A lower bound on the first spectral gap of Schrödinger operators with Kato class measures

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Abstract

We study Schrödinger operators on $\mathbb{R}^n$ formally given by $H_\mu = -\Delta - \mu$, where μ is a positive, compactly supported measure from the Kato class. Under the assumption that a certain condition on the μ-volume of balls is satisfied and that H_μ has at least two eigenvalues below the essential spectrum $\sigma_{\mathrm{ess}}(H_\mu) = [0, \infty)$, we derive a lower bound on the first spectral gap of H_μ. The assumption on the μ-volume of balls is in particular satisfied if μ is of the form $\mu = a\sigma_M$, where M is a compact $(n-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^n$, σ_M the surface measure on M, and $0 \leqslant a \in L_\infty(M)$.

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1 Introduction and main results

There is extensive literature on estimates for the first spectral gap of Schrödinger operators. Many papers concentrate on one of the following two extreme situations: On the one hand, for Schrödinger operators with convex potentials on convex domains the gap turns out to be relatively large (see, e.g., [SWYY85], [Smi96] and the references therein); on the other hand, in tunneling situations, where the potential has two or more separated wells (minima) of equal depth, the gap becomes exponentially small with respect to a separation parameter (see, e.g., [Har78], [Har80], [Sim84]).

For Schrödinger operators on $\mathbb{R}^n$ with bounded potentials, Kirsch and Simon proved in [KiSi87; Sec. 4] that the first spectral gap always admits a lower bound that is comparable to the gap size in tunnelling situations, without particular assumptions on the shape of V. In [KoVe07], Kondej and Veselic proved a similar result in dimension $n = 2$ for potentials given by certain measures supported on compact curves. This is the only gap estimate known to us for Schrödinger operators with a singular interaction given by a measure.

In the present paper we generalise the gap estimate of [KoVe07] to higher dimensions, for operators $H_\mu = -\Delta - \mu$ on $\mathbb{R}^n$ (defined as a form sum, cf. Section 2). We assume that μ is a positive, compactly supported measure on the Borel σ-algebra of $\mathbb{R}^n$ satisfying the volume bound

$$\mu(B(x, r)) \leqslant c_\mu r^{n-\alpha} \qquad (x \in \mathbb{R}^n,\ r > 0) \tag{1.1}$$

for some $c_\mu > 0$, $\alpha \in [0,2)$. (Up to a change in the constant c_μ, we could equivalently assume this bound for $r \leqslant 1$ only since μ has compact support.) Assumption (1.1) implies that μ is in the Kato class; conversely, μ being in the Kato class implies (1.1) for $\alpha = 2$ and some $c_\mu > 0$ (cf. Proposition 2.2). In Example 2.3(b) we will show that the surface measure on a compact $(n-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^n$ satisfies (1.1) with $\alpha = 1$.

The proof of the gap estimate in [KiSi87] uses the fact that the eigenfunctions of the Schrödinger operator are Lipschitz continuous if the potential is bounded. For singular potentials, Lipschitz continuity is no longer true in general. This difficulty is overcome in [KoVe07] by means of a method that relies on detailed knowledge of the geometry of $\operatorname{spt}\mu$, such as curvature bounds. Theorem 1.1 below provides a better (though not sharp) gap estimate, based only on the volume estimate (1.1).

Since μ has compact support, we have $\sigma_{\mathrm{ess}}(H_\mu) = [0,\infty)$ by [BEKŠ94; Thm. 3.1], so $\sigma(H_\mu) \setminus [0,\infty)$ consists of isolated eigenvalues of finite multiplicity. In the main results of this paper, Theorems 1.1 and 1.3 below, we will assume that

(A) H_μ has at least two negative eigenvalues; the lowest two are denoted by $\lambda_0 = -\kappa_0^2$ and $\lambda_1 = -\kappa_1^2\,(>\lambda_0)$.

The proofs of these theorems are given in Section 6.

1.1 Theorem. *Let $n \geqslant 2$ and suppose that $\mu \geqslant 0$ is a compactly supported measure on $\mathbb{R}^n$ satisfying (1.1) for some $c_\mu > 0$, $0 \leqslant \alpha < 2$. Let d denote the diameter of the smallest closed ball containing $\operatorname{spt}\mu$. If assumption (A) is satisfied then there exist $C, p, q, \beta > 0$ depending only on n and α such that*

$$\lambda_1 - \lambda_0 \geqslant \frac{C|\lambda_0|}{(c_\mu+1)^p(d+1)^q} e^{-\beta\kappa_0(d+1)}. \tag{1.2}$$

More precisely, for $\alpha < 1$ one can choose $\beta = n+1$, for $\alpha = 1$ any $\beta > n+1$, and $\beta = \frac{n-1}{2-\alpha} + 2$ for $\alpha > 1$. For $n \geqslant 5$, the factor $|\lambda_0|$ on the right hand side of (1.2) can be omitted.

1.2 Remarks. (a) In [KoVe07; Thm. 4.3], a gap estimate similar to the above is proved for the case $n = 2$, $\mu = c\sigma_{\operatorname{im}\gamma}$, where $c > 0$ and γ is a C^2-curve without self-intersections, parameterised by arc length (cf. Example 2.3(c)). An application of Theorem 1.1 in this situation yields a slightly better estimate: Firstly, in [KoVe07] the constant C depends on the curvature of γ, and β is not given explicitly. Secondly, there is a factor $|\lambda_1|$ in the gap estimate of [KoVe07]; replacing this factor by $|\lambda_0|$ is in fact achieved by a simple trick (see the argument leading to equation (6.3)). Thirdly, in [KoVe07] the estimate contains an additional factor that behaves like $|\lambda_0|^8$ for small $|\lambda_0|$.

(b) One can hardly expect that the value of β given in Theorem 1.1 is sharp. Computable examples of tunneling situations might lead to the conjecture that (1.2) always holds with $\beta = 1$, but it does not seem possible to prove this with the method presented in this paper.

(c) The attentive reader will note that the estimate (1.2) is not scaling invariant: It is easy to see that for $s > 0$ the operators H_μ and H_{μ_s} are unitarily equivalent, where μ_s

is defined by $\mu_s(A) := s^{2-n}\mu(sA)$. Moreover, under the assumptions of Theorem 1.1 one computes that μ_s satisfies (1.1) with $c_{\mu_s} = s^{2-\alpha}c_\mu$, that the smallest closed ball containing $\operatorname{spt}\mu_s$ has diameter d/s, and that the lowest two eigenvalues of H_{μ_s} are $s^2\lambda_0$ and $s^2\lambda_1$. Applying Theorem 1.1 to μ_s, with $s = \varepsilon d$ for some $\varepsilon > 0$, we thus obtain the scaling invariant estimate

$$\lambda_1 - \lambda_0 \geqslant \frac{C_\varepsilon|\lambda_0|}{\big((\varepsilon d)^{2-\alpha}c_\mu + 1\big)^p} e^{-(1+\varepsilon)\beta\kappa_0 d},$$

where $C_\varepsilon = C/(\frac{1}{\varepsilon}+1)^q$.

In dimension $n = 1$ we obtain a much simpler result.

1.3 Theorem. *Let $\mu \geqslant 0$ be a compactly supported measure on $\mathbb{R}$, d the diameter of* $\operatorname{spt}\mu$. *If assumption* (A) *is satisfied then*

$$\lambda_1 - \lambda_0 \geqslant \frac{|\lambda_0|}{d\|\mu\| + 1} e^{-2\kappa_0 d} \quad \textit{and} \quad \lambda_1 - \lambda_0 \geqslant \frac{\kappa_1}{d(\kappa_1 d + 1)} e^{-2\kappa_0 d}. \tag{1.3}$$

The paper is organised as follows. In Section 2 we recall some basic results on form small measures and the (extended) Kato class of measures. In Section 3 we use a ground state transformation to show the representation

$$\lambda_1 - \lambda_0 = \left\|\varphi_0 \nabla \tfrac{\varphi_1}{\varphi_0}\right\|_2^2 \|\varphi_1\|_2^{-2}$$

of the lowest spectral gap, where φ_j is an eigenfunction corresponding to the eigenvalue λ_j, for $j = 1, 2$. We demonstrate how this representation can be used to prove the main results, given the following two ingredients: (i) an estimate of the modulus of continuity of the eigenfunctions, (ii) a pointwise estimate from below for the ground state φ_0 of H_μ. These ingredients are provided in Sections 4 and 5, respectively. Since they are of independent interest, we will prove estimates that are sharper than necessary for the proof of Theorem 1.1. The proofs of Theorems 1.1 and 1.3 are given in Section 6. In the appendix we provide an estimate on the convolution kernel of $(\kappa^2 - \Delta)^{-1}$ (where $\kappa > 0$), needed in Section 5, that we did not find in the literature.

2 Form small measures and Kato class measures

Throughout this section let $\mu \geqslant 0$ be a measure on the Borel σ-algebra of $\mathbb{R}^n$. We recall the definition of the operator H_μ for form small μ and some results on the (extended) Kato class of measures. In the following we write for brevity L_p for $L_p(\mathbb{R}^n)$, and similarly W_2^1, C_c^∞, etc.

The measure μ is called *form small* with respect to the Laplacian on $\mathbb{R}^n$ if μ does not charge sets of zero capacity and there exist $\gamma \in [0, 1)$, $c \in \mathbb{R}$ such that

$$\int |u|^2 \, d\mu \leqslant \gamma \int |\nabla u|^2 \, dx + c \int |u|^2 \, dx \qquad (u \in W_2^1). \tag{2.1}$$

Here and in the following, we tacitly assume that a quasi-continuous representative of u is chosen if we write $u \in W_2^1$; then the integral $\int |u|^2\, d\mu$ is unambiguously defined. It is well-known that, under condition (2.1),

$$D(\tau_\mu) := W_2^1, \qquad \tau_\mu(u) := \int |\nabla u|^2\, dx - \int |u|^2\, d\mu \tag{2.2}$$

defines a closed quadratic form τ_μ in L_2. The domain $D(\tau_\mu)$ is dense in L_2, so we can define the Schrödinger operator H_μ as the selfadjoint operator in L_2 associated with τ_μ.

