The modulus semigroup for linear delay equations III

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Abstract

In this paper we describe the modulus semigroup of the C_0-semigroup associated with the linear differential equation with delay

$$\begin{cases} u'(t) = Au(t) + Lu_t \quad (t \geqslant 0), \\ u(0) = x \in X, \qquad u_0 = f \in L_p(-h,0;X), \end{cases}$$

in the Banach lattice $X \times L_p(-h,0;X)$, where X is a Banach lattice with order continuous norm. The progress with respect to previous papers is that A may be an unbounded generator of a C_0-semigroup possessing a modulus semigroup.

MSC 2000: 47D06, 47B60
Keywords: functional differential equation, delay equation, domination, modulus semigroup, Banach lattice

Introduction

In the present paper the results of the papers [3], [12] are further generalised. The main object is to obtain the modulus semigroup for a C_0-semigroup arising in the study of the initial value problem for a linear differential equation with delay,

$$\begin{cases} u'(t) = Au(t) + Lu_t \quad (t \geqslant 0), \\ u(0) = x, \quad u_0 = f, \end{cases} \tag{DE}$$

in the L_p-context, for $1 \leqslant p < \infty$, with initial values $x \in X$, $f \in L_p(-h,0;X)$. Here, X is a Banach lattice with order continuous norm, and $h = 1$ or $h = \infty$, corresponding to finite or infinite delay. Further, A is the (possibly unbounded) generator

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of a C_0-semigroup on X—the unboundedness of A is the important new feature in this paper—, and $L\colon C([-h,0];X) \to X$ is the bounded linear operator given by

$$Lf := \int_{[-h,0]} d\eta(\vartheta) f(\vartheta) \quad (f \in C([-h,0];X)),$$

where $\eta\colon [-h,0] \to L(X)$ is a function of bounded variation with no mass in zero. Also, for a function $u\colon (-h,\infty) \to X$, we recall the notation

$$u_t(\vartheta) := u(t+\vartheta) \quad (-h < \vartheta < 0),$$

for $t \geqslant 0$.

It is shown in [1] that the delay equation (DE) is equivalent to an abstract Cauchy problem

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) \quad (t \geqslant 0), \\ \mathcal{U}(0) = \binom{x}{\varphi} \end{cases}$$

on the space $X \times L_p(-h,0;X)$, where $\mathcal{A}$ is given by

$$\mathcal{A} := \begin{pmatrix} A & L \\ 0 & \frac{d}{d\vartheta} \end{pmatrix},$$

with domain

$$D(\mathcal{A}) := \left\{ (x,\varphi) \in D(A) \times W_p^1(-h,0;X)\,;\ \varphi(0) = x \right\}.$$

From [1], [4], [5], [8] it is known that the operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{T} := (e^{t\mathcal{A}})_{t\geqslant 0}$ on the Banach lattice $X \times L_p(-h,0;X)$.

Next, assume that the C_0-semigroup generated by A possesses a modulus semigroup, i.e., a smallest semigroup dominating $(e^{tA})_{t\geqslant 0}$, whose generator will be denoted by $A^\#$. Also, assume that η is 'of finite regular variation' (see Section 1.3 for details), which implies that the operator L possesses a modulus. It is the object of the paper to show that then the C_0-semigroup generated by

$$\widetilde{\mathcal{A}} := \begin{pmatrix} A^\# & |L| \\ 0 & \frac{d}{d\vartheta} \end{pmatrix},$$

with domain

$$D(\widetilde{\mathcal{A}}) := \left\{ (x,\varphi) \in D(A^\#) \times W_p^1(-h,0;X)\,;\ \varphi(0) = x \right\},$$

is the modulus semigroup of the C_0-semigroup generated by $\mathcal{A}$.

This result is shown in [3] for the case $X = \mathbb{R}^n$, where necessarily the generator A is a bounded operator. In [12] the result is generalised to the case of a Banach lattice X with order continuous norm, but still with a bounded generator A.

The first result on the subject is contained in [2], where the case $X = \mathbb{R}^n$ is treated in the framework of continuous functions (instead of $X \times L_p(-h, 0; X)$). We will also generalise this result to the case where X is a Banach lattice with order continuous norm; cf. Section 4.

For motivation why it is interesting to investigate modulus semigroups we refer to [2], [7], [12].

In Section 1 we recall certain notions and present some results needed in the sequel. We prove a 'domination lemma', and we introduce the delay semigroups in more detail. In particular, in the second part of Section 1.2 we indicate a new (simplified) method to treat the perturbed delay equation in the case $p = 1$.

