On the L_p-theory of C_0-semigroups associated with second order elliptic operators. I

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Abstract

We study L_p-theory of second order elliptic divergence type operators with measurable coefficients. To this end, we introduce a new method of constructing positive C_0-semigroups on L_p associated with sesquilinear (not necessarily sectorial) forms in L_2. A precise condition ensuring that the elliptic operator is associated with a quasi-contractive C_0-semigroup on L_p is established.

1 Introduction and main results

In this paper we study the L_p-theory of second order elliptic differential operators on an open set $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$, corresponding to the formal differential expression

$$\mathcal{L} = -\nabla \cdot (a\nabla) + b_1 \cdot \nabla + \nabla \cdot b_2 + V,$$

with singular measurable coefficients $a\colon \Omega \to \mathbb{R}^N \otimes \mathbb{R}^N$, $b_1, b_2\colon \Omega \to \mathbb{R}^N$, $V\colon \Omega \to \mathbb{R}$. The aim of the paper is to construct a quasi-contractive C_0-semigroup on $L_p := L_p(\Omega)$, whose generator is associated with $\mathcal{L}$ in a natural way which will be made precise below. As is well-known, this implies well-posedness of the corresponding Cauchy problem.

Elliptic operators in divergence form with measurable coefficients are usually defined by means of the form method. The form associated with the above differential expression is

$$\tau(u, v) := \langle a\nabla u, \nabla v\rangle + \langle \nabla u, b_1 v\rangle - \langle b_2 u, \nabla v\rangle + \langle V u, v\rangle \tag{1.1}$$

on a suitable domain $D(\tau)$ corresponding to the boundary conditions. (Here and in the sequel, $\langle f, g\rangle$ is defined as $\int_\Omega f(x)\cdot\overline{g}(x)\,dx$ whenever $f\cdot\overline{g}\in L_1$, for $f,g\colon \Omega\to\mathbb{C}$ or $f,g\colon\Omega\to\mathbb{C}^N$ measurable.)

The traditional way of constructing the corresponding C_0-semigroup is the following. If the form τ is densely defined, sectorial and closed then it is associated with an m-sectorial operator A in L_2 which generates a quasi-contractive analytic semigroup e^{-At} on L_2 (cf. [4, Thm. VI.2.1]). If $\left\|e^{-At}\restriction_{L_2\cap L_p}\right\|_{L_p\to L_p}\leqslant Me^{\omega_p t}$ for some $p\in[1,\infty)$, then the semigroup extends to a semigroup T_p on L_p. In this case we say that e^{-At} *extrapolates* to the semigroup T_p on L_p, which is *consistent* with e^{-At} in the sense that $e^{-At}\restriction_{L_2\cap L_p}=T_p(t)\restriction_{L_2\cap L_p}$ for all $t\geqslant 0$. For $p>1$, the semigroup T_p is always strongly continuous, whereas for $p=1$ this is the case if, e.g., T_1 is positive or quasi-contractive (see [19]). The above approach was used for constructing semigroups acting in all L_p, $1\leqslant p<\infty$ (this case is well-documented, see, e.g., [14] and [3]), as well as for constructing semigroups acting in L_p only for p from some subinterval of $[1,\infty)$ containing 2; see, e.g., [2], [6].

However, we do not assume τ to be a sectorial form in L_2; even its real part need not be bounded below, so that the traditional approach is not applicable. In the case $b_2=0$ and $V=0$, non-sectorial forms have been studied in [5], [6] where the coefficients of the first order terms of $\mathcal{L}$ are approximated in such a way that the approximating forms become sectorial in L_2 and the corresponding semigroups converge to a C_0-semigroup, in a suitable L_p.

In this paper we develop a new approach to the construction of a quasi-contractive C_0-semigroup associated with the form τ, which even in L_2 gives rise to a C_0-semigroup under assumptions when all known representation theorems break down. Our approach is based upon approximations by sectorial forms, however, not related to approximations of the coefficients of the first order terms.

Instead, we approximate the potential: we introduce a positive potential U which 'absorbs' all the singularities of the lower order terms of $\mathcal{L}$ in the sense that, being added to τ, it makes the sum sectorial in L_2. The sequence of the approximating semigroups T_m, which are associated with the sectorial forms $\tau_m:=\tau+U-U\wedge m$ ($m\in\mathbb{N}$), extrapolates to a suitable L_p and strongly converges to a quasi-contractive C_0-semigroup on L_p. The use of the perturbation theory of positive semigroups developed in [17], [18] is crucial for the realization of this idea.

The approach we present is in fact a general method of constructing positive C_0-semigroups on L_p corresponding to sesquilinear forms in L_2 (see Section 3 for details). In the context of Schrödinger operators with magnetic fields, and dominated semigroups with singular complex potentials, a similar approximation idea was used in [12] and in [7].

The result we obtain is sharp in the sense that, for a wide class of coefficients, the sufficient condition (see estimate (1.3) below) for the validity of our main theorem becomes necessary (see Section 6 for details).

We make the following qualitative assumptions on the coefficients of $\mathcal{L}$.

(a) $a \in L_{1,loc}$, a is a.e. invertible with $a^{-1} \in L_{1,loc}$, and a is uniformly sectorial, i.e.,

$$|\operatorname{Im} \zeta^* a\zeta| \leqslant \alpha \operatorname{Re} \zeta^* a\zeta \quad \text{a.e. } (\zeta \in \mathbb{C}^N)$$

for some $\alpha \geqslant 0$ (where ζ^* is the transpose of $\overline{\zeta}$). Let $a_s := \frac{a+a^\top}{2}$. Then

$$\tau_N(u,v) := \langle a\nabla u, \nabla v\rangle, \; D(\tau_N) := \left\{u \in W^1_{1,loc} \cap L_2; \, (\nabla u)^* a_s \nabla u \in L_1\right\}$$

defines a closed sectorial (non-symmetric) Dirichlet form in L_2 (for the closedness cf. [13, Theorem 3.2]). Let $\tau_a \subseteq \tau_N$ be a Dirichlet form.

(bV) The potentials $W_j := b_j^\top a_s^{-1} b_j$ $(j = 1,2)$ and $|V|$ are τ_a*-regular*, i.e., $D(\tau_a) \cap Q(W_j)$ and $D(\tau_a) \cap Q(|V|)$ are cores for τ_a. (For a potential $U \geqslant 0$, $Q(U) := \left\{u \in L_2; \, U|u|^2 \in L_1\right\}$ denotes the domain of the form $U(u) = \langle U|u|^2\rangle$ in L_2.)

We define the form τ on $D(\tau) := D(\tau_a) \cap Q(W_1 + W_2 + |V|)$ by (1.1). This is possible since for $u, v \in D(\tau)$ and $j = 1,2$ we have, by the Cauchy-Schwarz inequality,

$$|\nabla u \cdot b_j \overline{v}| = \left|a_s^{1/2}\nabla u \cdot a_s^{-1/2} b_j \overline{v}\right| \leqslant \left(a_s \nabla u \cdot \nabla \overline{u}\right)^{1/2} \left(W_j |v|^2\right)^{1/2} \in L_1. \tag{1.2}$$

Furthermore, $D(\tau)$ is dense in $D(\tau_a)$ as can be seen from Lemma 3.13 below. In particular, τ is densely defined.

Although the form τ itself need not be sectorial, the form $\tau + U$ with domain $D(\tau) \cap Q(U)$ is sectorial and closed for all $U \geqslant U_0 := W_1 + W_2 + 2V^-$ since the sum of the first order terms of τ is form small with respect to $\tau_a + W_1 + W_2$ by (1.2).

The only quantitative condition we need is obtained from the Lumer-Phillips theorem by a formal computation. Suppose τ is associated with a positive quasi-contractive C_0-semigroup $T_p(t) = e^{-A_p t}$ on L_p, for some $p \in [1,\infty)$. Then A_p is quasi-accretive which by the positivity of T_p is equivalent to $\langle A_p u, u^{p-1}\rangle \geqslant -\omega_p \|u\|_p^p$ in case $p > 1$, and to $\langle A_1 u\rangle \geqslant -\omega_1 \|u\|_1$ in case $p = 1$, for some $\omega_p \in \mathbb{R}$ and all $0 \leqslant u \in D(A_p)$. Formally, $A_p u = \mathcal{L}u$, $\nabla u^{p-1} = \frac{2}{p'} u^{p/2-1} \nabla u^{p/2}$, and $\nabla u = \frac{2}{p} u^{1-p/2} \nabla u^{p/2}$. Thus,

$$\begin{aligned}
\langle A_p u, u^{p-1}\rangle &= \left\langle -\nabla \cdot (a\nabla u) + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + Vu, u^{p-1}\right\rangle \\
&= \tfrac{4}{pp'}\langle a\nabla u^{p/2}, \nabla u^{p/2}\rangle + \langle (\tfrac{2}{p} b_1 - \tfrac{2}{p'} b_2) u^{p/2}, \nabla u^{p/2}\rangle + \langle V u^p\rangle
\end{aligned}$$

in case $p > 1$ and, in case $p = 1$,

$$\begin{aligned}
\langle A_1 u\rangle &= \left\langle -\nabla \cdot (a\nabla u) + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + Vu\right\rangle \\
&= 2\langle \nabla u^{1/2}, b_1 u^{1/2}\rangle + \langle Vu\rangle.
\end{aligned}$$

Now we define quadratic forms τ_p on $D(\tau_p) := D(\tau)$ $(1 \leqslant p < \infty)$,

$$\tau_p(u) := \tfrac{4}{pp'}\langle a_s \nabla u, \nabla u\rangle + \tfrac{2}{p}\langle \nabla|u|, b_1|u|\rangle - \tfrac{2}{p'}\langle b_2|u|, \nabla|u|\rangle + \langle V|u|^2\rangle \quad (p > 1),$$
$$\tau_1(u) := 2\langle \nabla|u|, b_1|u|\rangle + \langle V|u|^2\rangle.$$

Then the natural condition for L_p-accretivity is

$$\tau_p(u) \geqslant -\omega_p \|u\|_2^2 \quad \big(u \in D(\tau)\big), \tag{1.3}$$

i.e., τ_p is bounded from below. Note that $\tau_2 = \operatorname{Re}\tau$ (as to be expected), where the form $\operatorname{Re}\tau$ is defined by $(\operatorname{Re}\tau)(u,v) := \frac{1}{2}\big(\tau(u,v) + \overline{\tau(v,u)}\big)$ on $D(\operatorname{Re}\tau) := D(\tau)$.

The construction of the C_0-semigroup on L_p, corresponding to the formal differential expression $\mathcal{L}$ with boundary conditions prescribed by $D(\tau_a)$, is given in the following theorem, which constitutes a simplified version of the main result of the paper, Theorem 4.2.

Theorem 1.1. *Let assumptions* **(a)** *and* **(bV)** *be fulfilled. Let* $U_0 := W_1 + W_2 + 2V^-$*, and let* $T_{0,2}$ *be the* C_0*-semigroup on* L_2 *associated with the form* $\tau + U_0$*. Let* I *be the set of all* $p \in [1,\infty)$ *such that* $\omega_p := \inf\{\omega \in \mathbb{R};\ \tau_p \geqslant -\omega\} < \infty$*. Then the following assertions hold.*

(i) The set I *is an interval in* $[1,\infty)$*, and* $T_{0,2}$ *extrapolates to a* C_0*-semigroup* $T_{0,p}(t) = e^{-A_{0,p}t}$ *on* L_p*, for all* $p \in I$*.*

(ii) For all $p \in I$*, the sequence of* C_0*-semigroups* $T_{m,p}(t) = e^{-(A_{0,p} - U_0 \wedge m)t}$ *strongly converges in* L_p *to a* C_0*-semigroup* $T_p(t) = e^{-A_p t}$ *satisfying* $\|T_p(t)\| \leqslant e^{\omega_p t}$*. For* $p, q \in I$*, the semigroups* T_p *and* T_q *are consistent.*

(iii) For all $p \in I \setminus \{1\}$*, the form* τ_p *is closable. For all* $u \in D(A_p)$ *we have* $|u|^{p/2}\operatorname{sgn} u \in D(\overline{\tau_p})$ *and*

$$\operatorname{Re}\langle A_p u, u|u|^{p-2}\rangle \geqslant \overline{\tau_p}(|u|^{p/2}\operatorname{sgn} u).$$

(iv) If, in addition, we assume that

$$\big|\operatorname{Im}\langle (b_1 + b_2)u, \nabla u\rangle\big| \leqslant c_1 \tau_p(u) + c_2\|u\|_2^2 \quad \big(u \in D(\tau)\big)$$

for some $p \in \mathring{I}$*,* $c_1 \geqslant 0$*,* $c_2 \in \mathbb{R}$*, then* T_p *extends to a quasi-contractive analytic semigroup on* L_p *and* A_p *is an* m*-sectorial operator in* L_p*, for all* $p \in \mathring{I}$*.*

We shall call A_p the m-accretive operator in L_p, T_p the quasi-contractive C_0-semigroup on L_p associated with the form τ. The operator A_p is an L_p-realization of $\mathcal{L}$ with boundary conditions prescribed by $D(\tau_a)$.

