The regular part of symmetric forms associated with second order elliptic differential expressions

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Abstract

Let τ be a (not necessarily closable) positive symmetric form associated with a second order elliptic differential expression. We show that the regular part of τ (in the sense of B. Simon) can be obtained by modifying the coefficients of τ suitably; in particular, the regular part is again associated with a second order elliptic differential expression.

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Selfadjoint second order elliptic differential operators in divergence form are commonly defined by the form method: One starts with a positive symmetric form determined by the coefficients of the operator and the boundary conditions under consideration. If this form is closable, one defines the elliptic differential operator via Kato's representation theorem as the selfadjoint operator associated with the closure of the form.

In [ERSZ07], this approach is generalised as follows: Instead of assuming the form to be closable, the regular part of the form is taken; since the latter is always closable, one can then proceed as before. The aim of the present note is to demonstrate that by this approach one does not obtain new types of elliptic differential operators. Indeed, taking the regular part amounts to a modification of the coefficients of the form; see Theorem 1 below.

Before specifying our assumptions and notation, we recall the definition of the regular and singular parts of a positive symmetric form, which is due to Simon [Sim78; Sec. 2]. Let τ be a positive symmetric form in a Hilbert space $\mathcal{H}$, and endow $D(\tau)$ with the norm $\|\cdot\|_\tau$ given by $\|u\|_\tau^2 = \tau(u) + \|u\|_2^2$. Let $\widetilde{D(\tau)}$ be the completion of $D(\tau)$ and i_τ the continuous extension of the embedding $D(\tau) \hookrightarrow \mathcal{H}$ to $\widetilde{D(\tau)}$. Denote by Q_τ the orthogonal projection from $\widetilde{D(\tau)}$ onto the kernel of i_τ. Then the regular part τ_r and the singular part τ_s of τ are the symmetric forms in $\mathcal{H}$ defined by $D(\tau_\mathrm{r}) := D(\tau_\mathrm{s}) := D(\tau)$,

$$\tau_\mathrm{s}(u,v) := \langle Q_\tau u, v\rangle_{\widetilde{D(\tau)}}, \qquad \tau_\mathrm{r}(u,v) := \tau(u,v) - \tau_\mathrm{s}(u,v). \tag{1}$$

By [Sim78; Thms. 2.1 and 2.2], τ_r is the largest closable symmetric form less than τ, and τ_r is positive. (In [Sim78; Sec. 2], it is assumed that $\mathcal{H}$ is a complex Hilbert space and that τ is densely defined, but these assumptions are not needed for the definition and properties of τ_r and τ_s.)

Let now $\Omega \subseteq \mathbb{R}^n$ be an open set and μ a positive Radon measure on Ω with $\operatorname{spt}\mu = \Omega$. Let $a\colon \Omega \to \mathbb{R}^{n\times n}$ be a locally integrable function with values in the symmetric, positive semidefinite matrices. We define the symmetric form $\tau_{\max}$ in $L_2(\mu)$ by

$$D(\tau_{\max}) := \bigl\{u \in L_2(\mu) \cap W^1_{1,\mathrm{loc}}(\Omega);\ a\nabla u \cdot \nabla u \in L_1(\Omega)\bigr\},$$
$$\tau_{\max}(u,v) := \int_\Omega a\nabla u \cdot \nabla v.$$

(Throughout we assume the function spaces to be real vector spaces.) Let τ be a restriction of $\tau_{\max}$ with the following two properties:

(i) $D(\tau) \cap L_\infty(\mu)$ is invariant under multiplication with $C_\mathrm{c}^\infty(\Omega)$-functions,

(ii) there exists $\psi \in C_\mathrm{b}^1(\mathbb{R})$ with $\psi(0) = 0$ and $\psi'(0) = 1$ such that $\psi \circ u \in D(\tau)$ for all $u \in D(\tau)$.

One easily checks that these properties are satisfied, e.g., if $\tau = \tau_{\max}$, $D(\tau) = D(\tau_{\max}) \cap C^\infty(\Omega)$ or $D(\tau) = C_\mathrm{c}^\infty(\Omega)$, but also periodic type boundary conditions are covered.

Since a takes its values in the symmetric, positive semidefinite matrices, one can pointwise take the square root of a. The function $a^{1/2}$ thus obtained is measurable: For $f \in C(\mathbb{R})$ one sees by approximation with polynomials that the mapping $B \mapsto f(B)$, in the space of all selfadjoint operators $B \in \mathcal{L}(\mathbb{R}^N)$, is continuous. For $f(t) := (t^+)^{1/2}$ it follows that the function $x \mapsto a(x)^{1/2}$ on Ω is measurable.

