L_p-analyticity of Schrödinger semigroups on Riemannian manifolds

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Abstract

We address the problems of extrapolation, analyticity, and L_p-spectral independence for C_0-semigroups in the abstract context of metric spaces with exponentially bounded volume. The main application of the abstract result is L_p-analyticity of angle $\frac{\pi}{2}$ of Schrödinger semigroups on Riemannian manifolds with Ricci curvature bounded below, under the condition of form smallness of the negative part of the potential.

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The final aim of the present paper is the following result. Let M be a Riemannian manifold with Ricci curvature bounded below. Then a Schrödinger semigroup with form small negative part of the potential is analytic of angle $\frac{\pi}{2}$ on $L_p(M)$, for p from a certain subinterval of $[1,\infty)$ that contains 2 and is determined by the form bound (see Theorem 7 and Remark 8(a) below).

First we put the problem in a more abstract context. Let (M,μ) be a measure space, $\Omega \subseteq M$ a measurable subset, $s \in [1,\infty)$, and $T_s = \big(T_s(t); t \geqslant 0\big)$ a C_0-semigroup on $L_s(\Omega)$. Assume that $\|T_s(t)\restriction_{L_s\cap L_q}\colon L_q(\Omega) \to L_q(\Omega)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$, for some $s \neq q \in [1,\infty)$. Then the operators $T_s(t)\restriction_{L_s\cap L_q}$ extend to bounded operators $T_q(t)$, which define a semigroup on $L_q(\Omega)$. We say that T_s *extrapolates* to the semigroup T_q on $L_q(\Omega)$. By Riesz-Thorin interpolation we obtain a family of C_0-semigroups T_p on $L_p(\Omega)$, with p between s and q. The semigroups T_p are *consistent* in the sense that $T_{p_1}(t)\restriction_{L_{p_1}\cap L_{p_2}} = T_{p_2}(t)\restriction_{L_{p_1}\cap L_{p_2}}$ for all $t \geqslant 0$ and all p_1, p_2 between s and q.

If the initial semigroup T_s is known to be analytic, e.g. if $s=2$ and T_s is symmetric, then all the semigroups T_p for p strictly between q and s are analytic by Stein interpolation. But in general, the angle of analyticity tends to 0 as $p \to q$, and T_q is not analytic in general (see, e.g., [Dav89; Thm. 4.3.6], [Voi96] for the case $q = 1$).

We are thus led to the following problem: under which conditions does the semigroup T_s extrapolate to a consistent family of C_0-semigroups T_p on $L_p(\Omega)$ with p-independent angle of analyticity, for p from some subinterval of $[1,\infty)$ containing s? The conditions on both the space M and the semigroup T_s will be formulated in terms of a measurable semi-metric on M. We will also address the problem of p-independence of the spectra of the L_p-generators. The main results will only be stated; for the proofs we refer the reader to [Vog01].

Most of the known results concerning the problem of analyticity are about semigroups acting on the whole L_p-scale. Then the question of analyticity in L_1 is of particular interest. Starting from [Ama83], there are several specific results on certain classes of elliptic operators on domains of $\mathbb{R}^N$ stating that the semigroup on L_1 is analytic, but not giving the optimal angle ([Kat86], [CaVe88], [ArBa93], only to mention a few).

E.-M. Ouhabaz was the first to establish analyticity of angle $\frac{\pi}{2}$ in $L_1(\mathbb{R}^N)$. In his thesis ([Ouh92a]) he observed that a Gaussian upper bound on the semigroup kernel for complex times proved in [Dav89; Thm. 3.4.8] can be used to show the following. If T_2 is a symmetric sub-Markovian semigroup on $L_2(\mathbb{R}^N)$ satisfying a Gaussian upper bound then the corresponding consistent C_0-semigroups T_p on $L_p(\mathbb{R}^N)$ are analytic of angle $\frac{\pi}{2}$, for all $p \in [1,\infty)$. See [Ouh95] for a more general version not assuming the semigroup to be sub-Markovian.

With a similar method of proof based on kernel estimates, Ouhabaz' result was generalised in [Dav95a] from Euclidean space to metric spaces with polynomially bounded volume. Again in the context of Euclidean space, the symmetry assumption was dropped in [Hie96], with a result stating p-independence of the angle of analyticity. For a comprehensive discussion see [Are97].

