Equivalence of pointwise and global ellipticity estimates

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Abstract

Defining an elliptic operator $-\nabla \cdot (a\nabla)$ via the form method one normally imposes pointwise conditions on the matrix valued function a in order to get positivity, ellipticity and sectoriality of the form. In this note we show that the pointwise conditions on a are equivalent to the corresponding global ones on the form.

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Let Ω be an arbitrary open subset of $\mathbb{R}^d$, $a\colon \Omega \to \mathbb{C}^{d\times d}$ a locally integrable, hermitian matrix valued function. Define the symmetric form τ in $L_2(\Omega)$ by $\tau(u) := \int a\nabla u \cdot \nabla \overline{u}$ on $D(\tau) := C_c^\infty(\Omega)$. If τ is positive and closable then, by the form representation theorem, $\bar{\tau}$ is associated with a positive selfadjoint operator in $L_2(\Omega)$ (which corresponds to Dirichlet boundary conditions). The main aim of this note is to show that the positivity of the form τ is equivalent to the positivity of the function a, i.e., $a \geqslant 0$ in the matrix sense a.e.

A case of particular interest is the following: Let $a_1\colon \Omega \to \mathbb{R}^{d\times d}$ be locally integrable, symmetric matrix valued and locally strictly elliptic, i.e., for every compact set $K \subseteq \Omega$ there exists $\sigma > 0$ such that $a_1(x) \geqslant \sigma$ in the matrix sense for almost all $x \in K$. Then it is known that

$$D(\tau_N) := \Big\{u \in L_2(\Omega) \cap W^1_{2,loc}(\Omega);\, \tau_N(u) := \int a_1 \nabla u \cdot \nabla \overline{u} < \infty\Big\}$$

defines a symmetric Dirichlet form in $L_2(\Omega)$ (cf. [1, Thm. 1.3.9]; one can show the closedness of τ_N like the completeness of the Sobolev space $W_2^1(\Omega)$ because of the local strict ellipticity of a_1). The associated selfadjoint operator in $L_2(\Omega)$ corresponds to Neumann boundary conditions.

Notation. Let $|M|$ denote the Lebesgue measure of a measurable set $M \subseteq \mathbb{R}^d$, χ_M the characteristic function of M. S_{d-1} is the unit sphere of $\mathbb{R}^d$, for the spectral radius of a hermitian matrix $A \in \mathbb{C}^{d\times d}$ we write $|A|(= \sup_{\xi\in S_{d-1}} |A\xi \cdot \xi|)$. For a function $f\colon \Omega \to \mathbb{R}$ we use the shorthand $[f > 0]$ for the set $\{x \in \Omega;\, f(x) > 0\}$ (and similarly $[f < g]$ etc). $Q(f)$ denotes the form domain of the multiplication operator f in $L_2(\Omega)$.

Theorem. *Let $a\colon \Omega \to \mathbb{C}^{d\times d}$ be a locally integrable hermitian matrix valued function, $\tau(u) := \int a\nabla u \cdot \nabla \overline{u}$ for $u \in C_c^\infty(\Omega)$.*

(a) *The following are equivalent:*

   (i) $a \geqslant 0$ *a.e.,*

   (ii) $\tau(u) \geqslant 0$ *for all* $u \in C_c^\infty(\Omega)$*,*

   (iii) τ *is bounded from below, i.e., there exists* $c > 0$ *such that* $\tau(u) \geqslant -c\|u\|_2^2$ *for all* $u \in C_c^\infty(\Omega)$*.*

(b) *Let* D *be a sublattice of* $W^1_{1,loc}(\Omega) \cap L_2(\Omega)$ *with* $C_c^\infty(\Omega) \subseteq D \subseteq \{u; a\nabla u \cdot \nabla\overline{u} \in L_1(\Omega)\}$*. Let* $V\colon \Omega \to [0,\infty)$ *be a measurable function such that* $D \cap Q(V)$ *is dense in* $L_2(\Omega)$*. Define the form* τ_0 *in* $L_2(\Omega)$ *by* $\tau_0(u) := \int a\nabla u \cdot \nabla\overline{u} + \int V|u|^2$ *on* $D(\tau_0) := D \cap Q(V)$*. Then* τ_0 *is positive if and only if* $a \geqslant 0$ *in the matrix sense a.e.*

Remark. Let a_1 and τ_N be as in the introduction. Assume that $a_1\xi \cdot \xi \geqslant |a\xi \cdot \xi|$ a.e. for all $\xi \in \mathbb{R}^d$ (e.g. $a_1 = (|a| + 1)I$). Then $D := D(\tau_N)$ is an example of a sublattice D satisfying the assumption of part (b) of the theorem (or any sublattice D of $D(\tau_N)$ with $C_c^\infty(\Omega) \subseteq D$). A particular example is $D = W^1_{\infty,c}(\Omega)$, the space of Lipschitz continuous functions with compact support.

Proof of the Theorem. The only nontrivial part of (a) is showing that (iii) implies (i). For this purpose we assume that $[a \ngeqslant 0]$ has positive measure. In the following we construct a function violating (iii), later on we modify this function in order to show that the form τ_0 defined in (b) is not positive.