In accordance with [StVo96; p. 114] we say that μ is in the *extended Kato class* if there exists $\kappa > 0$ such that $G_\kappa * \mu \in L_\infty$, where G_κ is the convolution kernel of the free resolvent $(\kappa^2 - \Delta)^{-1}$. (It is automatic that then μ does not charge sets of zero capacity.) We say that μ is *Kato small* if $\lim_{\kappa\to\infty} \|G_\kappa * \mu\|_\infty < 1$. By [StVo96; Thm. 3.1], a Kato small measure is also form small. The measure μ is in the *(proper) Kato class* if $\|G_\kappa * \mu\|_\infty \to 0$ as $\kappa \to \infty$.

In the following let k_t denote the convolution kernel of $e^{t\Delta}$, i.e., $k_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ for $t > 0$, $x \in \mathbb{R}^n$. Moreover, for an operator B on L_2 and $p, q \in [1,\infty]$ we denote by $\|B\|_{p\to q}$ the norm of $B|_{L_p \cap L_2}$ regarded as an operator from L_p to L_q.

2.1 Proposition. *Let $\alpha > 0$, $\gamma \in [0,1)$ and assume that*

$$\Big\| \int_0^\alpha k_t * \mu\, dt \Big\|_\infty \leqslant \gamma.$$

Then μ is Kato small, and $\|e^{-tH_\mu}\|_{1\to1} \leqslant \frac{1}{1-\gamma}$ for all $t \in [0,\alpha]$.

Proof. Arguing as in [Voi86; Prop. 4.7(b)] one finds $\kappa > 0$ such that

$$\|G_\kappa * \mu\|_\infty = \Big\| \int_0^\infty e^{-\kappa^2 t} k_t * \mu\, dt \Big\|_\infty < 1,$$

so μ is Kato small. We recall the approximation of H_μ given in [StVo96]: For $j, m \in \mathbb{N}$, $m \geqslant \kappa$ let $\mu_j := \mathbb{1}_{B(0,j)}\mu$, $V_{j,m} := (m^2 - \kappa^2) G_m * \mu_j$. Then

$$\begin{aligned}\Big\| \int_0^\alpha e^{t\Delta} V_{j,m}\, dt \Big\|_\infty &= \Big\| (m^2 - \kappa^2) G_m * \int_0^\alpha k_t * \mu_j\, dt \Big\|_\infty \\ &\leqslant \|m^2 (m^2 - \Delta)^{-1}\|_{\infty\to\infty} \Big\| \int_0^\alpha k_t * \mu\, dt \Big\|_\infty \leqslant \gamma,\end{aligned}$$

and by [Voi77; Thm. 1(c) and the last formula line of part (i) of its proof] we obtain that $\|e^{-tH_{V_{j,m}}}\|_{1\to1} \leqslant \frac{1}{1-\gamma}$ for all $t \in [0,\alpha]$, $j, m \in \mathbb{N}$, $m \geqslant \kappa$. Moreover, $H_{V_{j,m}} \to H_{\mu_j}$ in the strong resolvent sense as $m \to \infty$, by [StVo96; Cor. 2.4(a)], and $H_{\mu_j} \to H_\mu$ in the strong resolvent sense as $j \to \infty$, by [StVo96; Thm. 3.3(a)]. We thus conclude that $\|e^{-tH_\mu}\|_{1\to1} \leqslant \frac{1}{1-\gamma}$ for all $t \in [0,\alpha]$; cf. [StVo96; proof of Cor. 2.4(b)]. ☐

In the next result we show the relation between the Kato class condition and the volume bound (1.1). We will need the following fact: Assume that $m\colon [0,\infty) \to [0,\infty)$ is increasing, $m(0) = 0$, $\mu(B(x,r)) \leqslant m(r)$ for all $x \in \mathbb{R}^n$, $r > 0$. Then for any decreasing function $f\colon [0,\infty) \to [0,\infty)$ and all $x \in \mathbb{R}^n$, one can estimate

$$\int_{\mathbb{R}^n} f(|x-y|)\, d\mu(y) \leqslant \int_0^\infty f(r)\, dm(r); \tag{2.3}$$

cf. [Dav95; p. 179].

2.2 Proposition. *Let $0 \leqslant \alpha \leqslant 2$. Assume that there exists $c \geqslant 0$ such that*

$$\Big\| \int_0^t k_s * \mu \, ds \Big\|_\infty \leqslant ct^{1-\frac{\alpha}{2}} \qquad (t > 0). \tag{2.4}$$

Then the volume estimate (1.1) holds with $c_\mu = c/g(e_1)$, where $g(x) := \int_0^1 k_s(x)\, ds$. Conversely, if $\alpha < 2$ ($\alpha \leqslant 1$ in the case $n = 1$) then (1.1) implies (2.4) with $c = \frac{2}{2-\alpha} 2^{-\alpha} \pi^{-\frac{n}{2}} \Gamma(\frac{n-\alpha}{2} + 1) c_\mu$; in particular, μ is in the Kato class.

Proof. Let $r > 0$. For $y \in \mathbb{R}^n \setminus \{0\}$ we compute, substituting $s = r^2 t$, that

$$\int_0^{r^2} k_s(y)\, ds = \int_0^1 (4\pi r^2 t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4r^2 t}}\, r^2 dt = r^{2-n} \int_0^1 k_t\big(\tfrac{y}{r}\big)\, dt = r^{2-n} g\big(\tfrac{y}{r}\big) =: g_r(y).$$

Since $g_r * \mu$ is lower semicontinuous, we thus obtain from assumption (2.4) that $\sup(g_r * \mu) \leqslant cr^{2-\alpha}$. Moreover, $g_r(y) \geqslant g_r(re_1) = r^{2-n} g(e_1)$ for $0 < |y| \leqslant r$, so we infer for all $x \in \mathbb{R}^n$ that

$$r^{2-n} g(e_1) \mu(B(x,r)) \leqslant \int_{B(x,r)} g_r(x-y)\, d\mu(y) \leqslant g_r * \mu(x) \leqslant cr^{2-\alpha},$$

and the first assertion follows.

Conversely, assume that (1.1) holds and that $\alpha < 2$. Then for $s > 0$ we obtain by (2.3), substituting $r = (4s\rho)^{\frac{1}{2}}$, that

$$\begin{aligned} \|k_s * \mu\|_\infty &\leqslant \int_0^\infty (4\pi s)^{-\frac{n}{2}} e^{-\frac{r^2}{4s}} c_\mu \, dr^{n-\alpha} = c_\mu (4\pi s)^{-\frac{n}{2}} \int_0^\infty e^{-\rho} (4s)^{\frac{n-\alpha}{2}} d\rho^{\frac{n-\alpha}{2}} \\ &= c_\mu (4s)^{-\frac{\alpha}{2}} \pi^{-\frac{n}{2}} \Gamma(\tfrac{n-\alpha}{2} + 1). \end{aligned}$$

(This computation is also valid in the case $n = \alpha = 1$ if one replaces $dr^{n-\alpha}$ and $d\rho^{\frac{n-\alpha}{2}}$ with the Dirac measure at 0.) Integration from 0 to t now yields the second assertion. $\square$

If $n = 1$ and μ is finite, then assumption (1.1) is trivially satisfied with $\alpha = 1$ and $c_\mu = \|\mu\|$. We now give examples in dimension $n \geqslant 2$ where assumption (1.1) is satisfied with $\alpha = 1$.

2.3 Example. (a) (cf. [BEKŠ94; Thm. 4.1(iv)]) Let $U \subseteq \mathbb{R}^{n-1}$ be open and bounded, $f\colon \overline{U} \to \mathbb{R}^n$ Lipschitz continuous, and

$$|f(x) - f(y)| \geqslant |x - y| \qquad (x, y \in \overline{U}).$$

Let $\mu = \sigma_{f(U)}$, i.e.,

$$\int \varphi \, d\mu = \int_U \varphi(f(y))g(y)\, dy \qquad (\varphi \in C_{\rm c}(\mathbb{R}^n)),$$

with $g := \big(\det(f'^\top f')\big)^{\frac{1}{2}}$ the square root of the Gram determinant. We show that then (1.1) holds with

$$\alpha = 1,\ c_\mu = 2^{n-1}\omega_{n-1}\|g\|_\infty,$$

where ω_{n-1} denotes the volume of the $(n-1)$-dimensional unit ball.

Indeed, let $x \in \mathbb{R}^n,\ r > 0$. There exists $y_0 \in \overline{U}$ such that $r_0 := \operatorname{dist}(x, f(\overline{U})) = |x - f(y_0)|$. If $r < r_0$ then $\mu(B(x,r)) = 0 \leqslant c_\mu r^{n-1}$. If $r \geqslant r_0$ then for $y \in U$ with $f(y) \in B(x,r)$ we have

$$|y - y_0| \leqslant |f(y) - f(y_0)| \leqslant |f(y) - x| + |x - f(y_0)| \leqslant r + r_0 \leqslant 2r,$$

and it follows that

$$\mu(B(x,r)) = \int_U \mathbb{1}_{B(x,r)}(f(y))g(y)\, dy \leqslant \int_{B(y_0,2r)} \|g\|_\infty \, dy = \omega_{n-1}(2r)^{n-1}\|g\|_\infty.$$

(b) Let M be a compact $(n-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^n$, σ_M the surface measure on M and $0 \leqslant a \in L_\infty(M)$. Then M can be covered by finitely many relatively open subsets that can be parameterised as in (a), so $\mu = a\sigma_M$ satisfies (1.1) with $\alpha = 1$.

