \leqslant 3\varepsilon\|\mu\|_{\mathrm{unif}}.$$

Indeed, for $\varepsilon \geqslant 1$ this is clear by part (a), and for $0<\varepsilon<1$ it follows from the estimate

$$\left|\int_{\mathbb{R}} u\, d\big(\mu-\mu(\cdot+\varepsilon)\big)\right| \leqslant \int_{\mathbb{R}} |u-u(\cdot-\varepsilon)|\, d|\mu| \leqslant \int_{\operatorname{spt} u+(0,\varepsilon)} \varepsilon\|u'\|_\infty\, d|\mu|.$$

The next lemma shows in particular that the norm $\|\cdot\|_{\mathbb{R}}$ induces the vague topology on $\big\{\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R});\ \|\mu\|_{\mathrm{unif}} \leqslant R,\ \operatorname{spt}\mu \subseteq [-R,R]\big\}$, for any $R>0$.

Lemma 2.4. *Let (μ_n) be a sequence in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Then the following are equivalent:*
(i) *$\mu_n \to \mu$ vaguely,*
(ii) *$(\mathbb{1}_I\mu_n)$ is $\|\cdot\|_{\mathrm{unif}}$-bounded in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $\|\mu_n-\mu\|_I \to 0$, for all bounded intervals $I \subseteq \mathbb{R}$.*

Proof. Without loss of generality assume that $\mu=0$. For a bounded open interval $I \subseteq \mathbb{R}$ we can regard $\{\mathbb{1}_I\mu;\ \mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})\}$ as the dual space of $C_0(I)$. Note that then $\mu_n \to 0$ vaguely if and only if
(i') $\mathbb{1}_I\mu_n \to 0$ in the weak* topology, for all bounded open intervals $I \subseteq \mathbb{R}$.
We now fix a bounded open interval $I \subseteq \mathbb{R}$.

(i') $\Rightarrow$ (ii): As a null sequence in the weak* topology, $(\mathbb{1}_I\mu_n)$ is bounded and hence $\|\cdot\|_{\mathrm{unif}}$-bounded; moreover, $(\mathbb{1}_I\mu_n)$ tends to 0 uniformly on compact subsets of $C_0(I)$. By the Arzelà-Ascoli theorem,

$$\big\{u \in W^1_\infty(\mathbb{R});\ \operatorname{spt} u \subseteq I,\ \operatorname{diam}\operatorname{spt} u \leqslant 2,\ \|u'\|_\infty \leqslant 1\big\}$$

is compact in $C_0(I)$, so we conclude that $\|\mu_n\|_I \to 0$.

(ii) $\Rightarrow$ (i'): It follows from Remark 2.2 that $\int u\, d\mu_n \to 0$ for all Lipschitz functions u in $C_0(I)$. Since the Lipschitz functions are dense in $C_0(I)$ and $(\mathbb{1}_I\mu_n)$ is bounded in $C_0(I)'$, this implies $\int u\, d\mu_n \to 0$ for all $u \in C_0(I)$. $\square$

An important property of the norm $\|\cdot\|_{\mathbb{R}}$ is that unlike $\|\cdot\|_{\mathrm{unif}}$ it allows approximation of measures in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ by absolutely continuous measures with C^∞-densities.

Proposition 2.5. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Then there exists a sequence (μ_n) in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ such that μ_n has a C^∞-density and $\|\mu_n\|_{\mathrm{unif}} \leqslant \|\mu\|_{\mathrm{unif}}$ for all $n \in \mathbb{N}$, $\|\mu - \mu_n\|_{\mathbb{R}} \to 0$, and $\limsup_{n\to\infty} |\mu_n|(I) \leqslant |\mu|(I)$ for all compact intervals $I \subseteq \mathbb{R}$.*

Proof. For $n \in \mathbb{N}$ let $\psi_n \in C_{\mathrm{c}}^\infty(\mathbb{R})$ satisfy $\operatorname{spt}\psi_n \subseteq (-\frac{1}{n}, \frac{1}{n})$, $\int \psi_n = 1$, $\psi_n(-\cdot) = \psi_n$ and $\psi_n \geqslant 0$. In particular, (ψ_n) is an approximation of the identity. For $n \in \mathbb{N}$ we define

$$f_n(x) := \int \psi_n(x-y)\,d\mu(y) = \psi_n * \mu(x) \qquad (x \in \mathbb{R});$$

then $f_n \in C^\infty(\mathbb{R})$. Set $\mu_n := f_n\lambda$.

Let $u \in W_\infty^1(\mathbb{R})$, $\operatorname{diam}\operatorname{spt} u \leqslant 2$, $\|u'\|_\infty \leqslant 1$. Then

$$\left|\int u\,d(\mu - \mu_n)\right| = \left|\int \big(u(y) - \psi_n * u(y)\big)\,d\mu(y)\right|.$$

Moreover,

$$\operatorname{diam}\operatorname{spt}(\psi_n * u) \leqslant 2 + \frac{2}{n}, \qquad \|u - \psi_n * u\|_\infty \leqslant \frac{1}{n}$$

since $\|u'\|_\infty \leqslant 1$ and $\operatorname{spt}\psi_n \subseteq (-\frac{1}{n}, \frac{1}{n})$. Thus, $\|\mu - \mu_n\|_{\mathbb{R}} \to 0$.

For $a, b \in \mathbb{R}$, $a < b$ we compute, applying Fubini's theorem twice and substituting $x = t + y$,

$$\begin{aligned}|\mu_n|([a,b]) &= \int_a^b |f_n(x)|\,dx \leqslant \int_a^b \int \psi_n(x-y)\,d|\mu|(y)\,dx \\ &= \int \int_{a-t}^{b-t} d|\mu|(y)\,\psi_n(t)\,dt = \int |\mu|\big((a-t, b-t]\big)\psi_n(t)\,dt.\end{aligned}$$

On the one hand, choosing $b = a + 1$ yields

$$\|\mu_n\|_{\mathrm{unif}} = \sup_{a\in\mathbb{R}} |\mu_n|([a, a+1]) \leqslant \|\mu\|_{\mathrm{unif}}.$$

On the other hand we obtain

$$\limsup_{n\to\infty} |\mu_n|([a,b]) \leqslant \limsup_{n\to\infty} \int |\mu|\big((a - \tfrac{1}{n}, b + \tfrac{1}{n}]\big)\psi_n(t)\,dt = |\mu|([a,b]). \qquad \square$$

Lemma 2.6. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $\psi \in W^1_\infty(\mathbb{R})$, and let $I \subseteq \mathbb{R}$ be an interval. Then*

$$\|\psi\mu\|_I \leqslant (\|\psi\|_\infty + \|\psi'\|_\infty)\|\mu\|_{I\cap \operatorname{spt}\psi}.$$

Proof. Let $u \in W^1_\infty(\mathbb{R})$ satisfy $\operatorname{spt} u \subseteq I$, $\operatorname{diam}\operatorname{spt} u \leqslant 2$ and $\|u'\|_\infty \leqslant 1$. Then $\psi u \in W^1_\infty(\mathbb{R})$, $\operatorname{spt}(\psi u) \subseteq I \cap \operatorname{spt}\psi$, $\operatorname{diam}\operatorname{spt}(\psi u) \leqslant 2$ and

$$\|(\psi u)'\|_\infty \leqslant \|\psi\|_\infty + \|\psi'\|_\infty.$$

Hence, the assertion follows. □

For $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ we define $\varphi_\mu \colon \mathbb{R} \to \mathbb{C}$ by

$$\varphi_\mu(t) := \int_0^t d\mu = \begin{cases} \mu\big((0,t]\big) & \text{if } t \geqslant 0,\\ -\mu\big((t,0]\big) & \text{if } t < 0. \end{cases}$$

The following result provides an alternative way for computing the $\|\cdot\|_I$-seminorm of a given measure.

Proposition 2.7. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $a \in \mathbb{R}$. If μ is real, then*

$$\|\mu\|_{[a-1,a+1]} = \min_{c\in\mathbb{R}} \int_{a-1}^{a+1} |\varphi_\mu(t) - c|\, dt.$$

If μ is complex, then

$$\|\mu\|_{[a-1,a+1]} \leqslant \min_{c\in\mathbb{C}} \int_{a-1}^{a+1} |\varphi_\mu(t) - c|\, dt \leqslant 2\|\mu\|_{[a-1,a+1]}.$$

(Note that the minimum exists since the integral depends continuously on c and tends to ∞ as $|c| \to \infty$.)