The generalisation to the case that the semigroup does not act on the whole L_p-scale requires a completely new approach, for lack of kernel estimates. E. B. Davies proved the following in [Dav95b]. If H is a selfadjoint superelliptic operator on $\mathbb{R}^N$ of order $2m < N$, with bounded measurable coefficients, then e^{-tH} extrapolates to an analytic semigroup on $L_p(\mathbb{R}^N)$ of angle $\frac{\pi}{2}$, for all $p \in [\frac{2N}{N+2m}, \frac{2N}{N-2m}]$. Analyticity of angle $\frac{\pi}{2}$ (but not extrapolation) was also shown in [Sch96; Sec. 3.3] in a more general setting assuming a generalised Gaussian upper bound (cf. the bound (7) below). On the other hand, extrapolation was studied in [Sem00] for generalised Schrödinger semigroups with form small negative part of the potential, but without showing analyticity of angle $\frac{\pi}{2}$. The proofs of these results are essentially restricted to Euclidean space since they rely on a discrete method where $\mathbb{R}^N$ is subdivided into congruent cubes.

In what follows we unify the above generalisations. We consider semigroups in measure spaces that do not necessarily act on the whole L_p-scale. Let $1 \leqslant p_0 < q_0 \leqslant \infty$ be fixed, and let (M,μ) be a σ-finite measure space with $\mu(M) > 0$. Let d be a measurable semi-metric on M, i.e., d: $M \times M \to [0,\infty)$ is measurable. Then $d(x,\cdot)$ is measurable for all $x \in M$ since the function $y \mapsto (x,y)$ is measurable. The open ball with respect to d with centre x and radius r will be denoted by $B(x,r)$. We assume $\mu(B(x,r)) < \infty$ for all $x \in M,\ r > 0$. Note that this property already implies that μ is σ-finite: for fixed $x_0 \in M$ we have $M = \bigcup_{n\in\mathbb{N}} B(x_0,n)$.

In the case $(p_0,q_0) \neq (1,\infty)$ let $v_r(x) := \mu(B(x,r))$ $(x \in M,\ r > 0)$, whereas in the case $(p_0,q_0) = (1,\infty)$ we only assume v_r: $M \to [0,\infty)$ to be measurable functions satisfying $\mu(B(x,r)) \leqslant v_r(x)$ for all $x \in M,\ r > 0$ and

$$v_r \leqslant v_R \text{ on } M\ (R > r > 0), \quad v_r(x) \leqslant v_{r+d(x,y)}(y)\ (x,y \in M,\ r > 0). \tag{1}$$

Note that (1) is automatically fulfilled if $v_r(x) = \mu(B(x,r))$ since $B(x,r) \subseteq B(y,r+d(x,y))$.

Fix a measurable subset $\Omega \subseteq M$. In the following we consider semigroups on $L_p(\Omega)$, $1 \leqslant p < \infty$. The reason for introducing the space M is that the functions $\mu(B(\cdot,r))$ on M can behave much better than the functions $\mu(B(\cdot,r)\cap\Omega)$ on Ω. An important example for this situation is $M = \mathbb{R}^N$ and an open subset $\Omega \subseteq \mathbb{R}^N$.

We assume two volume growth conditions,

$$v_r \leqslant c_0 e^{c_1 r} v_1 \text{ on } M \quad (r > 1), \tag{2}$$

$$v_{2r} \leqslant c_0 v_r \text{ on } M \quad (0 < r \leqslant \tfrac{1}{2}), \tag{3}$$

for some $c_0 \geqslant 1,\ c_1 > 0$. Condition (2) means that the volume of balls grows at most exponentially, condition (3) is the doubling property for small balls. The latter is known to be equivalent to

$$v_R \leqslant c_2 \left(\tfrac{R}{r}\right)^N v_r \quad (0 < r < R \leqslant 1) \tag{4}$$

for some $N > 0$. (In '(3) $\Longrightarrow$ (4)' one obtains $c_2 = c_0,\ N = \log_2 c_0$.)