In Section 2 we apply the 'domination lemma' of Section 1.1 in order to show that a semigroup dominating the perturbed (by the operator L) semigroup for the delay equation is also a dominating semigroup for the unperturbed semigroup.

In Section 3 we show the main result. Besides the new ideas prepared in Section 2 the proof relies heavily on results contained in [12].

In Section 4 we transfer the result to the framework of continuous functions, using consistent semigroups.

1 Preliminaries

1.1 The domination lemma

For use in Section 2 we single out the following 'domination lemma'. A version of this technical result was already used in [10; proof of Proposition 1.2].

1.1 Lemma. *Let X be a Banach lattice. Let T, S be C_0-semigroups on X, S positive, and assume that $R\colon [0,1] \to L(X)$ satisfies*

$$\frac{1}{t}\|R(t)\| \to 0 \qquad (t \to 0).$$

Assume that

$$|T(t)x| \leqslant S(t)|x| + |R(t)x| \tag{1.1}$$

for all $x \in X$, $0 \leqslant t \leqslant 1$.

Then T is dominated by S, i.e., $|T(t)x| \leqslant S(t)|x|$ ($x \in X$, $t \geqslant 0$).

Proof. Let $x \in X$. By induction, inequality (1.1) yields

$$|T(t)^n x| \leqslant S(t)^n|x| + \sum_{m=1}^{n} S(t)^{n-m}|R(t)T(t)^{m-1}x|$$

for all $0 \leqslant t \leqslant 1$, $n \in \mathbb{N}$. Replacing t by t/n we obtain

$$|T(t)x| \leqslant S(t)|x| + \sum_{m=1}^{n} S(\tfrac{n-m}{n}t)\left|R(\tfrac{t}{n})T(\tfrac{m-1}{n}t)x\right| \tag{1.2}$$

for all $n \in \mathbb{N}$, $0 \leqslant t \leqslant n$. With $c_t := \sup_{0\leqslant s\leqslant t}\|S(s)\|$, $d_t := \sup_{0\leqslant s\leqslant t}\|T(s)\|$ the last term in inequality (1.2) can be estimated as

$$\left\|\sum_{m=1}^{n} S(\tfrac{n-m}{n}t)\left|R(\tfrac{t}{n})T(\tfrac{m-1}{n}t)x\right|\right\| \leqslant nc_t\|R(\tfrac{t}{n})\|d_t\|x\| = c_t d_t\, t\, \tfrac{n}{t}\|R(\tfrac{t}{n})\|\|x\|\,.$$

Since this tends to zero as $n \to \infty$, inequality (1.2) yields the assertion. □

1.2 The delay semigroup

In this part we fix our assumptions concerning the delay semigroup. Assume that X is a Banach space. We assume that the operator A in X is the generator of a C_0-semigroup T. We assume that $h \in \{1, \infty\}$, we choose $p \in [1, \infty)$, and we denote by S the C_0-semigroup of left translation on $L_p(-h, 0; X)$,

$$S(t)\varphi(\vartheta) := \begin{cases} \varphi(t+\vartheta) & \text{for } -h < \vartheta < -t, \\ 0 & \text{for } -t < \vartheta < 0. \end{cases}$$

We recall that the operator $\mathcal{A}_0$ in $X \times L_p(-h, 0; X)$,

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}, \quad D(\mathcal{A}_0) := \{(x, \varphi) \in D(A) \times W_p^1(-h, 0; X);\ \varphi(0) = x\},$$

is the generator of a C_0-semigroup $\mathcal{T}_0$ which is given by

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & 0 \\ T_t & S(t) \end{pmatrix}. \tag{1.3}$$

Here, $T_t \in L(X, L_p(-h, 0; X))$ denotes the operator defined by

$$T_t x(\vartheta) := \begin{cases} 0 & \text{for } -h < \vartheta < -t, \\ T(t+\vartheta)x & \text{for } -t < \vartheta < 0. \end{cases}$$

For these statements we refer to [1; Proposition 3.1].