Proof. Let $u \in W^1_\infty(\mathbb{R})$ with $\operatorname{spt} u \subseteq [a-1,a+1]$ and $\|u'\|_\infty \leqslant 1$. Then

$$\begin{aligned}\int u\, d\mu &= \int_{a-1}^{a+1} \left(-\int_t^{a+1} u'(s)\, ds \right) d\mu(t)\\ &= -\int_{a-1}^{a+1} \int_{a-1}^{s} d\mu(t)\, u'(s)\, ds = -\int_{a-1}^{a+1} u'(s)\mu((a-1,s])\, ds.\end{aligned}$$

Since $\int_{a-1}^{a+1} u'(s)\,ds = 0$, it follows that

$$\int u\,d\mu = -\int_{a-1}^{a+1} u'(s)(\varphi_\mu(s) - c)\,ds$$

for all $c \in \mathbb{C}$. Thus,

$$\left|\int u\,d\mu\right| \leqslant \int_{a-1}^{a+1} |\varphi_\mu(s) - c|\,ds$$

and hence

$$\|\mu\|_{[a-1,a+1]} \leqslant \min_{c\in\mathbb{C}} \int_{a-1}^{a+1} |\varphi_\mu(s) - c|\,ds.$$

In case μ is real, we choose $c \in \mathbb{R}$ to be a "median" of φ_μ, i.e., both $\varphi_\mu \geqslant c$ and $\varphi_\mu \leqslant c$ hold on subsets of $[a-1, a+1]$ with Lebesgue measure at least 1. Then we can construct a real-valued function $u \in W^1_\infty(\mathbb{R})$ with $\operatorname{spt} u \subseteq [a-1, a+1]$ and $\|u'\|_\infty \leqslant 1$ such that

$$\int_{a-1}^{a+1} |\varphi_\mu(s) - c|\,ds = -\int_{a-1}^{a+1} u'(s)(\varphi_\mu(s) - c)\,ds = \int u\,d\mu.$$

Therefore,

$$\min_{c\in\mathbb{R}} \int_{a-1}^{a+1} |\varphi_\mu(s) - c|\,ds \leqslant \|\mu\|_{[a-1,a+1]}.$$

In case μ is complex, we have $\mu = \mu_{\mathrm{r}} + i\mu_{\mathrm{i}}$ with real measures $\mu_{\mathrm{r}}, \mu_{\mathrm{i}} \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Then by the above we find $c_{\mathrm{r}} \in \mathbb{R}$ and a real-valued function $u \in W^1_\infty(\mathbb{R})$ such that

$$\int_{a-1}^{a+1} |\varphi_{\mu_{\mathrm{r}}}(s) - c_{\mathrm{r}}|\,ds = \int u\,d\mu_r = \operatorname{Re}\int u\,d\mu \leqslant \|\mu\|_{[a-1,a+1]}.$$

In the same way we obtain $\int_{a-1}^{a+1} |\varphi_{\mu_{\mathrm{i}}}(s) - c_{\mathrm{i}}|\,ds \leqslant \|\mu\|_{[a-1,a+1]}$ for some $c_{\mathrm{i}} \in \mathbb{R}$. Since $\varphi_\mu = \varphi_{\mu_{\mathrm{r}}} + i\varphi_{\mu_{\mathrm{i}}}$, it follows that

$$\int_{a-1}^{a+1} \bigl|\varphi_\mu(s) - (c_{\mathrm{r}} + ic_{\mathrm{i}})\bigr|\,ds \leqslant 2\|\mu\|_{[a-1,a+1]}. \qquad \square$$

Remark 2.8. (a) According to Proposition 2.7, for any $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $a \in \mathbb{R}$ there exists $c_a = c_{\mu,a} \in \mathbb{C}$ such that

$$\int_{a-1}^{a+1} |\varphi_\mu(t) - c_a| \, dt \leqslant 2\|\mu\|_{[a-1,a+1]}. \tag{1}$$

Moreover, $c_{\mu,0}$ can always be chosen such that

$$|c_{\mu,0}| \leqslant \|\mu\|_{\mathrm{unif}}$$

since $|\varphi_\mu(t)| \leqslant \|\mu\|_{\mathrm{unif}}$ for $t \in [-1,1]$.

(b) We point out that $|c_{\mu,0}|$ can be large even if $\|\mu\|_{\mathbb{R}}$ is small. As an example consider $\mu = \delta_\varepsilon - \delta_{-\varepsilon}$ for small $\varepsilon > 0$, where one has $\|\mu\|_{\mathbb{R}} = 2\varepsilon$ and $\varphi_\mu = 1_{\mathbb{R}\setminus[-\varepsilon,\varepsilon)}$.

Lemma 2.9. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{Z}$, $\alpha \leqslant -1$, $\beta \geqslant 1$. Then*

$$\int_k^{k+1} |\varphi_\mu(t) - c_0| \, dt \leqslant 2\max\{k+1, -k\}\|\mu\|_{[\alpha,\beta]}$$

for all $k \in [\alpha, \beta - 1]$, with c_0 as in (1).

Proof. For $k = -1$ and $k = 0$ the assertion follows from (1). We now show the assertion for $k \geqslant 1$; the proof of the case $k \leqslant -2$ is analogous.

Using a telescope sum expansion, we estimate

$$\begin{aligned}|c_k - c_0| &\leqslant \sum_{j=1}^k \int_{j-1}^j \big(|c_j - \varphi_\mu(t)| + |\varphi_\mu(t) - c_{j-1}|\big) \, dt \\ &= \int_0^1 |\varphi_\mu(t) - c_0| \, dt + \sum_{j=1}^{k-1} \int_{j-1}^{j+1} |\varphi_\mu(t) - c_j| \, dt + \int_{k-1}^k |\varphi_\mu(t) - c_k| \, dt.\end{aligned}$$

By (1) we conclude that

$$\begin{aligned}\int_k^{k+1} \big|\varphi_\mu(t) - c_0\big| \, dt &\leqslant \int_k^{k+1} \big(|\varphi_\mu(t) - c_k| + |c_k - c_0|\big) \, dt \\ &\leqslant \sum_{j=0}^k \int_{j-1}^{j+1} |\varphi_\mu(t) - c_j| \, dt \leqslant (k+1) \cdot 2\|\mu\|_{[\alpha,\beta]}. \qquad \square\end{aligned}$$

3 Estimates on solutions

In this section we provide estimates on functions that satisfy the eigenvalue equation for H_μ (but that are not necessarily in L_2). These estimates enable us to show that the different versions for defining $-\Delta + \mu$ actually lead to the same operator.

Definition. For $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $z \in \mathbb{C}$ we say that $u \in L_{1,\mathrm{loc}}(\mathbb{R})$ is a (generalized) solution of the equation $H_\mu u = zu$ if $u \in C(\mathbb{R})$ and

$$-u'' + \mu u = zu$$

in the sense of distributions.

Remark 3.1. Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $z \in \mathbb{C}$.

(a) Note that u is a solution of $H_\mu u = zu$ if and only if u is a solution of $H_{\mu - z\lambda} u = 0$. Therefore, it is no loss of generality to assume $z = 0$ in this and the following section.

(b) Let u be a solution of the equation $H_\mu u = 0$. Then u'' is a local measure, so u' has locally bounded variation (and therefore its one-sided limits exist everywhere), and

$$u'(t+) - u'(s+) = \int_s^t u(r)\, d\mu(r)$$

for all $s, t \in \mathbb{R}$. In particular, any solution of $H_\mu u = 0$ is locally Lipschitz continuous.

Lemma 3.2. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, and let u be a solution of the equation $H_\mu u = 0$. Then*

$$|u(t)| + |u'(t+)| \leqslant \big(|u(0)| + |u'(0+)|\big) e^{\omega(|t|+1)} \qquad (t \in \mathbb{R}),$$

with $\omega = \|\mu\|_{\mathrm{unif}} + 1$.

Proof. Below we will use Gronwall's inequality in the following form: let $\varphi\colon [0,\infty) \to [0,\infty)$ be Borel measurable and locally bounded, $\varepsilon > 0$ and $0 \leqslant \nu \in \mathcal{M}_{\mathrm{loc}}([0,\infty))$ (a nonnegative Radon measure on $[0,\infty)$), with

$$\varphi(t) \leqslant \varepsilon + \int_{[0,t)} \varphi(s)\, d\nu(s) \qquad (t \geqslant 0).$$

Then $\varphi(t) \leqslant \varepsilon e^{\nu([0,t))}$ for all $t \geqslant 0$; see [4, Thm. 5.1, p. 498].

By Remark 3.1(b) we have

$$\begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} = \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix} + \begin{pmatrix} \int_0^t u'(s+)\,ds \\ \int_0^t u(s)\,d\mu(s) \end{pmatrix}$$

for all $t \in \mathbb{R}$. Thus, for $\varphi(t) := |u(t)| + |u'(t+)|$ and $\nu := \lambda + |\mu|$ we obtain

$$\varphi(t) \leqslant \varphi(0) + \int_{(t,0]} \varphi(s)\,d\nu(s) \qquad (t < 0).$$

By Gronwall's inequality we infer that

$$\varphi(t) \leqslant \varphi(0) e^{\nu((t,0])} \qquad (t \leqslant 0).$$

Thus, for $t \leqslant 0$ the assertion follows since $\|\nu\|_{\mathrm{unif}} \leqslant 1 + \|\mu\|_{\mathrm{unif}} = \omega$ and hence $\nu((t,0]) \leqslant \omega(|t|+1)$.