Remarks 1.2. (a) In fact, as it will be shown in the main body of the paper (see Corollary 4.4 below), the semigroups T_p constructed in the theorem do not depend on the approximating sequence of potentials. Furthermore, the assertions hold with U_0 replaced by any positive τ_a-regular potential U such that $\tau + U$ is sectorial and closable in L_2.

(b) The domain of τ_a determines the 'boundary conditions' under consideration. The standard examples are the case of Neumann boundary conditions $\tau_a = \tau_N$ and of Dirichlet boundary conditions $\tau_a = \tau_D := \overline{\tau_N \restriction_{C_c^\infty(\Omega)}}$. Assumption **(bV)** expresses that the lower order perturbations must not disturb the boundary conditions prescribed by $D(\tau_a)$. In the case of Dirichlet boundary conditions, assumption **(bV)** is fulfilled in particular if $W_1, W_2, V \in L_{1,loc}$.

Suppose that assumption **(bV)** is not fulfilled, but $D(\tau)$ is dense in L_2. Let $\widetilde{\tau}_a := \overline{\tau_N \restriction_{D(\tau)}}$ (note that $\widetilde{\tau}_a$ is a Dirichlet form). Then assumptions **(a)** and **(bV)** are fulfilled with $\widetilde{\tau}_a$ in place of τ_a, so Theorem 4.2 is still applicable to the form τ.

(c) If the form τ itself is sectorial then it is closable (see Lemma 3.5 below). In this case we have $2 \in I$, A_2 is the m-sectorial operator associated with $\overline{\tau}$ and, for $f \in L_2$, the function $u(t) := T_2(t)f$ is the weak solution of the Cauchy problem

$$\begin{cases} u_t &= -\mathcal{L}u, \\ u(0) &= f \end{cases}$$

with boundary conditions prescribed by $D(\tau)$.

(d) Let us point out that the interval I given in Theorem 4.2 is a set of $p \in [1, \infty)$ for which the form τ is associated with a *quasi-contractive* C_0-semigroup T_p on L_p ($I \setminus \{1\}$ is the maximal set of such $p \in (1, \infty)$ under the conditions of Corollary 6.4 below). The set of *all* $p \in [1, \infty)$ such that τ is associated with a C_0-semigroup T_p on L_p can be strictly larger than I, see [8].

The remainder of the paper is organized as follows. In Section 2 we give a brief account of Voigt's perturbation theory for positive semigroups. In Section 3 we show how to associate a positive C_0-semigroup on $L_p(\mu)$ with a sesquilinear form in $L_2(\mu)$. Section 4 contains the precise formulation of the main theorem and some useful consequences of it. The proof of the main theorem is given in Section 5. In Section 6 we discuss the sharpness of the main result.

2 Perturbations of positive C_0-semigroups by real-valued potentials

In this section we give a short survey of J. Voigt's perturbation theory for positive C_0-semigroups developed in [17], [18].

Let (Ω, μ) be a measure space, $1 \leqslant p < \infty$. Let T be a *positive* C_0-semigroup on $L_p(\mu)$, i.e., the semigroup operators $T(t)$ $(t \geqslant 0)$ are positivity preserving. Let

$-A$ be the generator of T and $V\colon \Omega \to \mathbb{R}$ a measurable function. If $V \in L_\infty(\mu)$ then T_V denotes the C_0-semigroup generated by $-(A+V)$.

The definition of T_V is extended to unbounded real-valued potentials by approximating V by $V^{(n)} := (V \wedge n) \vee (-n)$ and letting

$$T_V(t) := \operatorname*{s-lim}_{n\to\infty} T_{V^{(n)}}(t) \quad (t \geqslant 0) \tag{2.1}$$

if the limits exist. Obviously, T_V is a semigroup in this case. If $V \geqslant 0$ then $(T_{V^{(n)}})$ is a monotone decreasing sequence, for $V \leqslant 0$ it is monotone increasing. This leads to the following definition.

Definition 2.1. ([17, Def. 2.2], [18, Def. 2.5], [18, Def. 3.1])

(a) If $V \geqslant 0$ then the limit in (2.1) exists for all $t \geqslant 0$. If T_V is strongly continuous, V is called T-*admissible.* In this case, $T_{V^{(n)}} \to T_V$ as $n \to \infty$, i.e., $T(t)f = \lim_{n\to\infty} T_{V^{(n)}}(t)f$, uniformly for t in bounded subsets of $[0,\infty)$, for all $f \in L_p$.

(b) If $V \leqslant 0$ then V is called T-*admissible* if the limit in (2.1) exists for all $t \geqslant 0$ and defines a C_0-semigroup. In this case, $T_{V^{(n)}} \to T_V$ as $n \to \infty$.

By [18, Prop. 2.2], V is T-admissible if and only if $\sup_{0\leqslant t\leqslant 1,\, n\in\mathbb{N}} \|T_{V^{(n)}}(t)\| < \infty$.

(c) If $V \geqslant 0$ and V is T-admissible then $-V$ is T_V-admissible. If $T = (T_V)_{-V}$, then V is called T-*regular.*

The following result expresses, roughly speaking, that negative admissible potentials are always regular.

Lemma 2.2. *(cf. [18, Thm. 2.6, Prop. 3.3(b)]) Let* $V \geqslant 0$ *be measurable. If* $-V$ *is* T*-admissible, then* $(T_{-V})_V = T$*, and* V *is* T*-regular.*

Lemma 2.3. *([17, Prop. 3.1]) Let* $p, q \in [1,\infty)$*,* T_p*,* T_q *consistent positive* C_0*-semigroups on* $L_p(\mu)$*,* $L_q(\mu)$*, respectively,* $V \geqslant 0$ *measurable.*

(a) $(T_p)_V$ *and* $(T_q)_V$ *are consistent, and* V *is* T_p*-admissible if and only if* V *is* T_q*-admissible.*

(b) If $-V$ *is* T_p*- and* T_q*-admissible, then* $(T_p)_{-V}$ *and* $(T_q)_{-V}$ *are consistent.*

(c) V *is* T_p*-regular if and only if* V *is* T_q*-regular.*

We conclude the section with the following approximation result which we will use in Section 4 to show that the semigroup constructed in [6, Thm. 6] coincides with the semigroup constructed in Theorem 1.1.

Proposition 2.4. *Let* $p \in (1,\infty)$*. Let* T_n $(n \in \mathbb{N} \cup \{\infty\})$ *be positive* C_0*-semigroups on* $L_p(\mu)$ *with* $T_n \to T_\infty$*. Let* $0 \leqslant V \in \big(L_1 + L_\infty\big)(\mu)$ *such that* $-V$ *is* T_n*-admissible* $(n \in \mathbb{N} \cup \{\infty\})$*, and*

$$\|(T_n)_{-V}(t)\|_{p\to p} \leqslant e^{\omega t}, \quad \|(T_n)_{-V}(t)\|_{\infty\to\infty} \leqslant Ce^{\omega t} \quad (n \in \mathbb{N},\ t \geqslant 0).$$

for some $\omega \in \mathbb{R}$*,* $C \geqslant 1$*. Then* $(T_n)_{-V} \to (T_\infty)_{-V}$*.*

The crucial idea of the proof is to make use of the following result which gives an explicit rate of the convergence $T_{-V\wedge n} \to T_{-V}$.

Lemma 2.5. *Let $p \in (1,\infty)$, T be a positive C_0-semigroup on $L_p(\mu)$, and $0 \leqslant V \in \big(L_1+L_\infty\big)(\mu)$ such that $-V$ is T-admissible, and T_{-V} is contractive in $L_p(\mu)$ and bounded in $L_\infty(\mu)$. Let $-A$ be the generator of T, $-A_{-V}$ the generator of T_{-V}. Then*

$$\|(\lambda + A_{-V})^{-1}f - (\lambda + A - V\wedge n)^{-1}f\|_p \leqslant C\lambda^{-1-1/p}\|(V-n)^+\|_1^{1/p}\|f\|_\infty$$

for all $0 \leqslant f \in \big(L_p \cap L_\infty\big)(\mu)$, $\lambda > 0$ and $n \in \mathbb{N}$ such that $(V-n)^+ \in L_1(\mu)$, where C is the L_∞-bound of T_{-V}.

Proof. Let f, λ, n be given. For $m \in \mathbb{N}$ let $V_m := V \wedge m$. Then

$$u_m := (\lambda + A - V_m)^{-1}f \uparrow u := (\lambda + A_{-V})^{-1}f \quad \text{as } m \to \infty,$$

and $\|u\|_\infty \leqslant \frac{C}{\lambda}\|f\|_\infty$. For $m \in \mathbb{N}$ we have

$$(\lambda + A - V_m)^{-1} - (\lambda + A - V_n)^{-1} = (\lambda + A - V_m)^{-1}(V_m - V_n)(\lambda + A - V_n)^{-1}$$

and therefore $(\lambda + A - V_m)(u_m - u_n) = (V_m - V_n)u_n$. The contractivity of T_{-V} implies that $A - V_m$ is accretive, so we obtain, for $m \geqslant n$,

$$\begin{aligned}\lambda\|u_m - u_n\|_p^p &\leqslant \langle(\lambda + A - V_m)(u_m - u_n), (u_m - u_n)^{p-1}\rangle \\ &= \langle(V_m - V_n)u_n, (u_m - u_n)^{p-1}\rangle \\ &\leqslant \langle(V - V_n)u^p\rangle \leqslant \|(V-n)^+\|_1\|u\|_\infty^p.\end{aligned}$$

We conclude that $\|u_m - u_n\|_p^p \leqslant \lambda^{-p-1}\|(V-n)^+\|_1(C\|f\|_\infty)^p$, and $m \to \infty$ completes the proof. ☐

Proof of Proposition 2.4. Without restriction assume $\omega = 0$. Let $-A_n$, $-(A_n)_{-V}$ be the generators of T_n, $(T_n)_{-V}$, respectively. By the assumption, $A_n \to A_\infty$ in the strong resolvent sense as $n \to \infty$. So $A_n - V\wedge m \to A_\infty - V\wedge m$ in the strong resolvent sense as $n \to \infty$, for all $m \in \mathbb{N}$. By Lemma 2.5 we know that $A_n - V\wedge m \to (A_n)_{-V}$ in the strong resolvent sense as $m \to \infty$, uniformly in $n \in \mathbb{N}$. Since $A_\infty - V\wedge m \to (A_\infty)_{-V}$ in the strong resolvent sense, this yields the desired conclusion. ☐

3 The first Beurling-Deny criterion for sesquilinear forms

It is well-known that with every densely defined closed sectorial form in a Hilbert space H one can associate an analytic semigroup on H. In this section we are going to present a procedure how to associate a positive C_0-semigroup on $L_p(\mu)$ with a sesquilinear form in $L_2(\mu)$ fulfilling the first Beurling-Deny criterion ((Ω,μ) a measure space), even in cases when the form is not bounded below.

Definition 3.1. Let τ be a sesquilinear form in $L_2(\mu)$.

(a) τ is called *real* if $\operatorname{Re} u \in D(\tau)$ for all $u \in D(\tau)$, and $\tau(u,v) \in \mathbb{R}$ for all real-valued $u, v \in D(\tau)$.

(b) τ is said to *fulfill the first Beurling-Deny criterion* if τ is real and $u^+ \in D(\tau)$, $\tau(u^+, u^-) \leqslant 0$ for all real-valued $u \in D(\tau)$.

Note that, if τ fulfills the first Beurling-Deny criterion then so does $\operatorname{Re} \tau$.

The following proposition, due to Ouhabaz ([11, Prop. 2.2 and Thm. 2.4]), shows the relevance of these two notions.

Proposition 3.2. *Let τ be a densely defined closed sectorial form in $L_2(\mu)$, T the associated analytic semigroup on $L_2(\mu)$. Then T is real (i.e., all semigroup operators are reality preserving) if and only if τ is real, and T is positive if and only if τ fulfills the first Beurling-Deny criterion.*

The next lemma states that it suffices to verify the conditions of Definition 3.1 on a form core.

Lemma 3.3. *Let τ be a closable sectorial form. If τ fulfills the first Beurling-Deny criterion then so does $\overline{\tau}$.*

Proof. We first show that $\overline{\tau}$ is real. Without restriction $\operatorname{Re} \tau \geqslant 0$. Then

$$\tau(\operatorname{Re} u) \leqslant \tau(\operatorname{Re} u) + \tau(\operatorname{Im} u) = \operatorname{Re} \tau(u) \quad \big(u \in D(\tau)\big)$$

since τ is real. From this we easily deduce: if $u \in D(\overline{\tau})$, $(u_n) \subseteq D(\tau)$ with $u_n \to u$ in $D(\overline{\tau})$, then $\operatorname{Re} u \in D(\overline{\tau})$ and $\operatorname{Re} u_n \to \operatorname{Re} u$ in $D(\overline{\tau})$. By the latter we show that $\overline{\tau}(u,v) \in \mathbb{R}$ for all real-valued $u, v \in D(\overline{\tau})$, i.e., $\overline{\tau}$ is real.

