Elliptic operators generating stochastic semigroups

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Abstract

We use an intrinsic metric type approach to investigate when C_0-semigroups generated by second order elliptic differential operators are stochastic. We give a new condition for stochasticity that encompasses the volume growth conditions by Karp and Li and by Perelmuter and Semenov.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $a\colon \Omega \to \mathbb{R}^{n\times n}$ be a measurable function with values in the positive semidefinite matrices. We assume that a is sectorial, i.e., there exists $\alpha \geqslant 1$ such that

$$(a\xi\cdot\eta)^2 \leqslant \alpha(a\xi\cdot\xi)(a\eta\cdot\eta) \text{ a.e.} \qquad (\xi,\eta\in\mathbb{R}^n). \tag{1.1}$$

Let $m \in L_{1,\mathrm{loc}}(\Omega)$ such that $m > 0$ a.e. and ma is locally integrable, and let $\mu := m\lambda^n$, where λ^n is the Lebesgue measure on Ω. We are going to study conditions under which certain realisations of the operator $\frac{1}{m}\nabla\cdot(ma\nabla)$ generate stochastic C_0-semigroups on $L_1(\mu)$. Throughout the paper, the function spaces are assumed to be real vector spaces.

In order to give a precise meaning to the operator in question, we define the sectorial form $\tau_{0,\max}$ in $L_2(\mu)$ by

$$D(\tau_{0,\max}) := \bigl\{u \in L_2(\mu)\cap W^1_{1,\mathrm{loc}}(\Omega);\, a\nabla u\cdot\nabla u \in L_1(\mu)\bigr\},$$
$$\tau_{0,\max}(u,v) := \int_\Omega a\nabla u\cdot\nabla v\,d\mu.$$

Note that $C_c^\infty(\Omega) \subseteq D(\tau_{0,\max})$ since ma is locally integrable. Let τ_0 be a restriction of $\tau_{0,\max}$ that is densely defined and satisfies $F\circ u \in D(\tau_0)$ for all $u \in D(\tau_0)$ and all normal contractions $F \in C^\infty(\mathbb{R})$. Examples for $D(\tau_0)$ are $D(\tau_{0,\max})$ itself, $C_c^\infty(\Omega)$, and the intersection of $D(\tau_{0,\max})$ with $C^\infty(\Omega)$, $\bigl\{u|_\Omega;\, u\in C^\infty(\mathbb{R}^n)\bigr\}$ or $\bigl\{u|_\Omega;\, u \in C_c^\infty(\mathbb{R}^n)\bigr\}$.

We are particularly interested in the case that the form τ_0 is *Neumann type*, by which we mean that

$$D(\tau_{0,\max}) \cap \bigl\{u|_\Omega;\, u\in C_c^\infty(\mathbb{R}^n)\bigr\} \subseteq D(\tau_0). \tag{1.2}$$

This condition will be needed (in combination with the assumption $\mu = \lambda^n$) in Examples 3.2 and 3.5 and in Corollary 4.4.

We assume that τ_0 is closable; then $\tau := \overline{\tau_0}$ is a (non-symmetric) Dirichlet form (see, e.g., [MaRö92; Prop. I.4.10]). For conditions ensuring closability of τ_0 we refer to [RöWi85]; see also [VoVo03; Prop. A.1] for the case $\tau_0 = \tau_{0,\max}$. Let A be the m-sectorial operator in $L_2(\mu)$ associated with τ. Then $-A$ generates a C_0-semigroup T_2 on $L_2(\mu)$, and the Beurling-Deny criteria imply that T_2 extrapolates to a substochastic C_0-semigroup T_1 on $L_1(\mu)$ and to a sub-Markovian weak*-continuous semigroup T_∞ on $L_\infty(\mu)$.

Within the above framework, we are going to study when the semigroup T_1 is *stochastic*, i.e., when $\|T_1(t)f\|_1 = \|f\|_1$ for all $t \geqslant 0,\ 0 \leqslant f \in L_1(\mu)$. Related problems studied in the literature are conservativeness of the L_∞-semigroup and L_1-uniqueness. In the symmetric case it is clear that T_1 is stochastic if and only if T_∞ is *conservative*, i.e., $T_\infty \mathbb{1}_\Omega = \mathbb{1}_\Omega$ for all $t \geqslant 0$.

If $C_c^\infty(\Omega) \subseteq D(A_1)$, where A_1 is the generator of T_1, then $A_1|_{C_c^\infty(\Omega)}$ is said to be L_1-*unique* if $C_c^\infty(\Omega)$ is a core for A_1. For $\Omega = \mathbb{R}^n,\ D(\tau_0) = C_c^\infty(\mathbb{R}^n)$ and sufficiently regular coefficients, stochasticity of T_1 is equivalent to L_1-uniqueness of $A_1|_{C_c^\infty(\Omega)}$. It follows from [Sta99; Cor. 2.2] that this equivalence holds if $m = \varphi^2$ for some $\varphi \in W^1_{2,\mathrm{loc}}(\mathbb{R}^n),\ a \in L_{\infty,\mathrm{loc}}(\mathbb{R}^n)^{n\times n},\ \partial_j a \in L_{2,\mathrm{loc}}(\mu)^{n\times n}$ $(j = 1, \ldots, n)$ and a is locally Hölder continuous and locally uniformly elliptic. (These assumptions imply that A_1 is an extension of the operator $-\frac{1}{m}\nabla\cdot(ma\nabla)$ with domain $C_c^\infty(\Omega)$.)

There are two important means to approaching the above problems: Lyapunov functions and the intrinsic metric of τ. In [Dav85; Sec. 2] a Lyapunov function approach going back to [Kha60] is used to study conservativeness in the case of smooth symmetric coefficients (and manifolds, not just subdomains of $\mathbb{R}^n$). This approach has been generalised in [Sta99; Sec. I.1] to non-symmetric operators with non-smooth coefficients; there, stochasticity is studied for non-divergence form operators, but the operators can be written in divergence form.

1.1 Example. Let $\Omega = \mathbb{R}^n,\ D(\tau_0) = C_c^\infty(\mathbb{R}^n),\ m = \mathbb{1}_{\mathbb{R}^n}$ and $a(x) = a_0(|x|)I_n$ $(x \in \mathbb{R}^n)$, where $a_0\colon \mathbb{R} \to (0,\infty)$ is an even C^∞-function satisfying $\int_0^\infty \frac{r}{a_0(r)}\,dr = \infty$ and I_n is the identity matrix. Then from [Dav85; Thm. 2.4] it follows that T_1 is stochastic. Indeed, for the Lyapunov function

$$u(x) := n + \int_0^{|x|} \frac{r}{a_0(r)}\,dr \qquad (x \in \mathbb{R}^n)$$

we obtain that $u(x) \to \infty$ as $|x| \to \infty$ and $a\nabla u(x) = a_0(|x|)\cdot\frac{|x|}{a_0(|x|)}\frac{x}{|x|} = x$ for all $x \in \mathbb{R}^n$. Thus, $u \in C^\infty(\mathbb{R}^n)$ and $\nabla\cdot(a\nabla u) = n \leqslant u$, so T_∞ is conservative by [Dav85; Thm. 2.4].

The above might inspire the conjecture that T_1 is stochastic if instead of $a(x) = a_0(|x|)I_n$ one only assumes

$$\|a(x)\| \leqslant a_0(|x|) \qquad (x \in \mathbb{R}^n).$$

This conjecture is known to be true in dimension $n = 1$ (see Proposition 2.2), but in higher dimensions we can only prove it under the additional assumption that $r \mapsto r^{-2}a_0(r)$ is increasing on (R,∞) for some $R > 0$; see Corollary 4.4.

As pointed out in [PeSe87], Lyapunov functions are particularly useful for proving conservativeness of semigroups generated by non-divergence form operators. In the case of divergence form operators (or stochasticity and non-divergence form operators), the method has the disadvantage that it leads to assumptions on the derivative of the coefficient matrix a; see, e.g., [Dav85; Thm. 2.5] and [Sta99; Prop. 1.10(c)]. These assumptions can be avoided by working with the intrinsic metric (or related functions) as in [PeSe87] and [Stu94; Sec. 3]. We would like to point out, however, that with an intrinsic metric approach one can recover neither Example 1.1 nor the well-known criterion for stochasticity in the one-dimensional case; see Proposition 2.2 and Example 2.4 below.

In [Stu94], the question of conservativeness is studied in the general abstract context of strongly local symmetric regular Dirichlet forms, under the assumption that the intrinsic metric induces the topology of Ω. Sturm gives a condition for conservativeness in terms of the volume growth of intrinsic balls, which in the context of the canonical Dirichlet form on Riemannian manifolds is due to [KaLi83] and [Gri87]. (Note that in our framework, the form τ need neither be symmetric nor regular. Moreover, our assumptions do not imply that the intrinsic metric of τ induces the topology of Ω.)