Now let $t > 0$. As above we can estimate

$$\varphi_-(s) := |u(s)| + |u'(s-)| \leqslant \varphi(0) + \int_{(0,s)} \varphi_-(r)\,d\nu(r),$$

and Gronwall's inequality yields

$$|u(s)| + |u'(s-)| \leqslant \big(|u(0)| + |u'(0+)|\big) e^{\nu((0,s))} \qquad (s > 0).$$

For $s \downarrow t$ the assertion follows since $\nu((0,t]) \leqslant \omega(|t|+1)$. □

Remark 3.3. One could refine the above proof to obtain an estimate with $\omega = 2\|\mu\|_{\mathrm{unif}}^{1/2}$, at the expense of a multiplicative constant in the right hand side. By a bootstrap-like argument we will even improve the bound to $\omega = \|\mu\|_{\mathrm{unif}}^{1/2}$ in Proposition 4.5.

In the following elementary lemma we provide estimates on the derivative of a solution in terms of norms of the solution itself.

Lemma 3.4. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, and let $u \in C(\mathbb{R})$ be a solution of $H_\mu u = 0$. Then for any interval $I = [a,b] \subseteq \mathbb{R}$ of length 1 one has*

$$\|u'|_I\|_2 \leqslant \|u'|_I\|_\infty \leqslant M_\mu \|u|_I\|_\infty \leqslant \sqrt{3}\, M_\mu^{3/2} \|u|_I\|_2,$$

where $M_\mu := \|\mu\|_{\mathrm{unif}} + 2$. In particular, $u \in L_2(\mathbb{R})$ implies $u' \in L_2(\mathbb{R})$, and $u \in C_0(\mathbb{R})$ implies that $u'(t+) \to 0$ as $t \to \pm\infty$.

Proof. The first inequality is clear. We have

$$\left|\int_a^b u'(t+)\,dt\right| = \bigl|u(b) - u(a)\bigr| \leqslant 2\|u|_I\|_\infty,$$

so there exists $t_0 \in I$ with $|u'(t_0+)| \leqslant 2\|u|_I\|_\infty$. For $t \in I$ it follows with Remark 3.1(b) that

$$|u'(t+)| \leqslant |u'(t_0+)| + \left|\int_{t_0}^t u(s)\,d\mu(s)\right| \leqslant (2 + \|\mu\|_{\mathrm{unif}})\|u|_I\|_\infty,$$

which proves the second inequality.

Let now $s_0 \in I$ such that $|u(s_0)| = \|u|_I\|_\infty$. Then by the above we have $|u(s_0 + t)| \geqslant (1 - M_\mu|t|)\|u|_I\|_\infty$ for all $t \in \mathbb{R}$ with $s_0 + t \in I$. Therefore,

$$\|u|_I\|_2^2 \geqslant \int_0^{1/M_\mu} (M_\mu t)^2 \|u|_I\|_\infty^2\,dt = \frac{1}{3M_\mu}\|u|_I\|_\infty^2,$$

and the last inequality follows. ☐

Remark 3.5. As mentioned in the introduction, different methods for defining $-\Delta + \mu$ appear in the literature.

(a) In [1] the operator is defined along the lines of Sturm-Liouville differential operators. For $u \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ one defines $A_\mu u \in L_{1,\mathrm{loc}}(\mathbb{R})$ by

$$(A_\mu u)(t) := u'(t) - \int_0^t u(s)\,d\mu(s)$$

for a.a. $t \in \mathbb{R}$. (Here and in the following, we choose the continuous representative of u in the integral.)

Then we define $H_{\mu,\mathrm{SL}}$ as a realization of $-\Delta + \mu$ by

$$\begin{aligned} D(H_{\mu,\mathrm{SL}}) &:= \bigl\{u \in L_2(\mathbb{R}) \cap W^1_{1,\mathrm{loc}}(\mathbb{R});\ A_\mu u \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\ (A_\mu u)' \in L_2(\mathbb{R})\bigr\}, \\ H_{\mu,\mathrm{SL}} u &:= -(A_\mu u)'. \end{aligned}$$

(b) In [6] the form method is used to define the operator. It is well-known that μ defines an infinitesimally form small perturbation of the classical Dirichlet form τ_0 given by

$$D(\tau_0) := W_2^1(\mathbb{R}), \qquad \tau_0(u,v) := \int u'\overline{v}',$$

i.e., for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$\left| \int |u|^2 \, d\mu \right| \leqslant \delta\tau_0(u,u) + C_\delta \|u\|_2^2 \qquad (u \in D(\tau_0)).$$

Thus,

$$D(\tau_\mu) := W_2^1(\mathbb{R}), \qquad \tau_\mu(u,v) := \int u'\overline{v}' + \int u\overline{v}\, d\mu$$

defines a closed sectorial form τ_μ in $L_2(\mathbb{R})$. (For the second integral note that $u\overline{v}\mu$ defines a complex Radon measure for any $u, v \in W_2^1(\mathbb{R})$.)

We denote by $H_{\mu,\mathrm{F}}$ the m-sectorial operator associated with τ_μ, i.e.,

$$\begin{aligned} &D(H_{\mu,\mathrm{F}}) = \big\{u \in D(\tau_\mu);\ \exists f_u \in L_2(\mathbb{R})\ \forall v \in D(\tau_\mu)\colon \tau_\mu(u,v) = (f_u \,|\, v)\big\}, \\ &\quad H_{\mu,\mathrm{F}}u = f_u. \end{aligned}$$

Theorem 3.6. *With the above notation one has $H_\mu = H_{\mu,\mathrm{SL}} = H_{\mu,\mathrm{F}}$.*

Proof. It is easy to see that $H_{\mu,\mathrm{F}} \subseteq H_\mu$; we show the converse operator inclusion. Since $H_{\mu,\mathrm{F}}$ is sectorial, there exists $z \in \mathbb{C}$ such that $z + H_{\mu,\mathrm{F}}$ is injective and surjective. Let u be in the kernel of $z + H_\mu$; then u is a solution of $H_{\mu+z\lambda}u = 0$. Since $u \in L_2(\mathbb{R})$, Lemma 3.4 yields $u' \in L_2(\mathbb{R})$. For $\varphi \in C_\mathrm{c}^\infty(\mathbb{R})$ we obtain

$$(H_\mu u \,|\, \varphi) = -\,(u \,|\, \varphi'') + \int u\overline{\varphi}\, d\mu = (u' \,|\, \varphi') + \int u\overline{\varphi}\, d\mu = \tau_\mu(u,\varphi).$$

Since $C_\mathrm{c}^\infty(\mathbb{R})$ is dense in $W_2^1(\mathbb{R})$, it follows that $u \in D(H_{\mu,\mathrm{F}})$ and $H_{\mu,\mathrm{F}}u = H_\mu u$. Hence also $(z + H_{\mu,\mathrm{F}})u = 0$. Thus $u = 0$, so we have shown that $z + H_\mu$ is injective. Since surjective mappings cannot have proper injective extensions, we obtain $H_{\mu,\mathrm{F}} = H_\mu$.

Applying Fubini's theorem as in [7, Lemma 1.3], we obtain $H_{\mu,\mathrm{F}} \subseteq H_{\mu,\mathrm{SL}}$. For $u \in D(H_{\mu,\mathrm{SL}})$ and $\varphi \in C_\mathrm{c}^\infty(\mathbb{R})$ we compute, with the help of Fubini's theorem,

$$(H_{\mu,\mathrm{SL}}u \,|\, \varphi) = \int u'\overline{\varphi}' - \int\!\!\int_0^t u(s)\, d\mu(s)\, \overline{\varphi}'(t)\, dt = -\,(u \,|\, \varphi'') + \int u\overline{\varphi}\, d\mu.$$

Thus, $u \in D(H_\mu)$ and $H_\mu u = H_{\mu,\mathrm{SL}}u$. Hence, we obtain $H_{\mu,\mathrm{SL}} \subseteq H_\mu = H_{\mu,\mathrm{F}}$ and therefore the equality of all three operators. $\square$

4 Estimates on the difference of two solutions

Throughout this section let $\mu_1, \mu_2 \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, and let u_1, u_2 be solutions to

$$H_{\mu_1} u_1 = 0 \qquad \text{and} \qquad H_{\mu_2} u_2 = 0,$$

respectively. We set

$$v := u_1 - u_2 \qquad \text{and} \qquad \nu := \mu_1 - \mu_2,$$

and we aim for an estimate on v on an interval I in terms of $\|\nu\|_I$.

We first recall the definition and properties of the transfer matrices associated with a measure μ.