Next, let $\eta\colon [-h, 0] \to L(X)$ be a function of bounded variation (where, in the case of $h = \infty$, $[-h, 0]$ denotes the one point compactification of $(-\infty, 0]$). Then one can define an operator $L \in L(C([-h, 0]; X), X)$ by

$$L\varphi := \int d\eta(\vartheta)\varphi(\vartheta) \qquad (\varphi \in C([-h, 0]; X));$$

we refer to [12; Section 2] for details. We assume that η is left continuous, i.e.,

$$\eta(\vartheta) = \lim_{\vartheta' \to \vartheta -} \eta(\vartheta') \tag{1.4}$$

for all $\vartheta \in (-h, 0]$. For $\vartheta \in (-h, 0)$, this can always be achieved by redefining η, without changing L. For $\vartheta = 0$, however, this means that η does not give rise to mass at zero; we refer to [8; beginning of Section 2] for a short discussion concerning this assumption. We recall that, as a consequence, the variation

$$|\eta|([-\alpha, 0]) := \sup\Big\{ \sum_{j=1}^{n} \|\eta(\vartheta_j) - \eta(\vartheta_{j-1})\| \,;\, -\alpha = \vartheta_0 < \ldots < \vartheta_n = 0,\ n \in \mathbb{N} \Big\},$$

of η on $[-\alpha, 0]$ tends to zero as $\alpha \to 0+$; cf. [12; Lemma 2.1].

We are going to show that $\mathcal{B} := \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$ is a small Miyadera perturbation (cf. [9], [11], [6], [8]) of $\mathcal{A}_0$, for any $p \in [1, \infty)$, if the norm on $X \times L_p(-h, 0; X)$ is chosen suitably. For $1 < p < \infty$ this is known (and true for any of the norms on the product), whereas for $p = 1$, this is a new observation (making part of the paper [8] obsolete). We recall the following estimate from [1; Example 3.1]: With $M := \sup_{0 \leqslant s \leqslant 1} \|T(s)\|$, $\frac{1}{p'} + \frac{1}{p} = 1$, one has

$$\int_0^t \|L(T_s x + S(s) f)\| \, ds \leqslant tM|\eta|([-t, 0])\|x\| + t^{1/p'} |\eta|([-h, 0]) \, \|f\|_p \,, \tag{1.5}$$

for all $(x, f) \in D(\mathcal{A}_0)$, $0 \leqslant t \leqslant 1$. Note that $L(T_s x + S(s) f)$ is the first component of $\mathcal{B}\mathcal{T}_0(s)\binom{x}{f}$ (the second component being zero). If $1 < p < \infty$, then the coefficients of $\|x\|$ and $\|f\|_p$ tend to zero as $t \to 0$, and therefore $\mathcal{B}$ is an infinitesimally small Miyadera perturbation of $\mathcal{A}_0$. For $p = 1$, however, we choose a norm

$$\|(x, f)\|_c := \|x\| + c \, \|f\|_1 \,,$$

with $c > |\eta|([-h, 0])$. Then (1.5) shows

$$\begin{aligned} \int_0^t \|\mathcal{B}\mathcal{T}_0(s)\tbinom{x}{f}\|_c \, ds &\leqslant tM|\eta|([-t, 0])\|x\| + |\eta|([-h, 0]) \, \|f\|_1 \\ &\leqslant \max\big(tM|\eta|([-t, 0]), |\eta|([-h, 0])/c\big) \, \|(x, f)\|_c \,, \end{aligned}$$

for all $(x, f) \in D(\mathcal{A}_0)$, $0 \leqslant t \leqslant 1$, where $\max\big(tM|\eta|([-t, 0]), |\eta|([-h, 0])/c\big) < 1$ for small t, i.e., $\mathcal{B}$ is a small Miyadera perturbation of $\mathcal{A}_0$. These statements imply that $\mathcal{A} := \begin{pmatrix} A & L \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}$, with $D(\mathcal{A}) = D(\mathcal{A}_0)$, is the generator of a C_0-semigroup $\mathcal{T}$, for all $1 \leqslant p < \infty$. The semigroup $\mathcal{T}$ is associated with the Cauchy problem (DE).

1.3 The dominating delay semigroup

Additionally to the assumptions of Section 1.2 we now assume that X is a Banach lattice with order continuous norm, and that the C_0-semigroup T possesses a modulus semigroup $T^{\#}$, with generator $A^{\#}$. Applying the assertions of Section 1.2 we obtain that $\widetilde{\mathcal{A}}_0 := \begin{pmatrix} A^{\#} & 0 \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}$, with domain $D(\widetilde{\mathcal{A}}_0) := \{(x,\varphi) \in D(A^{\#}) \times W_p^1(-h,0;X)\,;\ \varphi(0) = x\}$, is the generator of a C_0-semigroup $\widetilde{\mathcal{T}}_0$.