From the above it follows that the set of all real-valued elements of $D(\tau)$ is dense in the set of all real-valued elements of $D(\overline{\tau})$. Now, for real-valued $u \in D(\tau)$, we have $\overline{\tau}(u^+, u - u^+) = -\overline{\tau}(u^+, u^-) \geqslant 0$ and $\overline{\tau}(u - u^+, u^+) = -\overline{\tau}\big((-u)^+, (-u)^-\big) \geqslant 0$. Thus, we can apply [9, Lemma I.4.9] to conclude that $u^+ \in D(\overline{\tau})$, $\overline{\tau}(u^+, u^-) \leqslant 0$ for all real-valued $u \in D(\overline{\tau})$. ☐

For the remainder of this section let τ be a densely defined sesquilinear form in $L_2(\mu)$ fulfilling the first Beurling-Deny criterion. The next result characterizes admissibility of potentials via form conditions, in the case of symmetric forms.

Proposition 3.4. *(cf. [17, Prop. 5.7, Prop. 5.8(a)]) Let τ be symmetric and closed, T the associated positive C_0-semigroup on $L_2(\mu)$, $V\colon \Omega \to [0, \infty)$ measurable.*

(a) The potential V is T-admissible if and only if $\tau + V$ is densely defined, and T_V is associated with $\tau + V$ in this case.

(b) The potential $-V$ is T-admissible if and only if $V \leqslant \tau + \omega$ for some $\omega \in \mathbb{R}$. In this case, $\tau - V$ is closable and T_{-V} is associated with $\overline{\tau - V}$.

Proof. All the assertions of the proposition, except for the closability of $\tau - V$, are shown in [17]. There the proof is given for the case of the diffusion semigroup on $\mathbb{R}^N$ only, but literally the same proof carries over to the general case. The closability of $\tau - V$ is due to A. Manavi ([10, Prop. 12.1.7]); we present his argument here.

Note that T_{-V} is a symmetric C_0-semigroup. Let $\tilde{\tau}$ be the densely defined, closed symmetric form in $L_2(\mu)$ associated with T_{-V}. By part (a) of the proposition, $(T_{-V})_V = T$ is associated with both $\tilde{\tau} + V$ and τ, taking into account Lemma 2.2 and the definition of T. Hence $\tilde{\tau} + V = \tau$. Since $Q(V) \supseteq D(\tau)$, this implies that $\tilde{\tau} \supseteq \tau - V$, i.e., $\tau - V$ has a closed extension. □

Proposition 3.4(a) is valid even for sectorial forms, see [10, Kor. 12.1.4(a)].

It is clear that a sesquilinear form τ fulfills the first Beurling-Deny criterion if and only if the same holds for $\tau + V$, for some measurable function $V\colon \Omega \to \mathbb{R}$ with $Q(V) \supseteq D(\tau)$. Surprisingly, a similar result holds for closability. It is a direct consequence of Proposition 3.4(b).

Corollary 3.5. *(cf. [10, Kor. 12.1.14]) Let τ be sectorial. Then τ is closable if and only if $\tau + V$ is closable for some measurable function $V \geqslant 0$ with $Q(V) \supseteq D(\tau)$.*

Proof. Without restriction τ is symmetric. Let $V \geqslant 0$ be measurable with $Q(V) \supseteq D(\tau)$. If τ is closable then it is clear that $\tau + V$ is closable. If $\tau + V$ is closable then $V \leqslant \overline{\tau + V} + \omega$ for some $\omega \in \mathbb{R}$. Proposition 3.4(b) implies that $\overline{\tau + V} - V$ is closable. Thus, τ is closable since $\tau \subseteq \overline{\tau + V} - V$. □

Definition 3.6. Let τ be sectorial and closable, $V \geqslant 0$ measurable. We say that V is *τ-regular* if $D(\tau + V)$ is a core for τ, i.e., $D(\tau) \cap Q(V)$ is dense in $D(\tau)$.

Remark 3.7. (a) For example, $V \in \big(L_1 + L_\infty\big)(\mu)$ is τ-regular if τ is a Dirichlet form, since $D(\tau) \cap L_\infty(\mu) \subseteq Q(V)$ is a core for τ.

(b) Obviously, if V is τ-regular then V is $\overline{\tau}$-regular, but the converse is not true in general ($D(\tau + V)$ may be $\{0\}$ although V is $\overline{\tau}$-regular, see [15]).

The following lemma states in particular that form regularity implies semigroup regularity.

Lemma 3.8. *Let τ be sectorial and closable, T the positive C_0-semigroup associated with $\overline{\tau}$, $V \geqslant 0$ τ-regular. Then V is T-regular, and T_V is associated with $\overline{\tau + V}$.*

Proof. Note that, by Lemma 3.3, $\overline{\tau + V}$ fulfills the first Beurling-Deny criterion. Let T_1 be the positive C_0-semigroups associated with $\overline{\tau + V}$.

Since $D(\tau + V)$ is a core for $\overline{\tau}$ and $(\tau + V - V \wedge n)(u) \to \overline{\tau}(u)$ for all $u \in D(\tau + V)$, we can use [4, Thm. VIII.3.6] to obtain $(T_1)_{-V\wedge n} \to T$. Thus, $-V$ is T_1-admissible, and $(T_1)_{-V} = T$. Lemma 2.2 implies that V is T_1-regular and that

$T_1 = T_V$. The latter shows the second assertion, and V is regular with respect to $T = (T_1)_{-V}$, by [18, Prop. 3.4(a)]. $\square$

In [10, Kor. 12.1.4(b)] it is shown that form regularity and semigroup regularity are actually equivalent, but we do not need this fact here.

Now we are ready to formulate the main result of this section. It is fundamental for Section 4.

Proposition 3.9. *Let $U \geqslant 0$ be measurable, $Q(U) \supseteq D(\tau)$, $\tau + U$ sectorial and closable, $T_{U,2}$ the positive C_0-semigroup associated with $\overline{\tau + U}$. Let $V \geqslant 0$ be $(\tau + U)$-regular, $\tau + V$ sectorial and closable, $T_{V,2}$ the positive C_0-semigroup associated with $\overline{\tau + V}$. Let $p \in [1, \infty)$.*

Assume that $T_{U,2}$ extrapolates to a positive C_0-semigroup $T_{U,p}$ on $L_p(\mu)$ and that $-U$ is $T_{U,p}$-admissible. Then the same holds with V in place of U, V is $(T_{U,p})_{-U}$-regular, and $(T_{U,p})_{-U} = (T_{V,p})_{-V}$.

Proof. Let $T_p := (T_{U,p})_{-U}$. It suffices to show that V is $T_{U,p}$-regular and that $T_{V,2}$, $(T_p)_V$ are consistent: then V is T_p-regular by [18, Prop. 3.4(a)] and thus $(T_{U,p})_{-U} = \big((T_p)_V\big)_{-V}$.

The potential U is $(\tau + V)$-regular since $Q(U) \supseteq D(\tau + V)$, and V is $(\tau + U)$-regular by the assumptions. Lemma 3.8 implies that both $\big(T_{V,2}\big)_U$ and $\big(T_{U,2}\big)_V$ are associated with $\overline{(\tau + V) + U} = \overline{(\tau + U) + V}$ and that U is $T_{V,2}$-regular. Therefore,

$$T_{V,2} = \big((T_{V,2})_U\big)_{-U} = \big((T_{U,2})_V\big)_{-U}.$$

Moreover, V is $T_{U,2}$-regular and hence $T_{U,p}$-regular by Lemma 2.3(c). Since $-U$ is $T_{U,p}$-admissible we obtain by [18, Thm. 2.6] that

$$(T_p)_V = \big((T_{U,p})_{-U}\big)_V = \big((T_{U,p})_V\big)_{-U}.$$

Now we combine the above two equalities and conclude by Lemma 2.3(a) and (b) that $T_{V,2}$ and $(T_p)_V$ are consistent. $\square$

Proposition 3.9 leads to the following definition. Recall that τ is a densely defined sesquilinear form fulfilling the first Beurling-Deny criterion.

Definition 3.10. Let $p \in [1, \infty)$. We say that τ is associated with a positive C_0-semigroup T_p on $L_p(\mu)$, $\tau \leftrightarrow T_p$ on $L_p(\mu)$ for short, if the following holds:

There exists $U \geqslant 0$ with $Q(U) \supseteq D(\tau)$ such that $\tau + U$ is sectorial and closable, the positive C_0-semigroup $T_{U,2}$ on $L_2(\mu)$ associated with $\overline{\tau + U}$ extrapolates to a C_0-semigroup $T_{U,p}$ on $L_p(\mu)$, $-U$ is $T_{U,p}$-admissible, and $T_p = (T_{U,p})_{-U}$.

According to Proposition 3.9, the semigroup T_p is uniquely determined by the form τ. If τ itself is sectorial and closable, we can choose $U = 0$. In this case $T_2(t) = e^{-At}$ where A is the m-sectorial operator associated with τ by the first representation theorem (see [4, Thm. VI.2.1]).

The following result is a generalization of Lemma 3.8.

Proposition 3.11. *Let $p \in [1, \infty)$ and assume that τ is associated with a positive C_0-semigroup T_p on $L_p(\mu)$. Let $U \geqslant 0$ with $Q(U) \supseteq D(\tau)$ be such that $\tau + U$ is sectorial and closable. If $V \geqslant 0$ is $(\tau + U)$-regular then V is T_p-regular, and $\tau + V \leftrightarrow (T_p)_V$.*

Proof. First assume that $V \geqslant U$. Then $\tau + V$ is a closable sectorial form. Let $T_{V,2}$ be the C_0-semigroup associated with $\overline{\tau + V}$. By Proposition 3.9 we obtain that $T_{V,2}$ extrapolates to a C_0-semigroup $T_{V,p}$ on L_p, $(T_{V,p})_{-V} = T_p$, and V is T_p-regular. Lemma 2.2 implies that $T_{V,p} = (T_p)_V$, i.e., $\tau + V \leftrightarrow (T_p)_V$.

In the general case we apply the above argument to $U + V$ in place of V. We conclude that $(\tau + V) + U \leftrightarrow (T_p)_{U+V}$ and that $U + V$ is T_p-regular. Thus, V is T_p-regular, by [18, Prop. 3.3(a)]. Moreover, $-U$ is admissible with respect to $(T_p)_{U+V}$ and $\big((T_p)_{U+V}\big)_{-U} = (T_p)_V$, by [18, Thm. 3.4]. Hence $\tau + V \leftrightarrow (T_p)_V$. ☐

Given τ, we consider the adjoint form τ^* which is defined by

$$\tau^*(u, v) := \overline{\tau(v, u)} \quad \text{on } D(\tau^*) := D(\tau).$$

Proposition 3.12. *Let $p \in (1, \infty)$ and assume that τ is associated with a positive C_0-semigroup T_p on $L_p(\mu)$. Then the form τ^* is associated with the adjoint semigroup T_p^* on $L_{p'}(\mu)$.*

Note that, since T_p is a real semigroup, it makes no difference whether the adjoint semigroup is taken with respect to the bilinear or with respect to the sesquilinear duality bracket.

Proof of Proposition 3.12. Let $U \geqslant 0$ with $Q(U) \supseteq D(\tau)$ such that $\tau + U$ is sectorial and closable, the positive C_0-semigroup $T_{U,2}$ on $L_2(\mu)$ associated with $\overline{\tau + U}$ extrapolates to a C_0-semigroup $T_{U,p}$ on $L_p(\mu)$, $-U$ is $T_{U,p}$-admissible, and $T_p = (T_{U,p})_{-U}$.

It is easy to see that $\tau^* + U$ is closable, fulfills the first Beurling-Deny criterion, and that $\overline{\tau^* + U} = (\overline{\tau + U})^*$. Thus, $\overline{\tau^* + U}$ is associated with the positive C_0-semigroup $T_{U,2}^*$ which in turn extrapolates to the semigroup $T_{U,p}^*$ on $L_{p'}(\mu)$. Moreover, $\big((T_{U,p}^*)_{-U\wedge n}\big)_{n\in\mathbb{N}}$ is an increasing sequence of semigroups, and

$$(T_{U,p}^*)_{-U\wedge n} = \big((T_{U,p})_{-U\wedge n}\big)^* \to T_p^* \quad \text{weakly as } n \to \infty$$

since $(T_{U,p})_{-U\wedge n} \to T_p$. We deduce that $(T_{U,p}^*)_{-U\wedge n} \to T_p^*$ strongly as $n \to \infty$. Hence, $-U$ is $T_{U,p}^*$-admissible and $(T_{U,p}^*)_{-U} = T_p^*$, i.e., τ^* is associated with T_p^*. ☐

We conclude the section by a result needed for applications of Proposition 3.9.

Lemma 3.13. *Let τ be sectorial and closable, $U, V \geqslant 0$ measurable. Assume that U is τ-regular. Then V is τ-regular if and only if V is $(\tau + U)$-regular. As a consequence, $U + V$ is τ-regular if U, V are τ-regular.*

Proof. Let V be $(\tau + U)$-regular. Then $D((\tau + U) + V)$ is a core for $\tau + U$ and hence a core for τ. Therefore, $D(\tau + V)$ is a core for τ, i.e., V is τ-regular.