Remark 4.1. Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, and let u be a solution of the equation $H_\mu u = 0$. Fix $s \in \mathbb{R}$. By Lemma 3.2, u is uniquely determined by $(u(s), u'(s+))$ (see also [1, Theorem 2.3]). Thus, for $t \in \mathbb{R}$ we can define the transfer matrix mapping the solution of $H_\mu u = 0$ at s to the solution at t, i.e.,

$$T_\mu(t,s)\colon \begin{pmatrix} u(s) \\ u'(s+) \end{pmatrix} \mapsto \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix}.$$

Note that $T_\mu(t,t) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $T_\mu(t,r) = T_\mu(t,s) T_\mu(s,r)$ for all $r, s, t \in \mathbb{R}$.

Let $u_{\mathrm{N},\mu}(\cdot, s), u_{\mathrm{D},\mu}(\cdot, s)$ be the solutions of $H_\mu u = 0$ satisfying Neumann and Dirichlet boundary conditions at s, respectively, i.e.,

$$\begin{aligned} u_{\mathrm{N},\mu}(s,s) &= 1, & u_D(s,s) &= 0, \\ \partial_1 u_{\mathrm{N},\mu}(s+, s) &= 0, & \partial_1 u_{\mathrm{D},\mu}(s+, s) &= 1. \end{aligned}$$

Then

$$T_\mu(t,s) = \begin{pmatrix} u_{\mathrm{N},\mu}(t,s) & u_{\mathrm{D},\mu}(t,s) \\ \partial_1 u_{\mathrm{N},\mu}(t+, s) & \partial_1 u_{\mathrm{D},\mu}(t+, s) \end{pmatrix}$$

and $\det T_\mu(t,s) = 1$ for all $t \in \mathbb{R}$, cf. [1, Proposition 2.5] and also [7, Remark 2.7]. It follows that

$$T_\mu(s,t) = T_\mu(t,s)^{-1} = \begin{pmatrix} \partial_1 u_{\mathrm{D},\mu}(t+, s) & -u_{\mathrm{D},\mu}(t,s) \\ -\partial_1 u_{\mathrm{N},\mu}(t+, s) & u_{\mathrm{N},\mu}(t,s) \end{pmatrix}, \tag{2}$$

in particular, $u_{\mathrm{D},\mu}(s,t) = -u_{\mathrm{D},\mu}(t,s)$ for all $t \in \mathbb{R}$. As a shorthand notation we will also write

$$u_{\mathrm{D},\mu}(t) := u_{\mathrm{D},\mu}(t,0), \qquad u_{\mathrm{N},\mu}(t) := u_{\mathrm{N},\mu}(t,0).$$

Next we provide a variation of constants type formula.

Lemma 4.2. *Let $s, t \in \mathbb{R}$. Then*

$$\begin{pmatrix} v(t) \\ v'(t+) \end{pmatrix} = T_{\mu_1}(t,s) \begin{pmatrix} v(s) \\ v'(s+) \end{pmatrix} + \int_s^t T_{\mu_1}(t,r) \begin{pmatrix} 0 \\ u_2(r) \end{pmatrix} d\nu(r).$$

Proof. (i) We will use the following integration by parts type formula: let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, u a solution of $H_\mu u = 0$ and $\tilde{u} \in W^1_{1,\mathrm{loc}}(\mathbb{R})$. Then

$$\int_s^t \tilde{u}(r) u(r)\, d\mu(r) = \tilde{u}(t) u'(t+) - \tilde{u}(s) u'(s+) - \int_s^t \tilde{u}'(r) u'(r)\, dr,$$

see [8, Lemma 1.3.3] (note that there $\int_s^t = \int_{[s,t]}$ for $s < t$, so the boundary values slightly change).

(ii) Without loss of generality let $s = 0$. We are going to show that

$$\begin{pmatrix} u_2(0) \\ u_2'(0+) \end{pmatrix} - T_{\mu_1}(t,0)^{-1} \begin{pmatrix} u_2(t) \\ u_2'(t+) \end{pmatrix} = \int_0^t T_{\mu_1}(r,0)^{-1} \begin{pmatrix} 0 \\ u_2(r) \end{pmatrix} d\nu(r).$$

Multiplying this identity by $T_{\mu_1}(t,0)$ yields the assertion since

$$T_{\mu_1}(t,0) \begin{pmatrix} v(0) \\ v'(0+) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ u_1'(t+) \end{pmatrix} - T_{\mu_1}(t,0) \begin{pmatrix} u_2(0) \\ u_2'(0+) \end{pmatrix}$$

and $T_{\mu_1}(t,0) T_{\mu_1}(r,0)^{-1} = T_{\mu_1}(t,r)$.

(iii) Using (2) we compute

$$\int_0^t T_{\mu_1}(r,0)^{-1} \begin{pmatrix} 0 \\ u_2(r) \end{pmatrix} d\nu(r) = \begin{pmatrix} -\int_0^t u_{\mathrm{D},\mu_1}(r) u_2(r)\, d\nu(r) \\ \int_0^t u_{\mathrm{N},\mu_1}(r) u_2(r)\, d\nu(r) \end{pmatrix}.$$

Integrating by parts as in (i) we obtain

$$\begin{aligned}
\begin{pmatrix} -\int_0^t u_{\mathrm{D},\mu_1} u_2\, d\nu \\ \int_0^t u_{\mathrm{N},\mu_1} u_2\, d\nu \end{pmatrix} &= \begin{pmatrix} -\int_0^t u_2 u_{\mathrm{D},\mu_1}\, d\mu_1 + \int_0^t u_{\mathrm{D},\mu_1} u_2\, d\mu_2 \\ \int_0^t u_2 u_{\mathrm{N},\mu_1}\, d\mu_1 - \int_0^t u_{\mathrm{N},\mu_1} u_2\, d\mu_2 \end{pmatrix} \\
&= \begin{pmatrix} -\big(u_2(t) u'_{\mathrm{D},\mu_1}(t+) - u_2(0)\big) + \big(u_{\mathrm{D},\mu_1}(t) u_2'(t+) - 0\big) \\ \big(u_2(t) u'_{\mathrm{N},\mu_1}(t+) - 0\big) - \big(u_{\mathrm{N},\mu_1}(t) u_2'(t+) - u_2'(0+)\big) \end{pmatrix} \\
&= \begin{pmatrix} u_2(0) \\ u_2'(0+) \end{pmatrix} - \begin{pmatrix} u'_{\mathrm{D},\mu_1}(t+) & -u_{\mathrm{D},\mu_1}(t) \\ -u'_{\mathrm{N},\mu_1}(t+) & u_{\mathrm{N},\mu_1}(t) \end{pmatrix} \begin{pmatrix} u_2(t) \\ u_2'(t+) \end{pmatrix},
\end{aligned}$$

which by (2) completes the proof. $\square$

Lemma 4.3. *Let* $c \in \mathbb{R}$. *Suppose that* $u_2(0) = u_1(0)$, $u_2'(0+) = u_1'(0+) + cu_1(0)$. *Then*

$$v(t) = \int_0^t \tfrac{d}{ds}\big(u_{\mathrm{D},\mu_1}(t,s)u_2(s)\big) \cdot (c - \varphi_\nu(s))\, ds \qquad (t \in \mathbb{R}).$$

Proof. Let $t \in \mathbb{R}$. Note that $v(0) = 0$, $v'(0+) = -cu_2(0)$. By Lemma 4.2 we thus obtain

$$v(t) = -u_{\mathrm{D},\mu_1}(t)cu_2(0) + \int_0^t u_{\mathrm{D},\mu_1}(t,r)u_2(r)\, d\nu(r).$$

Since $u_{\mathrm{D},\mu_1}(t,t) = 0$, we have

$$u_{\mathrm{D},\mu_1}(t,r)u_2(r) = -\int_r^t \tfrac{d}{ds}\big(u_{\mathrm{D},\mu_1}(t,s)u_2(s)\big)\, ds.$$

By Fubini's theorem it follows from the above two identities that

$$\begin{aligned} v(t) &= -u_{\mathrm{D},\mu_1}(t)cu_2(0) - \int_0^t \int_0^s d\nu(r)\, \tfrac{d}{ds}\big(u_{\mathrm{D},\mu_1}(t,s)u_2(s)\big)\, ds \\ &= \int_0^t (c - \varphi_\nu(s))\, \tfrac{d}{ds}\big(u_{\mathrm{D},\mu_1}(t,s)u_2(s)\big)\, ds. \end{aligned}$$

□

Now we can prove the desired estimate on $v = u_1 - u_2$ in terms of weak seminorms of $\nu = \mu_1 - \mu_2$.