We assume that the function η is 'of bounded regular variation', i.e., η takes its values in the regular operators,

$$\widetilde{\eta}(t) := \sup\Big\{ \sum_{j=1}^{n} |\eta(\vartheta_j) - \eta(\vartheta_{j-1})|\,;\ -h = \vartheta_0 < \cdots < \vartheta_n = t,\ n \in \mathbb{N} \Big\}$$

exists for all $-h \leqslant t \leqslant 0$, and $\widetilde{\eta}$ is of bounded variation. It has been shown in [12; Lemma 3.1] that then the function $\widetilde{\eta}$ is left continuous, in particular

$$\widetilde{\eta}(0) = \lim_{\vartheta \to 0} \widetilde{\eta}(\vartheta)\,.$$

Also, it has been shown in [12; Proposition 2.5] that the operator associated with the function $\widetilde{\eta}$ is the modulus $|L| \in L(C([-h,0];X),X)$ of L. Again, the operator $\widetilde{\mathcal{A}} := \begin{pmatrix} A^{\#} & |L| \\ 0 & \frac{d}{d\vartheta} \end{pmatrix}$, with $D(\widetilde{\mathcal{A}}) := D(\widetilde{\mathcal{A}}_0)$, generates a C_0-semigroup $\widetilde{\mathcal{T}}$.

1.2 Remarks. (a) From the expression (1.3) for the semigroup $\mathcal{T}_0$, and the corresponding expression for the semigroup $\widetilde{\mathcal{T}}_0$, it is immediate that $\widetilde{\mathcal{T}}_0$ dominates $\mathcal{T}_0$.

(b) Arguing as in [3; Lemma 2.1] one shows that $\widetilde{\mathcal{T}}$ dominates $\mathcal{T}$. (In fact, in view of the second part of Section 1.2 it is no longer necessary to treat the case $p = 1$ separately.)

(c) The Banach lattice $X \times L_p(-h,0;X)$ has order continuous norm. Therefore it follows from part (a) and [2; Theorem 2.1] that $\mathcal{T}_0$ possesses a modulus semigroup $\mathcal{T}_0^{\#}$, and $\mathcal{T}_0^{\#}(t) \leqslant \widetilde{\mathcal{T}}_0(t)\ \ (t \geqslant 0)$. In the same way, the C_0-semigroup $\mathcal{T}$ possesses a modulus semigroup $\mathcal{T}^{\#}$, and $\mathcal{T}^{\#}(t) \leqslant \widetilde{\mathcal{T}}(t)\ \ (t \geqslant 0)$.

2 Domination of unperturbed and perturbed delay semigroups

In the present section we assume that X is a Banach lattice, and that $A,\ L,\ 1 \leqslant p < \infty,\ \mathcal{A}_0,\ \mathcal{T}_0,\ \mathcal{A},\ \mathcal{T}$ are as in Section 1.2.

The following result is the main tool for helping to identify the domain of the generator of the modulus semigroup for the delay semigroup; cf. Section 3.

2.1 Proposition. *Let the notation be as above, and assume that $\mathcal{T}$ is dominated by a C_0-semigroup $\mathcal{S}$ on $X \times L_p(-h,0;X)$. Then $\mathcal{T}_0$ is dominated by $\mathcal{S}$ as well.*

Recall from Section 1.2 that $\mathcal{B} := \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$ is a (small) Miyadera perturbation of $\mathcal{A}$. Therefore, $\mathcal{T}$ can be represented as

$$\mathcal{T}(t) = \mathcal{T}_0(t) + \mathcal{R}_1(t)\,, \tag{2.1}$$

with

$$\mathcal{R}_1(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) = \int_0^t \mathcal{T}(t-s)\mathcal{B}\mathcal{T}_0(s)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) ds \qquad ((x,\varphi) \in D(\mathcal{A}_0))\,.$$

The procedure of the proof of Proposition 2.1 is similar to [10; proof of Proposition 1.2]. The method consists in finding parts in the representation (2.1) of $\mathcal{T}$ allowing to estimate $\mathcal{T}_0$, and other parts allowing an estimate needed for the application of Lemma 1.1. The difference to [10] is that in that paper one has to use an iterated form of (2.1).

2.2 Lemma. *There exists* $c \geqslant 0$ *such that*

$$\|\mathcal{R}_1(t)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)\| \leqslant ct|\eta|([-t,0])\|x\|\,,$$

for all $x \in D(A)$, $0 \leqslant t \leqslant 1$.