In Theorem 4.6, the main result of the paper, we will present a new condition for stochasticity that generalises the above mentioned volume growth condition. We will follow an intrinsic metric approach and work with a function ψ satisfying the condition

$$0 \leqslant \psi \in W^1_{1,\mathrm{loc}}(\Omega), \qquad a\nabla\psi\cdot\nabla\psi \leqslant 1 \text{ a.e.}, \qquad \rho\circ\psi \in D(\tau_0) \quad \bigl(0 \leqslant \rho \in C_{\mathrm{c}}^\infty[0,\infty)\bigr). \tag{1.3}$$

This is parallel to the use of the function U in [PeSe87].

1.2 Remarks. (a) Assume that $\mu(\Omega) < \infty$. Then $\psi = \mathbf{1}_\Omega$ satisfies condition (1.3) if and only if $\mathbf{1}_\Omega \in D(\tau_0)$. Observe that the latter already implies that $\int Au = \tau(u, \mathbf{1}_\Omega) = 0$ for all $u \in D(A)$ and hence that T_1 is stochastic.

(b) Condition (1.3) clearly depends on the choice of the form τ_0. Let $0 \leqslant \psi \in W^1_{1,\mathrm{loc}}(\Omega)$ such that $a\nabla\psi\cdot\nabla\psi \leqslant 1$ a.e. For $\tau_0 = \tau_{0,\max}$ it is straightforward that then condition (1.3) is satisfied if $\psi(x) \to \infty$ as $|x| \to \infty$ and $\mu\bigl(\Omega\cap B(0,r)\bigr) < \infty$ for all $r > 0$. If one only assumes τ_0 to be Neumann type (see (1.2)), then one needs in addition that ψ is the restriction of a C^∞-function on $\mathbb{R}^n$. In the case $D(\tau_0) = C_{\mathrm{c}}^\infty(\Omega)$, condition (1.3) is satisfied if and only if $\psi \in C^\infty(\Omega)$ and $\psi(x) \to \infty$ as $x \to \partial\Omega$.

(c) If ψ satisfies condition (1.3) then one has $\rho\circ\psi \in D(\tau) \cap D(\tau_{0,\max})$ for all $0 \leqslant \rho \in W^1_{\infty,\mathrm{c}}[0,\infty)$. Indeed, let $R > 0$ such that $\operatorname{spt}\rho \subseteq [0,R)$, and let $c := \|\rho'\|_\infty$. Choose $0 \leqslant \tilde\rho \in C_{\mathrm{c}}^\infty[0,\infty)$ such that $\tilde\rho(r) = c(R+1-r)$ for all $r \in [0,R]$. Then $\rho \leqslant \tilde\rho$, so $\tilde\rho\circ\psi \in L_2(\mu)$ implies $\rho\circ\psi \in L_2(\mu)$. Moreover,

$$a\nabla(\rho\circ\psi)\cdot\nabla(\rho\circ\psi) \leqslant c^2\mathbf{1}_{[\psi\leqslant R]}a\nabla\psi\cdot\nabla\psi \leqslant a\nabla(\tilde\rho\circ\psi)\cdot\nabla(\tilde\rho\circ\psi) \in L_1(\mu)$$

and hence $\rho\circ\psi \in D(\tau_{0,\max})$. Finally, let (ρ_k) be a sequence in $C_{\mathrm{c}}^\infty[0,\infty)$ converging uniformly to ρ that satisfies $\operatorname{spt}\rho_k \subseteq [0,R)$ and $\|\rho_k'\|_\infty \leqslant c$ for all $k \in \mathbb{N}$. Then as above we obtain that $\rho_k\circ\psi \leqslant \tilde\rho\circ\psi$ and $\tau_0(\rho_k\circ\psi) \leqslant \tau_0(\tilde\rho\circ\psi)$ for all $k \in \mathbb{N}$. Therefore, $(\rho_k\circ\psi)$ is a bounded sequence in $D(\tau)$ with $\rho_k\circ\psi \to \rho\circ\psi$ in $L_2(\mu)$ as $k \to \infty$, and from the closedness of τ we infer that $\rho\circ\psi \in D(\tau)$.

(d) We note the following important consequence of part (c): If $D(\tau_0) = D(\tau)\cap D(\tau_{0,\max})$ and ψ satisfies (1.3) then $\rho\circ\psi$ satisfies (1.3) for all $0 \leqslant \rho \in W^1_\infty[0,\infty)$ with $\|\rho'\|_\infty \leqslant 1$ and $\rho(r) \to \infty$ as $r \to \infty$.

The paper is organised as follows. In Section 2 we recall the well-known criterion for stochasticity in the one-dimensional case and give an example that illustrates the limitations of intrinsic metric approaches. In Section 3 we generalise the result of [PeSe87] to our framework and give a new proof. In Section 4 we present new conditions for stochasticity in terms of growth of the coefficient matrix a (in the case $\mu = \lambda^n$) and in terms of volume growth.

2 The one-dimensional case

In this section we assume that $n = 1$, $\Omega = (c,d)$ for some $-\infty \leqslant c < d \leqslant \infty$ and $C_{\mathrm{c}}^\infty(\Omega) \subseteq D(\tau_0)$. We first show that then $D(\tau)$ contains all compactly supported elements of $D(\tau_{0,\max})$.

2.1 Lemma. *Under the above assumptions one has* $D(\tau_{0,\max}) \cap W^1_{1,c}(\Omega) \subseteq D(\tau)$. *If, in addition,* $\mu\big((c,x_0)\big) = \mu\big((x_0,d)\big) = \infty$ *for some (and hence all)* $x_0 \in (c,d)$, *then* $D(\tau_{0,\max}) \subseteq D(\tau)$, *i.e.,* $C_c^\infty(\Omega)$ *is a core for* $\tau_{0,\max}$.

Proof. First observe that $W^1_{\infty,c}(\Omega) \subseteq D(\tau)$ by a standard convolution argument (recall $m, ma \in L_{1,\mathrm{loc}}(\Omega)$). Let $u \in D(\tau_{0,\max}) \cap W^1_{1,c}(\Omega)$. Let $c_0 := \min \operatorname{spt} u$, $d_0 := \max \operatorname{spt} u$ and $\varepsilon > 0$ such that $d_1 := d_0 + \varepsilon < d$. For $k \in \mathbb{N}$ define $u_k \in W^1_{\infty,c}(\Omega)$ by $u_k(x) := \int_c^x (u'(y) \wedge k) \vee (-k)\,dy$ for $x \leqslant d_0$, $u_k(x) := u_k(d_0)\big(1 - \varepsilon^{-1}(x - d_0)\big)$ for $d_0 < x < d_1$ and $u_k(x) := 0$ for $x \geqslant d_1$. Then $\operatorname{spt} u_k \subseteq [c_0, d_1]$ for all $k \in \mathbb{N}$ and $u_k \to u$ uniformly, hence $u_k \to u$ in $L_2(\mu)$ as $k \to \infty$. Moreover, $|u_k'| \leqslant |u'| + |u_k(d_0)|\varepsilon^{-1}\mathbb{1}_{[d_0,d_1]}$ for all $k \in \mathbb{N}$ and $u_k' \to u'$ a.e. as $k \to \infty$. We conclude that $u_k \to u$ in $D(\tau_{0,\max})$, and the first assertion follows.

Assume now that $\mu\big((c,x_0)\big) = \mu\big((x_0,d)\big) = \infty$, and let $u \in D(\tau_{0,\max})$. Then $u \in C(\Omega)$, and $\liminf_{x\to c} |u(x)| = \liminf_{x\to d} |u(x)| = 0$ since $u \in L_2(\mu)$. It follows that $u_\varepsilon := (|u| - \varepsilon)^+ \operatorname{sgn} u$ can be approximated in $D(\tau_{0,\max})$ by compactly supported elements of $D(\tau_{0,\max})$. Moreover, $u_\varepsilon \to u$ in $D(\tau_{0,\max})$ as $\varepsilon \to 0$, so the first assertion implies the second one. □

For the remainder of the section we assume that $(ma)^{-1} \in L_{1,\mathrm{loc}}(\Omega)$. Then the form $\tau_{0,\max}$ is closed: For a Cauchy sequence (u_k) in $D(\tau_{0,\max})$ we obtain that $u_k \to u$ in $L_2(\mu)$ and $(ma)^{1/2}u_k' \to f$ in $L_2(\Omega)$ for some $u \in L_2(\mu)$ and $f \in L_2(\Omega)$, hence $u_k' \to (ma)^{-1/2}f$ in $L_{1,\mathrm{loc}}(\Omega)$. It follows that $u \in W^1_{1,\mathrm{loc}}(\Omega)$, $(ma)^{1/2}u' = f \in L_2(\Omega)$, and thus $u \in D(\tau_{0,\max})$, $u_k \to u$ in $D(\tau_{0,\max})$.