Conversely, assume that V is τ-regular. Without restriction, τ is symmetric and $\tau \geqslant 0$. Let $0 \leqslant u \in D(\tau + U)$. There exists $(u_n) \subseteq D(\tau + V)$ such that $u_n \to u$ in $D(\tau)$ as $n \to \infty$. Let $v_n := (\operatorname{Re} u_n)^+$. Since τ fulfills the first Beurling-Deny criterion we have $\limsup_{n\to\infty} \tau(v_n) \leqslant \lim_{n\to\infty} \tau(u_n) = \tau(u)$. The lower semicontinuity of τ implies that $v_n \to u$ in $D(\tau)$ as $n \to \infty$. Moreover, $\tau((u - v_n)^+) \leqslant \tau(u - v_n) \to 0$ and thus $u \wedge v_n = u - (u - v_n)^+ \to u$ in $D(\tau)$ as $n \to \infty$. Finally, $u \wedge v_n \to u$ in $Q(U)$ by Lebesgue's dominated convergence theorem. We infer that $D((\tau + U) + V) \ni u \wedge v_n \to u$ in $D(\tau + U)$. This shows that $D((\tau + U) + V)$ is a core for $\tau + U$. □

4 L_p-properties of elliptic differential operators

In this section we formulate the main result of the paper and deduce some corollaries. We refer to Section 1 for the notation.

Recall that the form τ is defined on $D(\tau) := D(\tau_a) \cap Q(W_1 + W_2 + |V|)$ by (1.1). Since τ_a is a Dirichlet form, $(\operatorname{Re} u)^+ \in D(\tau)$ for all $u \in D(\tau)$. Therefore, τ fulfills the first Beurling-Deny criterion (we actually have $\tau(u^+, u^-) = 0$ for all real-valued $u \in D(\tau)$, and $\tau(u, v) \in \mathbb{R}$ for all real-valued $u, v \in D(\tau)$.) Further, $D(\tau)$ is a core for τ_a by Lemma 3.13, in particular, τ is densely defined.

The forms τ_p play a crucial role in all our results on elliptic operators. We will also make use of the symmetric form τ_∞ defined by

$$\tau_\infty(u) := -2\langle \nabla |u|, b_2 |u| \rangle + \langle V |u|^2 \rangle, \quad D(\tau_\infty) := D(\tau).$$

In the following proposition we collect several simple properties of the forms τ and τ_p which are important for the understanding of the subsequent results.

Proposition 4.1. *Assume that* **(a)** *and* **(bV)** *hold. Let I be the set of all $p \in [1, \infty)$ such that $\omega_p := \inf\{\omega \in \mathbb{R};\, \tau_p \geqslant -\omega\} < \infty$ (then $\tau_p \geqslant -\omega_p$ for all $p \in I$).*

(a) For all potentials $U \geqslant W_1 + W_2 + 2V^-$, the form $\tau + U$ is sectorial and closed. For all $1 < p < \infty$ and $U \geqslant p' W_1 + p W_2 + 2V^-$, the symmetric form $\tau_p + U$ is non-negative and closed. In particular, τ_p is closable for all $p \in I \setminus \{1\}$.

(b) The set I is an interval and, for all $p \in \mathring{I}$, there exist $\varepsilon_p > 0$, $c_p \in \mathbb{R}$ such that $\tau_p \geqslant \varepsilon_p \operatorname{Re} \tau_a - c_p$. If, for some $1 \leqslant p_0 < p < p_1 \leqslant \infty$, we have $\tau_{p_j} \geqslant -\omega_{p_j}$ ($j = 0, 1$) then we can choose $\varepsilon_p = 4(\frac{1}{p_0} - \frac{1}{p})(\frac{1}{p} - \frac{1}{p_1})$, $c_p = \theta \omega_{p_0} + (1 - \theta)\omega_{p_1}$, with $\theta = \frac{p_0^{-1} - p^{-1}}{p_0^{-1} - p_1^{-1}}$.

(c) For all $p, q \in \mathring{I}$, the norms on the Hilbert spaces $D(\overline{\tau_p})$ and $D(\overline{\tau_q})$ are equivalent.

Proof. (a) From (1.2) we deduce by Euclid's inequality ($|ab| \leqslant \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$ for all $a, b \in \mathbb{R},\ \varepsilon > 0$) that the sum of the first order terms of τ is form small with respect to $\tau_a + W_1 + W_2$. Thus, $\tau + U$ is a closed sectorial form for any potential $U \geqslant W_1 + W_2 + 2V^-$. The same argument works for τ_p if $1 < p < \infty$. By Corollary 3.5 we obtain that τ_p is closable if it is bounded below.

The proof of (b) and (c) relies on the following identity which results directly from the definition of the forms τ_p: for all $p_0, p_1 \in I$, $\theta \in (0,1)$ and p_θ defined by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$\tau_{p_\theta} = (1-\theta)\tau_{p_0} + \theta\tau_{p_1} + 4\left(\frac{1}{p_\theta p_\theta'} - \frac{1-\theta}{p_0 p_0'} - \frac{\theta}{p_1 p_1'}\right) \operatorname{Re}\tau_a. \tag{4.1}$$

In order to prove (b), it now suffices to show that

$$\frac{1}{p_\theta p_\theta'} - \frac{1-\theta}{p_0 p_0'} - \frac{\theta}{p_1 p_1'} = \left(\frac{1}{p_\theta'} - \frac{1}{p_0'}\right)\left(\frac{1}{p_\theta} - \frac{1}{p_1}\right) \left(= \left(\frac{1}{p_0} - \frac{1}{p_\theta}\right)\left(\frac{1}{p_\theta} - \frac{1}{p_1}\right)\right)$$

which in turn follows from the equality

$$\frac{1}{p_\theta p_0'} + \frac{1}{p_\theta' p_1} = \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)\frac{1}{p_0'} + \left(\frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}\right)\frac{1}{p_1} = \frac{1-\theta}{p_0 p_0'} + \frac{\theta}{p_1 p_1'} + \frac{1}{p_0' p_1}.$$

(c) By (4.1) we have $\tau_{p_\theta} \geqslant (1-\theta)\tau_{p_0} + \theta\tau_{p_1}$. We deduce that, for all $p, q \in \mathring{I}$, there exist $\varepsilon > 0$, $\omega \in \mathbb{R}$ such that $\tau_p \geqslant \varepsilon\tau_q - \omega$ and $\tau_q \geqslant \varepsilon\tau_p - \omega$. □

The form τ itself need not be sectorial. In fact, Theorem 4.2 includes cases where τ is not even bounded from the left. However, the form $\tau + W_1 + W_2 + 2V^-$ is sectorial and closed by Proposition 4.1(a). This enables us to make use of Definition 3.10 in the main result of the paper which reads as follows.

Theorem 4.2. *Assume that* (**a**) *and* (**bV**) *hold. Let I be the interval of all $p \in [1,\infty)$ such that $\omega_p := \inf\{\omega \in \mathbb{R};\ \tau_p \geqslant -\omega\} < \infty$. Then τ is associated with a consistent family of positive C_0-semigroups T_p on L_p with $\|T_p(t)\| \leqslant e^{\omega_p t}$ for all $p \in I,\ t \geqslant 0$.*

Let $-A_p$ be the generator of T_p ($p \in I$). Then, for all $p \in I \setminus \{1\}$ and $u \in D(A_p)$ we have $v_p := u|u|^{p/2-1} = |u|^{p/2}\operatorname{sgn} u \in D(\overline{\tau_p})$ and

$$\operatorname{Re}\langle A_p u, u|u|^{p-2}\rangle \geqslant \overline{\tau_p}(v_p). \tag{4.2}$$

If, in addition,

$$\left|\operatorname{Im}\langle (b_1 + b_2)u, \nabla u\rangle\right| \leqslant c_1\tau_p(u) + c_2\|u\|_2^2 \quad \big(u \in D(\tau)\big) \tag{4.3}$$

for some $p \in \mathring{I},\ c_1 \geqslant 0,\ c_2 \in \mathbb{R}$ then A_p is an m-sectorial operator for all $p \in \mathring{I}$, in particular, T_p extends to an analytic semigroup on L_p.

The proof of the theorem is delegated to Section 5.

Remarks 4.3. (a) We point out that the case $I = \{1\}$ is quite possible. By definition, $1 \in I$ if $\tau_1 \geqslant -\omega$ for some $\omega \in \mathbb{R}$. Note that the coefficient b_2 is not involved in this condition. In particular, if **(a)** holds, $b_1 = 0$ and $V \geqslant 0$ then τ is associated with a positive contractive C_0-semigroup on L_1, whenever ${b_2}^\top a_s^{-1} b_2$ is τ_a-regular.

(b) For the case $p = \infty$ we obtain the following by considering the adjoint picture in L_1. If $\tau_\infty \geqslant -\omega_\infty$ for some $\omega_\infty \in \mathbb{R}$ then we can associate a weak*-continuous quasi-contractive semigroup T_∞ on L_∞ with the form τ. Observe that the condition on τ_∞ imposes no additional restriction on b_1.

(c) Lemma 4.1(b) demonstrates the relevance of inequality (4.2): Assume that the domain of τ_a admits Sobolev imbedding, i.e., $D(\tau_a) \subseteq L_{2j}$ for some $j > 1$. Then it is easy to show that, for all $p \in \mathring{I}$,

$$\|(\lambda + A_p)^{-1}\|_{p \to pj} \leqslant c_p(\lambda - \omega_p)^{-\frac{1}{p}} \quad (\lambda > \omega_p).$$

In [6], an inequality similar to (4.2) was proved only for $|u|^{\frac{p}{2}}$ in place of $|u|^{\frac{p}{2}} \operatorname{sgn} u$.

Corollary 4.4. *Let the assumptions and notation be as in Theorem 4.2, $p \in I$. Let $(U_n)_{n \in \mathbb{N}_0}$ be a sequence of positive potentials such that U_0 is τ_a-regular, $U_n \leqslant U_0$, $\tau + U_n$ is sectorial ($n \in \mathbb{N}$) and $U_n \to 0$ a.e. ($n \to \infty$). Then $\tau + U_n$ is closable, the analytic semigroup $T_{U_n,2}$ associated with $\overline{\tau + U_n}$ extrapolates to a C_0-semigroup $T_{U_n,p}$ on L_p, and $T_{U_n,p} = (T_p)_{U_n} \to T_p$ as $n \to \infty$.*

Proof. Let $W := W_1 + W_2 + 2V^-$. Then $\tau + W$ is a closed sectorial form, by Proposition 4.1(a). Since $\tau + U_n + W$ is closed, the form $\tau + U_n$ is closable by Lemma 3.5. By Lemma 3.13, U_n is $(\tau_a + W)$-regular and hence $(\tau + W)$-regular. By Proposition 3.11, U_n is T_p-regular and $\overline{\tau + U_n} \leftrightarrow (T_p)_{U_n}$, i.e., $T_{U_n,2}$ and $(T_p)_{U_n}$ are consistent. Now, by [18, Cor. 3.6] we conclude that $(T_p)_{U_n} \to T_p$ as $n \to \infty$ since U_0 is T_p-regular. ☐

As a direct consequence of Theorem 4.2 we obtain a more explicit version of that theorem.

Corollary 4.5. *Let $V_+, V_- \geqslant 0$ be τ_a-regular with $V_+ - V_- = V$, and $\tau_+ := \operatorname{Re} \tau_a + V_+$. Assume that* (**a**) *and* (**bV**) *hold and that*

$$(-1)^j \langle b_j u, \nabla u \rangle \leqslant \beta_j \tau_+(u) + B_j \|u\|_2^2, \quad \langle V_- u^2 \rangle \leqslant \gamma \tau_+(u) + G\|u\|_2^2$$

($0 \leqslant u \in D(\tau) \cap Q(V_+)$, $j = 1, 2$) for some constants $\beta_1, \beta_2, \gamma \geqslant 0$, $B_1, B_2, G \in \mathbb{R}$. Let $I_0 := \big\{p \in [1, \infty);\ \frac{4}{pp'} - \frac{2}{p}\beta_1 - \frac{2}{p'}\beta_2 - \gamma \geqslant 0\big\}$. Then, with the notation of Theorem 4.2, $I \supseteq I_0$, and $\omega_p \leqslant \frac{2}{p} B_1 + \frac{2}{p'} B_2 + G$ for all $p \in I_0$. Moreover, for all $p \in \mathring{I}_0$ and $u \in D(A_p)$ we have $v_p := |u|^{\frac{p}{2}} \operatorname{sgn} u \in D(\tau_+)$ and

$$\operatorname{Re}\langle A_p u, u|u|^{p-2}\rangle \geqslant \Big(\tfrac{4}{pp'} - \tfrac{2}{p}\beta_1 - \tfrac{2}{p'}\beta_2 - \gamma\Big)\, \tau_+(v_p) - \Big(\tfrac{2}{p}B_1 + \tfrac{2}{p'}B_2 + G\Big)\, \|u\|_p^p.$$

If, in addition,

$$\big|\operatorname{Im}\langle (b_1+b_2)u, \nabla u\rangle\big| \leqslant c_1\tau_+(u) + c_2\|u\|_2^2 \quad \big(u\in D(\tau)\cap Q(V_+)\big)$$

for some $c_1 \geqslant 0,\ c_2 \in \mathbb{R}$ *then* T_p *extends to an analytic semigroup on* L_p *for all* $p \in \mathring{I}$.