The following is the main theorem of this note; its proof will be given below.

Theorem 1. *Under the above assumptions, the regular part τ_r of τ is given by*

$$\tau_\mathrm{r}(u,v) = \int a_\mathrm{r}\nabla u \cdot \nabla v \qquad \bigl(u, v \in D(\tau_\mathrm{r}) = D(\tau)\bigr),$$

where $a_\mathrm{r} = a^{1/2} p a^{1/2}$ for some measurable function $p\colon \Omega \to \mathbb{R}^{n\times n}$ with values in the orthogonal projection matrices.

Corollary 2. *Assume that $n = 1$ and $D(\tau) = C_\mathrm{c}^\infty(\Omega)$. Then the assertion of Theorem 1 holds with $a_\mathrm{r} = a\mathbf{1}_{\Omega_0}$, where*

$$\Omega_0 = \Bigl\{x \in \Omega;\ \exists\, \varepsilon > 0\colon \int_{x-\varepsilon}^{x+\varepsilon} a^{-1} < \infty\Bigr\}.$$

Proof. For $A \subseteq \Omega$ let τ_A be the symmetric form in $L_2(\mu)$ defined by $D(\tau_A) = C_\mathrm{c}^\infty(\Omega)$, $\tau_A(u,v) = \int_\Omega a\mathbf{1}_A u'v'$. It follows from [RöWi85; Thm. 1.1] that $A = \Omega_0$ is the largest set $A \subseteq \Omega$ such that τ_A is closable. By Theorem 1, this implies the assertion. $\square$

For the proof of Theorem 1 we need the following criterion for an orthogonal projection in a vector-valued L_2-space to be a multiplication operator.

Proposition 3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and H a separable Hilbert space. Let $\mathcal{G}$ be a closed subspace of $L_2(\Omega; H)$ satisfying $\varphi f \in \mathcal{G}$ for all $\varphi \in C_\mathrm{c}^\infty(\Omega)$, $f \in \mathcal{G}$. Then the orthogonal projection Q from $L_2(\Omega; H)$ onto $\mathcal{G}$ is given by*

$$(Qf)(x) = q(x)f(x) \qquad \bigl(f \in L_2(\Omega; H),\ x \in \Omega\bigr), \tag{2}$$

for some strongly measurable function $q\colon \Omega \to \mathcal{L}(H)$ with values in the orthogonal projections.

Proof. Below we will show that $Qf = 0$ a.e. on $[f = 0] := \{x \in \Omega;\ f(x) = 0\}$, for all $f \in L_2(\Omega; H)$. Then from [ArTh05; Thm. 2.3 and Cor. 2.4] (see also [Eva76; Thm. 5.7]) it follows that Q is given by (2), for some strongly measurable function $q\colon \Omega \to \mathcal{L}(H)$ with $\|q\|_{L_\infty(\Omega;\mathcal{L}_s(H))} = \|Q\| \leqslant 1$. For all $f \in L_2(\Omega; H)$ we obtain that $q^2 f = Q^2 f = Qf = qf$, so $q^2 = q$ a.e. Recalling that projections with norm less than or equal to 1 are orthogonal, we can thus assume without loss of generality that all the operators $q(x)$ are orthogonal projections, and the assertion is proved.

Let now $f \in L_2(\Omega; H)$; we show that $Qf = 0$ a.e. on $[f = 0]$. For $A := [f \neq 0]$ we find by convolution a sequence $(\varphi_k) \subseteq C_c^\infty(\Omega)$ such that $\varphi_k \to \mathbf{1}_A$ a.e. and $\|\varphi_k\|_\infty \leqslant 1$ for all $k \in \mathbb{N}$. Then $\varphi_k Qf \to \mathbf{1}_A Qf$ in $L_2(\Omega; H)$, so the assumption implies that $\mathbf{1}_A Qf \in \mathcal{G}$. Since

$$\|f - \mathbf{1}_A Qf\|_2 = \|\mathbf{1}_A(f - Qf)\|_2 \leqslant \|f - Qf\|_2$$

and Qf is the best approximation of f in $\mathcal{G}$, we conclude that $\mathbf{1}_A Qf = Qf$, i.e., $Qf = 0$ a.e. on $\Omega \setminus A = [f = 0]$. $\square$