It is well-known that in the case $D(\tau_0) = C_c^\infty(\Omega)$, the semigroup T_1 is stochastic if and only if the boundary points c and d are non-exit in the classification of Feller; see, e.g., [Osh92; Lemma 3.1] or [Aze74; Prop. 4.6]. The latter condition is still sufficient for stochasticity if as above we only assume $C_c^\infty(\Omega) \subseteq D(\tau_0)$. For the convenience of the reader we include a proof that is based on the same ansatz as the proofs in the next two sections.

2.2 Proposition. *Let* $x_0 \in (c,d)$, $M(x) := \int_{x_0}^x m(y)\,dy$ $(c < x < d)$ *and assume that*

$$\int_{x_0}^c \frac{M(x)}{m(x)a(x)}\,dx = \int_{x_0}^d \frac{M(x)}{m(x)a(x)}\,dx = \infty.$$

Then T_1 *is stochastic.*

Proof. Define $v \in W^1_{1,\mathrm{loc}}(\Omega)$ by $v(x) := \int_{x_0}^x \frac{M(y)}{m(y)a(y)}\,dy$. Then $v \geqslant 0$ and $v(x) \to \infty$ as $x \to c$ or $x \to d$. Thus, for $\varepsilon > 0$ we have $v_\varepsilon := (1 - \varepsilon v)^+ \in W^1_{1,c}(\Omega)$ and $0 \leqslant v_\varepsilon \leqslant 1$. Moreover, $v_\varepsilon' = -\varepsilon\frac{M}{ma}\mathbb{1}_{[\varepsilon v \leqslant 1]}$, hence $ma(v_\varepsilon')^2 = \frac{\varepsilon^2 M^2}{ma}\mathbb{1}_{[\varepsilon v \leqslant 1]}$ is integrable, and therefore $v_\varepsilon \in D(\tau_{0,\max})$. By Lemma 2.1 we obtain that $v_\varepsilon \in D(\tau)$ for all $\varepsilon > 0$.

Let now $0 \leqslant f \in L_1 \cap L_2(\mu)$ and $u := (1+A)^{-1}f$. For $\varepsilon > 0$ let $c_\varepsilon \in (c,x_0)$ and $d_\varepsilon \in (x_0,d)$ such that $\operatorname{spt} v_\varepsilon = [c_\varepsilon, d_\varepsilon]$. Then

$$\begin{aligned}\tau(u,v_\varepsilon) &= \int_c^d mau'v_\varepsilon' = -\varepsilon\int_{c_\varepsilon}^{d_\varepsilon} u'M = -\varepsilon uM\big|_{c_\varepsilon}^{d_\varepsilon} + \varepsilon\int_{c_\varepsilon}^{d_\varepsilon} um \\ &= -\varepsilon u(d_\varepsilon)M(d_\varepsilon) + \varepsilon u(c_\varepsilon)M(c_\varepsilon) + \varepsilon\int_{c_\varepsilon}^{d_\varepsilon} u\,d\mu \leqslant \varepsilon\|u\|_1\end{aligned}$$

since $u \geqslant 0$, $M(d_\varepsilon) > 0$ and $M(c_\varepsilon) < 0$. Using $v_\varepsilon \uparrow 1$ as $\varepsilon \to 0$, we conclude that

$$0 \leqslant \|f\|_1 - \|u\|_1 = \int (f-u)\,d\mu = \int Au\,d\mu = \lim_{\varepsilon\to 0} \tau(u, v_\varepsilon) \leqslant 0,$$

which implies the assertion. □

2.3 Remark. It is easy to see that the condition $\int_{x_0}^{c} \frac{M(x)}{m(x)a(x)}\,dx = \infty$ can be omitted in the above result if $\mu\big((c, x_0)\big) < \infty$ and the Neumann boundary condition is posed at c. An analogous observation holds for the right boundary point d.

The following example indicates that Proposition 2.2 cannot be proved by an intrinsic metric approach.

2.4 Example. Let $m := \mathbb{1}_{\mathbb{R}}$ and define $a \in L_{1,\mathrm{loc}}(\mathbb{R})$ by

$$a(x) := \begin{cases} 1 & \text{if } 2^k \leqslant |x| \leqslant 2^k + 2^{-k},\ k \in \mathbb{N}, \\ e^{|x|} & \text{otherwise.} \end{cases}$$

Then $M(x) = x$ for all $x \in \mathbb{R}$ and

$$\int_0^{-\infty} \frac{x}{a(x)}\,dx = \int_0^{\infty} \frac{x}{a(x)}\,dx \geqslant \sum_{k=1}^{\infty} \int_{2^k}^{2^k+2^{-k}} x\,dx = \infty,$$

so T_1 is stochastic by Proposition 2.2.

Let now $\psi \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ with $a(\psi')^2 \leqslant 1$ a.e. Then

$$\int_0^{\infty} |\psi'(x)|\,dx \leqslant \int_0^{\infty} a(x)^{-1/2}\,dx < \int_0^{\infty} e^{-|x|/2}\,dx + \sum_{k=1}^{\infty} 2^{-k} = 3,$$

in the same way $\int_{-\infty}^{0} |\psi'(x)|\,dx < 3$, and thus the intrinsic distance between 0 and $\pm\infty$ is less than 3. It also follows that there is no function ψ satisfying condition (1.3).

On the other hand, for $m = \mathbb{1}_{\mathbb{R}}$ and $a(x) = \big((1+|x|)\ln(2+|x|)\big)^2$ ($x \in \mathbb{R}$) one obtains that the intrinsic distance between 0 and $\pm\infty$ is ∞, yet T_1 is not stochastic (cf. [Dav85; Example B]). Therefore, the intrinsic distance between 0 and $\pm\infty$ yields no information about whether T_1 is stochastic or not.

3 A condition of Perelmuter and Semenov

We start with a simple condition for T_1 being stochastic, which is needed for the proof of Theorem 3.3.

3.1 Lemma. *Assume that there exists a sequence $(v_k) \subseteq D(\tau_0)$ such that*

$$\sup_{k\in\mathbb{N}} \tau(v_k) < \infty, \qquad \sup_{k\in\mathbb{N}} \|v_k\|_\infty < \infty$$

and $v_k|_{B_k} = 1$ a.e. for some sequence (B_k) of measurable subsets of Ω with $B_k \uparrow \Omega$ as $k \to \infty$. Then T_1 is stochastic.

Proof. For $u \in D(\tau_0)$ we have $(a\nabla u \cdot \nabla v_k)^2 \leqslant \alpha(a\nabla u \cdot \nabla u)(a\nabla v_k \cdot \nabla v_k)$ by (1.1) and hence

$$\tau(u, v_k)^2 \leqslant \alpha\tau(v_k) \int_{\Omega\setminus B_k} a\nabla u \cdot \nabla u \, d\mu \to 0 \qquad (k \to \infty).$$

Using $\sup_k \tau(v_k) < \infty$ again, we infer that $\tau(u, v_k) \to 0$ as $k \to \infty$ for all $u \in D(\tau)$. For $u \in (1 + A)^{-1}(L_1 \cap L_2(\mu))$ we conclude that

$$\int_\Omega Au \, d\mu = \lim_{k\to\infty} \int_\Omega Au \cdot v_k \, d\mu = \lim_{k\to\infty} \tau(u, v_k) = 0,$$

which implies the assertion. □

3.2 Example. Assume that $n \leqslant 2$, $\mu = \lambda^n$, τ_0 is Neumann type (i.e., (1.2) holds) and that a is bounded. Then the assumptions of Lemma 3.1 are satisfied for $v_k = v(\frac{\cdot}{k})|_\Omega$, where $v \in C_c^\infty(\mathbb{R}^n)$, $v = 1$ on $B(0, 1)$. Indeed,

$$\tau(v_k) = \tfrac{1}{k^2} \int_\Omega a\nabla v(\tfrac{\cdot}{k}) \cdot \nabla v(\tfrac{\cdot}{k}) \, d\lambda^n \leqslant k^{n-2}\|a\|_\infty \|\nabla v\|_\infty^2 \cdot \lambda^n(\operatorname{spt} v),$$

and $v_k = 1$ on $B_k = B(0, k) \cap \Omega$.

In the next theorem we give a much better (but not quite sharp) condition for T_1 being stochastic; see also Theorem 4.6 below. The result is essentially due to [PeSe87], except for the more restrictive framework of that paper. (Perelmuter and Semenov assumed that $\Omega = \mathbb{R}^n$, $a \in C^\infty$ and $\mu = \lambda^n$; the case of more general measures μ was studied in [Lis99; Thm. 4], for a the identity matrix. Our main contribution is a new and simpler proof.