Proof. Since $\tau_+(|u|) \leqslant \tau_+(u)$ for all $u \in D(\tau_+)$, and $1 \geqslant \frac{4}{pp'}$, the assumptions imply that

$$\begin{aligned}\tau_p(u) &= \tfrac{4}{pp'}\operatorname{Re}\tau_a(u) + \langle V_+|u|^2\rangle - \big(-\tfrac{2}{p}\langle b_1|u|, \nabla|u|\rangle\big) - \tfrac{2}{p'}\langle b_2|u|, \nabla|u|\rangle - \langle V_-|u|^2\rangle \\ &\geqslant \big(\tfrac{4}{pp'} - \tfrac{2}{p}\beta_1 - \tfrac{2}{p'}\beta_2 - \gamma\big)\tau_+(u) - \big(\tfrac{2}{p}B_1 + \tfrac{2}{p'}B_2 + G\big)\|u\|_2^2\end{aligned}$$

for all $p \in [1,\infty),\ u \in D(\tau)\cap Q(V_+)$. Let $W := W_1 + W_2 + |V|$. Then τ_p is a bounded form on $D(\tau_a + W)$. Since V_+ is (τ_a+W)-regular by Lemma 3.13, we deduce that $\tau_p \geqslant -\big(\frac{2}{p}B_1 + \frac{2}{p'}B_2 + G\big)$ for all $p \in I_0$. Thus, Theorem 4.2 implies the first two assertions. In order to obtain the remaining assertions, note that the above also implies that

$$\tau_p \geqslant \Big(\tfrac{4}{pp'} - \tfrac{2}{p}\beta_1 - \tfrac{2}{p'}\beta_2 - \gamma\Big)\tau_+ - \Big(\tfrac{2}{p}B_1 + \tfrac{2}{p'}B_2 + G\Big)$$

for all $p \in \mathring{I}_0$. □

For the remainder of the section, we are concerned with the case $b_2 = 0$, $V \geqslant 0$,

$$-\langle \nabla u, b_1 u\rangle \leqslant (\beta\tau_a + V + \omega)(u) \quad \big(0 \leqslant u \in D(\tau)\big)$$

for some $\beta < 2$, $\omega \in \mathbb{R}$. Then τ is associated with a consistent family of positive C_0-semigroups T_p on L_p, $p \geqslant \frac{2}{2-\beta}$, by Theorem 4.2. The semigroups are L_∞-contractive, by Remark 4.3(b).

In Corollary 4.4 we have shown that convergence of potentials implies strong convergence of the corresponding semigroups. Here we discuss approximation of the first order terms. For $n \in \mathbb{N}\cup\{\infty\}$, let $b_n\colon \Omega \to \mathbb{R}^N$ be measurable and define τ_n by

$$\tau_n(u,v) := \tau_a(u,v) + \langle \nabla u, b_n v\rangle + \langle Vu, v\rangle$$

on $D(\tau_n) := D(\tau_a)\cap Q({b_n}^\top a_s^{-1} b_n + V)$.

Proposition 4.6. *Let* **(a)** *hold and assume that* $b_n \to b_\infty$ *a.e.,* V *is* τ_a*-regular, and there exist* $0 < \beta < 2$, $\omega \in \mathbb{R}$, $0 \leqslant U_0 \in L_1 + L_\infty$ *such that, for all* $n \in \mathbb{N}\cup\{\infty\}$, *we have* ${b_n}^\top a_s^{-1} b_n \leqslant U_0$ *and*

$$-\langle \nabla u, b_n u\rangle \leqslant (\beta\tau_a + V + \omega)(u) \quad \big(0 \leqslant u \in D(\tau_n)\big).$$

Then, for all $p \geqslant \frac{2}{2-\beta}$, $\tau_n \leftrightarrow T_p^{(n)}$ *on* L_p $(n \in \mathbb{N}\cup\{\infty\})$, *and* $T_p^{(n)} \to T_p^{(\infty)}$ *as* $n \to \infty$.

For the proof of the proposition, we need the following elementary form convergence result which was proved in [16, Thm. A.1] for symmetric forms.

Lemma 4.7. *For $n \in \mathbb{N} \cup \{\infty\}$, let τ_n be a closed sectorial form in a Hilbert space H, and A_n the associated m-sectorial operator. Assume that, for some closed symmetric form $\mathfrak{h} \geqslant 1$ in H, and some $c \geqslant 1$, $\omega \in \mathbb{R}$ we have*

$$\frac{1}{c}\mathfrak{h} \leqslant \operatorname{Re}\tau_n + \omega \leqslant c\mathfrak{h} \quad (n \in \mathbb{N} \cup \{\infty\})$$

and

$$\sup_{\mathfrak{h}(v)\leqslant 1} \bigl|(\tau_\infty - \tau_n)(u,v)\bigr| \to 0 \quad \text{as } n \to \infty \quad \bigl(u \in D(\mathfrak{h})\bigr).$$

Then $A_n \to A_\infty$ in the strong resolvent sense.

Proof. Without restriction assume that $\omega = 0$. For all $f, g \in H$,

$$\langle A_n^{-1} f - A_\infty^{-1} f, g\rangle = (\tau_\infty - \tau_n)(A_\infty^{-1} f, (A_n^*)^{-1} g).$$

For all $g \in H$, $n \in \mathbb{N}$ we have $\mathfrak{h}\bigl((A_n^*)^{-1}g\bigr) \leqslant c \operatorname{Re}\tau_n\bigl((A_n^*)^{-1}g\bigr) \leqslant c^2\|g\|^2$ since $\|(A_n^*)^{-1}\| \leqslant c$. Hence

$$\|A_n^{-1} f - A_\infty^{-1} f\| = \sup_{\|g\|\leqslant 1} \bigl|\langle A_n^{-1} f - A_\infty^{-1} f, g\rangle\bigr| \leqslant \sup_{\mathfrak{h}(v)\leqslant c^2} \bigl|(\tau_\infty - \tau_n)(A_\infty^{-1} f, v)\bigr| \to 0.$$

□

Proof of Proposition 4.6. Let $q \in (1, \frac{2}{2-\beta})$, $U := q'U_0$. Then $\tau_q + U$ is nonnegative, by Proposition 4.1(a). Recall from Remark 3.7(a) that U is τ_a-regular. Let $p \geqslant \frac{2}{2-\beta}$. For $n \in \mathbb{N} \cup \{\infty\}$, let $T_p^{(n)}$ denote the positive C_0-semigroup on L_p associated with τ_n. Let $T_{U,2}^{(n)}$ denote the C_0-semigroup on L_2 associated with the closed sectorial form $\tau_n + U$. Since U is τ_a-regular, it is $T_p^{(n)}$-regular and $(\tau_n + U) \leftrightarrow (T_p^{(n)})_U$, by Corollary 4.4. Thus, $(T_p^{(n)})_U$ and $T_{U,2}^{(n)}$ are consistent.

We are going to show that $T_{U,2}^{(n)} \to T_{U,2}^{(\infty)}$ as $n \to \infty$. This will imply that $(T_p^{(n)})_U \to (T_p^{(\infty)})_U$ for all $p \geqslant \frac{2}{2-\beta}$ since $T_{U,2}^{(n)}$ is L_∞- and L_q-contractive. Then the assertion follows from Proposition 2.4.

Without restriction $U \geqslant 1$. Let $\mathfrak{h} := \tau_a + U + V$. It is straightforward that, for all $n \in \mathbb{N} \cup \{\infty\}$, we have $\frac{1}{2}\mathfrak{h} \leqslant \tau_n + U \leqslant 2\mathfrak{h}$. Moreover, for all $u, v \in D(\mathfrak{h})$,

$$\begin{aligned}\bigl|(\tau_\infty - \tau_n)(u,v)\bigr|^2 &= \bigl|\langle \nabla u, (b_n - b_\infty)v\rangle\bigr|^2 \\ &\leqslant \langle U^{-1}(b_n - b_\infty)^\top a_s^{-1}(b_n - b_\infty)(\nabla u)^* a \nabla u\rangle \langle U|v|^2\rangle.\end{aligned}$$

Therefore,

$$\sup_{\mathfrak{h}(v)\leqslant 1} \bigl|(\tau_\infty - \tau_n)(u,v)\bigr| \to 0 \quad \bigl(u \in D(\mathfrak{h})\bigr)$$

and hence $T_{U,2}^{(n)} \to T_{U,2}^{(\infty)}$, by Lemma 4.7. This completes the proof. □

Example 4.8. Here we give several examples of applications of Corollary 4.5 to the case $b_2 = 0$, $V = 0$.

(i) Assume $W_1 \leqslant \beta^2 \operatorname{Re} \tau_a + B$ for some $0 < \beta < 2$, $B \geqslant 0$, in the sense of quadratic forms on L_2. Then, by Euclid's inequality,

$$|\langle b_1 \nabla u, u\rangle| \leqslant \tfrac{1}{2\beta}\|W_1^{1/2}u\|_2^2 + \tfrac{\beta}{2}\|a_s^{1/2}\nabla u\|_2^2 \leqslant \beta \operatorname{Re}\tau_a(u) + \tfrac{B}{2\beta}\|u\|_2^2.$$

Hence, by Corollary 4.5, τ is associated with a family of consistent positive quasi-contractive C_0-semigroups T_p on L_p with growth bound less or equal $\frac{B}{p\beta}$, for all $p \geqslant \frac{2}{2-\beta}$. If $\beta < 1$ then τ sectorial and closed. In this case [6, Thm. 1], with use of [4, Thm. VI.2.1], associates τ with a family of consistent analytic quasi-contractive C_0-semigroups on L_p, $p \geqslant 2$, which coincide with T_p.

In [6, Thm. 6], under the additional condition that $W_1 \in L_1 + L_\infty$, τ was associated with a family of consistent C_0-semigroups on the same interval of the L_p-scale, by approximation of b by bounded vector fields in such a way that the corresponding semigroups converge in L_p. Proposition 4.6 shows that the limiting semigroup does not depend on the choice of the approximating sequence. This answers a question posed by V. Liskevich in a remark to [6, Thm. 6]. Moreover, it follows from Proposition 4.6 that the semigroup constructed in [6] coincides with the one constructed in Theorem 4.2.

(ii) Let $N \geqslant 2$, $\Omega = \mathbb{R}^N$, $a(x) = \mathrm{id}$, $D(\tau_a) = H^1$. Let $(e_j)_{j=1}^N$ be the canonical orthonormal basis in $\mathbb{R}^N$, $(x_n)_{n=1}^\infty = \mathbb{Q}^N$, $(c_n)_{n=1}^\infty \subseteq (0,\infty)$ be such that the potential $U(x) = \sum_n c_n^2 |x - x_n|^{-n}$ is τ_a-regular (see [15] for details of the construction). Let $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \setminus \{0\}$ be such that

$$|\beta|^2 := \sum_n \beta_n^2 < \infty$$

Let $b_1 := \sum_{n=1}^\infty b_{1n}$, where

$$b_{1n}(x) = c_n |x - x_n|^{-\frac{n}{2}} \beta_n \Big(\tfrac{\partial |x-x_n|}{\partial x_1} e_2 - \tfrac{\partial |x - x_n|}{\partial x_2} e_1 \Big).$$

We show that $\langle b_{1n} \nabla u, u\rangle = 0$ for all $n \in \mathbb{N}$, $u \in H^1 \cap Q(|b_{1n}|^2)$. For $u \in C_c^1\big(\mathbb{R}^N \setminus \{x_n\}\big)$, the equality follows by integration by parts. For general $u \in H^1 \cap Q(|b_{1n}|^2)$, it then follows from the fact that $C_c^1\big(\mathbb{R}^N \setminus \{x_n\}\big)$ is dense in $H^1 \cap Q(|b_{1n}|^2)$ and that the form $(u, v) \mapsto \langle b_{1n}\nabla u, v\rangle$ is bounded on $H^1 \cap Q(|b_{1n}|^2)$.

The drift b_1 is nowhere integrable on $\mathbb{R}^N$. However, by the Cauchy-Schwarz inequality,

$$|b_1|(x) \leqslant \sum_{n=1}^\infty |b_{1n}|(x) \leqslant \sum_{n=1}^\infty c_n |x - x_n|^{-\frac{n}{2}} \cdot 2|\beta_n| \leqslant 2|\beta| U^{\frac{1}{2}}(x).$$

Hence $W_1 = |b_1|^2$ is τ_a-regular and $\langle b_1 \nabla u, u\rangle = 0$ for all $u \in H^1 \cap Q(U)$.

Thus, τ is associated with a consistent family of positive contractive C_0-semigroups T_p on L_p, $p \geqslant 1$.

(iii) Let $N \geqslant 2$, $\Omega = \mathbb{R}^N$, $a(x) = \mathrm{id}$, $D(\tau_a) = H^1$. Let $b_1(x) = cx|x|^\alpha$ for some $c, \alpha \in \mathbb{R}$. Then $|b_1|^2$ is τ_a-regular. Moreover,

$$-\langle b_1 \nabla u, u\rangle = \frac{c}{2}(N+\alpha)\langle r^\alpha u^2\rangle \quad \big(0 \leqslant u \in C_c^1(\mathbb{R}^N \setminus 0)\big).$$

Hence, if $c(N+\alpha) \leqslant 0$ then τ is associated with a consistent family of positive contractive C_0-semigroups on L_p, $p \geqslant 1$. If $N \geqslant 3$ we can use the Hardy inequality $\|\frac{u}{r}\|_2^2 \leqslant \frac{4}{(N-2)^2}\|\nabla u\|_2^2$ to treat the case $c(N+\alpha) > 0$ with $-2 \leqslant \alpha \leqslant 0$. For $\alpha = -2$, τ is associated with a quasi-contractive C_0-semigroup on some L_p if (and only if, see Remark 6.5 below) $c < N-2$, and then τ is associated with a consistent family of positive contractive C_0-semigroups on L_p, $p \geqslant \frac{N-2}{N-2-c}$. For $\alpha \in (-2, 0]$, we use the fact that $r^\alpha \leqslant \varepsilon r^{-2} + C_{\alpha,\varepsilon}$ for all $r, \varepsilon > 0$ with some constant $C_{\alpha,\varepsilon}$ to conclude that in this case τ is associated with a consistent family of positive quasi-contractive C_0-semigroups on L_p, $p > 1$. If $\alpha = 0$ then the semigroup extrapolates also to L_1.