Proposition 4.4. *Suppose that* $u_2(0) = u_1(0)$, $u_2'(0+) = u_1'(0+) + c_{\nu,0}u_1(0)$, *where* $c_{\nu,0}$ *is as in* (1). *Let* $\alpha, \beta \in \mathbb{Z}$, $\alpha \leqslant -1$, $\beta \geqslant 1$. *Let* $c, \omega > 0$ *such that*

$$|\partial_1 u_{\mathrm{D},\mu_1}(t+,s)| \leqslant ce^{\omega|t-s|} \qquad (s,t \in \mathbb{R}). \tag{3}$$

Then

$$|v(t)| \leqslant Cce^{\omega|t|}\big\|u_2|_{[\alpha,\beta]}\big\|_\infty \|\nu\|_{[\alpha,\beta]} \qquad (t \in [\alpha,\beta]),$$

with a constant $C > 0$ *depending only on* ω *and* $\|\mu_2\|_{\mathrm{unif}}$.

Observe that (3) is always satisfied with $\omega = \|\mu_1\|_{\mathrm{unif}} + 1$ and $c = e^\omega$; this follows from Lemma 3.2 since $u_{\mathrm{D},\mu_1}(s,s) = 0$ and $\partial_1 u_{\mathrm{D},\mu_1}(s+,s) = 1$ for all $s \in \mathbb{R}$.

Proof. From $u_{\mathrm{D},\mu_1}(s,s) = 0$ and the assumed estimate on $\partial_1 u_{\mathrm{D},\mu_1}$ we derive $|u_{\mathrm{D},\mu_1}(t,s)| \leqslant \frac{c}{\omega}e^{\omega|t-s|}$ for all $s,t \in \mathbb{R}$. Moreover, by Lemma 3.4 we have $\|u_2'|_{[\alpha,\beta]}\|_\infty \leqslant (\|\mu_2\|_{\mathrm{unif}} + 2)\|u_2|_{[\alpha,\beta]}\|_\infty$. Recalling from (2) that $u_{\mathrm{D},\mu_1}(t,s) = -u_{\mathrm{D},\mu_1}(s,t)$, we thus obtain

$$
\begin{aligned}
\left|\tfrac{d^+}{ds}\big(u_{\mathrm{D},\mu_1}(t,s)u_2(s)\big)\right| &= \left|-\partial_1 u_{\mathrm{D},\mu_1}(s+,t)u_2(s) + u_{\mathrm{D},\mu_1}(t,s)u_2'(s+)\right| \\
&\leqslant C_0 c e^{\omega|t-s|}\|u_2|_{[\alpha,\beta]}\|_\infty
\end{aligned}
\tag{4}
$$

for all $s,t \in [\alpha,\beta]$, with $C_0 = 1 + \frac{1}{\omega}(\|\mu_2\|_{\mathrm{unif}} + 2)$.

Let $t \in [0,\beta]$. By Lemma 4.3 we have

$$
|v(t)| \leqslant \int_0^t \left|\tfrac{d^+}{ds}\big(u_{\mathrm{D},\mu_1}(t,s)u_2(s)\big)\right| \cdot |\varphi_\nu(s) - c_{\nu,0}|\,ds.
$$

By (4) and Lemma 2.9 we conclude that

$$
\begin{aligned}
|v(t)| &\leqslant C_0 c\|u_2|_{[\alpha,\beta]}\|_\infty \sum_{k=1}^{\beta}\int_{k-1}^{k} e^{\omega(t-s)}|\varphi_\nu(s) - c_{\nu,0}|\,ds \\
&\leqslant C_0 c\|u_2|_{[\alpha,\beta]}\|_\infty \sum_{k=1}^{\beta} e^{\omega(t+1-k)} \cdot 2k\|\nu\|_{[\alpha,\beta]} \leqslant Cc\|u_2|_{[\alpha,\beta]}\|_\infty e^{\omega t}\|\nu\|_{[\alpha,\beta]},
\end{aligned}
$$

with $C := C_0\sum_{k=1}^\infty 2ke^{-\omega(k-1)}$. The proof in the case $t \in [\alpha,0)$ is analogous. $\square$

Using the previous result we can sharpen the estimate in Lemma 3.2.

Proposition 4.5. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and let u be a solution of the equation $H_\mu u = 0$. Then for $t \in \mathbb{R}$ one has*

$$
\big(\|\mu\|_{\mathrm{unif}}|u(t)|^2 + |u'(t+)|^2\big)^{1/2} \leqslant \big(\|\mu\|_{\mathrm{unif}}|u(0)|^2 + |u'(0+)|^2\big)^{1/2} e^{\|\mu\|_{\mathrm{unif}}^{1/2}(|t|+1/2)}.
$$

Proof. The case $\mu = 0$ is trivial, since then u is affine. So, let $\mu \neq 0$.

(i) First, assume that $\mu = \rho\lambda$ with a density $\rho \in C(\mathbb{R})$. Then $u \in C^2(\mathbb{R})$, $u'' = \rho u$. Let $\omega > 0$. Then for $\varphi(t) := \omega^2|u(t)|^2 + |u'(t)|^2$ we obtain

$$
\begin{aligned}
|\varphi'(t)| &= \left|2\operatorname{Re}\big((\omega^2 + \rho)u(t)\overline{u'(t)}\big)\right| \\
&\leqslant (\omega^2 + |\rho|)\big(\omega|u(t)|^2 + \tfrac{1}{\omega}|u'(t)|^2\big) = (\omega + \tfrac{1}{\omega}|\rho|)\varphi(t).
\end{aligned}
$$

It follows that $\varphi(t) \leqslant \varphi(s)\exp\bigl(\omega(t-s) + \frac{1}{\omega}\int_s^t |\rho(r)|\,dr\bigr)$ and hence

$$\omega^2|u(t)|^2 + |u'(t)|^2 \leqslant \bigl(\omega^2|u(s)|^2 + |u'(s)|^2\bigr)\exp\bigl(\omega(t-s) + \tfrac{1}{\omega}|\mu|([s,t])\bigr)$$

for all $s, t \in \mathbb{R}$, $s < t$.

(ii) By Proposition 2.5 there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ such that μ_n has a smooth density and $\|\mu_n\|_{\mathrm{unif}} \leqslant \|\mu\|_{\mathrm{unif}}$ for all $n \in \mathbb{N}$, $\|\mu_n - \mu\|_{\mathbb{R}} \to 0$ and $\limsup_{n\to\infty} |\mu_n|(I) \leqslant |\mu|(I)$ for all compact intervals $I \subseteq \mathbb{R}$. Lemma 2.4 implies $\mu_n \to \mu$ vaguely and, hence, $1_{[a,b]}\mu_n \to 1_{[a,b]}\mu$ weakly for all $a, b \in \mathbb{R}$ such that $\mu(\{a\}) = \mu(\{b\}) = 0$. (Note that μ has at most countable many atoms.)

(iii) For $n \in \mathbb{N}$ let u_n be the solution of $H_{\mu_n}u_n = 0$ such that $u_n(0) = u(0)$ and $u_n'(0+) = u'(0+) + c_{\mu-\mu_n,0}u(0)$. Then (u_n) is uniformly bounded on any compact interval by Lemma 3.2 and Remark 2.8(a), and Proposition 4.4 implies that $u_n \to u$ locally uniformly. Let $s, t \in \mathbb{R}$ such that $\mu(\{s\}) = \mu(\{t\}) = 0$. Then

$$u_n'(t) - u_n'(s) = \int_s^t u_n(r)\,d\mu_n(r) \to \int_s^t u(r)\,d\mu(r) = u'(t+) - u'(s+)$$

as $n \to \infty$ since $u_n \to u$ uniformly on $[s,t]$ and $1_{[s,t]}\mu_n \to 1_{[s,t]}\mu$ weakly. It follows from Lemma 3.4 that (u_n') is uniformly bounded on $[0,1]$. Thus, integrating s from 0 to 1 we conclude by Lebesgue's dominated convergence theorem that

$$u_n'(t) - \bigl(u_n(1) - u_n(0)\bigr) \to u'(t+) - \bigl(u(1) - u(0)\bigr)$$

and hence $u_n'(t) \to u'(t+)$.