3.3 Theorem. *Assume that there exists ψ satisfying condition* (1.3) *such that*

$$\int_{[r_k \leqslant \psi \leqslant r_k + d_k]} a\nabla\psi \cdot \nabla\psi \, d\mu \leqslant e^{cd_k^2} - 1 \qquad (k \in \mathbb{N}) \tag{3.1}$$

for some $c > 0$ and $(r_k), (d_k) \subseteq (0, \infty)$ with $r_k \to \infty$ as $k \to \infty$. Then T_1 is stochastic.

3.4 Remarks. (a) Because of $a\nabla\psi \cdot \nabla\psi \leqslant 1$, assumption (3.1) is in particular satisfied if

$$\mu\big([r_k \leqslant \psi \leqslant r_k + d_k]\big) \leqslant e^{cd_k^2} - 1 \qquad (k \in \mathbb{N}). \tag{3.2}$$

It is straightforward that the latter holds, e.g., if $\liminf_{r\to\infty} \mu\big([r \leqslant \psi \leqslant 2r]\big)^{1/r^2} < \infty$.

(b) In [PeSe87], the (equivalent, see introduction) question of L_1-uniqueness is studied, and instead of condition (3.1) the following is assumed: There exist sequences $(t_k), (\varepsilon_k), (\beta_k) \subseteq (0, \infty)$ such that $t_k \to \infty$, $\varepsilon_k \to 0$ as $k \to \infty$ and

$$M := \sup_{k\in\mathbb{N}} \tfrac{\varepsilon_k}{\beta_k} \sqrt{t_k}\, \mu\big([1 \leqslant \varepsilon_k\psi \leqslant 1 + \beta_k]\big)^{1/(t_k+1)} < \infty \tag{3.3}$$

(cf. [PeSe87; Rem. 3]). We now show: If (3.3) is satisfied then condition (3.2) holds for $r_k = 1/\varepsilon_k$, $d_k = \beta_k/\varepsilon_k$, $c = M^2$ and all k with $t_k \geqslant 1$. Observe that (3.3) is equivalent to

$$\mu\big([r_k \leqslant \psi \leqslant r_k + d_k]\big) = \mu\big([1 \leqslant \varepsilon_k\psi \leqslant 1 + \beta_k]\big) \leqslant \left(\frac{Md_k}{\sqrt{t_k}}\right)^{t_k+1} \qquad (k \in \mathbb{N}),$$

so we only have to show that $\big(Md_k/\sqrt{t}\big)^{t+1} \leqslant e^{M^2d_k^2}-1$ for all $t \geqslant 1$. With $x = M^2d_k^2/t$ this inequality reads $x^{(t+1)/2} \leqslant e^{tx}-1$. The latter is clear for $0 \leqslant x \leqslant 1$: then $x^{(t+1)/2} \leqslant x \leqslant e^x - 1 \leqslant e^{tx}-1$. For $x > 1$ we can estimate

$$x^{(t+1)/2} \leqslant x^t = e^{t\ln x} \leqslant e^{t(x-1)} \leqslant e^{-1}e^{tx} \leqslant e^{tx}-1$$

since $e^{tx} \geqslant e$.

Conversely, it is easy to see that (3.2) implies (3.3) with $t_k = d_k^2$, $\varepsilon_k = 1/r_k$ and $\beta_k = d_k/r_k$, so the assumption in [PeSe87; Rem. 3] is satisfied if (3.2) holds with $d_k \to \infty$.

3.5 Example (cf. the corollary in [PeSe87; p. 719])**.** Assume that $\mu = \lambda^n$ and that τ_0 is Neumann type (i.e., (1.2) holds), and let $R > 1$. Define $\psi \in C^\infty(\Omega)$ by $\psi(x) := \sqrt{\rho(|x|)}$, where $\rho \in C^\infty[0,\infty)$ is increasing and satisfies $\rho(r) = \ln R$ for $r \leqslant R$ and $\rho(r) = \ln r$ for $r \geqslant R+1$. Then $\nabla\psi = 0$ on $B(0,R)\cap\Omega$ and $\nabla\psi(x) = \frac{1}{2}(\ln|x|)^{-1/2}|x|^{-1}\frac{x}{|x|}$ for $x \in \Omega\setminus B(0,R{+}1)$. Thus, if there exists $c > 0$ such that

$$a_{rr}(x) := a(x)\tfrac{x}{|x|}\cdot\tfrac{x}{|x|} \leqslant c|x|^2\ln|x| \qquad (x \in \Omega,\ |x| > R) \tag{3.4}$$

then $\delta\psi$ satisfies condition (1.3) for some $\delta > 0$ (cf. Remark 1.2(b)). Moreover, for $x \in \Omega$ and $r > R+1$ we have $\psi(x) \leqslant 2r$ if and only if $|x| \leqslant e^{4r^2}$, so

$$\lambda^n\big([r \leqslant \delta\psi \leqslant 2r]\big) \leqslant \omega_n e^{4nr^2/\delta^2} \qquad \big(r > \delta(R+1)\big),$$

where $\omega_n = \lambda^n\big(B(0,1)\big)$. Therefore, Theorem 3.3 is applicable if (3.4) is satisfied.

It is shown in [Dav85; Example B] that T_1 is not stochastic if $(\Omega,\mu) = (\mathbb{R}^n,\lambda^n)$, $D(\tau_0) = C_c^\infty(\mathbb{R}^n)$, $\varepsilon > 0$ and $a(x) = (1+|x|)^2\big(\ln(1+|x|)\big)^{1+\varepsilon}I_n$ for all $x \in \mathbb{R}^n$, where I_n is the identity matrix. In this sense condition (3.4) is rather sharp; however, in Example 4.5 below we will show that the condition can be slightly relaxed.

Since we do not assume τ_0 to be closed, we need the following auxiliary lemma for the proof of Theorem 3.3. We note that the assertion of part (a) is a rather strong result; for our purposes weak convergence would suffice.

3.6 Lemma. (a) *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, $\varphi(0) = 0$. Let $u \in D(\tau)$, (u_k) a sequence in $D(\tau)$ such that $u_k \to u$ in $D(\tau)$. Then $\varphi\circ u_k \to \varphi\circ u$ in $D(\tau)$.*

(b) *Let $0 \leqslant u \in D(\tau)\cap L_\infty(\mu)$. Then there exists a sequence $(u_k) \subseteq D(\tau_0)$ such that $u_k \to u$ in $D(\tau)$ and $0 \leqslant u_k \leqslant \|u\|_\infty$ a.e. for all $k \in \mathbb{N}$.*

Proof. (a) follows from [Anc76; Théorème 10].

(b) Let (v_j) be a sequence in $D(\tau_0)$ with $v_j \to u$ in $D(\tau)$, and let $(\rho_k) \subseteq C^\infty(\mathbb{R})$ be a sequence of normal contractions satisfying $0 \leqslant \rho_k(t) \uparrow t^+ \wedge \|u\|_\infty =: \rho(t)$ as $k \to \infty$, for all $t \in \mathbb{R}$. Then by part (a) we have $D(\tau_0) \ni \rho_k\circ v_j \to \rho_k\circ u$ in $D(\tau)$ as $j \to \infty$, for all $k \in \mathbb{N}$. Moreover, $\rho_k\circ u \to \rho\circ u = u$ in $D(\tau)$ (see, e.g., [Anc76; Prop. 11]), so the assertion follows. □

We point out that in the case that (d_k) is not bounded, Theorem 3.3 can be deduced from our main result, Theorem 4.6 below (see Remark 4.8(c)). Here we give an independent proof that is much more elementary.

Proof of Theorem 3.3. Let $\rho \in C^\infty(\mathbb{R})$ be a decreasing function such that $\rho = 1$ on $(-\infty, 0)$ and $\rho = 0$ on $(1, \infty)$. For $k \in \mathbb{N}$ define $\rho_k \in C_c^\infty[0,\infty)$ by $\rho_k(t) := \rho\big(\frac{t-r_k}{d_k}\big)$; then $v_k := \rho_k \circ \psi \in D(\tau_0)$ by condition (1.3) and $v_k = 1$ on $[\psi \leqslant r_k]$. Moreover, with $C := \|\rho'\|_\infty^2$ and $w := a\nabla\psi \cdot \nabla\psi \,(\leqslant 1)$ we obtain that

$$a\nabla v_k \cdot \nabla v_k = (\rho_k' \circ \psi)^2 a\nabla\psi \cdot \nabla\psi \leqslant C d_k^{-2} \mathbb{1}_{[r_k \leqslant \psi \leqslant r_k + d_k]} w \qquad (k \in \mathbb{N}). \tag{3.5}$$

Suppose first that (d_k) is bounded. Then

$$\tau(v_k) \leqslant C d_k^{-2} \int_{[r_k \leqslant \psi \leqslant r_k + d_k]} w \, d\mu \leqslant C d_k^{-2} \big(e^{cd_k^2} - 1\big) \qquad (k \in \mathbb{N})$$

implies $\sup_k \tau(v_k) < \infty$, and the assertion follows from Lemma 3.1.