(c) Let $n = 2,\ N \in \mathbb{N},\ \gamma_1, \dots, \gamma_N$ Lipschitz curves in $\mathbb{R}^2$, $\gamma_j \colon I_j \to \mathbb{R}^2$ parameterised by arc length on a compact interval $I_j \subseteq \mathbb{R}$ $(j = 1, \dots, N)$, and assume that $|\gamma_j(s) - \gamma_j(t)| \geqslant \frac{1}{2}|s - t|$ for all $j \in \{1, \dots, N\},\ s,t \in I_j$. In particular, each curve γ_j is intersection free, but the different curves may intersect. If, e.g., $\gamma \colon I \to \mathbb{R}^2$ is a curve (with a compact interval $I \subseteq \mathbb{R}$) that is piecewise C^1 and parameterised by arc length, then γ can be split into finitely many parts $\gamma_1, \dots, \gamma_N$ satisfying the above.

Let now

$$\Gamma := \sum_{j=1}^N \gamma_j, \quad a \in L_\infty(\operatorname{im}\Gamma), \quad \mu := a\sigma_{\operatorname{im}\Gamma}.$$

By part (a), applied with $f_j(t) := \gamma_j(2t)$, we obtain that then (1.1) holds with $\alpha = 1$, $c_\mu = N\|a\|_\infty 2^1\omega_1 \cdot 2 = 8N\|a\|_\infty$. It follows that the spectral gap estimate of [KoVe07] can be obtained as a special case of Theorem 1.1.

The following representation of the eigenfunctions of H_μ is extracted from [BEKŠ94; proof of Cor. 2.3]. We include the proof for the reader's convenience.

2.4 Lemma. *Assume that μ is form small, φ an eigenfunction of H_μ, $H_\mu\varphi = -\kappa^2\varphi$ for some $\kappa > 0$. Then $\varphi = G_\kappa * (\varphi\mu)$.*

Proof. The form smallness of μ implies that $D(H_\mu) \subseteq W_2^1 \subseteq L_2(\mu)$, so $\varphi \in L_2(\mu)$. By [BEKŠ94; Lemma 2.2] we obtain that $u := G_\kappa * (\varphi\mu) \in W_2^1$ and

$$\langle u, v\rangle_\kappa := \langle \nabla u, \nabla v\rangle + \kappa^2 \langle u, v\rangle = \int \varphi \overline{v}\, d\mu \qquad (v \in W_2^1). \tag{2.5}$$

Moreover, for $v \in W_2^1$ we have

$$0 = \langle (H_\mu + \kappa^2)\varphi, v\rangle = (\tau_\mu + \kappa^2)(\varphi, v) = \langle \varphi, v\rangle_\kappa - \int \varphi \overline{v}\, d\mu. \tag{2.6}$$

Combining (2.5) and (2.6) we conclude that $\langle \varphi, v\rangle_\kappa = \langle u, v\rangle_\kappa$ for all $v \in W_2^1$ and therefore $\varphi = u$. □

We use the above representation for proving the following estimates of the L_2-norm against the L_∞-norm of the eigenfunctions of H_μ (cf. [KoVe07; Lemma 5.5]); these estimates will be needed in the proofs of the main results.

2.5 Proposition. *Assume that μ is form small, φ a bounded eigenfunction of H_μ, $H_\mu\varphi = -\kappa^2\varphi$ for some $\kappa > 0$.*
(a) *If μ is finite then $\|\varphi\|_2^2 \leqslant \kappa^{-2}\|\mu\|\|\varphi\|_\infty^2$.*
(b) *If $n = 1$ and $\operatorname{spt}\mu \subseteq B[0, R]$ for some $R > 0$ then $\|\varphi\|_2^2 \leqslant (2R + \frac{1}{\kappa})\|\varphi\|_\infty^2$.*
(c) *If $n \geqslant 5$ and $\operatorname{spt}\mu \subseteq B[0, R]$ for some $R > 0$ then $\|\varphi\|_2^2 \leqslant \frac{2n-4}{n-4}\omega_n R^n \|\varphi\|_\infty^2$.*

Proof. Without loss of generality assume that $\|\varphi\|_\infty = 1$. By Lemma 2.4 we have $\varphi = G_\kappa * (\varphi\mu)$.

(a) Since $\|G_\kappa\|_1 = \|(\kappa^2 - \Delta)^{-1}\|_{1\to 1} = \kappa^{-2}$, we obtain that

$$\|\varphi\|_2^2 \leqslant \|\varphi\|_1 \|\varphi\|_\infty = \|G_\kappa * (\varphi\mu)\|_1 \leqslant \kappa^{-2}\|\mu\|.$$

(b) Outside $\operatorname{spt}\mu$ we have $(\kappa^2 - \Delta)\varphi = 0$ and hence $(\kappa^2 - \Delta)|\varphi| \leqslant 0$ in the distributional sense, i.e., $|\varphi|$ is a subsolution of the equation $(\kappa^2 - \Delta)u = 0$. Moreover, $u(x) := e^{-\kappa(|x| - R)}$ defines a solution of this equation on $\mathbb{R} \setminus \{0\}$, and $|\varphi| \leqslant u$ on $[-R, R]$. Since $\varphi = G_\kappa * (\varphi\mu)$ vanishes at ∞, we conclude that $|\varphi| \leqslant u$ on $\mathbb{R}$ and therefore

$$\|\varphi\|_2^2 \leqslant 2\int_0^R 1\, dx + 2\int_R^\infty e^{-2\kappa(x-R)}\, dx = 2R + \tfrac{1}{\kappa}.$$

(c) Arguing as in (b) we obtain that $|\varphi|$ is subharmonic on $\mathbb{R}^n \setminus B[0, R]$ and vanishes at ∞. Therefore, $|\varphi(x)| \leqslant \big(\frac{R}{|x|}\big)^{n-2}$ for $|x| \geqslant R$ and hence

$$\begin{aligned}\|\varphi\|_2^2 &\leqslant \int \big(\tfrac{R}{|x|} \wedge 1\big)^{2(n-2)}\, dx = \omega_n R^n + n\omega_n \int_R^\infty R^{2(n-2)} r^{-2(n-2)}\, r^{n-1} dr \\ &= \omega_n R^n + n\omega_n R^{2(n-2)} \tfrac{1}{n-4} R^{4-n} = \big(1 + \tfrac{n}{n-4}\big)\omega_n R^n.\end{aligned}$$

□

3 The ground state transformation

Throughout this section, $\mu \geqslant 0$ is a Kato class measure on $\mathbb{R}^n$. It follows from [BlMa90; Thm. 3.2.(ii)] that then the eigenfunctions of H_μ are in $C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at ∞.

Assume that $\lambda_0 := \inf \sigma(H_\mu) < 0$ is an eigenvalue of H_μ. It is well-known that λ_0 is non-degenerate and that a corresponding eigenfunction φ_0 can be chosen such that $\varphi_0 \geqslant 0$. Lemma 2.4 implies that then $\inf_{B[0,R]} \varphi_0 > 0$ for all $R > 0$. We define the unitarily transformed (and shifted) form $\widetilde{\tau}_\mu$ in $L_2(\varphi_0^2) := L_2(\mathbb{R}^n, \varphi_0^2 dx)$ by

$$D(\widetilde{\tau}_\mu) := \bigl\{u \in L_2(\varphi_0^2);\, \varphi_0 u \in D(\tau_\mu)\bigr\}, \quad \widetilde{\tau}_\mu(u) := (\tau_\mu - \lambda_0)(\varphi_0 u).$$

In the case $\mu = V dx$, with V from the Kato class, the following result is already proved in [DaSi84; Prop. 4.4S]; see [KoVe07; Thm. 3.1] for the case $\mu = c\sigma_M$, with $M \subseteq \mathbb{R}^n$ a compact C^2-manifold of codimension 1 (cf. Example 2.3(b)).

3.1 Proposition. *For all $u \in D(\widetilde{\tau}_\mu)$ one has $\nabla u \in L_2(\varphi_0^2)^n$ and*

$$\widetilde{\tau}_\mu(u) = \int |\nabla u|^2 \varphi_0^2 dx. \tag{3.1}$$

Proof. We first assume that $u \in W_2^1 \cap L_\infty$. Then $\varphi_0 u \in W_2^1$ and $\varphi_0 u\overline{u} \in W_2^1$ since $\varphi_0 \in W_2^1 \cap L_\infty$. Therefore, by the product rule,

$$\int \nabla(\varphi_0 u) \cdot \nabla(\overline{\varphi_0 u})\, dx = \int |\nabla u|^2 \varphi_0^2 dx + \int \nabla\varphi_0 \cdot \nabla(\varphi_0 u\overline{u})\, dx,$$

and hence

$$\begin{aligned}
\widetilde{\tau}_\mu(u) - \int |\nabla u|^2 \varphi_0^2 dx &= \tau_\mu(\varphi_0 u) - \lambda_0 \int |\varphi_0 u|^2\, dx - \int |\nabla u|^2 \varphi_0^2 dx \\
&= \int \nabla\varphi_0 \cdot \nabla(\varphi_0 u\overline{u})\, dx - \int \varphi_0 \cdot \varphi_0 u\overline{u}\, d\mu - \lambda_0 \int \varphi_0 \cdot \varphi_0 u\overline{u}\, dx \\
&= (\tau_\mu - \lambda_0)(\varphi_0, \varphi_0\overline{u}u) = \int (H_\mu - \lambda_0)\varphi_0 \cdot \varphi_0 u\overline{u}\, dx = 0.
\end{aligned}$$

Thus we have shown the assertion for $u \in D := W_2^1 \cap L_\infty$.