(iv) Let $t > s > 0$ such that $\mu(\{s\}) = \mu(\{t\}) = 0$. By (i) we obtain

$$\omega^2|u_n(t)|^2 + |u_n'(t)|^2 \leqslant \bigl(\omega^2|u_n(s)|^2 + |u_n'(s)|^2\bigr)\exp\bigl(\omega(t-s) + \tfrac{1}{\omega}|\mu_n|([s,t])\bigr)$$

for all $n \in \mathbb{N}$. For $n \to \infty$ we infer, using part (iii) and the estimate $\limsup |\mu_n|([s,t]) \leqslant |\mu|([s,t])$, that

$$\omega^2|u(t)|^2 + |u'(t+)|^2 \leqslant \bigl(\omega^2|u(s)|^2 + |u'(s+)|^2\bigr)\exp\bigl(\omega(t-s) + \tfrac{1}{\omega}|\mu|([s,t])\bigr).$$

Finally, let $t > 0$. Then there exist sequences (s_n) in $[0,t)$ and (t_n) in $[t,\infty)$ such that $s_n \to 0$, $t_n \to t$ and $\mu(\{s_n\}) = \mu(\{t_n\}) = 0$ for all $n \in \mathbb{N}$, and from the above we deduce that

$$\omega^2|u(t)|^2 + |u'(t+)|^2 \leqslant \bigl(\omega^2|u(0)|^2 + |u'(0+)|^2\bigr)\exp\bigl(\omega t + \tfrac{1}{\omega}|\mu|((0,t])\bigr).$$

Choosing $\omega = \|\mu\|_{\mathrm{unif}}^{1/2}$ yields the assertion for $t \geqslant 0$ since then $|\mu|((0,t]) \leqslant \omega^2(t+1)$. The case $t < 0$ is proved analogously. $\square$

5 Gordon's Theorem

We are now in the position to formulate the weak Gordon condition and to prove our main theorem. We will then apply the result to quasiperiodic measures.

Definition. Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $C > 0$. We say that μ is a *weak Gordon measure* with weight C if there exists a sequence (p_m) in $(0, \infty)$ with $p_m \to \infty$ such that

$$\lim_{m\to\infty} e^{Cp_m} \|\mu - \mu(\cdot + p_m)\|_{[-p_m,p_m]} = 0.$$

The following lemma relates our notion of weak Gordon measures with previous versions for Gordon potentials/measures (see [5, 2, 7]).

Lemma 5.1. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $C > 0$. Then the following are equivalent:*

(i) *μ is a weak Gordon measure with weight C.*

(ii) *There exist sequences (p_m) in $(0, \infty)$ and (μ_m) in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ such that μ_m is periodic with period p_m $(m \in \mathbb{N})$, $p_m \to \infty$ and*

$$e^{Cp_m} \|\mu - \mu_m\|_{[-p_m,2p_m]} \to 0 \qquad (m \to \infty).$$

Moreover, the measures μ_m in (ii) *can be chosen such that*

$$\mathbb{1}_{[a_m,p_m-a_m]}\mu_m = \mathbb{1}_{[a_m,p_m-a_m]}\mu, \quad \|\mu_m\|_{\mathrm{unif}} \leqslant \big(1 + \tfrac{1}{2a_m}\big)\|\mu\|_{\mathrm{unif}}$$

for all $m \in \mathbb{N}$, with $0 < a_m \leqslant \frac{p_m}{2}$ and $\inf_{m\in\mathbb{N}} a_m > 0$.

Proof. (i) $\Rightarrow$ (ii): Choose a sequence (p_m) as in the above definition, and let (a_m) be as in the additional statement of the lemma. Let $m \in \mathbb{N}$, and let $\psi_m : \mathbb{R} \to \mathbb{R}$ be the piecewise affine function with $\operatorname{spt} \psi_m = [-a_m, p_m + a_m]$, $\psi_m = 1$ on $[a_m, p_m - a_m]$ and $\|\psi'\|_\infty = \frac{1}{2a_m}$. Then $\sum_{k\in\mathbb{Z}} \psi_m(\cdot + kp_m) = 1_{\mathbb{R}}$, the measure

$$\mu_m := \sum_{k\in\mathbb{Z}} (\psi_m\mu)(\cdot + kp_m)$$

is p_m-periodic and $\mathbb{1}_{[a_m,p_m-a_m]}\mu_m = \mathbb{1}_{[a_m,p_m-a_m]}\mu$.

With $I_m := [-p_m, 2p_m]$ we obtain

$$\begin{aligned}\|\mu - \mu_m\|_{I_m} &\leqslant \sum_{k\in\mathbb{Z}} \big\|\psi_m(\cdot + kp_m)\big(\mu - \mu(\cdot + kp_m)\big)\big\|_{I_m} \\ &\leqslant \big(1 + \tfrac{1}{2a_m}\big) \sum_{k\in\mathbb{Z}} \|\mu - \mu(\cdot + kp_m)\|_{I_m\cap[-a_m-kp_m,p_m+a_m-kp_m]}\end{aligned}$$

by Lemma 2.6. For $|k| > 2$ the set $I_m \cap [-a_m - kp_m, p_m + a_m - kp_m]$ is empty; for $|k| \leqslant 2$ one easily shows the estimate

$$\|\mu - \mu(\cdot + kp_m)\|_{I_m \cap [-a_m - kp_m, p_m + a_m - kp_m]} \leqslant |k| \cdot \|\mu - \mu(\cdot + p_m)\|_{[-p_m, p_m]},$$

where for $|k| = 2$ one has to use the triangle inequality. It follows that $\|\mu - \mu_m\|_{I_m} \leqslant 6\big(1 + \frac{1}{2a_m}\big)\|\mu - \mu(\cdot + p_m)\|_{[-p_m,p_m]}$, which implies (ii).

Finally, assume without loss of generality that $p_m \geqslant 2$ for all $m \in \mathbb{N}$. Then for all $a \in \mathbb{R}$ we have

$$|\mu_m|\big((a, a+1]\big) \leqslant \sum_{k\in\mathbb{Z}} \big\|\psi_m(\cdot + kp_m)|_{(a,a+1]}\big\|_\infty \|\mu\|_{\mathrm{unif}} \leqslant \big(1 + \tfrac{1}{2a_m}\big)\|\mu\|_{\mathrm{unif}},$$

due to the particular construction of ψ_m and the fact that at most 3 summands in the second term are nonzero.

(ii) $\Rightarrow$ (i): Since μ_m is p_m-periodic, we have

$$\begin{aligned}
&\|\mu - \mu(\cdot + p_m)\|_{[-p_m,p_m]} \\
&\qquad \leqslant \|\mu - \mu_m\|_{[-p_m,p_m]} + \|\mu_m(\cdot + p_m) - \mu(\cdot + p_m)\|_{[-p_m,p_m]} \\
&\qquad \leqslant 2\|\mu - \mu_m\|_{[-p_m,2p_m]},
\end{aligned}$$

and the assertion follows. ☐

Remark 5.2. The above lemma shows that a weak Gordon measure can (locally on three periods) be very well approximated by periodic measures (where the approximation depends on C). In [7] a local measure was called Gordon measure if the condition of Lemma 5.1(ii) is satisfied for all $C > 0$, where $\|\cdot\|_{[-p_m,2p_m]}$ is replaced by the total variation norm on $[-p_m, 2p_m]$.

The definition in [7] extends the previous definitions for bounded ([5]) and locally integrable potentials ([2]), so our notion generalizes and refines the Gordon condition from those papers.

We will need the following estimate for solutions to periodic measures.

Lemma 5.3 (see [7, Lemma 2.8]). *Let μ be p-periodic and $z \in \mathbb{C}$. Let u be a solution of $H_\mu u = zu$. Then*

$$\left\|\begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix}\right\| \leqslant 2\max\left\{\left\|\begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix}\right\|;\ t \in \{-p, p, 2p\}\right\}.$$

The following is our main result on absence of eigenvalues.

Theorem 5.4. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be a weak Gordon measure with weight $C > 0$. Then H_μ does not have any eigenvalues with modulus less than $C^2 - \|\mu\|_{\mathrm{unif}}$.*

Proof. Let (p_m) and (μ_m) be as in Lemma 5.1(ii), without loss of generality $p_m \geqslant 2$ for all $m \in \mathbb{N}$. Let the additional statement of the lemma be satisfied with a sequence (a_m) satisfying $p_m + a_m \in \mathbb{N}$ for all $m \in \mathbb{N}$, $a_m \to \infty$ and $\frac{a_m}{p_m} \to 0$.

Assume that $z \in \mathbb{C}$ with $|z| < C^2 - \|\mu\|_{\mathrm{unif}}$ is an eigenvalue of H_μ. Let u be a corresponding eigenfunction; then $u \in W_2^1(\mathbb{R}) \subseteq C_0(\mathbb{R})$.