If (d_k) is not bounded then we can assume without loss of generality that $d_k \to \infty$ as $k \to \infty$. Let $\delta > 0$ and $v \in D(\tau_0)$. For $0 \leqslant u \in D(\tau_0) \cap L_\infty(\mu)$ we have $\ln(1 + u/\delta) \in D(\tau_0)$, $\nabla \ln(1 + u/\delta) = (u + \delta)^{-1} \nabla u$ and hence

$$(a\nabla u \cdot \nabla v)^2 \leqslant \alpha\big(a\nabla u \cdot \nabla \ln(1 + u/\delta)\big) \cdot (u + \delta)(a\nabla v \cdot \nabla v),$$

so we obtain the estimate

$$\tau(u, v)^2 \leqslant \alpha\tau\big(u, \ln(1 + u/\delta)\big) \int (u + \delta)(a\nabla v \cdot \nabla v) \, d\mu.$$

By Lemma 3.6 it follows that this estimate is valid for all $0 \leqslant u \in D(\tau) \cap L_\infty(\mu)$.

Let now $0 \leqslant f \in L_1(\mu) \cap L_\infty(\mu)$, $u := (1 + A)^{-1} f$. Then $0 \leqslant u \in D(\tau) \cap L_1(\mu) \cap L_\infty(\mu)$, so due to (3.5) we infer for all $k \in \mathbb{N}$ that

$$\begin{aligned}
\langle Au, v_k \rangle^2 = \tau(u, v_k)^2 &\leqslant \alpha\big\langle Au, \ln(1 + u/\delta)\big\rangle \cdot C d_k^{-2} \int_{[r_k \leqslant \psi \leqslant r_k + d_k]} (u + \delta) w \, d\mu \\
&\leqslant \alpha \|Au\|_1 \ln(1 + \|u\|_\infty/\delta) \cdot C d_k^{-2} \Big(\int_{[r_k \leqslant \psi \leqslant r_k + d_k]} u \, d\mu + \delta(e^{cd_k^2} - 1) \Big).
\end{aligned}$$

Choosing $\delta = e^{-2cd_k^2}$ we conclude that $\langle Au, v_k \rangle^2 \to 0$ as $k \to \infty$, and hence $\int Au \, d\mu = 0$. Thus we have shown $\int Au \, d\mu = 0$ for all $u \in (1 + A)^{-1}(L_1 \cap L_\infty(\mu))$, and the assertion follows. $\square$

4 New conditions for stochasticity

In this section we present new conditions in terms of volume growth and of growth of the coefficient matrix a that imply T_1 being stochastic. All the results are based on the following abstract condition.

4.1 Proposition. *Assume that there exist an integrable function $\varphi\colon [0,1] \to [1, \infty]$ and a sequence $(v_k) \subseteq D(\tau_0)$ such that $\sup_k \|v_k\|_\infty < \infty$, $v_k \to 1$ a.e. and*

$$\int_\Omega \frac{a\nabla v_k \cdot \nabla v_k}{\varphi \circ u} \, d\mu \to 0 \qquad (k \to \infty) \tag{4.1}$$

for all $u = (1 + A)^{-1} f$, where $0 \leqslant f \in L_{\infty,c}(\Omega)$, $\|f\|_\infty \leqslant 1$. Then T_1 is stochastic.

Proof. Let $0 \leqslant f \in L_{\infty,\mathrm{c}}(\Omega)$, $\|f\|_\infty \leqslant 1$ and $u := (1+A)^{-1}f$. We are going to show that

$$\tau(u,v)^2 \leqslant \alpha\|\varphi\|_1\|Au\|_1 \int_\Omega \frac{a\nabla v\cdot\nabla v}{\varphi\circ u}\,d\mu \qquad \big(v \in D(\tau_0)\big). \tag{4.2}$$

Then by (4.1) we obtain that

$$\int_\Omega Au\,d\mu = \lim_{k\to\infty}\int_\Omega Au\cdot v_k\,d\mu = \lim_{k\to\infty}\tau(u,v_k) = 0,$$

hence $\int Au\,d\mu = 0$ for all $u \in (1+A)^{-1}(L_{\infty,\mathrm{c}}(\Omega))$, and the assertion follows.

Let $\varepsilon > 0$. Then there exists an increasing sequence $(\varphi_j) \subseteq C^\infty[0,1]$ such that $\varphi_j \geqslant 1$, $\|\varphi_j\|_1 \leqslant (1+\varepsilon)\|\varphi\|_1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty}\varphi_j \geqslant \varphi$ pointwise. Let $j \in \mathbb{N}$, define $h_j \in C^\infty[0,1]$ by $h_j(t) := \int_0^t \varphi_j(s)\,ds$, and fix $v \in D(\tau_0)$. For $0 \leqslant u \in D(\tau_0)$ with $\|u\|_\infty \leqslant 1$ we obtain that $h_j\circ u \in D(\tau_0)$, $\nabla(h_j\circ u) = (\varphi_j\circ u)\nabla u$ and thus

$$(a\nabla u\cdot\nabla v)^2 \leqslant \alpha\big(a\nabla u\cdot\nabla(h_j\circ u)\big)\cdot(\varphi_j\circ u)^{-1}(a\nabla v\cdot\nabla v),$$

so we can estimate

$$\tau(u,v)^2 \leqslant \alpha\tau(u,h_j\circ u)\int_\Omega(\varphi_j\circ u)^{-1}(a\nabla v\cdot\nabla v)\,d\mu. \tag{4.3}$$

Let now $0 \leqslant f \in L_{\infty,\mathrm{c}}(\Omega)$, $\|f\|_\infty \leqslant 1$ and $u := (1+A)^{-1}f$. Then $0 \leqslant u \in D(\tau)$ and $\|u\|_\infty \leqslant 1$. By Lemma 3.6 there exists a sequence $(u_k) \subseteq D(\tau_0)$ such that $0 \leqslant u_k \leqslant 1$ for all $k \in \mathbb{N}$, $u_k \to u$ a.e. and in $D(\tau)$, and $h_j\circ u_k \to h_j\circ u$ in $D(\tau)$ as $k\to\infty$. Moreover, $(\varphi_j\circ u_k)^{-1} \to (\varphi_j\circ u)^{-1}$ a.e. as $k\to\infty$ and $\|(\varphi_j\circ u_k)^{-1}\|_\infty \leqslant 1$ for all $k \in \mathbb{N}$, so from (4.3) it follows that

$$\tau(u,v)^2 \leqslant \alpha\tau(u,h_j\circ u)\int_\Omega(\varphi_j\circ u)^{-1}(a\nabla v\cdot\nabla v)\,d\mu \leqslant \alpha\|Au\|_1\|h_j\circ u\|_\infty\int_\Omega\frac{a\nabla v\cdot\nabla v}{\varphi_j\circ u}\,d\mu.$$

Finally, observe that $\|h_j\circ u\|_\infty \leqslant \|h_j\|_\infty = \|\varphi_j\|_1 \leqslant (1+\varepsilon)\|\varphi\|_1$. Therefore, letting first $j\to\infty$ and then $\varepsilon\to 0$, we conclude that (4.2) holds, and the proof is complete. □

Remark. The above proof can be simplified if the function φ is continuous. In the applications below, φ will even be continuous and decreasing.

Proposition 4.1 will be applied via the following result, which is a preliminary version of our main result, Theorem 4.6 below.

4.2 Proposition. *Assume that there exist ψ satisfying condition* (1.3) *and a decreasing integrable function φ: $[0,1]\to[0,\infty]$ such that*

$$\exp\big(-\varphi\circ(a\nabla\psi\cdot\nabla\psi)\big) \in L_1(\mu).$$

Then T_1 is stochastic.

Proof. First observe that we can assume without loss of generality that φ is continuous and that $\varphi(1) = 0$. Moreover, by replacing φ with $\varphi - \ln$ we can assume that $\varphi(0) = \infty$, $\varphi(t) \geqslant \ln t^{-1}$ for all $t \in (0,1)$ and that φ is strictly decreasing and hence bijective.