Proof. This inequality is shown in the same way as [3; inequality (2.4) in Lemma 2.2(a)]. □

2.3 Remark. In the proof of Proposition 2.1 we will need the following general fact about delay semigroups. For $f \in L_p(-h,0;X)$, $0 \leqslant t < h$ one has

$$\mathbf{1}_{(-h,-t)} P_2 \mathcal{T}(t)\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) = S(t)f$$

(where P_2 is the projection onto the second component of $X \times L_p(-h,0;X)$), and this implies

$$|\mathcal{T}_0(t)\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)| = |\left(\begin{smallmatrix} 0 \\ S(t)f \end{smallmatrix}\right)| \leqslant |\mathcal{T}(t)\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)|\,.$$

Proof of Proposition 2.1. Let $(x,f) \in X \times L_p(-h,0;X)$. We estimate (using Remark 2.3 in the second estimate)

$$\begin{aligned} |\mathcal{T}_0(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)| &\leqslant |\mathcal{T}(t)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)| + |(\mathcal{T}_0(t) - \mathcal{T}(t))\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)| + |\mathcal{T}_0(t)\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)| \\ &\leqslant \mathcal{S}(t)|\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)| + |\mathcal{R}_1(t)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)| + \mathcal{S}(t)|\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)| \\ &= \mathcal{S}(t)|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)| + |\mathcal{R}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)|\,, \end{aligned}$$

where $\mathcal{R}(t) := \mathcal{R}_1(t)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By Lemma 2.2 we have $\|\mathcal{R}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\| \leqslant ct|\eta|([-t,0])\|x\|$, and thus

$$\frac{1}{t}\|\mathcal{R}(t)\| \to 0 \qquad (t \to 0)\,.$$

(Recall that $|\eta|([-t,0]) \to 0$ because η is assumed to induce no mass at 0.) By Lemma 1.1 we obtain that $\mathcal{T}_0$ is dominated by $\mathcal{S}$. □

3 The modulus semigroup

In this section we assume that X is a Banach lattice with order continuous norm.

3.1 Theorem. *Let X, $\mathcal{A}$, and $\widetilde{\mathcal{A}}$ be as introduced in Section 1.3. Then $\mathcal{A}^{\#} = \widetilde{\mathcal{A}}$.*

The following result will serve as a final preparation for the proof.

3.2 Proposition. *With the previous hypotheses and notations, we have:*
(a) $\mathcal{T}_0^{\#}(t) \leqslant \mathcal{T}^{\#}(t) \leqslant \widetilde{\mathcal{T}}(t)$, *for all* $t \geqslant 0$.
(b) $\mathcal{A}_0^{\#} = \widetilde{\mathcal{A}}_0$.

Proof. (a) The first inequality follows from Proposition 2.1 since $\mathcal{T}^{\#}$ is a C_0-semigroup dominating $\mathcal{T}$. The second inequality was mentioned in Remark 1.2(c).

(b) From Section 1.2 we recall the representation (1.3), and correspondingly,

$$\widetilde{\mathcal{T}}_0(t) = \begin{pmatrix} T^{\#}(t) & 0 \\ (T^{\#})_t & S(t) \end{pmatrix}. \tag{3.1}$$

The inequalities $|\mathcal{T}_0(t)| \leqslant \mathcal{T}_0^{\#}(t) \leqslant \widetilde{\mathcal{T}}_0(t)$ (for the second of these inequalities we refer to Remark 1.2(c)) show that $\mathcal{T}_0^{\#}$ is of the form

$$\mathcal{T}_0^{\#}(t) = \begin{pmatrix} T_{11}^{\#}(t) & 0 \\ V(t) & S(t) \end{pmatrix},$$

with positive operators $T_{11}^{\#}(t), V(t)$ satisfying

$$|T(t)| \leqslant T_{11}^{\#}(t) \leqslant T^{\#}(t), \tag{3.2}$$

$$|T_t| \leqslant V(t) \leqslant (T^{\#})_t, \tag{3.3}$$

for all $t \geqslant 0$.

From the semigroup property of $\mathcal{T}_0^{\#}$ one obtains that $(T_{11}^{\#}(t))_{t\geqslant 0}$ is a C_0-semigroup on X, and therefore (3.2) implies $T_{11}^{\#} = T^{\#}$.

Inequality (3.3) implies $\operatorname{spt} V(t)x \subseteq [-t, 0)$, for all $x \in X$. Let $(x, \varphi) \in D(\mathcal{A}_0^{\#})$, $0 < t < h$. Then $\mathcal{T}_0^{\#}(t)\binom{x}{\varphi} = \binom{T^{\#}(t)x}{\widetilde{\varphi}_t}$, where

$$\widetilde{\varphi}_t(\vartheta) = \begin{cases} \varphi(t+\vartheta) & \text{for } -h < \vartheta \leqslant -t, \\ V(t)x(\vartheta) & \text{for } -t < \vartheta < 0. \end{cases}$$