Now observe that $D \supseteq \varphi_0^{-1} C_c^\infty$ ($\psi \in C_c^\infty$ implies $\varphi_0^{-1}\psi \in D$ since $\varphi_0^{-1} \in W_{2,\mathrm{loc}}^1 \cap L_{\infty,\mathrm{loc}}$). Since C_c^∞ is a core for τ_μ, it follows that D is a core for $\widetilde{\tau}_\mu$. Let $u \in D(\widetilde{\tau}_\mu)$, $(u_k) \subseteq D$, $u_k \to u$ in $D(\widetilde{\tau}_\mu)$. Then $u_k \to u$ in $L_2(\varphi_0^2)$, and by (3.1) applied to $u_k - u_{k'} \in D$ we obtain that $(\nabla u_k)_k$ is a Cauchy sequence in $L_2(\varphi_0^2)^n$. This implies $\nabla u \in L_2(\varphi_0^2)^n$ in the distributional sense, $\nabla u_k \to \nabla u$ in $L_2(\varphi_0^2)^n$, and we conclude that (3.1) holds for all $u \in D(\widetilde{\tau}_\mu)$. ☐

In the following assume in addition that μ has compact support and that $\lambda_1 := \inf\bigl(\sigma(H_\mu) \setminus \{\lambda_0\}\bigr) < 0$; then λ_1 is an eigenvalue of H_μ since $\sigma_{\mathrm{ess}}(H_\mu) = [0, \infty)$ by [BEKŠ94; Thm. 3.1]. Let φ_1 be an associated eigenfunction, φ_1 real-valued. By Proposition 3.1 we then obtain

$$(\lambda_1 - \lambda_0)\|\varphi_1\|_2^2 = (\tau_\mu - \lambda_0)(\varphi_1) = \widetilde{\tau}_\mu\bigl(\tfrac{\varphi_1}{\varphi_0}\bigr) = \bigl\|\varphi_0 \nabla \tfrac{\varphi_1}{\varphi_0}\bigr\|_2^2. \tag{3.2}$$

We now describe how this formula can be used for the estimation of $\lambda_1 - \lambda_0$. The ansatz is largely the same as in [KiSi87] and [KoVe07]; we will indicate the differences below.

As in [KiSi87; p. 405] we normalise φ_0, φ_1 such that $\|\varphi_0\|_\infty = \|\varphi_1\|_\infty = 1$, $\sup \varphi_1 = 1$. Note that then $\inf \varphi_1 < 0$ since φ_0, φ_1 are orthogonal. Since $\varphi_1 \in C_0(\mathbb{R}^n)$, there exist $x_0, x_1 \in \mathbb{R}^n$ such that $\varphi_1(x_0) = \min \varphi_1$, $\varphi_1(x_1) = \max \varphi_1$. By Lemma 2.4 we have $\varphi_1 = G_{\sqrt{|\lambda_1|}} * (\varphi_1 \mu)$, and hence $(|\lambda_1| - \Delta)\varphi_1 = 0$ on $\mathbb{R}^n \setminus \operatorname{spt} \mu$. This implies that φ_1 has no positive maxima and no negative minima outside $\operatorname{spt} \mu$, and thus $x_0, x_1 \in \operatorname{spt} \mu$.

In the following we first assume that $n \geqslant 2$. Let R be the radius of the smallest closed ball containing $\operatorname{spt} \mu$; for simplicity suppose that $\operatorname{spt} \mu \subseteq B[0, R]$. Let $\varepsilon := \frac{1}{4} \inf_{B(0,R+1)} \varphi_0$. Then $0 < \varepsilon \leqslant \frac{1}{4}$. Since $\varphi_1 \in C_0(\mathbb{R}^n)$, there exists $\delta \in (0, 1]$ such that $|\varphi_1(x) - \varphi_1(y)| \leqslant \varepsilon$ for $|x - y| \leqslant \delta$. (Explicit estimates from below for ε and δ will be given in the next two sections.) It follows that

$$\tfrac{\varphi_1}{\varphi_0} \leqslant \tfrac{\varepsilon}{\varphi_0} \leqslant \tfrac{1}{4} \quad \text{on } B(x_0, \delta), \qquad \tfrac{\varphi_1}{\varphi_0} \geqslant \varphi_1 \geqslant 1 - \varepsilon \geqslant \tfrac{3}{4} \quad \text{on } B(x_1, \delta). \tag{3.3}$$

In [KiSi87] and [KoVe07], similar estimates were proved by means of gradient estimates on φ_0 and φ_1.

Let now $\widetilde{T}$ be the convex hull of $B(x_0, \delta) \cup B(x_1, \delta)$, and $T := \big\{x \in \widetilde{T};\ 0 \leqslant \langle x - x_0, x_1 - x_0\rangle \leqslant |x_1 - x_0|^2\big\}$. Then T is a tube connecting the two points x_0 and x_1. By the Cauchy-Schwarz inequality we obtain from (3.2) that

$$(\lambda_1 - \lambda_0)\|\varphi_1\|_2^2 \geqslant \int_T \varphi_0^2 \big|\nabla \tfrac{\varphi_1}{\varphi_0}\big|^2 \geqslant \frac{1}{|T|} \inf_T \varphi_0^2 \Big(\int_T \big|\nabla \tfrac{\varphi_1}{\varphi_0}\big|\Big)^2,$$

where $|T|$ denotes the volume of T. (This estimate leads to better results than the corresponding estimate in [KiSi87; eq. (4.4)] and [KoVe07; eq. (16)].) Using the fundamental theorem of calculus, we infer from (3.3) that $\int_T \big|\nabla \frac{\varphi_1}{\varphi_0}\big| \geqslant \frac{1}{2}\omega_{n-1}\delta^{n-1}$. Moreover, $|T| = |x_1 - x_0|\omega_{n-1}\delta^{n-1}$, so we conclude that

$$(\lambda_1 - \lambda_0)\|\varphi_1\|_2^2 \geqslant \frac{1}{2R\omega_{n-1}\delta^{n-1}} (4\varepsilon)^2 \big(\tfrac{1}{2}\omega_{n-1}\delta^{n-1}\big)^2 = \frac{2\omega_{n-1}}{R}\varepsilon^2\delta^{n-1}. \tag{3.4}$$

For $n = 1$, the above estimation simplifies considerably: If $\operatorname{spt} \mu \subseteq [-R, R]$ then from (3.2) we obtain, again using the Cauchy-Schwarz inequality and the fundamental theorem of calculus, that

$$(\lambda_1 - \lambda_0)\|\varphi_1\|_2^2 \geqslant \int_{-R}^{R} \varphi_0^2 \big|\nabla \tfrac{\varphi_1}{\varphi_0}\big|^2 \geqslant \frac{1}{2R} \inf_{[-R,R]} \varphi_0^2 \Big(\int_{-R}^{R} \big|\nabla \tfrac{\varphi_1}{\varphi_0}\big|\Big)^2 \geqslant \frac{1}{2R} \inf_{[-R,R]} \varphi_0^2. \tag{3.5}$$

4 Hölder continuity of the eigenfunctions

Throughout this section we assume that $n \geqslant 2$ and that $\mu \geqslant 0$ is a Kato class measure on $\mathbb{R}^n$. Then the eigenfunctions of H_μ are uniformly continuous since they are in $C_0(\mathbb{R}^n)$ by [BlMa90]. In the main result of this section, Theorem 4.3 below, we will show that the eigenfunctions are Hölder continuous if the volume bound (1.1) is satisfied for some $c_\mu > 0$, $\alpha \in [0, 2)$, with a Hölder exponent depending on α.

We start with an estimate of the modulus of continuity of the eigenfunctions of H_μ that is rather simple but holds in great generality. Let G_0 be the fundamental solution of $-\Delta$ given by

$$G_0(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x|} & \text{if } n = 2, \\ \frac{1}{(n-2)\sigma_{n-1}} |x|^{2-n} & \text{if } n \geqslant 3. \end{cases}$$

For $r_0 > 0$ let

$$\rho_{1,r_0} := \big(G_0 - G_0(r_0 e_1)\big)^+, \quad \rho_{2,r_0} := |\nabla(G_0 - \rho_{1,r_0})|.$$

Then $\rho_{2,r_0}(x) = \frac{1}{\sigma_{n-1}}|x|^{1-n}$ for $|x| > r_0$, $\rho_{2,r_0}(x) = 0$ otherwise.

4.1 Proposition. *Let φ be an eigenfunction of H_μ, $H_\mu \varphi = -\kappa^2 \varphi$ for some $\kappa > 0$. Let $\varepsilon > 0$. If $r_0 > 0$ satisfies $\|\rho_{1,r_0} * \mu\|_\infty \leqslant \frac{\varepsilon}{4}$ then*

$$|\varphi(x) - \varphi(y)| \leqslant \varepsilon \|\varphi\|_\infty \quad \text{for all } x, y \in \mathbb{R}^n \text{ such that } |x - y| \leqslant \tfrac{\varepsilon}{2} \|\rho_{2,r_0} * \mu\|_\infty^{-1}.$$

Proof. Let $g_1 := \big(G_\kappa - G_\kappa(r_0 e_1)\big)^+$, $g_2 := G_\kappa - g_1$. By Lemma 2.4 we have

$$\varphi = G_\kappa * (\varphi\mu) = (g_1 + g_2) * (\varphi\mu).$$

Since $g_1 * (\varphi\mu) = \varphi - g_2 * (\varphi\mu)$ is continuous, we can estimate $|g_1 * (\varphi\mu)(x) - g_1 * (\varphi\mu)(y)| \leqslant 2\|g_1 * (\varphi\mu)\|_\infty$ and hence

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leqslant |g_2 * (\varphi\mu)(x) - g_2 * (\varphi\mu)(y)| + 2\|g_1 * (\varphi\mu)\|_\infty \\ &\leqslant |x - y| \cdot \||\nabla g_2| * (\varphi\mu)\|_\infty + 2\|g_1 * (\varphi\mu)\|_\infty \\ &\leqslant \big(|x - y| \cdot \||\nabla g_2| * \mu\|_\infty + 2\|g_1 * \mu\|_\infty\big)\|\varphi\|_\infty \end{aligned} \tag{4.1}$$

for all $x, y \in \mathbb{R}^n$. Observe that

$$\nabla G_\kappa(z) = \int_0^\infty (4\pi t)^{-\frac{n}{2}} \big(-\tfrac{2z}{4t}\big) e^{-\frac{|z|^2}{4t} - \kappa^2 t} dt$$

for all $z \in \mathbb{R}^n \setminus \{0\}$, and that this formula also holds for $\kappa = 0$. (For $n \geqslant 3$ the latter is clear; for $n = 2$ it can easily be seen by computing the integral.) We infer that $|\nabla g_2| \leqslant \rho_{2,r_0}$ and $g_1 \leqslant \rho_{1,r_0}$. By the assumption on r_0 we thus obtain from (4.1) that

$$|\varphi(x) - \varphi(y)| \leqslant \big(|x - y| \cdot \|\rho_{2,r_0} * \mu\|_\infty + 2 \cdot \tfrac{\varepsilon}{4}\big)\|\varphi\|_\infty,$$

and the assertion follows. ☐

4.2 Remarks. (a) The above estimate of the modulus of continuity does not depend on κ. It is clear that for large κ one obtains a better estimate using (4.1) without further estimating g_1 and $|\nabla g_2|$.