We are going to apply Proposition 4.1 with $\varphi_1\colon [0,1] \to [1,\infty]$ defined by $\varphi_1(0) := \infty$ and $\varphi_1(t) := \frac{1}{t}\varphi^{-1}\big(\frac{1}{2}\ln\frac{1}{t}\big)$ for $0 < t \leqslant 1$. Substituting $s = \frac{1}{2}\ln\frac{1}{t}$ we compute

$$\int_0^1 \varphi_1(t)\,dt = \int_0^1 \varphi^{-1}\big(\tfrac{1}{2}\ln\tfrac{1}{t}\big)\,\tfrac{dt}{t} = 2\int_0^\infty \varphi^{-1}(s)\,ds = 2\int_0^1 \varphi(r)\,dr < \infty,$$

so φ_1 is integrable. Next we show that

$$\frac{x}{\varphi_1(t)} \leqslant t + \exp(-\varphi(x)) \qquad (t, x \in [0,1]). \tag{4.4}$$

This inequality is trivial in the cases $t = 0$ and $x \leqslant t\varphi_1(t) = \varphi^{-1}\big(\frac{1}{2}\ln\frac{1}{t}\big)$, so let $0 < t \leqslant 1$ and $\varphi^{-1}\big(\frac{1}{2}\ln\frac{1}{t}\big) < x \leqslant 1$. Then $\varphi(x) < \frac{1}{2}\ln\frac{1}{t}$ and hence $\exp(-\varphi(x)) > t^{1/2}$. Moreover, $\varphi(t^{1/2}) \geqslant \ln t^{-1/2}$ (see the beginning of the proof), thus $\varphi^{-1}(\ln t^{-1/2}) \geqslant t^{1/2}$, and we conclude that

$$\frac{x}{\varphi_1(t)} = \frac{tx}{\varphi^{-1}(\ln t^{-1/2})} \leqslant t^{1/2} < \exp(-\varphi(x)).$$

Let now $\rho \in C_c^\infty[0,\infty)$ be a decreasing function with $\rho(0) = 1$. For $k \in \mathbb{N}$ let $v_k := \rho \circ (\psi/k)$. Then $v_k \in D(\tau_0)$ by condition (1.3), $\|v_k\|_\infty \leqslant 1$ for all $k \in \mathbb{N}$ and $v_k \to 1$ a.e. as $k \to \infty$. Moreover, by (4.4) we obtain for $0 \leqslant f \in L_{\infty,c}(\Omega)$, $\|f\|_\infty \leqslant 1$ and $u := (1+A)^{-1}f$ that $\frac{a\nabla\psi\cdot\nabla\psi}{\varphi_1\circ u} \leqslant u + \exp\big(-\varphi\circ(a\nabla\psi\cdot\nabla\psi)\big) \in L_1(\mu)$ and hence

$$\int_\Omega \frac{a\nabla v_k\cdot\nabla v_k}{\varphi_1\circ u}\,d\mu \leqslant \|\rho'\|_\infty^2 k^{-2}\int_\Omega \frac{a\nabla\psi\cdot\nabla\psi}{\varphi_1\circ u}\,d\mu \to 0 \qquad (k\to\infty),$$

so the assertion follows from Proposition 4.1. □

Remark. The function φ_1 used in the above proof is strictly decreasing: Without loss of generality it was assumed that $\varphi + \ln$ is decreasing; thus $2\varphi + \ln = 2(\varphi + \ln) - \ln$ and hence $r \mapsto re^{2\varphi(r)}$ is strictly decreasing. Since φ is strictly decreasing, this implies that $s \mapsto \varphi^{-1}(s)e^{2s}$ is strictly increasing on $(0,\infty)$, so $t \mapsto \varphi_1(t) = \varphi^{-1}\big(\frac{1}{2}\ln\frac{1}{t}\big)\exp\big(2\cdot\frac{1}{2}\ln\frac{1}{t}\big)$ is strictly decreasing on $(0,1)$.

In the next two results, which deal with the special case $\mu = \lambda^n$, we give conditions for T_1 being stochastic in terms of the growth of the coefficient matrix a at ∞.

4.3 Corollary. *Let $\mu = \lambda^n$ and assume that there exist ψ satisfying condition* (1.3) *and a decreasing function $h\colon [0,\infty) \to (0,1]$ such that*

$$\int_1^\infty \frac{h(r)}{r}\,dr < \infty, \qquad (a\nabla\psi\cdot\nabla\psi)(x) \leqslant h(|x|) \quad (x \in \Omega).$$

Then T_1 is stochastic.

Proof. Assume without loss of generality that $h|_{[0,1]} = 1$ and that $h\colon [1,\infty) \to (0,1]$ is bijective. Then $\ln\circ h^{-1}\colon (0,1] \to [0,\infty)$ is decreasing, and

$$\int_0^1 \ln h^{-1}(t)\,dt = \int_0^\infty h(e^s)\,ds = \int_1^\infty h(r)\,\tfrac{dr}{r} < \infty.$$

Thus, $\varphi := (n+1)\ln\circ h^{-1}$ (with $\varphi(0) := \infty$) is decreasing and integrable. Moreover,

$$\exp\big(-\varphi\circ(a\nabla\psi\cdot\nabla\psi)\big)(x) \leqslant \exp\big(-\varphi(h(|x|))\big) = |x|^{-n-1}\wedge 1 \qquad (x \in \Omega),$$

so the assertion follows from Proposition 4.2. □

The following result generalises Example 3.5; its proof illustrates the possible interplay between the functions ψ and h in Corollary 4.3.

4.4 Corollary. *Assume that* $\mu = \lambda^n$, τ_0 *is Neumann type (i.e.,* (1.2) *holds) and that there exist an open set* $X \subseteq (0,\infty)$ *and an increasing function* $g\colon (0,\infty) \to [1,\infty)$ *such that*

$$\int_1^\infty \frac{dr}{rg(r)} = \infty, \qquad a_{rr}(x) = a(x)\tfrac{x}{|x|} \cdot \tfrac{x}{|x|} \leqslant |x|^2 g(|x|) \quad (x \in \Omega,\ |x| \in X)$$

and $\liminf_{r\to\infty} \lambda\big(X \cap (0,r)\big)/r > 0$. *Then* T_1 *is stochastic.*

If $\tau_0 = \tau_{0,\max}$ *then the assumption on* X *to be open can be omitted.*

Proof. Without loss of generality we can assume that $g \in C^\infty(0,\infty)$ (convolution with a $C_c^\infty(-1,0)$-function) and that $(0,1) \subseteq (0,\infty) \setminus X$. Let $\varepsilon, R > 0$ such that $\lambda\big(X \cap (0,r)\big) \geqslant 2\varepsilon r$ for all $r > R$. Since X is open, there exists a C^∞-function $\chi\colon (0,\infty) \to [0,1]$ such that $\chi = 0$ on $(0,\infty) \setminus X$ and $\int_0^r \chi(s)\,ds \geqslant \varepsilon r$ for all $r > R$.

Define $f\colon [0,\infty) \to [1,\infty)$ by $f(r) = 1 + \int_0^r \frac{\chi(s)}{sg(s)}\,ds$ and $0 \leqslant \psi \in C^\infty(\Omega)$ by $\psi(x) := \ln f(|x|)$. With $c := 2/\varepsilon$ we obtain that

$$\int_r^{cr} \chi(s)\,ds \geqslant \int_0^{cr} \chi(s)\,ds - r \geqslant \varepsilon cr - r = r = \frac{1}{c^2 - c}\int_{cr}^{c^2 r} ds \qquad (r > R). \tag{4.5}$$

Since g is increasing, we deduce that

$$\int_R^\infty \frac{\chi(s)}{sg(s)}\,ds \geqslant \frac{1}{c^2-c}\int_{cR}^\infty \frac{1}{sg(s)}\,ds = \infty,$$

so $f(r) \to \infty$ as $r \to \infty$. Moreover,

$$(a\nabla\psi \cdot \nabla\psi)(x) = \left(\frac{\chi(|x|)}{f(|x|)\cdot|x|g(|x|)}\right)^2 a_{rr}(x) \leqslant \frac{1}{f(|x|)^2 g(|x|)} \leqslant 1$$

for all $x \in \Omega$, hence by Remark 1.2(b) it follows that ψ satisfies condition (1.3). Finally, $h := \frac{1}{f^2 g}$ is decreasing and as above we deduce from (4.5) that

$$\frac{1}{c^2-c}\int_{cR}^\infty \frac{h(r)}{r}\,dr \leqslant \int_R^\infty \frac{\chi(r)h(r)}{r}\,dr = \int_R^\infty \frac{f'(r)}{f(r)^2}\,dr = -\frac{1}{f(r)}\bigg|_R^\infty = \frac{1}{f(R)} < \infty,$$

so the assertion follows from Corollary 4.3.