Let (ξ_k) be a dense sequence in S_{d-1}. If $a(x) \ngeqslant 0$ for some $x \in \Omega$ then there exists $k \geqslant |a(x)|$ such that $a(x)\xi_k \cdot \xi_k \leqslant -1/k$. We obtain

$$[a \ngeqslant 0] = \bigcup_{k\in\mathbb{N}} \big([a\xi_k \cdot \xi_k \leqslant -1/k] \cap [|a| \leqslant k]\big),$$

so we can fix $k \in \mathbb{N}$ such that $F := [a\xi_k \cdot \xi_k \leqslant -1/k] \cap [|a| \leqslant k]$ has positive measure.

Now we localize F: Almost all $x \in \Omega$ are Lebesgue points of the function χ_F, so since F has positive measure there exists $x \in \Omega$ satisfying

$$\frac{1}{|B_\delta(x)|}\int_{B_\delta(x)} \chi_F \to \chi_F(x) = 1 \quad \text{for } \delta \to 0.$$

Let $\varepsilon \in (0,1)$ be given. Fix $\delta > 0$ in such a way that $B := B_\delta(x) \subseteq \overline{B_\delta(x)} \subseteq \Omega$ and $|B \cap F| = \int_B \chi_F \geqslant (1-\varepsilon)|B|$. Since the function $|a|$ restricted to B is integrable, there exists $\alpha > 0$ such that for all measurable sets $M \subseteq B$ with $|M| \leqslant \alpha$ we have $\int_M |a| \leqslant \varepsilon|B|$. By the regularity of the Lebesgue measure there exists an open set Ω_0, $B \cap F \subseteq \Omega_0 \subseteq B$, with $|\Omega_0 \setminus F| \leqslant \alpha$. It follows that

$$\int_{\Omega_0\setminus F} |a| \leqslant \varepsilon|B| \quad \text{and} \quad |\Omega_0| \geqslant |B \cap F| \geqslant (1-\varepsilon)|B|. \tag{1}$$

We can choose functions $u_n \in C_c^\infty(\Omega_0)$ satisfying

$$0 \leqslant u_n \leqslant \frac{1}{n}, \quad \|\nabla u_n\|_\infty \leqslant 1, \quad \text{and} \quad |[\nabla u_n \neq \pm\xi_k] \cap \Omega_0| \leqslant \varepsilon|B|$$

for all $n \in \mathbb{N}$. (The reader should think of pieces of hyperplanes orthogonal to ξ_k which have distance $\frac{2}{n}$ to each other and to the boundary of Ω_0. On these hyperplanes set $u_n := \frac{1}{n}$ and extend like a wash board.) The important point is that on the set $G := F \cap [\nabla u_n = \pm\xi_k]$ the function $a\nabla u_n \cdot \nabla u_n$ is less or equal $-1/k$.

Let $H := \Omega_0 \setminus G$. Then $|H| \leqslant |\Omega_0 \setminus F| + |\Omega_0 \cap [\nabla u_n \neq \pm\xi_k]| \leqslant 2\varepsilon|B|$ since $|\Omega_0 \setminus F| \leqslant |B \setminus F| \leqslant \varepsilon|B|$ by (1). We can estimate

$$\tau(u_n) = \int_G a\xi_k \cdot \xi_k + \int_H a\nabla u_n \cdot \nabla u_n \leqslant -\frac{1}{k}|G| + \int_{H\cap F} |a| + \int_{H\setminus F} |a|.$$

By (1) we obtain $|G| = |\Omega_0| - |H| \geqslant (1-3\varepsilon)|B|$ and (since $|a| \leqslant k$ on F)

$$\tau(u_n) \leqslant -\tfrac{1}{k}|G| + k|H \cap F| + \textstyle\int_{\Omega_0\setminus F} |a| \leqslant \left(-\tfrac{1}{k}(1-3\varepsilon) + 2k\varepsilon + \varepsilon\right)|B|. \quad (2)$$

Note that $\|u_n\|_2 \leqslant \frac{1}{n}\|\chi_B\|_2$ since $0 \leqslant u_n \leqslant \frac{1}{n}$. Therefore, given $c > 0$, it is easy to choose first ε and then n in such a way that $\tau(u_n) + c\|u_n\|_2^2 < 0$. This completes the proof of (a).