In order to obtain L_p-spectral independence, the exponential volume growth condition (2) is too weak. In fact, it is known that in the case of exponential volume growth the

L_p-spectrum of the semigroup generators typically does depend on p (see, e.g., [Stu93; Prop. 2(b)]). Instead, we need the more restrictive subexponential volume growth condition

$$\forall\, \varepsilon > 0\, \exists\, c_\varepsilon > 0\, \forall\, r > 1: v_r \leqslant c_\varepsilon e^{\varepsilon r} v_1, \tag{5}$$

as in [Stu93; p. 443].

In the case $M = \mathbb{R}^N$, d the supremum metric (for this choice see Remark 4 below) and μ the Lebesgue measure, conditions (2) to (5) are trivially fulfilled with $v_r(x) = \mu(B(x,r)) = (2r)^N$. If M is a complete Riemannian manifold with Ricci curvature bounded below, d the Riemannian distance and μ the Riemannian volume, then (2) and (4) hold for $v_r(x) = \mu(B(x,r))$ and N the dimension of M, by Bishop's comparison principle (see, e.g., [GHL90; Thm. 4.19]). Condition (5) holds, e.g., if the Ricci curvature of M is bounded below by a function $K\colon M \to \mathbb{R}$ satisfying $\liminf_{x\to\infty} K(x) \geqslant 0$ (see [Stu93; Prop. 1(a)]). Note that, in the case of a Riemannian manifold, v_r is a function heavily depending on the space variable: in contrast to the flat space case it is not bounded away from zero in general.

In order to formulate our results we need the following notation. By means of the semi-metric d, we define weight functions $\rho_{\gamma,y}$ on M,

$$\rho_{\gamma,y}(x) := e^{-\gamma d(x,y)} \quad (x, y \in M,\ \gamma \in \mathbb{R}).$$

Let B be a linear operator in $L_1(\Omega) + L_\infty(\Omega)$, and $1 \leqslant p \leqslant q \leqslant \infty$. We denote the norm of B as an operator from L_p to L_q by

$$\|B\|_{p\to q} := \sup\bigl\{\|Bf\|_q;\ f \in L_p(\Omega) \cap D(B),\ \|f\|_p \leqslant 1\bigr\} \in [0,\infty],$$

and for $\gamma \in \mathbb{R}$ we define the weighted operator norm

$$\begin{aligned}\|B\|_{p\to q,\gamma} &:= \sup_{y\in M} \|\rho_{\gamma,y} B \rho_{\gamma,y}^{-1}\restriction_{\{f\in L_p(\Omega);\ \rho_{\gamma,y}^{-1} f \in D(B)\}}\|_{p\to q} \\ &= \inf\bigl\{c \geqslant 0;\ \forall\, f \in D(B),\ y \in M\colon \|\rho_{\gamma,y} Bf\|_q \leqslant c \|\rho_{\gamma,y} f\|_p\bigr\} \in [0,\infty].\end{aligned}$$

The following is our main abstract result.

Theorem 1. *([Vog01; Thm. 2.1, Thm. 2.4]) Assume that (M,d) is separable and that (2) and (3) hold. Let $p_0 \leqslant s \leqslant q_0$ and T_s a C_0-semigroup on $L_s(\Omega)$. Assume that there exist $m > 1$, $t_0 > 0$ and $\alpha_0, \beta_0 \geqslant 0$ with $\alpha_0 + \beta_0 = p_0^{-1} - q_0^{-1}$ such that*

$$\sup_{0<t\leqslant t_0} \|v_{t^{1/m}}^{\alpha_0} T_s(t) v_{t^{1/m}}^{\beta_0}\|_{p_0\to q_0, t^{-1/m}} < \infty. \tag{6}$$

(a) *Then T_s extrapolates to a consistent family of C_0-semigroups $T_p(t) = e^{-tA_p}$ on $L_p(\Omega)$, $p \in [p_0, q_0] \setminus \{\infty\}$, with angle of analyticity not depending on p.*

(b) *If the subexponential volume growth condition (5) holds instead of (2), then the spectrum $\sigma(A_p)$ does not depend on $p \in [p_0, q_0] \setminus \{\infty\}$, and the operators A_p have consistent resolvents.*

By the phrase 'angle of analyticity not depending on p' we mean the following. If one of the semigroups T_p is analytic of angle θ, then all of them are analytic of angle θ; if one of the semigroups is *not* analytic, then none of them is analytic. We point out that part (a) of the above theorem contributes to the solution of *two* problems, extrapolation as well as analyticity of semigroups. So to say, it deals with extension of the L_p-scale as well as extension of the time scale.