In the case $\tau_0 = \tau_{0,\max}$, the above proof works with $\chi = \mathbb{1}_X$. Then we only have $\psi \in W^1_{\infty,c}(\Omega)$, but ψ still satisfies condition (1.3). □

4.5 Example. The condition of Corollary 4.4 is satisfied, e.g., if there exists $c > 0$ such that

$$a_{rr}(x) \leqslant c|x|^2 \ln|x| \ln\ln|x| \quad (x \in \Omega,\ |x| > 10).$$

The following is the main result of the paper; the proof will be given at the end of the section.

4.6 Theorem. *Assume that there exist* ψ *satisfying condition* (1.3)*, a decreasing integrable function* $\varphi\colon [0,1] \to [0,\infty]$ *and an increasing function* $v\colon (0,\infty) \to (3,\infty)$ *such that*

$$\int_0^\infty \frac{r}{\ln v(r)}\,dr = \infty, \qquad \int_{[\psi\leqslant r]} \exp\bigl(-\varphi\circ(a\nabla\psi\cdot\nabla\psi)\bigr)\,d\mu \leqslant v(r) \quad (r>0). \tag{4.6}$$

Then T_1 *is stochastic.*

Setting $\varphi = 0$ in this theorem, we obtain the following volume growth condition for stochasticity.

4.7 Corollary. *Assume that there exist* ψ *satisfying condition* (1.3) *and an increasing function* $v\colon (0,\infty) \to (3,\infty)$ *such that*

$$\int_0^\infty \frac{r}{\ln v(r)}\,dr = \infty, \qquad \mu\bigl([\psi\leqslant r]\bigr) \leqslant v(r) \quad (r>0).$$

Then T_1 *is stochastic.*

4.8 Remarks. (a) In the context of stochastical completeness of Riemannian manifolds, the volume growth condition in Corollary 4.7 is due to [KaLi83] and [Gri87]. In [Gri89] it is shown that the condition is sharp in the following sense: Given a smooth increasing function $v\colon (0,\infty)\to(3,\infty)$ such that $\int_0^\infty \frac{r}{\ln v(r)}\,dr < \infty$, there exists a complete Riemannian manifold M such that $\mu\bigl(B(x,r)\bigr) \leqslant v(r)$ for all $r>0$ and some $x\in M$, and M is not stochastically complete, i.e., the L_1-semigroup associated with the canonical Dirichlet form on M is not stochastic. (Here, $\mu\bigl(B(x,r)\bigr)$ denotes the Riemannian volume of the geodesic ball with centre x and radius r.) This does not mean, however, that the condition of Corollary 4.7 is necessary; see Example 4.9 below.

(b) The condition $\int_1^\infty \frac{r}{\ln v(r)}\,dr = \infty$ is satisfied, e.g., if $v(r) = e^{cr^2}$ or $v(r) = e^{cr^2\ln r}$ for some $c>0$. Observe that in the former case, Corollary 4.7 is already covered by Theorem 3.3 (see Remark 3.4(a)), but it is not hard to see that in the latter case it is not.

(c) As pointed out in Section 3, Theorem 3.3 can be deduced from Theorem 4.6 if (d_k) is not bounded. Note that in this case we can assume without loss of generality that (d_k) is increasing and that $r_k+d_k < r_{k+1}$ for all $k\in\mathbb{N}$. Moreover, $D(\tau_0) = D(\tau)\cap D(\tau_{0,\max})$ without loss of generality.

Let $\rho\colon [0,\infty)\to[0,\infty)$ be such that $\rho(0)=0$, and ρ has slope 1 on the intervals $[r_k, r_k+d_k]$ and slope 0 otherwise. Then $\rho(r)\to\infty$ as $r\to\infty$, so $\tilde\psi := \rho\circ\psi$ satisfies condition (1.3) by Remark 1.2(d). Moreover, for $k\in\mathbb{N}$ and $\tilde r_k := \sum_{j=1}^k d_j$ we obtain that

$$\int_{[\psi\leqslant\tilde r_k]} a\nabla\tilde\psi\cdot\nabla\tilde\psi\,d\mu = \sum_{j=1}^k \int_{[r_j\leqslant\psi\leqslant r_j+d_j]} a\nabla\psi\cdot\nabla\psi\,d\mu \leqslant \sum_{j=1}^k e^{cd_j^2} \leqslant ke^{cd_k^2}.$$

We now define the functions φ and v by $\varphi(0) := \infty$, $\varphi(t) := \ln\frac{1}{t}$ for $0<t\leqslant 1$ and $v(r) := ke^{cd_k^2}$ for $\tilde r_{k-1} < r \leqslant \tilde r_k$, $k\in\mathbb{N}$ (where $\tilde r_0 := 0$). Then

$$\int_{[\psi\leqslant r]} \exp\bigl(-\varphi\circ(a\nabla\psi\cdot\nabla\psi)\bigr)\,d\mu = \int_{[\psi\leqslant r]} a\nabla\psi\cdot\nabla\psi\,d\mu \leqslant v(r) \qquad (r>0)$$

and

$$\int_0^\infty \frac{r}{\ln v(r)}\,dr = \sum_{k=1}^\infty \int_{\tilde r_{k-1}}^{\tilde r_k} \frac{r}{\ln k + cd_k^2}\,dr \geqslant \sum_{k=1}^\infty \frac{d_k^2/2}{\ln k + cd_k^2} = \infty.$$

Thus, we can apply Theorem 4.6 to obtain that T_1 is stochastic.

We now present two examples for situations in which Theorem 4.6 is applicable, but Corollary 4.7 is not. These examples illustrate that if one has good control of the coefficients a and m on a suitable small portion of Ω then no control on the remainder of Ω is needed.

4.9 Example. We are going to apply Theorem 4.6 with $\varphi|_{(0,1]} = 0$, $\varphi(0) = \infty$. Let $(r_k) \subseteq (0, \infty)$ be a sequence with $r_{k+1} \geqslant r_k + 1$ for all $k \in \mathbb{N}$.

(a) Let $\Omega := \mathbb{R} \times (0, 1) \subseteq \mathbb{R}^2$, and for $k \in \mathbb{N}$ let $\Omega_k := \{x \in \Omega;\ r_k < |x_1| < r_k + 1\}$. Assume that $m \leqslant 1$ and $a_{11} \leqslant k^2$ on Ω_k, for all $k \in \mathbb{N}$, and that

$$D(\tau_0) \supseteq \{u|_\Omega;\ u \in C^\infty(\mathbb{R}^2),\ u(x_1, 0) = u(x_1, 1)\ (x_1 \in \mathbb{R})\}.$$

(This includes Neumann and periodic boundary conditions.) There exists $0 \leqslant \psi_0 \in C^\infty(\mathbb{R})$ such that $\psi_0(x) \geqslant \frac{1}{2} \ln k$ and $|\psi_0'(x)| \leqslant \frac{1}{k}$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}$ with $r_k \leqslant |x| \leqslant r_k + 1$, $\psi_0'(x) = 0$ otherwise. Then for $\psi \in C^\infty(\Omega)$ defined by $\psi(x) := \psi_0(x_1)$ we obtain that ψ satisfies condition (1.3) ($(a\nabla\psi \cdot \nabla\psi)(x) = a_{11}(x)\psi_0'(x_1)^2 \leqslant 1$ for all $x \in \Omega$.)

Let now $r > 0$ and choose $k \in \mathbb{N}$ such that $\frac{1}{2} \ln k \leqslant r < \frac{1}{2} \ln(k + 1)$. Then $[\psi \leqslant r]$ is contained in $(-r_{k+1}, r_{k+1}) \times (0, 1)$, hence

$$\int_{[\psi \leqslant r]} \exp\bigl(-\varphi \circ (a\nabla\psi \cdot \nabla\psi)\bigr)\, d\mu \leqslant \sum_{j=1}^{k} \mu(\Omega_j) \leqslant 2k \leqslant 2e^{2r},$$

so from Theorem 4.6 it follows that T_1 is stochastic. Observe that we have no control of $\mu([\psi \leqslant r])$ in this example, so we cannot apply Corollary 4.7.

(b) Let $\Omega := \mathbb{R}^n$, and for $k \in \mathbb{N}$ let $\Omega_k := \{x \in \mathbb{R}^n;\ r_k < |x| < r_k + \frac{1}{k}\}$. Assume that $\mu(\Omega_k) \leqslant \exp\bigl((\ln k)^2\bigr)$ and $a_{rr} \leqslant 1$ on Ω_k, for all $k \in \mathbb{N}$, and that $D(\tau_0) \supseteq C_c^\infty(\mathbb{R}^n)$. There exists a radially symmetric function $0 \leqslant \psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \geqslant \frac{1}{2} \ln k$ and $|\nabla\psi| \leqslant 1$ on Ω_k for all $k \in \mathbb{N}$, $\nabla\psi = 0$ otherwise. Then ψ satisfies condition (1.3), and with the same argument as in part (a) one shows that

$$\int_{[\psi \leqslant r]} \exp\bigl(-\varphi \circ (a\nabla\psi \cdot \nabla\psi)\bigr)\, d\mu \leqslant e^{2r + 4r^2} \qquad (r > 0),$$

so T_1 is stochastic by Theorem 4.6.