(b) Let $r_0 > 0$ be as in Proposition 4.1 and assume that μ is finite. Obviously, $\|\rho_{2,r_0} * \mu\|_\infty \leqslant \|\rho_{2,r_0}\|_\infty \|\mu\| = \frac{1}{\sigma_{n-1}} r_0^{1-n} \|\mu\|$, so it follows that

$$|\varphi(x) - \varphi(y)| \leqslant \varepsilon \|\varphi\|_\infty \quad \text{if } |x - y| \leqslant \frac{\sigma_{n-1}}{2\|\mu\|} \varepsilon r_0^{n-1}.$$

For the proof of Theorem 1.1 we will need the following application of Proposition 4.1. In the case of Example 2.3(c) (where $n = 2$ and $\alpha = 1$), a related result is given in [KoVe07; Prop. 6.8].

4.3 Theorem. *Assume that the measure μ is finite and that (1.1) holds for some $c_\mu > 0$, $0 \leqslant \alpha < 2$. Let φ be an eigenfunction of H_μ and $\varepsilon > 0$. Then*

$$|\varphi(x) - \varphi(y)| \leqslant \varepsilon \|\varphi\|_\infty \quad \text{for all } |x - y| \leqslant \delta,$$

where

$$\delta = \begin{cases} \frac{\sigma_{n-1}}{2} \frac{1-\alpha}{n-\alpha} \|\mu\|^{-\frac{1-\alpha}{n-\alpha}} c_\mu^{-\frac{n-1}{n-\alpha}} \varepsilon & \text{if } \alpha < 1, \\ \frac{\sigma_{n-1}}{2n} \frac{\varepsilon}{c_\mu} \Big[\ln\big(1 + \frac{4}{\sigma_{n-1}} \|\mu\|^{\frac{1}{n-1}} c_\mu^{\frac{n-2}{n-1}} \frac{1}{\varepsilon}\big)\Big]^{-1} & \text{if } \alpha = 1, \\ \frac{1}{2} (\sigma_{n-1})^{\frac{1}{2-\alpha}} \frac{\alpha-1}{n-1} \big(\frac{2-\alpha}{4}\big)^{\frac{\alpha-1}{2-\alpha}} \big(\frac{\varepsilon}{c_\mu}\big)^{\frac{1}{2-\alpha}} & \text{if } \alpha > 1. \end{cases}$$

Proof. Let $r_0 > 0$. In order to apply Proposition 4.1, we have to estimate $\|\rho_{1,r_0} * \mu\|_\infty$ and $\|\rho_{2,r_0} * \mu\|_\infty$. Let $m(r) := c_\mu r^{n-\alpha}$ $(r \geqslant 0)$. Using (2.3) we estimate

$$\begin{aligned} \|\rho_{1,r_0} * \mu\|_\infty &\leqslant \int_0^{r_0} \rho_{1,r_0}(re_1)\, dm(r) = \rho_{1,r_0}(re_1) m(r)\big|_{0+}^{r_0} - \int_0^{r_0} m(r)\, d\rho_{1,r_0}(r) \\ &= \int_0^{r_0} m(r) \tfrac{1}{\sigma_{n-1}} r^{1-n}\, dr = \tfrac{c_\mu}{\sigma_{n-1}} \int_0^{r_0} r^{1-\alpha}\, dr = \tfrac{c_\mu}{\sigma_{n-1}(2-\alpha)} r_0^{2-\alpha}. \end{aligned}$$

Setting

$$r_0 := \big(\tfrac{2-\alpha}{4} \sigma_{n-1} \tfrac{\varepsilon}{c_\mu}\big)^{\frac{1}{2-\alpha}} \tag{4.2}$$

we thus obtain $\|\rho_{1,r_0} * \mu\|_\infty \leqslant \frac{\varepsilon}{4}$, so

$$|\varphi(x) - \varphi(y)| \leqslant \varepsilon \|\varphi\|_\infty \quad \text{for all } |x - y| \leqslant \tfrac{\varepsilon}{2} \|\rho_{2,r_0} * \mu\|_\infty^{-1} \tag{4.3}$$

by Proposition 4.1.

Recall that $\sigma_{n-1} \rho_{2,r_0}(x) \leqslant (|x| \vee r_0)^{1-n}$ for all $x \in \mathbb{R}^n$. In the case $\alpha > 1$ we obtain by (2.3) that

$$\begin{aligned} \sigma_{n-1} \|\rho_{2,r_0} * \mu\|_\infty &\leqslant \int_0^\infty (r \vee r_0)^{1-n}\, dm(r) = r_0^{1-n} m(r_0) + \int_{r_0}^\infty r^{-\alpha} (n-\alpha) c_\mu\, dr \\ &= c_\mu r_0^{1-\alpha} + c_\mu \tfrac{n-\alpha}{\alpha-1} r_0^{1-\alpha} = c_\mu \tfrac{n-1}{\alpha-1} r_0^{1-\alpha}, \end{aligned}$$

hence

$$\tfrac{\varepsilon}{2} \|\rho_{2,r_0} * \mu\|_\infty^{-1} \geqslant \tfrac{1}{2} \tfrac{\alpha-1}{n-1} \sigma_{n-1} \tfrac{\varepsilon}{c_\mu} r_0^{\alpha-1} = \tfrac{1}{2} \tfrac{\alpha-1}{n-1} \big(\tfrac{2-\alpha}{4}\big)^{\frac{\alpha-1}{2-\alpha}} \big(\sigma_{n-1} \tfrac{\varepsilon}{c_\mu}\big)^{1+\frac{\alpha-1}{2-\alpha}}$$

by (4.2), and the assertion follows from (4.3).

Let now $r_1 := \big(\frac{\|\mu\|}{c_\mu}\big)^{\frac{1}{n-\alpha}}$. Then $m(r_1) = c_\mu r_1^{n-\alpha} = \|\mu\|$ and hence, again by (2.3),

$$\sigma_{n-1} \|\rho_{2,r_0} * \mu\|_\infty \leqslant \int_0^{r_1} (r \vee r_0)^{1-n}\, dm(r).$$

In the case $\alpha < 1$ we estimate

$$\sigma_{n-1}\|\rho_{2,r_0} * \mu\|_\infty \leqslant \int_0^{r_1} r^{1-n}\,dm(r) = \int_0^{r_1} r^{-\alpha}(n-\alpha)c_\mu\,dr = c_\mu \tfrac{n-\alpha}{1-\alpha} r_1^{1-\alpha} = \tfrac{n-\alpha}{1-\alpha}\|\mu\|^{\frac{1-\alpha}{n-\alpha}} c_\mu^{\frac{n-1}{n-\alpha}},$$

so as above the assertion follows from (4.3).

Finally, let $\alpha = 1$. Then

$$\begin{aligned}\sigma_{n-1}\|\rho_{2,r_0} * \mu\|_\infty &\leqslant r_0^{1-n} m(r_0 \wedge r_1) + \int_{r_0\wedge r_1}^{r_1} r^{-1}(n-1)c_\mu\,dr \\ &= r_0^{1-n} c_\mu (r_0\wedge r_1)^{n-1} + (n-1)c_\mu \ln \tfrac{r_1}{r_0\wedge r_1} \\ &= c_\mu\Big[\big(\tfrac{r_1}{r_0}\wedge 1\big)^{n-1} + (n-1)\ln\big(\tfrac{r_1}{r_0}\vee 1\big)\Big] = c_\mu f\big(\tfrac{r_1}{r_0}\big),\end{aligned}$$

with $f(x) := (x\wedge 1)^{n-1} + (n-1)\ln(x \vee 1)$. We show that $f(x) \leqslant n\ln(1+x)$ by distinguishing the cases $x < 1$ and $x \geqslant 1$: If $x < 1$ then

$$f(x) = x^{n-1} + 0 \leqslant x \leqslant \tfrac{1}{\ln 2}\ln(1+x) \leqslant n\ln(1+x)$$

since $n \geqslant 2$. If $x \geqslant 1$ then $f(x) = 1 + (n-1)\ln x$, so we have to show that $h_n(x) := n\ln(1+x) - (n-1)\ln x \geqslant 1$. Observe that $h_n(x) \geqslant h_2(x)$ and that h_2 is increasing on $[1,\infty)$, hence $h_n(x) \geqslant h_2(1) = 2\ln 2 - \ln 1 \geqslant 1$. We conclude that

$$\|\rho_{2,r_0} * \mu\|_\infty \leqslant \tfrac{c_\mu}{\sigma_{n-1}} f\big(\tfrac{r_1}{r_0}\big) \leqslant \tfrac{nc_\mu}{\sigma_{n-1}} \ln\big(1 + \tfrac{r_1}{r_0}\big).$$

By (4.3) this implies the assertion since $r_1 = \big(\frac{\|\mu\|}{c_\mu}\big)^{\frac{1}{n-1}}$ and $r_0 = \frac{\sigma_{n-1}}{4}\frac{\varepsilon}{c_\mu}$. □

4.4 Remark. Theorem 4.3 implies that the eigenfunctions of H_μ are Lipschitz continuous if $\alpha < 1$. In general, the eigenfunctions are not Lipschitz continuous if $\alpha = 1$: One can show that for $n \geqslant 2$ and $\mu = c\sigma_M$, where $M = B_{\mathbb{R}^{n-1}}(0,1) \times \{0\}$ and $c > 0$ is large enough for the ground state φ_0 to exist, the logarithmic factor given in Theorem 4.3 reflects the correct behaviour of the modulus of continuity of φ_0.