For $m \in \mathbb{N}$ let u_m be the solution of $H_{\mu_m} u_m = z u_m$ satisfying $u_m(a_m) = u(a_m)$, $u_m'(a_m+) = u'(a_m+)$. Then $u_m = u$ on $[a_m, p_m - a_m]$ since $\mu_m = \mu$ on this interval. We now apply Proposition 4.4 with $\nu = (\mu_m - z\lambda) - (\mu - z\lambda)$. Note that $\mathbb{1}_{[a_m, a_m+2]}\nu = 0$, so one has $c_{\nu, a_m+1} = 0$ in (1). Setting $\omega_m := \|\mu_m - z\lambda\|_{\mathrm{unif}}^{1/2}$ and taking into account Proposition 4.5, we obtain

$$|u(t) - u_m(t)| \leqslant C_m e^{\omega_m |t - (a_m+1)|} \|\mu - \mu_m\|_{[-p_m, a_m+1]} \quad \big(m \in \mathbb{N},\ t \in [-p_m, a_m]\big)$$

with $C_m > 0$ depending only on ω_m, and similarly for $t \in [p_m - a_m, 2p_m]$. Therefore,

$$\sup_{t \in [-p_m, 2p_m]} |(u - u_m)(t)| \leqslant C_m e^{\omega_m (p_m + a_m + 1)} \|\mu - \mu_m\|_{[-p_m, 2p_m]}. \tag{5}$$

By Lemma 5.1 and the assumption on z we have

$$\omega_m^2 \leqslant \|\mu_m\|_{\mathrm{unif}} + |z| \leqslant \big(1 + \tfrac{1}{2a_m}\big)\|\mu\|_{\mathrm{unif}} + |z| \longrightarrow \|\mu\|_{\mathrm{unif}} + |z| < C^2.$$

Hence, for large m we obtain

$$\omega_m (p_m + a_m + 1) = \omega_m p_m \big(1 + \tfrac{a_m+1}{p_m}\big) \leqslant C p_m.$$

We conclude that the right hand side in (5) tends to zero as $m \to \infty$. Therefore, given $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $|u(t) - u_m(t)| \leqslant \varepsilon$ for all $m \geqslant m_0$ and $t \in [-p_m, 2p_m]$. Since $u \in C_0(\mathbb{R})$, we can choose m_0 such that $|u(t)| \leqslant \varepsilon$ for $|t| \geqslant p_{m_0} - 1 =: t_0$. Then $|u_m| \leqslant 2\varepsilon$ on $[-p_m, 2p_m] \setminus (-t_0, t_0)$. By Lemma 3.4 it follows that $|u_m'| \leqslant 2\varepsilon(2 + \|\mu_m\|_{\mathrm{unif}})$ on that set.

Thus we have shown

$$\big(u_m(\pm p_m), u_m'(\pm p_m+)\big), \big(u_m(2p_m), u_m'(2p_m+)\big) \to 0 \qquad (m \to \infty).$$

By Lemma 5.3 we conclude that $\big(u_m(0), u_m'(0+)\big) \to 0$ as $m \to \infty$. For any compact interval $I \subseteq \mathbb{R}$, Lemma 3.2 yields $u_m \to 0$ uniformly on I. Since $u_m \to u$ uniformly on I by (5), we obtain $u = 0$, a contradiction. $\square$

As an immediate consequence of Theorem 5.4 we obtain the following version of Gordon's theorem.

Corollary 5.5. *Let μ be a weak Gordon measure with weight C for all $C > 0$. Then H_μ does not have any eigenvalues.*

By a scaling argument we can improve the eigenvalue bound $C^2 - \|\mu\|_{\mathrm{unif}}$ from Theorem 5.4.

Corollary 5.6. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be a weak Gordon measure with weight $C > 0$, and let $r > 0$. Then H_μ does not have any eigenvalues with modulus less than $C^2 - \|\mu\|_{\mathrm{unif},r}$, where*

$$\|\mu\|_{\mathrm{unif},r} := \tfrac{1}{r} \sup_{a\in\mathbb{R}} |\mu|((a, a + r]).$$

(Note that $\|\mu\|_{\mathrm{unif},1} = \|\mu\|_{\mathrm{unif}}$.)

Proof. Let $\mu_r := r\mu(r\cdot)$, and define $B_r \colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by $B_r u := u(\frac{\cdot}{r})$. A straightforward computation shows that $B_r^{-1} H_\mu B_r = r^{-2} H_{\mu_r}$.

We now show that μ_r is a weak Gordon measure with weight Cr. Let (p_m) be a sequence of periods for the weak Gordon measure μ. For $u \in W_\infty^1(\mathbb{R})$ with $\operatorname{spt} u \subseteq [-\frac{p_m}{r}, \frac{p_m}{r}]$, $\operatorname{diam} \operatorname{spt} u \leqslant 2$ and $\|u'\|_\infty \leqslant 1$ we have

$$\int_{-p_m/r}^{p_m/r} u\, d\big(\mu_r - \mu_r(\cdot + \tfrac{p_m}{r})\big) = \int_{-p_m}^{p_m} ru(\tfrac{\cdot}{r})\, d\big(\mu - \mu(\cdot + p_m)\big).$$

Since $u_r := ru(\frac{\cdot}{r})$ satisfies $\operatorname{diam} \operatorname{spt} u_r \leqslant 2r$ and $\|u_r'\|_\infty \leqslant 1$, we infer by Remark 2.2 that

$$\big\|\mu_r - \mu_r(\cdot + \tfrac{p_m}{r})\big\|_{[-p_m/r, p_m/r]} \leqslant (r + 1)^2 \|\mu - \mu(\cdot + p_m)\|_{[-p_m, p_m]}.$$

Multiplying by $\exp(Cr\frac{p_m}{r})$ we deduce that μ_r is indeed a weak Gordon measure with weight Cr. Therefore, by Theorem 5.4, H_{μ_r} has no eigenvalues z with $|z| < (Cr)^2 - \|\mu_r\|_{\mathrm{unif}}$.

To conclude the assertion for $H_\mu = r^{-2} B_r H_{\mu_r} B_r^{-1}$, it remains to observe that $\|\mu_r\|_{\mathrm{unif}} = r^2 \|\mu\|_{\mathrm{unif},r}$. $\square$

Remark 5.7. (a) The supremum of all $C > 0$ such that μ is a weak Gordon measure with weight C is given by

$$C_\mu := -\liminf_{p\to\infty} \tfrac{1}{p} \ln \|\mu - \mu(\cdot + p)\|_{[-p,p]}$$

if $C_\mu > 0$.

(b) Assume that $C_\mu > 0$. Then by part (a) and Corollary 5.6, H_μ does not have any eigenvalues z with

$$|z| < E_\mu := C_\mu^2 - \inf_{r>0} \|\mu\|_{\mathrm{unif},r}.$$

It is easy to see that $\|\mu\|_{\mathrm{unif},r} \leqslant (1 + \frac{s}{r})\|\mu\|_{\mathrm{unif},s}$ for all $r > s > 0$. Thus, $\lim_{r\to\infty} \|\mu\|_{\mathrm{unif},r} = \inf_{r>0} \|\mu\|_{\mathrm{unif},r}$.

In the next section we will show that $-E_\mu$ can occur as an eigenvalue, so the above bound is sharp.

(c) In the proof of Theorem 5.4 we have shown that $H_\mu u = zu$ does not have any solutions in $C_0(\mathbb{R})$, for z with small modulus. Hence, we also obtain absence of eigenvalues for H_μ considered as an operator in $L_p(\mathbb{R})$ with $1 \leqslant p < \infty$.

We now apply our main result to quasiperiodic measures.

Example 5.8. Let $\mu_1, \mu_2 \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be 1- and α-periodic, respectively, where $\alpha > 0$ is irrational. Then $\mu := \mu_1 + \mu_2$ is a quasiperiodic measure. Under a suitable assumption on α we are going to prove that H_μ has no eigenvalues. Analogous results have been shown in [2] under an additional assumption on the oscillation of V_2, where $\mu_2 = V_2\lambda$, and in [7] under a Hölder condition on the translates of μ_2. Using our weak Gordon condition, we can remove these additional assumptions on μ_2 altogether.

As in [2, 7] we assume that there exist $B > 0$ and a sequence $(\frac{p_m}{q_m})$ in $\mathbb{Q}$ such that

$$\left|\alpha - \frac{p_m}{q_m}\right| \leqslant Bm^{-q_m} \qquad (m \in \mathbb{N});$$

in particular, α is a Liouville number. As for the Liouville numbers one sees that the set of all numbers α with the above property is a dense G_δ set.

Using the periodicity of μ_1, μ_2 and Remark 2.3(b), we now show that μ is a weak Gordon measure with weight C, for all $C > 0$:

$$\begin{aligned}\|\mu - \mu(\cdot + p_m)\|_{[-p_m,p_m]} &= \|\mu_2 - \mu_2(\cdot + p_m - \alpha q_m)\|_{[-p_m,p_m]} \\ &\leqslant 3|p_m - \alpha q_m| \|\mu_2\|_{\mathrm{unif}} = 3q_m\left|\tfrac{p_m}{q_m} - \alpha\right| \|\mu_2\|_{\mathrm{unif}}\end{aligned}$$

and hence

$$e^{Cp_m}\|\mu-\mu(\cdot+p_m)\|_{[-p_m,p_m]} \leqslant \exp\bigl(Cq_m\tfrac{p_m}{q_m}\bigr)\cdot 3q_mBm^{-q_m}\|\mu_2\|_{\mathrm{unif}} \to 0$$

as $m\to\infty$. By Corollary 5.5 it follows that H_μ does not have any eigenvalues.