5 Proof of the main theorem

We separate the core of the proof of Theorem 4.2 into a lemma. Let $p \in (1, \infty)$. For $u \in L_{1,loc}$, $n \in \mathbb{N}$ let $u_{n,p} := \big(|u|^{\frac{p}{2}-1}\big) \wedge n$, $v_{n,p} := u u_{n,p}$, $w_{n,p} := u u_{n,p}^2$, $v_p(u) := u|u|^{\frac{p}{2}-1}$ and $w_p(u) := u|u|^{p-2}$.

Lemma 5.1. *Let τ be a densely defined sesquilinear form in L_2 fulfilling the first Beurling-Deny criterion. Let $\mathfrak{h}$ be a closed symmetric form in L_2, $\mathfrak{h} \geqslant -\omega$ for some $\omega \in \mathbb{R}$. Assume that there exists a sequence $(U_n)_{n \in \mathbb{N}_0}$ of positive potentials such that $D(U_0) \supseteq D(\tau)$, $\tau + U_0$ is sectorial and closed, $U_n \downarrow 0$ $(n \to \infty)$, and*

$$w_{n,p} \in D(\tau),\ v_{n,p} \in D(\mathfrak{h}),\ \operatorname{Re}\tau(u, w_{n,p}) \geqslant \mathfrak{h}(v_{n,p}) - \langle U_n |v_{n,p}|^2\rangle \tag{5.1}$$

for all $u \in D(\tau)$, $n \in \mathbb{N}$.

(a) Then τ is associated with a positive C_0-semigroup $T_p(t) = e^{-A_p t}$ on L_p with $\|T_p(t)\| \leqslant e^{\omega t}$ $(t \geqslant 0)$, and for all $u \in D(A_p)$ we have $v_p(u) \in D(\mathfrak{h})$ and

$$\operatorname{Re}\langle A_p u, w_p(u)\rangle \geqslant \mathfrak{h}(v_p(u)). \tag{5.2}$$

(b) If, in addition,

$$|\operatorname{Im}\tau(u, w_{n,p})| \leqslant M(\operatorname{Re}\tau + U_n + \tilde{\omega})(u, w_{n,p}) \quad (u \in D(\tau),\ n \in \mathbb{N}) \tag{5.3}$$

for some $M \geqslant 0$, $\tilde{\omega} \in \mathbb{R}$, then A_p is m-sectorial of angle $\arctan M$*. In particular, T_p is an analytic semigroup.*

Proof. (a) Without restriction assume $\omega = 0$. The proof is divided into three steps. In step (i) we consider the m-sectorial operator A_0 in L_2, associated with $\tau + U_0$, and show that $e^{-A_0 t}$ extrapolates to a contractive C_0-semigroup $T_{0,p}(t) = e^{-A_{0,p}t}$ on L_p. In step (ii) we show that $-U_0$ is $T_{0,p}$-admissible and $(T_{0,p})_{-U_0}$ is a contractive C_0-semigroup. This proves the first assertion of (a). The second assertion is proved in step (iii).

(i) By the exponential formula, it suffices to show that, given $f \in L_2 \cap L_p$ and $0 < \lambda \in \rho(-A_0)$, one has $\|(\lambda + A_0)^{-1} f\|_p \leqslant \frac{1}{\lambda}\|f\|_p$. Let $u := (\lambda + A_0)^{-1} f$. Then $u \in D(\tau + U_0) = D(\tau)$. This implies that $v_{n,p} \in Q(U_0)$. By assumption (5.1) and the equality $u\overline{w}_{n,p} = |v_{n,p}|^2$ we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}\lambda\|v_{n,p}\|_2^2 + (\mathfrak{h} + U_0 - U_n)(v_{n,p}) &\leqslant \lambda\langle u, w_{n,p}\rangle + \operatorname{Re}(\tau + U_0)(u, w_{n,p}) \\ &= \operatorname{Re}\big\langle(\lambda + A_0)u, w_{n,p}\big\rangle \leqslant \|f\|_p\|w_{n,p}\|_{p'}.\end{aligned} \tag{5.4}$$

Observe that $|w_{n,p}|^{p'} = |u|^{p'} u_{n,p}^{2p'} \leqslant |v_{n,p}|^2$. Hence $\|w_{n,p}\|_{p'} \leqslant \|v_{n,p}\|_2^{\frac{2}{p'}}$, and from estimate (5.4) we obtain that

$$\|v_{n,p}\|_2^{\frac{2}{p}} \leqslant \tfrac{1}{\lambda}\|f\|_p \quad (n \in \mathbb{N}).$$

Since $|v_{n,p}| \uparrow |v_p(u)|$ we conclude by the Beppo Levi theorem that $v_p(u) \in L_2$, and

$$\|(\lambda + A_0)^{-1} f\|_p = \|v_p(u)\|_2^{\frac{2}{p}} \leqslant \tfrac{1}{\lambda}\|f\|_p.$$

(ii) With the quantities introduced in (i) we proceed as follows. By Lebesgue's dominated convergence theorem, $v_{n,p} \to v_p(u)$ in L_2 and $w_{n,p} \to w_p(u)$ in $L_{p'}$. Further, $A_0 u = f - \lambda u \in L_p$. From estimate (5.4) we obtain

$$\liminf_{n\to\infty}\big(\mathfrak{h}(v_{n,p}) + \langle(U_0 - U_n)|v_{n,p}|^2\rangle\big) \leqslant \lim_{n\to\infty} \operatorname{Re}\langle A_0 u, w_{n,p}\rangle = \operatorname{Re}\langle A_0 u, w_p(u)\rangle.$$

By the Beppo Levi theorem, $(U_0 - U_n)|v_{n,p}|^2 \uparrow U_0|v_p(u)|^2$ in L_1. Hence the left hand side of the previous inequality equals $\liminf_n \mathfrak{h}(v_{n,p}) + \langle U_0|v_p(u)|^2\rangle$. The lower semicontinuity of $\mathfrak{h}$ implies that

$$v_p(u) \in D(\mathfrak{h}),\ \ (\mathfrak{h} + U_0)(v_p(u)) \leqslant \operatorname{Re}\langle A_0 u, w_p(u)\rangle. \tag{5.5}$$

So far we have proved inequality (5.5) for all u from the core $D := (\lambda + A_0)^{-1}(L_2 \cap L_p)$ of $A_{0,p}$, where $\lambda > 0$ is some element of $\rho(-A_0)$.

Let now $u \in D(A_{0,p})$. Choose $(u^{(m)}) \subseteq D$ such that $u^{(m)} \to u$ in $D(A_{0,p})$. Then $v_p(u^{(m)}) \to v_p(u)$ in L_2 and $w_p(u^{(m)}) \to w_p(u)$ in $L_{p'}$. From (5.5) we conclude that

$$\liminf_{m\to\infty}(\mathfrak{h} + U_0)(v_p(u^{(m)})) \leqslant \lim_{m\to\infty} \operatorname{Re}\langle A_{0,p}u^{(m)}, w_p(u^{(m)})\rangle = \operatorname{Re}\langle A_{0,p}u, w_p(u)\rangle.$$

The lower semicontinuity of $\mathfrak{h} + U_0$ implies that (5.5) holds for all $u \in D(A_{0,p})$.

For $m \in \mathbb{N}$, let $A_m := A_{0,p} - U_0 \wedge m$. Then A_m is a closed operator and, by (5.5), $\operatorname{Re}\langle A_m u, w_p(u)\rangle \geqslant 0$ for all $u \in D(A_m) = D(A_{0,p})$. By the Lumer-Phillips theorem, $e^{-A_m t} = (T_{0,p})_{-U_0 \wedge m}(t)$ is a contractive C_0-semigroup on L_p and, by [18, Prop. 2.2] (see Definition 2.1(b)), we conclude that $-U_0$ is $T_{0,p}$-admissible and that $T_p := (T_{0,p})_{-U_0}$ is a contractive C_0-semigroup on L_p.

(iii) Let $-A_p$ be the generator of T_p. By (ii), $A_m \to A_p$ in the strong resolvent sense. Let $u \in D(A_p)$. Then $u^{(m)} := (1 + A_m)^{-1}(1 + A_p)u \to u$ in L_p as $m \to \infty$. Since

$$u^{(m)} + A_m u^{(m)} = u + A_p u,$$

we also have $A_m u^{(m)} \to A_p u$ in L_p. Furthermore, $v_p(u^{(m)}) \to v_p(u)$ in L_2 and $w_p(u^{(m)}) \to w_p(u)$ in $L_{p'}$ as $m \to \infty$. Hence, estimate (5.5) yields

$$\liminf_m \mathfrak{h}(v_p(u^{(m)})) \leqslant \lim \langle A_m u^{(m)}, w_p(u^{(m)})\rangle = \langle A_p u, w_p(u)\rangle.$$

The lower semicontinuity of $\mathfrak{h}$ implies (5.2).

(b) Let $u \in D(A_0) \cap D(A_{0,p})$. Then, since $u\overline{w}_{n,p}$ is real,

$$\operatorname{Im}\langle A_m u, w_{n,p}\rangle = \operatorname{Im}\langle (A_0 - U_0 \wedge m)u, w_{n,p}\rangle = \operatorname{Im}\tau(u, w_{n,p}).$$

By (5.5) we know that $U_n|u\overline{w}_{n,p}| \leqslant U_0|v_p(u)|^2 \in L_1$. Thus, $\langle U_n u, w_{n,p}\rangle \to 0$ by Lebesgue's dominated convergence theorem. By (5.3) we conclude that

$$\begin{aligned}|\operatorname{Im}\langle A_m u, w_p(u)\rangle| &= \lim_{n\to\infty} |\operatorname{Im}\tau(u, w_{n,p})| \\ &\leqslant \lim_{n\to\infty} M(\operatorname{Re}\tau + (U_0 - m)^+ + U_n + \tilde{\omega})(u, w_{n,p}) = M \operatorname{Re}\langle (A_m + \tilde{\omega})u, w_p(u)\rangle.\end{aligned}$$

This estimate carries over to all $u \in D(A_m)$ since $D(A_0) \cap D(A_{0,p})$ is a core for A_m. Let now $u \in D(A_p)$ and $u^{(m)}$ be as in the beginning of step (iii). Then

$$\begin{aligned}|\operatorname{Im}\langle A_p u, w_p(u)\rangle| &= \lim_m |\operatorname{Im}\langle A_m u^{(m)}, w_p(u^{(m)})\rangle| \\ &\leqslant \lim_m M \operatorname{Re}\langle (A_m + \tilde{\omega})u^{(m)}, w_p(u^{(m)})\rangle = M \operatorname{Re}\langle (A_p + \tilde{\omega})u, w_p(u)\rangle,\end{aligned}$$

which shows the m-sectoriality of A_p with angle $\arctan M$. □

For the application of Lemma 5.1 in the proof of Theorem 4.2 we need to compute the gradient of $v_{n,p}$ and $w_{n,p}$.

Lemma 5.2. *For $\alpha \in \mathbb{R}$, $r > 0$, $z \in \mathbb{C}$, denote $z_{\alpha,r} := |z|^\alpha \wedge r$ if $\alpha \neq 0$ and $z_{0,r} := 1 \wedge r$. Let $\varphi\colon \mathbb{C} \to \mathbb{C}$, $\varphi(z) = z z_{\alpha,r}$. Then, for all (complex valued) $u \in W^1_{1,loc}$, $v = \varphi \circ u \in W^1_{1,loc}$ and*

$$\nabla v = u_{\alpha,r}\big(\nabla u + \alpha \mathbb{1}_{\{|u|^\alpha < r\}} \operatorname{sgn} u \nabla |u|\big).$$

Proof. It is easy to see that φ is a Lipschitz continuous function. So $v = \varphi \circ u$ is in $W^1_{1,loc}$. If $\alpha \notin (0,1)$ then the function $[0,\infty) \ni t \mapsto t^\alpha \wedge r$ is Lipschitz continuous too, hence $u_{\alpha,r} \in W^1_{1,loc}$, $\nabla u_{\alpha,r} = \alpha \mathbb{1}_{\{|u|^\alpha<r\}} |u|^{\alpha-1}\nabla|u|$ and the second statement of the lemma follows from the general product rule.