For the proof of Theorem 4.6 we need the following technical result.

4.10 Lemma. *Let γ: $(0, \infty) \to (1, \infty)$ be a measurable function with $\int_0^\infty \frac{r}{\gamma(r)}\, dr = \infty$.*

(a) *If γ is increasing then $\int_0^\infty \frac{r}{\gamma(r) + r^2}\, dr = \infty$.*

(b) *If $\gamma(r) \geqslant r^2$ for all $r > 0$ then there exists a decreasing and bijective function g: $(0, \infty) \to (0, 1)$ such that $g \notin L_1(0, \infty)$ and $\gamma \circ (g^2)^{-1} \in L_1(0, 1)$.*

Proof. (a) Let $\tilde\gamma(r) := \gamma(r) + r^2$ for all $r > 0$. If there exists $r_0 > 0$ such that $\gamma(r) \geqslant r^2$ for all $r \geqslant r_0$ then $\int_0^\infty \frac{r}{\tilde\gamma(r)}\, dr \geqslant \int_{r_0}^\infty \frac{r}{2\gamma(r)}\, dr = \infty$. If there exists no such r_0 then there exists a sequence $(r_k) \subseteq (1, \infty)$ such that $\gamma(r_k) < r_k^2$ and $r_{k+1} \geqslant 2r_k$ for all $k \in \mathbb{N}$. We conclude that $\tilde\gamma(r) \leqslant \tilde\gamma(r_k) < 2r_k^2$ for all $k \in \mathbb{N}$ and all $r \leqslant r_k$, and therefore

$$\int_0^\infty \frac{r}{\tilde\gamma(r)}\, dr \geqslant \sum_{k=1}^\infty \frac{1}{2r_k^2} \int_{r_k/2}^{r_k} r\, dr = \sum_{k=1}^\infty \frac{1}{2r_k^2} \cdot \tfrac{1}{2}(r_k^2 - r_k^2/4) = \infty.$$

(b) We define $f, F, \tilde{g}$: $(0,\infty) \to (0,\infty)$ by

$$f(r) := \frac{r}{\gamma(r)}, \qquad F(r) := 2 + \int_0^r f(s)\,ds, \qquad \tilde{g}(r) := \int_r^\infty \frac{ds}{\gamma(s)F(s)}.$$

(The function $\tilde{g}$ is defined since $\gamma(s) \geqslant s^2$ for all $s > 0$.) Then $\tilde{g}$ is strictly decreasing and continuous, and

$$\int_0^\infty \tilde{g}(r)\,dr = \int_0^\infty \frac{1}{\gamma(s)F(s)} \int_0^s dr\,ds = \int_0^\infty \frac{f(s)}{F(s)}\,ds = \ln F(s)\big|_0^\infty = \infty$$

since $F(r) \to \infty$ as $r \to \infty$ by the assumption.

Let now $c := 1/\tilde{g}(0+)$ and $g := c\tilde{g}$. Then g: $(0,\infty) \to (0,1)$ is decreasing and bijective. It remains to show that $\gamma \circ (g^2)^{-1} \in L_1(0,1)$. Because of $F(r) \geqslant 2$ and $\gamma(r) \geqslant r^2$ we have

$$\frac{2rf(r)}{F(r)^2} = \frac{2r^2}{\gamma(r)F(r)^2} \leqslant \frac{1}{F(r)}$$

and hence

$$\frac{1}{F(r)} \leqslant \frac{2}{F(r)} - \frac{2rf(r)}{F(r)^2} = \frac{d}{dr}\frac{2r}{F(r)}$$

for all $r > 0$. Thus, $\int_0^s \frac{dr}{F(r)} \leqslant \frac{2r}{F(r)}\big|_0^s = \frac{2s}{F(s)}$ for all $s > 0$. We conclude that

$$\begin{aligned}
\int_0^1 \gamma \circ (g^2)^{-1}(r)\,dr &= \int_\infty^0 \gamma(r)(g^2)'(r)\,dr = -2c^2 \int_0^\infty \gamma(r)\tilde{g}(r)\tilde{g}'(r)\,dr \\
&= 2c^2 \int_0^\infty \frac{\tilde{g}(r)}{F(r)}\,dr = 2c^2 \int_0^\infty \frac{1}{\gamma(s)F(s)} \int_0^s \frac{dr}{F(r)}\,ds \\
&\leqslant 2c^2 \int_0^\infty \frac{2s}{\gamma(s)F(s)^2}\,ds = -\frac{4c^2}{F(s)}\bigg|_0^\infty = \frac{4c^2}{F(0+)} = 2c^2.
\end{aligned}$$

□

Proof of Theorem 4.6. Define γ: $(0,\infty) \to (1,\infty)$ by $\gamma(r) := \ln v(r+1) + (r+1)^2$. Then

$$\int_0^\infty \frac{r}{\gamma(r)}\,dr = \int_1^\infty \frac{r-1}{\ln v(r) + r^2}\,dr = \infty$$

by Lemma 4.10(a) since v is increasing, and $\gamma(r) \geqslant r^2$ for all $r > 0$. By Lemma 4.10(b) there exists a decreasing and bijective function g: $(0,\infty) \to (0,1)$ such that $g \notin L_1(0,\infty)$ and $\varphi_0 := \gamma \circ (g^2)^{-1} \in L_1(0,1)$.

Without loss of generality we assume that $D(\tau_0) = D(\tau) \cap D(\tau_{0,\max})$. Let $\rho(r) := \int_0^r g(s)\,ds$ for all $r > 0$. Then $\|\rho'\|_\infty \leqslant 1$ and $\rho(r) \to \infty$ as $r \to \infty$, so $\tilde{\psi} := \rho \circ \psi$ satisfies condition (1.3) by Remark 1.2(d). Moreover,

$$a\nabla\tilde{\psi} \cdot \nabla\tilde{\psi} = (g \circ \psi)^2 a\nabla\psi \cdot \nabla\psi.$$

Let now $w := \exp\bigl(-\varphi \circ (a\nabla\psi \cdot \nabla\psi)\bigr)$ and $\tilde{\varphi} := \varphi_0 + \varphi$. Since both φ_0 and φ are decreasing and $(g \circ \psi)^2 \leqslant 1$, $a\nabla\psi \cdot \nabla\psi \leqslant 1$ a.e., we infer that

$$\tilde{\varphi} \circ (a\nabla\tilde{\psi} \cdot \nabla\tilde{\psi}) \geqslant \varphi_0 \circ (g \circ \psi)^2 + \varphi \circ (a\nabla\psi \cdot \nabla\psi) = \gamma \circ \psi - \ln w.$$

We conclude that

$$\begin{aligned}\int_\Omega \exp\bigl(-\tilde\varphi\circ(a\nabla\tilde\psi\cdot\nabla\tilde\psi)\bigr)\,d\mu &\leqslant \sum_{k=1}^\infty \int_{[k-1\leqslant\psi<k]} \exp\bigl(-\gamma\circ\psi+\ln w\bigr)\,d\mu\\ &\leqslant \sum_{k=1}^\infty \exp\bigl(-\gamma(k-1)\bigr)\int_{[\psi\leqslant k]} w\,d\mu \leqslant \sum_{k=1}^\infty \exp(-k^2)<\infty,\end{aligned}$$

where we have used that $\exp\bigl(-\gamma(k-1)\bigr)=\exp(-k^2)/v(k)$ for all $k\in\mathbb{N}$. Since $\tilde\varphi$ is decreasing and integrable, the assertion follows from Proposition 4.2. □

4.11 Remark. In the above proof, only the estimate

$$\int_{[k-1\leqslant\psi<k]} \exp\bigl(-\varphi\circ(a\nabla\psi\cdot\nabla\psi)\bigr)\,d\mu\leqslant v(k)\quad (k\in\mathbb{N})$$

was needed. This observation, however, is not useful for weakening the assumptions of Theorem 4.6. Indeed, if the above estimate is satisfied for some increasing function $v\colon(0,\infty)\to(3,\infty)$ such that $\int_0^\infty \frac{r}{\ln v(r)}\,dr=\infty$ then a straightforward computation shows that (4.6) holds with $\tilde v(r):=(r+1)v(r+1)$ in place of $v(r)$ (take into account Lemma 4.10(a)).

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