Proof of Theorem 1. We endow $D(\tau)$ with the norm $\|\cdot\|_\tau$ given by $\|u\|_\tau^2 = \tau(u) + \|u\|_2^2$. Then the embedding

$$D(\tau) \hookrightarrow L_2(\mu) \times L_2(\Omega)^n, \qquad u \mapsto (u, a^{1/2}\nabla u)$$

is isometric, so we can consider the completion $\widetilde{D(\tau)}$ of $D(\tau)$ as a subspace of $L_2(\mu) \times L_2(\Omega)^n$. The continuous extension of the embedding $D(\tau) \hookrightarrow L_2(\mu)$ to $\widetilde{D(\tau)}$ has the kernel $\{0\} \times \mathcal{H}_s$, where

$$\mathcal{H}_s := \bigl\{w \in L_2(\Omega)^n;\ (0, w) \in \widetilde{D(\tau)}\bigr\}.$$

Let Q be the orthogonal projection from $L_2(\Omega)^n$ onto $\mathcal{H}_s$. Then the orthogonal projection from $\widetilde{D(\tau)}$ onto $\{0\} \times \mathcal{H}_s$ is given by $(u, w) \mapsto (0, Qw)$. Thus, by (1) the singular part τ_s of τ is given by

$$\tau_s(u, v) = \bigl\langle (0, Q(a^{1/2}\nabla u)), (v, a^{1/2}\nabla v)\bigr\rangle_{\widetilde{D(\tau)}} = \int_\Omega Q(a^{1/2}\nabla u) \cdot a^{1/2}\nabla v$$

for all $u, v \in D(\tau_s) = D(\tau)$.

Below we will show that $\varphi f \in \mathcal{H}_s$ for all $\varphi \in C_c^\infty(\Omega)$, $f \in \mathcal{H}_s$. Then by Proposition 3 we obtain that the projection Q is given by

$$(Qf)(x) = q(x)f(x) \qquad \bigl(f \in L_2(\Omega)^n,\ x \in \Omega\bigr),$$

for some measurable function $q\colon \Omega \to \mathbb{R}^{n\times n}$ with values in the orthogonal projection matrices. We conclude that $\tau_s(u, v) = \int_\Omega a^{1/2} q a^{1/2} \nabla u \cdot \nabla v$ for all $u, v \in D(\tau_s)$, and the assertion follows (with $p = \mathrm{id} - q$) since $\tau_r = \tau - \tau_s$.

Let $f \in \mathcal{H}_s$. Then there exists a sequence $(\tilde{u}_k) \subseteq D(\tau)$ such that

$$\tilde{u}_k \to 0 \quad \text{in } L_2(\mu), \qquad a^{1/2}\nabla \tilde{u}_k \to f \quad \text{in } L_2(\Omega)^n \qquad (k \to \infty),$$

without loss of generality $\tilde{u}_k \to 0$ a.e. Let $\psi \in C_b^1(\mathbb{R})$ be as in property (ii) of τ. Then $u_k := \psi \circ \tilde{u}_k \in D(\tau)$ for all $k \in \mathbb{N}$. Moreover, the sequence $(\psi' \circ \tilde{u}_k)_k$ is bounded in $L_\infty(\Omega)$ and $\psi' \circ \tilde{u}_k \to \psi'(0) = 1$ a.e., so we obtain that

$$u_k \to 0 \quad \text{in } L_2(\mu), \qquad a^{1/2}\nabla u_k = (\psi' \circ \tilde{u}_k) a^{1/2} \nabla \tilde{u}_k \to f \quad \text{in } L_2(\Omega)^n \qquad (k \to \infty).$$

Let now $\varphi \in C_c^\infty(\Omega)$. Then

$$\|u_k a^{1/2}\nabla\varphi\|_{L_2(\Omega)^n}^2 = \int_\Omega |u_k|^2 a\nabla\varphi \cdot \nabla\varphi \to 0 \qquad (k \to \infty)$$

by the dominated convergence theorem (use $\|\psi\|_\infty^2 a\nabla\varphi \cdot \nabla\varphi$ as a dominating function), and hence

$$a^{1/2}\nabla(\varphi u_k) = \varphi a^{1/2}\nabla u_k + u_k a^{1/2}\nabla\varphi \to \varphi f \quad \text{in } L_2(\Omega)^n \qquad (k \to \infty).$$

Moreover, $\varphi u_k \in D(\tau)$ for all $k \in \mathbb{N}$ by property (i) of τ and $\varphi u_k \to 0$ in $L_2(\mu)$ as $k \to \infty$, so we conclude that $\varphi f \in \mathcal{H}_s$. □

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