5 Estimate of φ_0 from below

In this section we prove a pointwise estimate of the ground state φ_0 of H_μ from below. Throughout the section we assume that

$$\Big\|\int_0^t k_s * \mu\,ds\Big\|_\infty \leqslant ct^\theta \qquad (t > 0) \tag{5.1}$$

for some $c > 0,\ \theta \in (0,1]$ (cf. Proposition 2.2); in particular, μ is in the Kato class.

As mentioned in Section 3, μ being in the Kato class implies that the eigenfunctions of H_μ are in $C_0(\mathbb{R}^n)$. The following result provides an estimate of the L_∞-norm of the eigenfunctions.

5.1 Proposition. *Assume that* (5.1) *holds for some* $c > 0$, $\theta \in (0,1]$. *Let* φ *be an eigenfunction of* H_μ, $H_\mu\varphi = -\kappa^2\varphi$ *for some* $\kappa > 0$. *If* $\int |\varphi|\,d\mu < \infty$ *then*

$$\|\varphi\|_\infty \leqslant c_n(c,\theta,\kappa)\int |\varphi|\,d\mu,$$

where

$$\begin{aligned}
c_1(c,\theta,\kappa) &= \tfrac{1}{2\kappa},\\
c_2(c,\theta,\kappa) &= \tfrac{e}{2\pi}\ln\big(1+(4c)^{\frac{1}{\theta}}\kappa^{-2}\big),\\
c_n(c,\theta,\kappa) &= \tfrac{1}{n-2}\omega_n^{-2/n}(4\pi)^{-\frac{n-2}{2}}(4c)^{\frac{n-2}{2\theta}} \qquad (n \geqslant 3).
\end{aligned}$$

Before proving this proposition, we state and prove the main result of this section, the announced pointwise estimate of φ_0.

5.2 Theorem. *Assume that* $\operatorname{spt}\mu \subseteq B[0,R]$ *for some* $R > 0$ *and that* (5.1) *holds for some* $c > 0$, $\theta \in (0,1]$. *Let* $0 \leqslant \varphi_0 \in D(H_\mu)$ *be the ground state of* H_μ, $\|\varphi_0\|_\infty = 1$, $\kappa > 0$ *such that* $H_\mu\varphi_0 = -\kappa^2\varphi_0$.

(a) *If* $n = 1$ *then* $\varphi_0(x) \geqslant e^{-\kappa(|x|+R)}$ *for all* $x \in \mathbb{R}$.

(b) *If* $n = 2$ *then*

$$\varphi_0(x) \geqslant \frac{1}{2e}\Big(\big[(4c)^{-\frac{1}{2\theta}}(|x|+R)^{-1}\big] \wedge 1\Big)e^{-\kappa(|x|+R)} \qquad (x \in \mathbb{R}^2).$$

(c) *If* $n \geqslant 3$ *then*

$$\varphi_0(x) \geqslant c_n(4c)^{-\frac{n-2}{2\theta}}(|x|+R)^{2-n}e^{-\kappa(|x|+R)} \qquad (x \in \mathbb{R}^n),$$

where $c_n = \frac{1}{\sigma_{n-1}}\omega_n^{2/n}(4\pi)^{\frac{n-2}{2}} = 2^{n-3}\big(\frac{2}{n}\big)^{\frac{2}{n}}\Gamma\big(\frac{n}{2}\big)^{\frac{n-2}{n}}$.

Proof. Lemma 2.4 implies that

$$\varphi_0(x) = \int_{B(0,R)} G_\kappa(x-y)\varphi_0(y)\,d\mu(y) \geqslant G_\kappa\big((|x|+R)e_1\big)\int \varphi_0\,d\mu \tag{5.2}$$

for all $x \in \mathbb{R}^n$.

(a) For $n = 1$ we have $G_\kappa(x) = \frac{1}{2\kappa}e^{-\kappa|x|}$ $(x \in \mathbb{R})$, and $\int \varphi_0\,d\mu \geqslant 2\kappa$ by Proposition 5.1, so the assertion follows from (5.2).

(b) Let $x \in \mathbb{R}^2 \setminus \{0\}$. Proposition A.1(b) from the appendix and Proposition 5.1 yield

$$G_\kappa(x)\int \varphi_0\,d\mu \geqslant \frac{\frac{1}{2\pi}\ln\big(1+\frac{1}{\kappa|x|}\big)e^{-\kappa|x|}}{\frac{e}{2\pi}\ln\big(1+(4c)^{\frac{1}{2\theta}}\frac{1}{\kappa}\big)^2} = \frac{1}{2e}\frac{\ln\big(1+\frac{a}{\kappa}\big)}{\ln\big(1+\frac{b}{\kappa}\big)}e^{-\kappa|x|},$$

with $a = \frac{1}{|x|}$ and $b = (4c)^{\frac{1}{2\theta}}$. By (5.2) the assertion follows if $\ln(1+x)/\ln(1+y) \geqslant \frac{x}{y} \wedge 1$ for all $x, y > 0$. In the case $x \geqslant y$ the latter inequality is clear; for $x < y$ it is equivalent to $\frac{1}{x}\ln(1+x) \geqslant \frac{1}{y}\ln(1+y)$, which is a consequence of the concavity of ln, so the proof of (b) is complete.

(c) Let $x \in \mathbb{R}^n \setminus \{0\}$. Proposition A.1(a) from the appendix and Proposition 5.1 yield

$$G_\kappa(x)\int \varphi_0\,d\mu \geqslant \tfrac{1}{(n-2)\sigma_{n-1}}|x|^{2-n}e^{-\kappa|x|}\cdot(n-2)\omega_n^{2/n}(4\pi)^{\frac{n-2}{2}}(4c)^{-\frac{n-2}{2\theta}},$$

so the assertion follows from (5.2). $\square$

For the proof of Proposition 5.1 we need the following estimate of $\|T_\mu(t)\|_{p\to\infty}$.

5.3 Lemma. *Assume that* (5.1) *holds for some* $c > 0,\ \theta \in (0,1]$. *Let* $t_0 := 2(4c)^{-\frac{1}{\theta}}$. *Then*

$$\|T_\mu(t_0)\|_{p\to\infty} \leqslant 2(2\pi t_0)^{-\frac{n}{2p}} = 2(4\pi)^{-\frac{n}{2p}}(4c)^{\frac{n}{2\theta p}} \qquad (1 \leqslant p \leqslant \infty).$$

Proof. Let $t_1 := \frac{1}{2}t_0 = (4c)^{-\frac{1}{\theta}}$. By assumption (5.1) we have

$$\Big\| \int_0^{t_1} k_t * \mu \, dt \Big\|_\infty \leqslant c t_1^\theta = \tfrac{1}{4}.$$

From Proposition 2.1 it follows that $\|T_{a\mu}(t_1)\|_{1\to1} \leqslant 1/(1-\frac{a}{4})$ for all $0 \leqslant a < 4$. Moreover, $\|T_{0\mu}(t_1)\|_{1\to\infty} = \|e^{t_1\Delta}\|_{1\to\infty} = (4\pi t_1)^{-\frac{n}{2}}$, so by the Stein interpolation theorem we infer as in [StVo96; Thm. 5.1] that

$$\|T_\mu(t_1)\|_{1\to2} \leqslant \|T_{2\mu}(t_1)\|_{1\to1}^{\frac{1}{2}} \|T_{0\mu}(t_1)\|_{1\to\infty}^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}}(4\pi t_1)^{-\frac{n}{4}}.$$

Hence, by duality,

$$\|T_\mu(t_0)\|_{1\to\infty} \leqslant \|T_\mu(t_1)\|_{1\to2}^2 \leqslant 2(4\pi t_1)^{-\frac{n}{2}} = 2(2\pi t_0)^{-\frac{n}{2}}.$$

The assertion now follows from Riesz-Thorin interpolation with the estimate

$$\|T_\mu(t_0)\|_{\infty\to\infty} \leqslant \|T_\mu(t_1)\|_{1\to1}^2 \leqslant \big(\tfrac{1}{1-1/4}\big)^2 < 2. \qquad \square$$

Proof of Proposition 5.1. By Lemma 2.4 we have $\varphi = G_\kappa * (\varphi\mu) = (\kappa^2 - \Delta)^{-1}(\varphi\mu)$. In the case $n = 1$, the assertion follows since $\|G_\kappa\|_\infty = \frac{1}{2\kappa}$. For $n \geqslant 2$ we estimate, with $t_0 = 2(4c)^{-\frac{1}{\theta}}$ as in Lemma 5.3,

$$\|\varphi\|_\infty = \|e^{-\kappa^2 t_0} T_\mu(t_0)\varphi\|_\infty \leqslant \big\|T_\mu(t_0)(\kappa^2-\Delta)^{-1} \colon M(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)\big\| \int |\varphi| \, d\mu,$$

where $M(\mathbb{R}^n)$ is the Banach space of finite Borel measures on $\mathbb{R}^n$. We now have to show that the norm of $T_\mu(t_0)(\kappa^2-\Delta)^{-1}$ is less or equal $c_n(c,\theta,\kappa)$.