The existence of $\lim_{t\to 0} \frac{1}{t}\big(\mathcal{T}_0^{\#}(t)\binom{x}{\varphi} - \binom{x}{\varphi}\big)$ shows $x \in D(A^{\#})$, $\varphi \in W_p^1(-h, 0; X)$, and we obtain $\mathcal{A}_0^{\#}\binom{x}{\varphi} = \binom{A^{\#}x}{\varphi'}$. Since $\mathcal{T}_0^{\#}$ leaves $D(\mathcal{A}_0^{\#})$ invariant, we also obtain $\widetilde{\varphi}_t \in W_p^1(-h, 0; X)$, and hence

$$\varphi(0) = \widetilde{\varphi}_t(-t) = \lim_{\vartheta\to(-t)+} \widetilde{\varphi}_t(\vartheta) = \lim_{\vartheta\to(-t)+} V(t)x(\vartheta) = x,$$

where the last equality holds because of (3.3). (In fact, this last equality is first shown for $x \geqslant 0$, and then carries over to general x.)

Thus we have shown $\mathcal{A}_0^\# \subseteq \widetilde{\mathcal{A}}_0$. Since both of these operators are generators we conclude $\mathcal{A}_0^\# = \widetilde{\mathcal{A}}_0$. □

Proof of Theorem 3.1. From Proposition 3.2(a) and [10; Proposition A.1] we obtain $D(\widetilde{\mathcal{A}})$ $(= D(\mathcal{A}_0^\#)) \subseteq D(\mathcal{A}^\#)$. For $(x,\varphi) \in D(\widetilde{\mathcal{A}})_+$ we have

$$\mathcal{T}_0^\#(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \leqslant \mathcal{T}^\#(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \leqslant \widetilde{\mathcal{T}}(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \qquad (t \geqslant 0),$$

and this implies

$$\begin{pmatrix} A^\# x \\ \varphi' \end{pmatrix} = \mathcal{A}_0^\# \begin{pmatrix} x \\ \varphi \end{pmatrix} \leqslant \mathcal{A}^\# \begin{pmatrix} x \\ \varphi \end{pmatrix} \leqslant \widetilde{\mathcal{A}} \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} A^\# x + |L|\varphi \\ \varphi' \end{pmatrix}. \tag{3.4}$$

We define

$$L^\#\varphi := P_1 \mathcal{A}^\#\left(\begin{smallmatrix} \varphi(0) \\ \varphi \end{smallmatrix}\right) - A^\#\varphi(0)$$

(where P_1 is the projection onto the first component of $X \times L_p(-h,0;X)$). Then (3.4) implies

$$\mathcal{A}^\# \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} A^\# x + L^\#\varphi \\ \varphi' \end{pmatrix}$$

and

$$0 \leqslant L^\#\varphi \leqslant |L|\varphi. \tag{3.5}$$

If additionally $\varphi(0) = 0$ then we obtain

$$L^\#\varphi = P_1\mathcal{A}^\#\left(\begin{smallmatrix} 0 \\ \varphi \end{smallmatrix}\right) = \lim_{t\to 0} \frac{1}{t} P_1 \mathcal{T}^\#(t)\left(\begin{smallmatrix} 0 \\ \varphi \end{smallmatrix}\right). \tag{3.6}$$

Now, let $\psi \in W_p^1(-h,0;X)$, $\psi(0) = 0$. Then $(0,\psi) \in D(\mathcal{A})$, $|\psi| \in W_p^1(-h,0;X)$ (by [12; Theorem 1.1]), and therefore $(0,|\psi|) \in D(\mathcal{A}^\#)$. Now (3.6), the corresponding equality for L and $\mathcal{T}$, and $\left|\mathcal{T}(t)\left(\begin{smallmatrix} 0 \\ \psi \end{smallmatrix}\right)\right| \leqslant \mathcal{T}^\#(t)\left|\left(\begin{smallmatrix} 0 \\ \psi \end{smallmatrix}\right)\right|$ $(t \geqslant 0)$ imply

$$|L\psi| \leqslant L^\#|\psi|. \tag{3.7}$$

We are going to show that (3.5), (3.7) imply equality in (3.5). First observe that, since $\tilde{\eta}$ does not give rise to mass at 0, there exists a sequence (φ_k) in $W_p^1(-h,0;X)$, $\varphi_k(0) = 0$, $0 \leqslant \varphi_k \leqslant \varphi$ $(k \in \mathbb{N})$ such that $|L|\varphi_k \to |L|\varphi$ $(k \to \infty)$. For $k \in \mathbb{N}$ the application of [12; Theorem 1.1 and Remark 1.2] yields

$$\begin{aligned} |L|\varphi_k &= \sup\{|L\psi|\,;\, \psi \in W_p^1(-h,0;X),\ |\psi| \leqslant \varphi_k\} \\ &\leqslant \sup\bigl\{L^\#|\psi|\,;\, \psi \in W_p^1(-h,0;X),\ |\psi| \leqslant \varphi_k\bigr\} \leqslant L^\#\varphi_k \leqslant L^\#\varphi. \end{aligned}$$

For $k \to \infty$ we conclude $|L|\varphi \leqslant L^\#\varphi$.