Now we use the sequence (u_n) in order to construct a function $u \in D(\tau_0)$ with $\tau_0(u) < 0$, thus proving (b). Let $\varepsilon := 1/(2k^2 + 5k + 3)$. Since $D(\tau_0)$ is a dense sublattice of $L_2(\Omega)$, there exists a function $0 \leqslant \varphi \in D(\tau_0)$ satisfying $|[\varphi < \chi_B]| < \alpha$. Hence

$$\int_{[\varphi<\chi_B]} |a| \leqslant \varepsilon|B| \quad (3)$$

according to the choice of α. Since $\varphi \in Q(V)$ and $0 \leqslant u_n \leqslant \frac{1}{n}$ $(n \in \mathbb{N})$ there exists $n_0 \in \mathbb{N}$ with

$$\int V|\varphi \wedge u_n|^2 < \varepsilon|B| \quad \text{for all } n \geqslant n_0. \quad (4)$$

By the assumptions on D we have $a\nabla\varphi \cdot \nabla\varphi \in L_1(\Omega)$, so there exists $\alpha_2 > 0$ such that for all measurable sets $M \subseteq \Omega$ with $|M| \leqslant \alpha_2$ we have $\int_M |a\nabla\varphi \cdot \nabla\varphi| \leqslant \varepsilon|B|$. Obviously $\bigcap_{n\in\mathbb{N}}[0 < \varphi < 1/n] = \varnothing$, so since $u_n \leqslant 1/n$ there exists $n \geqslant n_0$ with $|[0 < \varphi < u_n]| \leqslant \alpha_2$. Noting $\nabla\varphi = 0$ a.e. on $[\varphi = 0]$, we obtain

$$\int_{[\varphi<u_n]} |a\nabla\varphi \cdot \nabla\varphi| = \int_{[0<\varphi<u_n]} |a\nabla\varphi \cdot \nabla\varphi| \leqslant \varepsilon|B|. \quad (5)$$

For $u := \varphi \wedge u_n \in D(\tau_0)$ we estimate (noting $\nabla u = \nabla u_n$ a.e. on $[u = u_n]$ and $\nabla u = \nabla\varphi$ a.e. on $[u < u_n] = [\varphi < u_n]$)

$$\begin{aligned}\tau(u) &= \int_{[u=u_n]} a\nabla u \cdot \nabla u + \int_{[u<u_n]} a\nabla u \cdot \nabla u \\ &\leqslant \int a\nabla u_n \cdot \nabla u_n + \int_{[\varphi<u_n]} |a\nabla u_n \cdot \nabla u_n| + \int_{[\varphi<u_n]} |a\nabla\varphi \cdot \nabla\varphi|.\end{aligned}$$

By (3) and (5) we conclude $\tau(u) \leqslant \tau(u_n) + 2\varepsilon|B|$. From (4) recall $\int V|u|^2 \leqslant \varepsilon|B|$. According to (2) and the choice of ε it follows that

$$\tau_0(u) \leqslant \tau(u_n) + 3\varepsilon|B| \leqslant -\varepsilon|B| < 0.$$

This completes the proof of (b). □

Now let $a\colon \Omega \to \mathbb{C}^{d\times d}$ be a locally integrable matrix valued function (no longer assumed to be hermitian). Define the sesquilinear form τ_a in $L_2(\Omega)$ by $\tau_a(u) := \int a\nabla u \cdot \nabla \overline{u}$ on $C_c^\infty(\Omega)$. We call the function a *sectorial* if $|\operatorname{Im}(a\xi \cdot \bar{\xi})| \leqslant \alpha \operatorname{Re}(a\xi \cdot \bar{\xi})$ a.e. for all $\xi \in \mathbb{C}^d$ and some $\alpha \geqslant 0$ and *strictly elliptic* if $\operatorname{Re}(a\xi \cdot \bar{\xi}) \geqslant \sigma|\xi|^2$ a.e. for all $\xi \in \mathbb{C}^d$ and some $\sigma > 0$. The form τ_a is called *sectorial* if $|\operatorname{Im}\tau_a(u)| \leqslant \alpha \operatorname{Re}\tau_a(u) + c\|u\|_2^2$ and *strictly elliptic* if $\operatorname{Re}\tau_a(u) \geqslant \sigma\|\nabla u\|_2^2 - c\|u\|_2^2$ for all $u \in C_c^\infty(\Omega)$ and some $c \in \mathbb{R}$, $\alpha \geqslant 0$, $\sigma > 0$.

Corollary. *The form τ_a is sectorial (strictly elliptic) if and only if a is sectorial (strictly elliptic).*

Proof. Let $a(x)^*$ denote the adjoint matrix of $a(x)$. Then $a_R := (a + a^*)/2$ and $a_I := (a - a^*)/2i$ are hermitian matrix valued functions. One calculates

$$\operatorname{Re}(a\xi \cdot \bar{\xi}) = a_R\xi \cdot \bar{\xi} \quad \text{and} \quad \operatorname{Im}(a\xi \cdot \bar{\xi}) = a_I\xi \cdot \bar{\xi} \tag{6}$$

for all $\xi \in \mathbb{C}^d$, so $\operatorname{Re}\tau_a = \tau_{a_R}$ and $\operatorname{Im}\tau_a = \tau_{a_I}$. The form τ_a is sectorial with α as above if and only if the forms $\alpha \operatorname{Re}\tau_a \pm \operatorname{Im}\tau_a = \tau_{\alpha a_R \pm a_I}$ are bounded from below. By the above theorem this is equivalent to $\alpha a_R \pm a_I \geqslant 0$ a.e., i.e. to the sectoriality of a according to (6).

For the second part of the Corollary just apply the theorem to the function $a_R - \sigma$. □

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References

[1] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press, 1989.

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