An Eberlein-Šmulian type result for the weak* topology

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Abstract. We present a result on relative weak* compactness in the dual of a Banach space X that allows a short proof of both the Eberlein-Šmulian theorem and Šmulian's characterisation of weak compactness of closed convex subsets of X.

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A subset A of a Banach space X is called conditionally countably weakly compact if every sequence in A has a weak accumulation point in X. The difficult implication of the Eberlein-Šmulian theorem states that such a set is already relatively weakly compact. This implication was proved by W. Eberlein in [Ebe47]; an elementary proof was given by R. Whitley [Whi67]. H. Cohen observed in [Coh67] that Whitley's argument can also be used to show that a relatively weakly compact subset A of X is sequentially dense in the weak closure of A.

The aim of the present note is twofold. On the one hand, we apply Whitley's argument and a version of Cohen's argument to obtain an Eberlein-Šmulian type result for the weak* topology of the dual X' (Proposition 2), thus avoiding the use of the bidual X'' in these arguments. On the other hand, we add a further equivalence to the Eberlein-Šmulian theorem (condition (iv) in Theorem 3) that allows us to obtain Šmulian's characterisation of weak compactness of closed convex subsets of X as an immediate corollary. In fact, the implication (iv) $\Rightarrow$ (ii) of Theorem 3 is implicit in the proof of Šmulian's theorem in [DuSc88; Thm. V.6.2].

The following lemma was already used in [Whi67]; the short proof is included for the convenience of the reader.

1. Lemma. *Let X be a Banach space, and let Z be a finite dimensional subspace of X'. Then there exists a finite subset F of B_X such that*

$$\|x'\| \leqslant 2 \max_{x \in F} |x'(x)| \qquad (x' \in Z).$$

Proof. The unit sphere S_Z of Z is compact, so there exist $x'_1, \ldots, x'_n \in S_Z$ such that $S_Z \subseteq \bigcup_{k=1}^{n} B(x'_k, \frac{1}{4})$. Let now $x_1, \ldots, x_n \in B_X$ such that $x'_k(x_k) \geqslant \frac{3}{4}$ for

$k = 1, \ldots, n$. Then for all $x' \in S_Z$ there exists $k \in \{1, \ldots, n\}$ such that

$$|x'(x_k)| \geqslant x'_k(x_k) - |(x'_k - x')(x_k)| \geqslant \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2}\|x'\|,$$

and the assertion follows with $F = \{x_1, \ldots, x_n\}$. □

The following result in particular gives a condition for a subset A of a dual space X' to be relatively weakly* compact in the norm-closed linear hull of A.

2. Proposition. *Let X be a Banach space, and let $A \subseteq X'$. Assume that for every sequence (a'_n) in A one has*

$$\bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}}\{a'_k;\, k \geqslant n\} \neq \varnothing.$$

Then A is bounded, the weak closure $\operatorname{cl}_{w^*} A$ of A is contained in $\overline{\operatorname{conv}} A$, and A is sequentially dense in $\operatorname{cl}_{w^*} A$.*

Proof. Assume without loss of generality that X is a real Banach space. If A is not bounded, then by the uniform boundedness principle there exist $x \in X$ and a sequence (a'_n) in A such that $a'_n(x) \to \infty$ as $n \to \infty$. This contradicts the assumption that $\bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}}\{a'_k;\, k \geqslant n\} \neq \varnothing$.

Let now $x'_0 \in \operatorname{cl}_{w^*} A$. Then by the definition of the weak*-neighbourhoods of x'_0 and by Lemma 1 we can recursively choose an increasing sequence $(F_n)_{n\in\mathbb{N}_0}$ of finite subsets of B_X and a sequence $(a'_n)_{n\in\mathbb{N}}$ in A such that $F_0 = \varnothing$,

$$|(x'_0 - a'_n)(x)| \leqslant \tfrac{1}{n} \qquad (x \in F_{n-1})$$

and

$$\|x'\| \leqslant 2 \max_{x\in F_n} |x'(x)| \qquad \big(x' \in \operatorname{lin}\{x'_0, a'_1, \ldots, a'_n\}\big).$$

Let $M := \bigcup_{n\in\mathbb{N}} F_n$ and $Y := \operatorname{lin}\big(\{x'_0\} \cup \{a'_n;\, n \in \mathbb{N}\}\big)$. Then

$$\|x'\| \leqslant 2 \sup_{x\in M} |x'(x)|$$

holds for all $x' \in Y$ and hence for all $x' \in \overline{Y}$.