Having established equality in (3.5) we have shown $\widetilde{\mathcal{A}} \subseteq \mathcal{A}^\#$. Since both of these operators are generators we obtain $\widetilde{\mathcal{A}} = \mathcal{A}^\#$. □

4 The modulus semigroup in the space of continuous functions

We assume that all the quantities are as in Section 1.3. We want to treat the delay semigroup in the space of continuous functions and to show properties analogous to those of the preceding section.

For convenience, we only treat the case $h = 1$ and refer to Remark 4.3(c) for the necessary modifications for $h = \infty$. The delay semigroup $\mathcal{T}_C$ in $C([-1,0];X)$, associated with the Cauchy problem (DE) is generated by the operator $\mathcal{A}_C$,

$$\begin{aligned} D(\mathcal{A}_C) &:= \bigl\{\varphi \in C^1([-1,0];X)\,;\ \varphi(0)\in D(A),\ \varphi'(0) = A\varphi(0) + L\varphi\bigr\}, \\ \mathcal{A}_C\varphi &:= \varphi'; \end{aligned}$$

cf. [6; Chap. VI, Sec. 6].

For the remainder of this section we fix $1 \leqslant p < \infty$. The operator $\mathcal{J}_p\colon C([-1,0];X) \to X \times L_p(-h,0;X)$, $\mathcal{J}_p\varphi := (\varphi(0),\varphi)$, is continuous. For $\varphi \in D(\mathcal{A}_C)$, the function $u(t) := \mathcal{T}_C(t)\varphi$ $(t \geqslant 0)$ is the unique solution of the Cauchy problem for the delay differential equation

$$u'(t) = Au(t) + Lu_t\,, \quad u_0 = \varphi.$$

It is easy to see that this implies that $t \mapsto \mathcal{J}_p u(t) =: u_p(t)$ is a solution of

$$u_p'(t) = \mathcal{A}_p u_p(t)\,, \quad u_p(0) = (\varphi(0),\varphi)\,,$$

and therefore $\mathcal{T}_p(t)\bigl(\begin{smallmatrix}\varphi(0)\\ \varphi\end{smallmatrix}\bigr) = u_p(t)$. These considerations show the following result.

4.1 Proposition. (a) *The semigroups $\mathcal{T}_C$ and $\mathcal{T}_p$ are consistent, in the sense that $\mathcal{J}_p\mathcal{T}_C(t) = \mathcal{T}_p(t)\mathcal{J}_p$ $(t \geqslant 0)$.*

(b) *The semigroups $\widetilde{\mathcal{T}}_C$ and $\widetilde{\mathcal{T}}_p$ are consistent.*

For the proof of Theorem 4.2 below we recall how the modulus semigroup $\mathcal{T}_p^{\#}$ can be obtained. We denote by Γ the set of all subdivisions of 1 by positive reals,

$$\Gamma = \{\gamma \in (0,1]^n;\ \gamma_1 + \cdots + \gamma_n = 1,\ n \in \mathbb{N}\}.$$

For $t \geqslant 0$, $\gamma = (\gamma_1,\ldots,\gamma_n) \in \Gamma$ we define

$$(\mathcal{T}_p)_\gamma(t) := |\mathcal{T}_p(\gamma_n t)| \cdots |\mathcal{T}_p(\gamma_1 t)|$$

and, for $(x,\varphi) \in X_+ \times L_p(-h,0;X)_+$, obtain

$$\mathcal{T}_p^{\#}(t)\bigl(\begin{smallmatrix}x\\ \varphi\end{smallmatrix}\bigr) = \sup_{\gamma\in\Gamma}(\mathcal{T}_p)_\gamma(t)\bigl(\begin{smallmatrix}x\\ \varphi\end{smallmatrix}\bigr) = \lim_{\gamma\in\Gamma}(\mathcal{T}_p)_\gamma(t)\bigl(\begin{smallmatrix}x\\ \varphi\end{smallmatrix}\bigr).$$

These statements are proved in [2; proof of Theorem 2.1].