6 Sharpness of the condition

In this section we show that our results are sharp in the following sense: we construct a measure $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ such that the operator H_μ has -1 as an eigenvalue, μ is a weak Gordon measure with weight C for all $C<1$, and $\inf_{r>0}\|\mu\|_{\mathrm{unif},r}=0$. With the notation of Remark 5.7 we then have $C_\mu=1$ and $E_\mu=1$.

The construction will be such that the Gordon condition is satisfied with arbitrarily many periods, not just with three periods. In fact, we construct a symmetric eigenfunction to the eigenvalue -1 in the following way. We first fix u on an infinite discrete set $T\subseteq\mathbb{R}$ and then choose the continuous continuation to $\mathbb{R}$ satisfying $u''=u$ on $\mathbb{R}\setminus T$. We will determine T and also u on T recursively.

Choose a sequence $(l_m)_{m\in\mathbb{N}}$ in $(0,\infty)$ satisfying $l_m\to\infty$. Set $p_0:=0$, $T_0:=\{0\}$ and $u(0):=1$. Recursively we define, for $m\in\mathbb{N}$,

$$\begin{aligned} p_m &:= 2(m-1)p_{m-1}+l_m\\ T_m &:= \{jp_m+t;\ |j|\leqslant m,\ t\in T_{m-1}\}\cap[-mp_m,mp_m],\\ u(jp_m+t) &:= 2^{-m|j|}u(t) \qquad \bigl(jp_m+t\in T_m\setminus T_{m-1}\bigr). \end{aligned}$$

Let $T:=\bigcup_{m\in\mathbb{N}_0}T_m$. Then u is fixed on the discrete set T. Consider the unique continuous continuation $u\colon\mathbb{R}\to(0,\infty)$ satisfying $u''=u$ on $\mathbb{R}\setminus T$. It is easy to see that $u\in L_2(\mathbb{R})$. Moreover, note that the derivative of u has jumps precisely at T. Thus, u solves $-u''+\mu u=-u$ for some pure-point measure μ with $\operatorname{spt}\mu=T$.

We now assume that

$$\frac{2(m-1)p_{m-1}+m\ln 2}{l_m}\to 0 \qquad (m\to\infty). \tag{6}$$

Since then $2^m\leqslant e^{l_m}$ for large m, a straightforward computation yields boundedness of the masses of single points (see also Remark 6.2 below). Furthermore, $T=\operatorname{spt}\mu$ is uniformly discrete (i.e., there exists a minimal distance

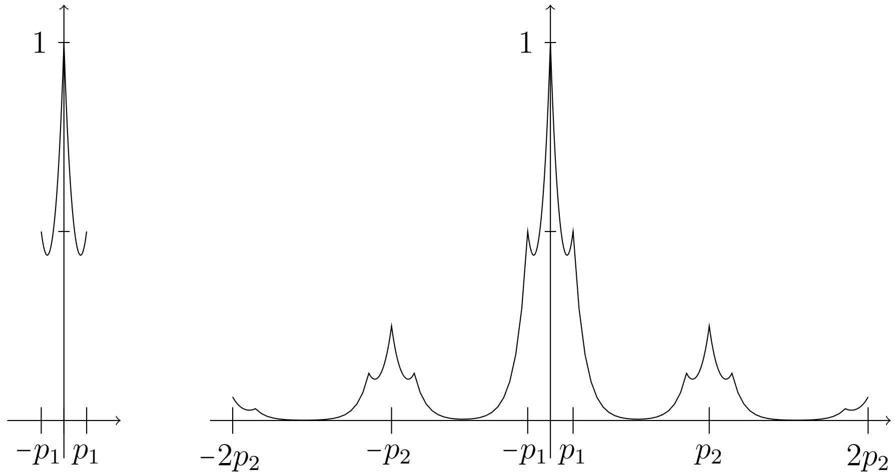

Figure 1: u defined on the intervals corresponding to T_1 and to T_2.

between any two points in T). These two facts imply $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Hence, -1 is an eigenvalue of H_μ (with eigenfunction u).

We will now investigate μ and show that μ is indeed a weak Gordon measure with weight C for all $C < 1$. Let $s_m := (m-1)p_{m-1}$ and $t_m := s_m + l_m = p_m - s_m$. By construction, the measures

$$\mathbb{1}_{(-mp_m,(m-1)p_m)}\mu \quad \text{and} \quad \mathbb{1}_{(-mp_m,(m-1)p_m)}\big(\mu(\cdot + p_m)\big)$$

differ only at the points $-s_m$ and $-t_m$, and there the corresponding point masses are interchanged. (It suffices to consider the open intervals here since the test functions for computing the $\|\cdot\|_I$-norm are zero at the boundary.) With the help of the next lemma we can compute the difference $\mu(\{-s_m\}) - \mu(\{-t_m\})$.

Lemma 6.1. *Let $l, L > 0$, and let $a, b, c, d > 0$ satisfy $\frac{a}{b} = \frac{d}{c}$. Let $u \in C[-l, L+l]$ such that $u(-l) = a$, $u(0) = b$, $u(L) = c$ and $u(L+l) = d$. Assume that u is twice continuously differentiable on $(-l, L+l) \setminus \{0, L\}$, with $u'' = u$. Then the measure μ defined by $-u'' + \mu u = -u$ satisfies* $\operatorname{spt}\mu \subseteq \{0, L\}$ *and*

$$\mu(\{0\}) - \mu(\{L\}) = \frac{2\frac{c}{b} - 2\frac{b}{c}}{e^L - e^{-L}}\,.$$

Proof. First observe that $\frac{1}{b}u(-t) = \frac{1}{c}u(L+t)$ holds for $t = 0$ and for $t = l$, and hence for all $t \in [-l, 0]$ by uniqueness of the boundary value problem for

$u'' = u$. It follows that $\frac{u'(0-)}{u(0)} + \frac{u'(L+)}{u(L)} = 0$. Therefore,

$$\begin{aligned} \mu(\{0\}) - \mu(\{L\}) &= \frac{u'(0+) - u'(0-)}{u(0)} - \frac{u'(L+) - u'(L-)}{u(L)} \\ &= \frac{1}{b} u'(0+) + \frac{1}{c} u'(L-). \end{aligned} \tag{7}$$

A straightforward computation yields

$$u(t) = \frac{c - e^{-L} b}{e^L - e^{-L}} e^t + \frac{e^L b - c}{e^L - e^{-L}} e^{-t} \tag{8}$$

for all $t \in [0, L]$, so from (7) we deduce that

$$\begin{aligned} \mu(\{0\}) - \mu(\{L\}) &= \frac{1}{b} \cdot \frac{2c - (e^L + e^{-L})b}{e^L - e^{-L}} + \frac{1}{c} \cdot \frac{(e^L + e^{-L})c - 2b}{e^L - e^{-L}} \\ &= \frac{2\frac{c}{b} - 2\frac{b}{c}}{e^L - e^{-L}} . \end{aligned}$$

□

Remark 6.2. If $e^{-L} \leqslant \frac{c}{b} \leqslant e^L$, then the two coefficients in the right hand side of (8) are non-negative, and this implies $\left|\frac{u'(0+)}{u(0)}\right|, \left|\frac{u'(L-)}{u(L)}\right| \leqslant 1$. (In the following we will have $\frac{c}{b} = 2^{-m}$.)

We now apply Lemma 6.1 with $l = l_1$ and $L = l_m$, for $m \geqslant 2$. Note that $u(t_m) = u(1p_m - s_m) = 2^{-m \cdot 1} u(-s_m) = 2^{-m} u(s_m)$. Therefore,

$$\mu(\{s_m\}) - \mu(\{t_m\}) = \frac{2 \cdot 2^{-m} - 2 \cdot 2^m}{e^{l_m} - e^{-l_m}} ,$$

and for $m \geqslant 2$ with $l_m \geqslant 2$ it follows that

$$\|\mu - \mu(\cdot + p_m)\|_{[-mp_m, (m-1)p_m]} = \frac{2 \cdot 2^m - 2 \cdot 2^{-m}}{e^{l_m} - e^{-l_m}} \leqslant 4e^{m \ln 2 - l_m}.$$

For $C < 1$ we infer by (6) that

$$e^{Cp_m} \|\mu - \mu(\cdot + p_m)\|_{[-mp_m, (m-1)p_m]} \leqslant 4e^{C2(m-1)p_{m-1} + m \ln 2 - (1-C) l_m} \to 0.$$

In particular, μ is a weak Gordon measure with weight C, and hence $C_\mu \geqslant 1$.

Furthermore we have $\inf_{r>0} \|\mu\|_{\mathrm{unif},r} = 0$ since $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and

$$\tfrac{1}{r} \sup_{a \in \mathbb{R}} \#\big(\operatorname{spt} \mu \cap (a, a + r]\big) \to 0$$

as $r \to \infty$, where $\#$ denotes the counting measure. Thus, $E_\mu = C_\mu^2 = 1$.

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