By the assumption there exists $y' \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}}\{a'_k;\, k \geqslant n\}$. Then $y' \in \overline{\operatorname{conv}} A$, so the second assertion is proved if we show that $y' = x'_0$. Let $x \in M$. Then there exists $n_0 \in \mathbb{N}$ such that $x \in F_{n_0}$. For $n > n_0$ we have $x \in F_{n-1}$ and hence $|(x'_0 - a'_n)(x)| \leqslant \frac{1}{n}$, so that $|(x'_0 - x')(x)| \leqslant \frac{1}{n}$ for all $x' \in \overline{\operatorname{conv}}\{a'_k;\, k \geqslant n\}$. It follows that $(x'_0 - y')(x) = 0$, and as $x'_0 - y' \in \overline{Y}$ we conclude that

$$\|x'_0 - y'\| \leqslant 2 \sup_{x\in M} |(x'_0 - y')(x)| = 0.$$

Note that above we have shown that $\bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}}\{a'_k;\, k \geqslant n\} = \{x'_0\}$. In order to prove that A is sequentially dense in $\operatorname{cl}_{w^*} A$, we now show that in fact $a'_n \to x'_0$ weakly*. By the assumption we obtain for any subsequence (a'_{n_j}) of (a'_n) that

$$\varnothing \neq \bigcap_{k\in\mathbb{N}} \overline{\operatorname{conv}}\{a'_{n_j};\, j \geqslant k\} \subseteq \bigcap_{k\in\mathbb{N}} \overline{\operatorname{conv}}\{a'_n;\, n \geqslant k\} = \{x'_0\}$$

and hence $x_0' \in \overline{\mathrm{conv}}\{a_{n_j}'; j \in \mathbb{N}\}$. Given $x \in X$, we infer that

$$x_0'(x) \in \overline{\mathrm{conv}}\{a_{n_j}'(x); j \in \mathbb{N}\}$$

for any subsequence $(a_{n_j}'(x))$ of $(a_n'(x))$. From this property of the sequence $(a_n'(x))$ (of real numbers) one easily deduces that $a_n'(x) \to x_0'(x)$. □

With the above, the proof of the difficult implication of the Eberlein-Šmulian theorem is easy. We change perspective and apply Proposition 2 to the dual of a Banach space.

3. Theorem. *(Eberlein-Šmulian) Let A be a subset of a Banach space X. Then the following are equivalent.*

(i) *A is relatively weakly compact;*
(ii) *A is conditionally sequentially weakly compact;*
(iii) *A is conditionally countably weakly compact;*
(iv) *for every sequence (a_n) in A one has*

$$\bigcap_{n\in\mathbb{N}} \overline{\mathrm{conv}}\{a_k; k \geqslant n\} \neq \varnothing.$$

Moreover, if one of the above conditions holds then the weak closure of A is equal to the weak sequential closure of A.

Proof. (i) ⇒ (ii). Let (a_n) be a sequence in A. Then $X_0 := \overline{\mathrm{lin}}\{a_n; n \in \mathbb{N}\}$ is weakly closed, hence $A_0 := A \cap X_0$ is a relatively weakly compact subset of X_0. Since X_0 is separable, we thus obtain that the weak topology on A_0 is metrisable. It follows that (a_n) has a subsequence that converges weakly in X_0 and hence also weakly in X.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (iv) follows from the fact that the weak accumulation points of a sequence (a_n) in A are contained in $\bigcap_{n\in\mathbb{N}} \overline{\mathrm{conv}}\{a_k; k \geqslant n\}$.

(iv) ⇒ (i). By Proposition 2 we obtain that the weak* closure of A in X'' is a bounded subset of $\overline{\mathrm{conv}} A \subseteq X$. By the Banach-Alaoglu theorem this implies that A is relatively weakly compact. The proposition also yields equality of the weak closure of A and the weak sequential closure of A. □

As an immediate consequence of implication (iv) ⇒ (i) of the above theorem we obtain the following result.

4. Theorem. *(Šmulian, [Smu39]) Let A be a closed convex subset of a Banach space X. Then A is weakly compact if and only if every decreasing sequence of non-empty closed convex subsets of A has a non-empty intersection.*

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