4.2 Theorem. *$\widetilde{\mathcal{T}}_C$ is the modulus semigroup of $\mathcal{T}_C$.*

Proof. The property that $\widetilde{\mathcal{T}}_p$ dominates $\mathcal{T}_p$ clearly shows that $\widetilde{\mathcal{T}}_C$ dominates $\mathcal{T}_C$ as well. Assume that $\mathcal{S}$ is a C_0-semigroup on $C([-1,0];X)$ dominating $\mathcal{T}_C$.

Let $\varphi \in C([-1,0];X)_+$. Then $\mathcal{S}(s)\varphi \geqslant |\mathcal{T}_C(s)\psi|$ for all $s \geqslant 0$, $\psi \in C([-1,0];X)$, $|\psi| \leqslant \varphi$. This shows $\mathcal{J}_p\mathcal{S}(s)\varphi \geqslant |\mathcal{T}_p(s)|\mathcal{J}_p\varphi$ for all $s \geqslant 0$. Let $t \geqslant 0$. Then, for $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma$, we obtain

$$\begin{aligned}\mathcal{J}_p\mathcal{S}(t)\varphi &= \mathcal{J}_p\mathcal{S}(\gamma_n t)\cdots\mathcal{S}(\gamma_1 t)\varphi \geqslant |\mathcal{T}_p(\gamma_n t)|\mathcal{J}_p\mathcal{S}(\gamma_{n-1}t)\cdots\mathcal{S}(\gamma_1 t)\varphi \\ &\geqslant \cdots \geqslant |\mathcal{T}_p(\gamma_n t)|\cdots|\mathcal{T}_p(\gamma_1 t)|\mathcal{J}_p\varphi = (\mathcal{T}_p)_\gamma(t)\mathcal{J}_p\varphi.\end{aligned}$$

Taking the supremum over $\gamma \in \Gamma$ we conclude

$$\mathcal{J}_p\mathcal{S}(t)\varphi \geqslant \mathcal{T}_p^{\#}(t)\mathcal{J}_p\varphi = \widetilde{\mathcal{T}}_p(t)\mathcal{J}_p\varphi = \mathcal{J}_p\widetilde{\mathcal{T}}_C(t)\varphi,$$

and therefore $\mathcal{S}(t)\varphi \geqslant \widetilde{\mathcal{T}}_C(t)\varphi$.

So we have shown that $\mathcal{S}$ dominates $\widetilde{\mathcal{T}}_C$. This shows the assertion. □

4.3 Remarks. (a) For the case $X = \mathbb{R}^n$, the result of Theorem 4.2 was shown in [2; Proposition 3.3].

(b) The result of Theorem 4.2 is less general than one might hope to show. Namely, in the space of continuous functions, the delay semigroup can be defined under weaker conditions than assumed in the present paper. Indeed, instead of being defined by a function η of bounded variation, one may just assume $L\colon C([-1,0];X) \to X$ to be continuous, in order to obtain the C_0-semigroup $\mathcal{T}$; cf. [6; Chap. VI, Sec. 6]. Then, assuming L to have a modulus $|L|$, and assuming that L and $|L|$ do not have mass at zero, one obtaines that the corresponding C_0-semigroup $\widetilde{\mathcal{T}}_C$ dominates $\mathcal{T}_C$; cf. [7], [2; Proposition 3.2]. Our method of proof does not yield the conjectured result that also in this case the modulus semigroup of $\mathcal{T}_C$ is given by $\widetilde{\mathcal{T}}_C$.

(c) In the case of $h = \infty$ we note that results corresponding to Proposition 4.1 and Theorem 4.2 can be shown in the space

$$C_0((-\infty,0];X) = \{\varphi \in C((-\infty,0];X);\ \lim_{\vartheta\to-\infty}\varphi(\vartheta) = 0\}.$$

In this case the mapping $\mathcal{J}_p$ used above does no longer exist. However, on the dense subspace $C_c((-\infty,0];\,X)$ ($= \{\varphi \in C((-\infty,0];\,X);\ \operatorname{spt}\varphi$ compact$\}$) the mapping $\mathcal{J}_p$ exists, and the restriction of $\mathcal{J}_p$ to subspaces $C_0((\vartheta_0,0];\,X)$ is continuous with respect to the supremum norm, for all $\vartheta_0 \in (-\infty,0)$. Also, $\mathcal{T}_C(t)(C_0((\vartheta_0,0];\,X)) \subseteq C_0((\vartheta_0-t,0];\,X)$ for all $t \geqslant 0$. These observations can be used to carry out the proof in an analogous way as for the case of $h = 1$.

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