First suppose that $n = 2$. Then for all $p \in [1,\infty)$ we have

$$\begin{aligned} \|(\kappa^2-\Delta)^{-1} \colon M(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| &\leqslant \int_0^\infty e^{-\kappa^2 t} \|e^{t\Delta} \colon M(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| \, dt \\ &\leqslant \int_0^\infty (4\pi t)^{-\frac{2}{2}(1-\frac{1}{p})} e^{-\kappa^2 t} \, dt = \kappa^{-\frac{2}{p}} (4\pi)^{\frac{1}{p}-1} \int_0^\infty s^{\frac{1}{p}-1} e^{-s} \, ds. \end{aligned}$$

Since $\int_0^\infty s^{\frac{1}{p}-1} e^{-s} \, ds = \Gamma(\frac{1}{p}) = p\Gamma(1+\frac{1}{p}) \leqslant p$, we conclude by Lemma 5.3 that

$$\begin{aligned} \|T_\mu(t_0)(\kappa^2-\Delta)^{-1} \colon M(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)\| &\leqslant 2(4\pi)^{-\frac{2}{2p}}(4c)^{\frac{2}{2\theta p}} \cdot \kappa^{-\frac{2}{p}}(4\pi)^{\frac{1}{p}-1} p \\ &= \tfrac{1}{2\pi} p \big((4c)^{\frac{1}{\theta}} \kappa^{-2}\big)^{\frac{1}{p}} =: f(p). \end{aligned}$$

For the proof of the case $n=2$ it remains to show that there exists $p\in[1,\infty)$ such that $f(p)\leqslant c_2(c,\theta,\kappa)$. With $a:=(4c)^{\frac{1}{\theta}}\kappa^{-2}$ this inequality reads $pa^{\frac{1}{p}}\leqslant e\ln(1+a)$. If $a\leqslant e$ then $a\leqslant e\ln(1+a)$, so the inequality holds with $p=1$. For $a>e$ we take $p=\ln a$; then

$$pa^{\frac{1}{p}}=pe^{\frac{1}{p}\ln a}=\ln a\cdot e^1\leqslant e\ln(1+a).$$

Let now $n\geqslant 3$. For $m\in M(\mathbb{R}^n)$ we have $(\kappa^2-\Delta)^{-1}m=\int G_\kappa(\cdot-y)\,dm(y)$ (Bochner integral of $y\mapsto G_\kappa(\cdot-y)\in L_1(\mathbb{R}^n)$) and hence

$$\|T_\mu(t_0)(\kappa^2-\Delta)^{-1}m\|_\infty\leqslant\int\|T_\mu(t_0)G_\kappa(\cdot-y)\|_\infty\,dm(y)\leqslant\sup_{y\in\mathbb{R}^n}\|T_\mu(t_0)G_0(\cdot-y)\|_\infty\cdot\|m\|.$$

Using Lemma 5.4 below and Lemma 5.3, we deduce that

$$\begin{aligned}\|T_\mu(t_0)(\kappa^2-\Delta)^{-1}\colon M(\mathbb{R}^n)\to L_\infty(\mathbb{R}^n)\|&\leqslant\tfrac{1}{2(n-2)}\omega_n^{-\frac{2}{n}}\|T_\mu(t_0)\|_{1\to\infty}^{\frac{n-2}{n}}\|T_\mu(t_0)\|_{\infty\to\infty}^{\frac{2}{n}}\\&\leqslant\tfrac{1}{2(n-2)}\omega_n^{-\frac{2}{n}}\cdot 2(4\pi)^{-\frac{n-2}{2}}(4c)^{\frac{n-2}{2\theta}}\end{aligned}$$

and conclude the proof. □

In the following interpolation lemma we implicitly use the fact that G_0 is in the weak Lebesgue space $L_{\frac{n}{n-2},w}$.

5.4 Lemma. *Let $n\geqslant 3$. Let $T\colon L_1+L_\infty(\mathbb{R}^n)\to L_\infty(\mathbb{R}^n)$ be a bounded linear operator, $y\in\mathbb{R}^n$, $f(x):=(n-2)G_0(x-y)=\frac{1}{\sigma_{n-1}}|x-y|^{2-n}$ ($x\in\mathbb{R}^n\setminus\{y\}$). Then*

$$\|Tf\|_\infty\leqslant\tfrac12\omega_n^{-\frac{2}{n}}\|T\|_{1\to\infty}^{\frac{n-2}{n}}\|T\|_{\infty\to\infty}^{\frac{2}{n}}.$$

Proof. Without loss of generality assume that $y=0$. Let $r_0>0$, $f_1:=\big(f-f(r_0e_1)\big)^+$, $f_2:=f-f_1$. Then $\|f_2\|_\infty=f(r_0e_1)=\frac{1}{\sigma_{n-1}}r_0^{2-n}$,

$$\|f_1\|_1=\int_0^{r_0}(r^{2-n}-r_0^{2-n})\,r^{n-1}dr=\int_0^{r_0}(r-r_0^{2-n}r^{n-1})\,dr=\big(\tfrac12-\tfrac1n\big)r_0^2,$$

and hence

$$\|Tf\|_\infty\leqslant\|Tf_1\|_\infty+\|Tf_2\|_\infty\leqslant\|T\|_{1\to\infty}\tfrac{n-2}{2n}r_0^2+\|T\|_{\infty\to\infty}\tfrac{1}{\sigma_{n-1}}r_0^{2-n}.$$

Let now $a:=\frac1n\|T\|_{1\to\infty}$, $b:=\frac{1}{\sigma_{n-1}}\|T\|_{\infty\to\infty}$ and $r_0:=\big(\frac{b}{a}\big)^{1/n}$ (without loss of generality $T\neq 0$, so $a\neq 0$). Then we obtain that

$$\|Tf\|_\infty\leqslant r_0^2\big(\tfrac{n-2}{2}a+br_0^{-n}\big)=r_0^2\big(\tfrac{n-2}{2}a+a\big)=\left(\frac{n}{\sigma_{n-1}}\frac{\|T\|_{\infty\to\infty}}{\|T\|_{1\to\infty}}\right)^{\frac{2}{n}}\frac{n}{2}\frac{1}{n}\|T\|_{1\to\infty},$$

and the assertion follows. □

6 Proofs of the main results

We are now in a position to prove our main theorems stated in Section 1.

Proof of Theorem 1.1. Let $R := \frac{d}{2}$ and assume without loss of generality that $\operatorname{spt}\mu \subseteq B[0,R]$. As in Section 3 we normalise φ_0, φ_1 such that $\|\varphi_0\|_\infty = \|\varphi_1\|_\infty = 1$ and $\sup\varphi_0 = \sup\varphi_1 = 1$. Let $\varepsilon := \frac{1}{4}\inf_{B[0,R+1]}\varphi_0$. Then $\varepsilon \in (0, \frac{1}{4}]$. Let $\tilde\delta > 0$ be as in Theorem 4.3, so that $|\varphi_1(x) - \varphi_1(y)| \leqslant \varepsilon$ for $|x-y| \leqslant \tilde\delta$, and let $\delta := \tilde\delta \wedge 1$. According to (3.4) we can estimate

$$\lambda_1 - \lambda_0 \geqslant \frac{2\omega_{n-1}}{R}\varepsilon^2\delta^{n-1}\|\varphi_1\|_2^{-2}. \tag{6.1}$$

By Proposition 2.5(a) we have $\|\varphi_1\|_2^{-2} \geqslant |\lambda_1|\|\mu\|^{-1}$. Moreover,

$$\|\mu\| = \mu(B[0,R]) \leqslant c_\mu R^{n-\alpha} \tag{6.2}$$

and hence

$$\kappa_0^2 - \kappa_1^2 = \lambda_1 - \lambda_0 \geqslant 2\omega_{n-1}|\lambda_1|c_\mu^{-1}R^{\alpha-n-1}\varepsilon^2\delta^{n-1} \geqslant \gamma\kappa_1^2,$$

where $\gamma := 2\omega_{n-1}(c_\mu+1)^{-1}(d+1)^{\alpha-n-1}\varepsilon^2\delta^{n-1}$. This implies $\kappa_0^2 \geqslant (1+\gamma)\kappa_1^2$, and therefore $\kappa_0^2 - \kappa_1^2 \geqslant \kappa_0^2 - \frac{1}{1+\gamma}\kappa_0^2 = \frac{\gamma}{1+\gamma}\kappa_0^2$. Since $\gamma \leqslant 2\omega_{n-1}(\frac{1}{4})^2 \leqslant 1$, we conclude that

$$\lambda_1 - \lambda_0 \geqslant \tfrac{\gamma}{1+\gamma}\kappa_0^2 \geqslant \tfrac{\gamma}{2}\kappa_0^2 = \omega_{n-1}|\lambda_0|(c_\mu+1)^{-1}(d+1)^{\alpha-n-1}\varepsilon^2\delta^{n-1}. \tag{6.3}$$

In the case $n \geqslant 5$ we have $\|\varphi_1\|_2^2 \leqslant 6\omega_n R^n$ by Proposition 2.5(c); then (6.1) yields

$$\lambda_1 - \lambda_0 \geqslant \frac{2\omega_{n-1}}{6\omega_n}(d+1)^{-n-1}\varepsilon^2\delta^{n-1}. \tag{6.4}$$

We are going to show that

$$\varepsilon^2\delta^{n-1} \geqslant C_0(c_\mu+1)^{-p_0}(d+1)^{-q_0}e^{-\beta\kappa_0(d+1)}, \tag{6.5}$$

with β as in the assertion of the theorem and constants $C_0, p_0, q_0 > 0$ depending only on n and α. Then (6.3) implies (1.2) with $C = \omega_{n-1}C_0$, $p = p_0 + 1$, $q = q_0 + n + 1 - \alpha$, and the additional assertion for $n \geqslant 5$ follows from (6.4).