Let now $0 < \alpha < 1$. We denote $z_{\delta,\alpha,r} = (|z| + \delta)^\alpha \wedge r$ and approximate φ with the functions φ_δ, $\varphi_\delta(z) := z z_{\delta,\alpha,r}$. The function $[0,\infty) \ni t \mapsto (t+\delta)^\alpha \wedge r$ is Lipschitz continuous and

$$\nabla u_{\delta,\alpha,r} = \alpha \mathbb{1}_{\{(|u|+\delta)^\alpha<r\}} (|u| + \delta)^{\alpha-1} \nabla|u|.$$

So, by the general product rule,

$$\nabla(\varphi_\delta \circ u) = u_{\delta,\alpha,r}\bigl(\nabla u + \alpha \tfrac{u}{|u|+\delta} \mathbb{1}_{\{(|u|+\delta)^\alpha<r\}} \nabla|u|\bigr).$$

Finally, $\varphi_\delta \circ u \to \varphi \circ u$ and $\nabla(\varphi_\delta \circ u) \to u_{\alpha,r}\bigl(\nabla u + \alpha \mathbb{1}_{\{|u|^\alpha<r\}} \operatorname{sgn} u \nabla|u|\bigr)$ in $L_{1,loc}$ by Lebesgue's dominated convergence theorem, which implies the assertion. $\square$

Proof of Theorem 4.2. Let $p \in I$, i.e., $\tau_p \geqslant -\omega_p$. Let $U_0 := W_1 + W_2 + 2V^-$. By Proposition 4.1(a), $\tau + U_0$ is a closed sectorial form.

First we study the case $p > 1$. Let $u \in D(\tau)$. Then $v_{n,p}, w_{n,p} \in D(\tau)$ as multiples of normal contractions of u. At the end of the proof we will show that

$$\operatorname{Re} \tau(u, w_{n,p}) \geqslant \tau_p(v_{n,p}) - \tfrac{1}{2}\langle \mathbb{1}_n (W_1 + W_2)|v_{n,p}|^2\rangle, \tag{5.6}$$

where $\mathbb{1}_n$ is the indicator of the set $\{x; |u|^{\frac{p-2}{2}} \geqslant n\}$. Applying Lemma 5.1(a) with $\mathfrak{h} = \overline{\tau_p}$ and $U_n = \frac{1}{2}\mathbb{1}_n(W_1 + W_2)$ $(n \in \mathbb{N})$, we obtain all the assertions of Theorem 4.2 except for the analyticity of T_p.

Let now assumption (4.3) hold for some $p \in \mathring{I}$. Then it holds for all $p \in \mathring{I}$, by Proposition 4.1(c). To prove the analyticity of T_p, we need the inequality

$$|\operatorname{Im} \tau(u, w_{n,p})| \leqslant |\operatorname{Im} \tau_a(v_{n,p})| + |\tfrac{1}{p} - \tfrac{1}{p'}| \operatorname{Re} \tau_a(v_{n,p}) + |\operatorname{Im}\langle (b_1 + b_2)v_{n,p}, \nabla v_{n,p}\rangle| \tag{5.7}$$

which is also shown at the end of the proof. The first term in the right hand side of (5.7) can be estimated by $\alpha \operatorname{Re} \tau_a(v_{n,p})$, due to assumption **(a)**. Thus, by (4.3) we obtain that

$$|\operatorname{Im} \tau(u, w_{n,p})| \leqslant \bigl(\alpha + |\tfrac{1}{p} - \tfrac{1}{p'}|\bigr) \operatorname{Re} \tau_a(v_{n,p}) + c_1 \tau_p(v_{n,p}) + c_2 \|v_{n,p}\|^2.$$

By Proposition 4.1(b) we have $\operatorname{Re} \tau_a(v_{n,p}) \leqslant C(\tau_p + \tilde{\omega}_1)(v_{n,p})$ for some $\tilde{\omega}_1 \in \mathbb{R}$, $C > 0$ depending on p. Moreover, $\tau_p(v_{n,p}) \leqslant (\operatorname{Re} \tau + U_n)(u, w_{n,p})$ by (5.6). We conclude that

$$|\operatorname{Im} \tau(u, w_{n,p})| \leqslant \bigl[C\bigl(\alpha + |\tfrac{1}{p} - \tfrac{1}{p'}|\bigr) + c_1\bigr](\operatorname{Re} \tau + U_n + \tilde{\omega}_2)(u, w_{n,p})$$

for some $\tilde{\omega}_2 \in \mathbb{R}$, so Lemma 5.1(b) implies that A_p is an m-sectorial operator.

The proof for the case $p = 1$ is based on the assertions of the theorem in the case $p > 1$. Let U_0 be as above. Then $\tilde\tau := \tau + U_0$ is a closed sectorial form in L_2. Let T_0 be the associated analytic semigroup on L_2. Let $1 < p < \infty$ and $\tilde\tau_p := \tau_p + U_0$. For all $0 \leqslant u \in D(\tilde\tau) = D(\tau)$ we have

$$\tilde\tau_p(u) = \tfrac{4}{pp'}\tau_a(u) - \tfrac{2}{p'}\langle u, b_2\nabla u\rangle + \tfrac{1}{p}\big(2\langle b_1\nabla u, u\rangle + \langle V u^2\rangle\big) + \langle(\tfrac{1}{p'}V + U_0)u^2\rangle.$$

We apply Euclid's inequality to the second term, and the estimate

$$\tau_1(u) = 2\langle b_1\nabla u, u\rangle + \langle V u^2\rangle \geqslant -\omega_1\|u\|_2^2$$

to the third term in the right hand side, to obtain

$$\begin{aligned}\tilde\tau_p(u) &\geqslant \tfrac{4}{pp'}\tau_a(u) - \tfrac{2}{p'}\big(\tfrac12\tau_a(u) + \tfrac12\langle W_2u^2\rangle\big) - \tfrac{\omega_1}{p}\|u\|_2^2 + \langle(U_0 - \tfrac{1}{p'}V^-)u^2\rangle\\ &= \tfrac{1}{p'}(\tfrac{4}{p} - 1)\tau_a(u) - \tfrac{\omega_1}{p}\|u\|_2^2 + \langle(U_0 - \tfrac{1}{p'}(V^- + W_2))u^2\rangle.\end{aligned}$$

For $1 < p \leqslant 4$, Theorem 4.2 applied to $\tilde\tau$ implies: T_0 extrapolates to a C_0-semigroup $T_{0,p}$ on L_p, and for the generator $-A_{0,p}$ of $T_{0,p}$ we have

$$\langle A_{0,p}u, u^{p-1}\rangle \geqslant \langle(U_0 - \tfrac{1}{p'}(V^- + W_2))u^p\rangle - \tfrac{\omega_1}{p}\|u\|_p^p \quad \big(0 \leqslant u \in D(A_{0,p})\big). \tag{5.8}$$

In particular, $\|T_{0,p}(t)\|_{p\to p} \leqslant e^{\frac{\omega_1}{p}t}$ for all $t \geqslant 0$, $1 < p \leqslant 4$. Since T_0 is a positive C_0-semigroup, [19] implies that T_0 extrapolates to a C_0-semigroup $T_{0,1}$ on L_1.

Let now $U_{n,m} := \big(U_0 - \frac{1}{m}(V^- + W_2)\big) \wedge n$ for $n, m \in \mathbb{N}$. It follows from (5.8) that

$$\|(T_{0,p})_{-U_{n,m}}(t)\|_{p\to p} \leqslant e^{\frac{\omega_1}{p}t} \quad (t \geqslant 0)$$

for all $n \in \mathbb{N}$, $m \geqslant 2$ and $1 < p \leqslant \frac{m}{m-1}$ (i.e., $\frac{1}{p'} \leqslant \frac{1}{m}$). Since $(T_{0,p})_{-U_{n,m}}$ and $(T_{0,1})_{-U_{n,m}}$ are consistent by Lemma 2.3(b), we obtain $\|(T_{0,1})_{-U_{n,m}}(t)\|_{1\to1} \leqslant e^{\omega_1 t}$ for all $t \geqslant 0$, $n \in \mathbb{N}$, $m \geqslant 2$. Since $U_{n,m} \uparrow U_0 \wedge n$ as $m \to \infty$, we have $(T_{0,1})_{-U_{n,m}} \to (T_{0,1})_{-U_0\wedge n}$ for all $n \in \mathbb{N}$, by [17, Prop. A.2]. Hence

$$\sup_{n\in\mathbb{N}} \|(T_{0,1})_{-U_0\wedge n}(t)\|_{1\to1} \leqslant e^{\omega_1 t} \quad (t \geqslant 0).$$

Finally, [18, Prop. 2.2] implies that $-U_0$ is $T_{0,1}$-admissible, and we obtain $\tau \leftrightarrow (T_{0,1})_{-U_0} =: T_1$, with $\|T_1(t)\|_{1\to1} \leqslant e^{\omega_1 t}$ for all $t \geqslant 0$.

To complete the proof it remains to show inequalities (5.6) and (5.7). Let $\mathbb{1}_n^c := 1 - \mathbb{1}_n$, i.e., the indicator of the set $\{x; |u|^{\frac{p-2}{2}} < n\}$. We write $u_n = u_{n,p}$, $v_n = v_{n,p}$ $\big(= u(|u|^{\frac{p-2}{2}} \wedge n)\big)$ and $w_n = w_{n,p}$ $\big(= u(|u|^{p-2} \wedge n^2)\big)$ for short. Lemma 5.2 implies that

$$\nabla v_n = u_n(\nabla u + \tfrac{p-2}{2}\mathbb{1}_n^c \operatorname{sgn} u \nabla|u|) = \operatorname{sgn} u(u_n \operatorname{sgn}\overline{u}\nabla u + \tfrac{p-2}{2}\mathbb{1}_n^c u_n\nabla|u|).$$

Let $\varphi_n := u_n \operatorname{Re}(\operatorname{sgn} \overline{u} \nabla u) = u_n \nabla |u|$ and $\psi_n := u_n \operatorname{Im}(\operatorname{sgn} \overline{u} \nabla u)$. Then we have

$$\operatorname{sgn} \overline{u} \nabla v_n = \varphi_n + i\psi_n + \tfrac{p-2}{2} \mathbb{1}_n^c \varphi_n = (\tfrac{p}{2} \mathbb{1}_n^c + \mathbb{1}_n)\varphi_n + i\psi_n.$$

In the same way, with $\rho_n = (p-1)\mathbb{1}_n^c + \mathbb{1}_n$, we have

$$\nabla \overline{w}_n = u_n^2 (\nabla \overline{u} + (p-2)\mathbb{1}_n^c \operatorname{sgn} \overline{u} \nabla |u|) = u_n \operatorname{sgn} \overline{u}(\rho_n \varphi_n - i\psi_n).$$

Now we compute the different terms occurring in $\tau(u, w_n)$ and $\tau_p(v_n)$ separately.

$$\begin{aligned} a\nabla u \cdot \nabla \overline{w}_n &= a(u_n \operatorname{sgn} \overline{u} \nabla u) \cdot (\rho_n \varphi_n - i\psi_n) = a(\varphi_n + i\psi_n)(\rho_n \varphi_n - i\psi_n), \\ a\nabla v_n \cdot \nabla \overline{v}_n &= a(\operatorname{sgn} \overline{u} \nabla v_n) \cdot (\operatorname{sgn} u \nabla \overline{v}_n) \\ &= (\tfrac{p^2}{4} \mathbb{1}_n^c + \mathbb{1}_n) a_s \varphi_n \cdot \varphi_n + a_s \psi_n \cdot \psi_n + i(a - a_s)\psi_n \cdot (p\mathbb{1}_n^c + 2\mathbb{1}_n)\varphi_n. \end{aligned} \tag{5.9}$$

Therefore $\operatorname{Re} a\nabla u \cdot \nabla \overline{w}_n = \big((p-1)\mathbb{1}_n^c + \mathbb{1}_n\big) a_s \varphi_n \cdot \varphi_n + a_s \psi_n \cdot \psi_n$. Note that $\frac{4}{pp'} \frac{p^2}{4} = p - 1$. Hence, we obtain

$$\operatorname{Re} \tau_a(u, w_n) = \tfrac{4}{pp'} \operatorname{Re} \tau_a(v_n) + (1 - \tfrac{4}{pp'}) \langle \mathbb{1}_n a_s \varphi_n \cdot \varphi_n + a_s \psi_n \cdot \psi_n \rangle.$$

For the first order terms we compute

$$\begin{aligned} \overline{v}_n \nabla v_n &= |v_n| \big((\tfrac{p}{2} \mathbb{1}_n^c + \mathbb{1}_n)\varphi_n + i\psi_n \big), \\ \overline{w}_n \nabla u &= |v_n| u_n \operatorname{sgn} \overline{u} \nabla u = |v_n|(\varphi_n + i\psi_n), \\ u \nabla \overline{w}_n &= |v_n|(\rho_n \varphi_n - i\psi_n). \end{aligned} \tag{5.10}$$

Thus, $\operatorname{Re} \overline{v}_n \nabla v_n = |v_n|(\frac{p}{2} \mathbb{1}_n^c + \mathbb{1}_n)\varphi_n$. We obtain that

$$\operatorname{Re} \overline{w}_n \nabla u = |v_n| \varphi_n = \tfrac{2}{p} \operatorname{Re}(\overline{v}_n \nabla v_n) + (1 - \tfrac{2}{p}) \mathbb{1}_n |v_n| \varphi_n$$

and, since $\frac{2}{p'} \frac{p}{2} = p - 1$,

$$\operatorname{Re} u \nabla \overline{w}_n = \big((p-1)\mathbb{1}_n^c + \mathbb{1}_n\big)|v_n|\varphi_n = \tfrac{2}{p'} \operatorname{Re}(\overline{v}_n \nabla v_n) + (1 - \tfrac{2}{p'}) \mathbb{1}_n |v_n| \varphi_n.$$

Let now $\varepsilon_p := \frac{1}{p'} - \frac{1}{p} = 1 - \frac{2}{p} = -(1 - \frac{2}{p'})$. Then $\varepsilon_p^2 = 1 - \frac{4}{pp'}$. We get

$$\begin{aligned} \operatorname{Re} \tau(u, w_n) &= \operatorname{Re} \tau_a(u, w_n) + \operatorname{Re}\langle \nabla u, b_1 w_n \rangle - \operatorname{Re}\langle b_2 u, \nabla w_n \rangle + \langle V u, w_n \rangle \\ &= \tau_p(v_n) + \varepsilon_p^2 \langle \mathbb{1}_n a_s \varphi_n \cdot \varphi_n + a_s \psi_n \cdot \psi_n \rangle + \varepsilon_p \langle \mathbb{1}_n (b_1 + b_2)|v_n| \cdot \varphi_n \rangle. \end{aligned}$$

This implies (5.6) since $\varepsilon_p \mathbb{1}_n |(b_1 + b_2)v_n \cdot \varphi_n| \leqslant \varepsilon_p^2 \mathbb{1}_n a_s \varphi_n \cdot \varphi_n + \frac{1}{2} \mathbb{1}_n (W_1 + W_2)|v_n|^2$, by Euclid's inequality.