Recall from Proposition 2.2 that (5.1) holds with $\theta = 1 - \frac{\alpha}{2}$ and $c = \frac{2}{2-\alpha}2^{-\alpha}\pi^{-\frac{n}{2}}\Gamma(\frac{n-\alpha}{2}+1)c_\mu$. Hence, by Theorem 5.2 and the definition of ε we have

$$\varepsilon \geqslant \begin{cases} \frac{1}{8e}(4c+1)^{-\frac{1}{2-\alpha}}(2R+1)^{-1}e^{-\kappa_0(2R+1)} & \text{if } n = 2, \\ \frac{c_n}{4}(4c)^{-\frac{n-2}{2-\alpha}}(2R+1)^{2-n}e^{-\kappa_0(2R+1)} & \text{if } n \geqslant 3. \end{cases}$$

We infer that there exists a constant $c_{n,\alpha} > 0$ (depending only on n and α) such that

$$\varepsilon \geqslant c_{n,\alpha}(c_\mu+1)^{-\frac{n-1}{2-\alpha}}(d+1)^{-(n-1)}e^{-\kappa_0(d+1)}. \tag{6.6}$$

Assume now that $\alpha < 1$. Then $\|\mu\|^{-\frac{1-\alpha}{n-\alpha}}c_\mu^{-\frac{n-1}{n-\alpha}} \geqslant R^{-(1-\alpha)}c_\mu^{-1}$ by (6.2). According to Theorem 4.3 we thus have $\tilde\delta \geqslant \tilde c_{n,\alpha}(c_\mu+1)^{-1}(d+1)^{-(1-\alpha)}\varepsilon =: \delta_1$, with $\tilde c_{n,\alpha} > 0$ depending only on n and α. Without loss of generality $\tilde{c}_{n,\alpha} \leqslant 4$; then $\delta_1 \leqslant 1$ since $\varepsilon \leqslant \frac{1}{4}$. We obtain that

$$\varepsilon^2 \delta^{n-1} \geqslant \tilde{c}_{n,\alpha}^{n-1} (c_\mu + 1)^{-(n-1)} (d+1)^{-(n-1)(1-\alpha)} \varepsilon^{n+1},$$

so from (6.6) we deduce that (6.5) holds with $\beta = n+1$, $p_0 = \frac{n^2-1}{2-\alpha} + n - 1$, $q_0 = n^2 - 1 + (n-1)(1-\alpha)$ and $C_0 = \tilde{c}_{n,\alpha}^{n-1} c_{n,\alpha}^{n+1}$.

In the case $\alpha > 1$ we have $\tilde{\delta} \geqslant \tilde{c}_{n,\alpha} \big(\frac{\varepsilon}{c_\mu+1}\big)^{\frac{1}{2-\alpha}} =: \delta_2$. Again, a suitable choice of $\tilde{c}_{n,\alpha}$ ensures that $\delta_2 \leqslant 1$, and in the same way as above we deduce (6.5), now with $\beta = \frac{n-1}{2-\alpha} + 2$, $p_0 = \beta^2 - \beta - 2$, $q_0 = (n-1)\beta$ and $C_0 = \tilde{c}_{n,\alpha}^{n-1} c_{n,\alpha}^{\beta}$.

Finally assume that $\alpha = 1$. Given $\beta > n+1$, there exists $\tilde{\alpha} \in (1,2)$ such that $\beta = \frac{n-1}{2-\tilde{\alpha}} + 2$. Then $\mu(B(x,r)) \leqslant c_\mu (r \wedge R)^{n-1} \leqslant c_\mu R^{\tilde{\alpha}-1} r^{n-\tilde{\alpha}}$ for all $x \in \mathbb{R}^n$, $r > 0$, and from the preceding paragraph we obtain that

$$\varepsilon^2 \delta^{n-1} \geqslant \tilde{c}_{n,\tilde{\alpha}}^{n-1} c_{n,\tilde{\alpha}}^{\beta} (R^{\tilde{\alpha}-1} c_\mu + 1)^{-(\beta^2-\beta-2)} (d+1)^{-(n-1)\beta} e^{-\beta\kappa_0(d+1)}.$$

Using the inequality $R^{\tilde{\alpha}-1} c_\mu + 1 \leqslant (R+1)^{\tilde{\alpha}-1}(c_\mu + 1)$, we deduce by straightforward computation that (6.5) holds with $p_0 = \beta^2 - \beta - 2$, $q_0 = \beta^2 - \beta - n - 1$ and $C_0 = \tilde{c}_{n,\tilde{\alpha}}^{n-1} c_{n,\tilde{\alpha}}^{\beta}$. ☐

Proof of Theorem 1.3. Normalising μ, φ_0 and φ_1 as in the proof of Theorem 1.1, we obtain from (3.5) and Theorem 5.2(a) that

$$\lambda_1 - \lambda_0 \geqslant \frac{1}{d} \inf_{[-d/2,d/2]} \varphi_0^2 \|\varphi_1\|_2^{-2} \geqslant \frac{1}{d} e^{-2\kappa_0 d} \|\varphi_1\|_2^{-2}.$$

The second assertions thus follows from Proposition 2.5(b). Proposition 2.5(a) implies that $\lambda_1 - \lambda_0 \geqslant \frac{|\lambda_1|}{d\|\mu\|} e^{-2\kappa_0 d}$, so the first assertion follows as in the argument leading to equation (6.3). ☐

A Appendix

Here we prove the estimate of G_κ from below that is needed in the proof of Theorem 5.2. We assume that $n \geqslant 2$ and recall that $\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the $(n-1)$-dimensional volume of the unit sphere in $\mathbb{R}^n$.

A.1 Proposition. *Let $\kappa > 0$, and let G_κ be the convolution kernel of $(\kappa^2 - \Delta)^{-1}$ on $L_2(\mathbb{R}^n)$.*

(a) *If $n \geqslant 3$ then $G_\kappa(x) \geqslant \frac{1}{(n-2)\sigma_{n-1}} |x|^{2-n} e^{-\kappa|x|}$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

(b) *If $n = 2$ then $G_\kappa(x) \geqslant \frac{1}{2\pi} \ln\big(1 + \frac{1}{\kappa|x|}\big) e^{-\kappa|x|}$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.*

Proof. By [Sch66; eq. (VII,10;15)] we have

$$G_\kappa(x) = (2\pi)^{-\frac{n}{2}} \kappa^{\frac{n-2}{2}} |x|^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(\kappa|x|) \qquad \big(x \in \mathbb{R}^n \setminus \{0\}\big), \tag{A.1}$$

where K_ν is the modified Bessel function of the third kind of order $\nu \geqslant 0$. In particular we see that $G_\kappa(x) = \kappa^{n-2} G_1(\kappa x)$, and that it therefore suffices to show the assertion for $\kappa = 1$.

(a) By [AbSt72; 9.6.9] we obtain from (A.1) that

$$G_1(x) = \frac{1}{(n-2)\sigma_{n-1}}|x|^{2-n} + o(|x|^{2-n}) \qquad \text{as } x \to 0.$$

Let $\varepsilon > 0$,

$$\varphi_\varepsilon(x) := \frac{1-\varepsilon}{(n-2)\sigma_{n-1}}|x|^{2-n}e^{-|x|} - G_1(x) \qquad \big(x \in \mathbb{R}^n \setminus \{0\}\big).$$

Then there exists $\delta > 0$ such that $\varphi_\varepsilon < 0$ on $B(0,\delta) \setminus \{0\}$, and $\varphi_\varepsilon(x) \to 0$ as $|x| \to \infty$. Moreover, a straightforward computation shows that $(1-\Delta)\varphi_\varepsilon(x) = -\frac{1-\varepsilon}{(n-2)\sigma_{n-1}}(n-3)|x|^{1-n}e^{-|x|} \leqslant 0$ for $x \neq 0$. Thus φ_ε has no positive maxima. We conclude that $\varphi_\varepsilon < 0$ on $\mathbb{R}^n \setminus \{0\}$ for all $\varepsilon > 0$, and for $\varepsilon \to 0$ we obtain (a).

(b) By (A.1) we have $G_1(x) = \frac{1}{2\pi}K_0(|x|)$, so we must show that $K_0(r) \geqslant \ln(1+\frac{1}{r})e^{-r}$ for all $r > 0$. Let $a := 2e^{-\gamma}$, where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. (Then $1 < a < 2$.) From [AbSt72; 9.6.13] it follows that $K_0(r) = \ln\frac{a}{r} + o(r)$ as $r \to 0$.

Let $f(r) := \ln(1+\frac{a}{r})e^{-r}$ for all $r > 0$. One easily sees that $f(r) = (1-r)\ln\frac{a}{r} + O(r)$ as $r \to 0$, so $g := f - K_0 < 0$ on $(0,\varepsilon)$ for some $\varepsilon > 0$. We will prove that $g < 0$ on $(0,\infty)$; then the claim follows. A straightforward computation shows that

$$g''(r) + \tfrac{1}{r}g'(r) - g(r) = f''(r) + \tfrac{1}{r}f'(r) - f(r) = \tfrac{a}{r(a+r)}h(\tfrac{r}{a})e^{-r} \tag{A.2}$$

for all $r > 0$, with $h(s) := \frac{1}{a}\frac{1}{1+s} + 2 - (1+s)\ln(1+\frac{1}{s})$, and that

$$h''(s) = \frac{1}{(1+s)^3}\left(\frac{2}{a} - \Big(1+\frac{1}{s}\Big)^2\right) \qquad (s > 0).$$

We obtain that h is concave on $(0,s_0)$ and convex on (s_0,∞), where $s_0 = \big(\sqrt{\frac{2}{a}} - 1\big)^{-1}$. Since $h(s) \to -\infty$ as $s \to 0$ and $h(s) \to 1$ as $s \to \infty$, we infer that there exists $s_1 \in (0,s_0)$ such that $h < 0$ on $(0,s_1)$ and $h > 0$ on (s_1,∞).

Now, if g has a positive maximum at $r > 0$ then the left hand side of (A.2) is negative, and hence $\frac{r}{a} \in (0,s_1)$. Similarly, if g has a negative minimum at $r > 0$ then $\frac{r}{a} \in (s_1,\infty)$. Thus, to the left of a positive maximum there cannot be any negative minimum. Since $g < 0$ on $(0,\varepsilon)$ and $g(r) \to 0$ as $r \to 0$, we conclude that g has no positive maximum. This implies that $g < 0$ on $(0,\infty)$ since $g(r) \to 0$ as $r \to \infty$. ☐

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