To prove (5.7), we first compute $\operatorname{Im} \tau_a(u, w_n)$.

$$\begin{aligned} \operatorname{Im}(a\nabla u \cdot \nabla \overline{w}_n) &= ((p-1)\mathbb{1}_n^c + \mathbb{1}_n) a\psi_n \cdot \varphi_n - a\varphi_n \cdot \psi_n \\ &= (p-2)\mathbb{1}_n^c a_s \psi_n \cdot \varphi_n + (p\mathbb{1}_n^c + 2\mathbb{1}_n)(a - a_s)\psi_n \cdot \varphi_n. \end{aligned}$$

The second term in the right hand side equals $\operatorname{Im}(a\nabla v_n \cdot \nabla\overline{v}_n)$, by (5.9). The first term we estimate, using Euclid's inequality and (5.9), as follows:

$$\begin{aligned}|(p-2)\mathbb{1}_n^c a_s\psi_n\cdot\varphi_n| &\leqslant |p-2|\mathbb{1}_n^c\big(\tfrac{p}{4}a_s\varphi_n\cdot\varphi_n+\tfrac{1}{p}a_s\psi_n\cdot\psi_n\big)\\ &= |1-\tfrac{2}{p}|\mathbb{1}_n^c\big(\tfrac{p^2}{4}a_s\varphi_n\cdot\varphi_n+a_s\psi_n\cdot\psi_n\big)\leqslant|\tfrac{1}{p}-\tfrac{1}{p'}|\operatorname{Re}(a\nabla v_n\cdot\nabla\overline{v_n}).\end{aligned}$$

For the first order terms we have, by (5.10),

$$\operatorname{Im}\big(\langle\nabla u, b_1w_n\rangle-\langle b_2u,\nabla w_n\rangle\big)=\langle(b_1+b_2)|v_n|,\psi_n\rangle=-\operatorname{Im}\langle(b_1+b_2)v_n,\nabla v_n\rangle.$$

Thus, inequality (5.7) follows. □

6 Sharpness of the result

In this section we show that, under some conditions additional to **(a)** and **(bV)**, if $\tau \leftrightarrow T_p$ on L_p for some $p\in(1,\infty)$, with $\|T_p(t)\|\leqslant e^{\omega_p t}$ for some $\omega_p\in\mathbb{R}$, then estimate (1.3) holds.

Lemma 6.1. *Let $1<p<\infty$, T_p a positive contractive C_0-semigroup on L_p. Let $U\geqslant 0$ be a T_p-admissible potential, $-A_U$ the generator of $(T_p)_U$. Then*

$$\operatorname{Re}\langle A_Uu, u|u|^{p-2}\rangle\geqslant\langle U|u|^p\rangle\quad\big(u\in D(A_U)\big).$$

Proof. Let $-A$ be the generator of T_p. For $m\in\mathbb{N}$ let $U_m=U\wedge m$. Let $u\in D(A_U)$ and $u_m:=(1+A+U_m)^{-1}(1+A_U)u$. Since A is accretive, we have

$$\operatorname{Re}\langle(1+A_U)u,u_m|u_m|^{p-2}\rangle=\operatorname{Re}\langle(1+A+U_m)u_m,u_m|u_m|^{p-2}\rangle\geqslant\langle(1+U_m)|u_m|^p\rangle.$$

Since $u_m\to u$ in L_p and $U_m\uparrow U$, we complete the proof by an application of Fatou's lemma. □

The following theorem is the main part of our sharpness result.

Theorem 6.2. *Let* **(a)**, **(bV)** *hold and assume that $\tau\leftrightarrow T_p$ on L_p for some $p\geqslant 2$, with $\|T_p(t)\|\leqslant e^{\omega_p t}$ $(t\geqslant 0)$ for some $\omega_p\in\mathbb{R}$. If there exists a τ_a-regular potential $U\geqslant 0$ such that $\|(T_p)_U(t)\|_{\infty\to\infty}\leqslant C$ for all $t\geqslant 0$ then estimate* (1.3) *holds.*

If $\langle\nabla u, b_2u\rangle\leqslant\omega\|u\|_2^2$ $(u\in D(\tau))$ and $U\geqslant V^-+\omega$, then $\|(T_p)_U(t)\|_{\infty\to\infty}\leqslant 1$ for all $t\geqslant 0$, by Remark 4.3(b) and Proposition 3.11.

The proof of Theorem 6.2 is based on the following lemma.

Lemma 6.3. *Let (M,μ) be a measure space, $\mathfrak{h}$ a Dirichlet form in $L_2(\mu)$ and $r\geqslant 1$.*

(a) Then $D_1:=\big\{0\leqslant u\in D(\mathfrak{h})\cap L_\infty(\mu);\, u^{1/r}\in D(\mathfrak{h})\big\}$ is dense in $D(\mathfrak{h})_+$, the set of positive elements of $D(\mathfrak{h})$.

(b) Let $\mathfrak{h}_1$ be a densely defined closed sectorial form in $L_2(\mu)$ fulfilling the first Beurling-Deny criterion, A the m-sectorial operator associated with $\mathfrak{h}_1$. Assume that $D(\mathfrak{h}_1) = D(\mathfrak{h})$, and $\|e^{-At}\|_{\infty\to\infty} \leqslant C$ $(0 \leqslant t \leqslant 1)$ for some $C > 0$. Then $D_2 := \{u^r; 0 \leqslant u \in D(A) \cap L_\infty(\mu), Au \in L_\infty(\mu)\}$ is dense in $D(\mathfrak{h})_+$.

Proof. (a) For $n \in \mathbb{N}$ define φ_n: $[0,\infty) \to [0,n]$ by $\varphi_n(s) := s \wedge (ns^r) \wedge n$. It is easy to show that the functions φ_n are Lipschitz continuous with constant r, $\varphi_n^{1/r}$ are Lipschitz continuous and that $\varphi_n(s) \to s$ $(s \geqslant 0)$ as $n \to \infty$. For $u \in D(\mathfrak{h})_+$ we conclude that $\varphi_n(u) \in D_1$, and from [1, Prop. 11] we deduce that $\varphi_n(u) \to u$ in $D(\mathfrak{h})$ as $n \to \infty$.

(b) By (a), it remains to show that D_2 is dense in D_1. Let $u \in D_1$ and $v := u^{1/r}$. Then $v \in D(\mathfrak{h}) \cap L_\infty(\mu)$. By [9, Thm. I.2.13(ii)] we have $v_\lambda := \lambda(\lambda + A)^{-1}v \to v$ in $D(\mathfrak{h}_1)$ and thus in $D(\mathfrak{h})$ as $\lambda \to \infty$. The assumption on A implies that $v_\lambda \in D(A) \cap L_\infty$ and $\|v_\lambda\|_\infty \leqslant 2C\|v\|_\infty$ for large λ. Moreover, we have $Av_\lambda = \lambda(v - v_\lambda) \in L_\infty$. Therefore, $v_\lambda^r \in D_2$ and, by [1, Théorème 10], $v_\lambda^r \to v^r = u$ in $D(\mathfrak{h})$ as $\lambda \to \infty$. □

Proof of Theorem 6.2. Without restriction assume $U \geqslant U_0 := W_1 + W_2 + 2|V|$ (see Lemma 3.13). We have to prove $\tau_p \geqslant -\omega_p$ on $D(\tau_p) = D(\tau_a + U_0)$. Notice that τ_p is a bounded form on $D(\tau_a + U_0)$. Since U is $(\tau_a + U_0)$-regular, by Lemma 3.13, it therefore suffices to show $\tau_p(u) \geqslant -\omega_p\|u\|_2^2$ for all $u \in D(\tau_a + U)$. Since τ_p fulfills the first Beurling-Deny criterion we can restrict ourselves to $u \geqslant 0$.

By Proposition 4.1(a), $\tau + U$ is a closed sectorial form in L_2. Let A_U be the m-sectorial operator in L_2 associated with $\tau + U$. Then the assumptions of Lemma 6.3(b) are fulfilled with $\mathfrak{h} = \tau_a + U$, $\mathfrak{h}_1 = \tau + U$, $A = A_U$ since $e^{-A_U t}$ and $(T_p)_U$ are consistent by Corollary 4.4. Below we show that

$$\tau_p(u^{\frac{p}{2}}) \geqslant -\omega_p\|u^{\frac{p}{2}}\|_2^2 \tag{6.1}$$

for all $0 \leqslant u \in D(A_U) \cap L_\infty$ with $A_U u \in L_\infty$. Then, an application of Lemma 6.3(b) shows that $\tau_p(u) \geqslant -\omega_p\|u\|_2^2$ for all $0 \leqslant u \in D(\tau_a + U)$, and the proof is complete.

So, let $0 \leqslant u \in D(A_U) \cap L_\infty$ with $A_U u \in L_\infty$. Then $u \in D(\tau_a + U) \cap L_\infty$ and hence $u^r \in D(\tau_a + U) \cap L_\infty$, $\nabla u^r = ru^{r-1}\nabla u$ for all $r \geqslant 1$. From this we easily obtain $\tau(u, u^{p-1}) = \tau_p(u^{\frac{p}{2}})$ (cf. the computation on page 3) and thus, by the definition of A_U, $(\tau_p + U)(u^{\frac{p}{2}}) = \langle A_U u, u^{p-1}\rangle$.

Since $e^{-A_U t}$ and $e^{-A_{p,U} t} := (T_p)_U$ are consistent and $u, A_U u \in L_2 \cap L_\infty \subseteq L_p$, we obtain $u \in D(A_{p,U})$ and $A_{p,U}u = A_U u$. By Lemma 6.1 we infer that

$$(\tau_p + U)(u^{\frac{p}{2}}) = \langle A_{p,U}u, u^{p-1}\rangle \geqslant \langle (U - \omega_p)u^p\rangle,$$

i.e., (6.1) holds.

□

By Proposition 3.12 we easily obtain the following corollary.

Corollary 6.4. *Let* **(a)**, **(bV)** *hold and assume that, for some* τ_a*-regular potential* $U \geqslant 0$*, the form* $\tau + U$ *is sectorial and closable and the associated semigroup* T_U *satisfies* $\|T_U(t)\|_{1\to 1} \leqslant C$*,* $\|T_U(t)\|_{\infty\to\infty} \leqslant C$ $(t \geqslant 0)$*. If* $\tau \leftrightarrow T_p$ *on* L_p *for some* $p \in (1, \infty)$*, with* $\|T_p(t)\| \leqslant e^{\omega_p t}$ $(t \geqslant 0)$ *for some* $\omega_p \in \mathbb{R}$*, then estimate* (1.3) *holds.*

Remark 6.5. The previous result is in particular applicable in the case of weakly differentiable b_1 and b_2. For $j = 1, 2$, we assume that b_j is of τ_a*-regular divergence*, i.e., there exists a measurable function div b_j such that $|\text{div } b_j|$ is τ_a-regular and

$$2\langle b_j u, \nabla u\rangle = -\langle (\text{div } b_j)u^2\rangle \quad \big(0 \leqslant u \in D(\tau) \cap Q(|\text{div } b_j|)\big).$$

Let $U := V^- + |\text{div } b_1| + |\text{div } b_2|$. Then

$$(\tau_1 + U)(u) = \big\langle (-\text{div } b_1 + V + U)u^2 \big\rangle \geqslant 0,$$
$$(\tau_\infty + U)(u) = \big\langle (\text{div } b_2 + V + U)u^2 \big\rangle \geqslant 0$$

for all $0 \leqslant u \in D(\tau + U)$, so $(T_p)_U$ is L_1- and L_∞-contractive.

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