Estimate (6) is equivalent to the validity of the seemingly stronger bound

$$\|v_{t^{1/m}}^{\alpha} T_s(t) v_{t^{1/m}}^{\beta}\|_{p\to q,\gamma} \leqslant C e^{\omega t + \nu\gamma^m t} \quad (t > 0,\ \gamma \geqslant 0) \tag{7}$$

for all $p_0 \leqslant p \leqslant q \leqslant q_0$, $\alpha, \beta \geqslant 0$ with $\alpha + \beta = p^{-1} - q^{-1}$, for some constants $C > 0$, $\omega \in \mathbb{R}$, $\nu \geqslant 0$. The latter can be considered as a generalised Gaussian upper bound (cf. [Sch96; p. 44]) since we have the following.

Proposition 2. *Assume that (M,d) is separable. Let T_2 be a C_0-semigroup on $L_2(\Omega)$. Then the estimates in (7) hold with $\alpha = \beta = \frac{1}{2}$, $p = 1$, $q = \infty$ if and only if the semigroup operators $T_2(t)$ have integral kernels k_t satisfying the Gaussian upper bound*

$$|k_t(x,y)| \leqslant C\big(v_{t^{1/m}}(x)v_{t^{1/m}}(y)\big)^{-\frac{1}{2}} \exp\big(\omega t - c_m\big(\tfrac{d(x,y)^m}{\nu t}\big)^{\frac{1}{m-1}}\big) \quad (t>0,\ x,y\in\Omega), \quad (8)$$

with $c_m = (m-1)m^{-\frac{m}{m-1}}$.

Proof. Let $D \subseteq M$ be countable and dense, $t > 0$. By the Dunford-Pettis theorem, (7) holds if and only if $T_2(t)$ has an integral kernel k_t satisfying

$$\begin{aligned}|k_t(x,y)| &\leqslant \inf_{w\in D} C\rho_{\gamma,w}(x)^{-1}v_{t^{1/m}}(x)^{-\frac{1}{2}}v_{t^{1/m}}(y)^{-\frac{1}{2}}\rho_{\gamma,w}(y)e^{\omega t+\nu\gamma^m t}\\ &= C\big(v_{t^{1/m}}(x)v_{t^{1/m}}(y)\big)^{-\frac{1}{2}}e^{\omega t+\nu\gamma^m t-\gamma d(x,y)}\end{aligned}$$

for almost all $x, y \in \Omega$ and all rational $\gamma \geqslant 0$. We now optimise with respect to γ (Davies' trick): setting $\gamma = \big(\frac{d(x,y)}{m\gamma t}\big)^{\frac{1}{m-1}}$ yields the desired conclusion. ☐

By the above proposition, condition (6) in Theorem 1 is satisfied if the semigroup satisfies a Gaussian upper bound of order m. Gaussian upper bounds are known to hold for wide classes of uniformly elliptic operators, e.g. for

(a) second order uniformly elliptic operators in divergence form on $\mathbb{R}^N$
with real coefficients [Aro67],
with complex coefficients in dimensions 1 and 2 [AMT98],
with uniformly continuous complex coefficients in higher dimensions [Aus96];

(b) superelliptic operators of order $2m$ in dimensions $N < 2m$ [Dav95b];

(c) second order uniformly elliptic operators in divergence form on Riemannian manifolds with Ricci curvature bounded below [Sal92].

For more detailed discussions of examples for which Gaussian upper bounds are valid, we refer to [Kun99].

Recall that $v_r = (2r)^N$ in the case of Euclidean space $\mathbb{R}^N$ endowed with the supremum metric and the Lebesgue measure. So we immediately obtain from Theorem 1:

Corollary 3. *Let $\Omega \subseteq \mathbb{R}^N$ be measurable, $p_0 \leqslant s \leqslant q_0$, and T_s a C_0-semigroup on $L_s(\Omega)$. Assume that there exist $m > 1$, $t_0, C > 0$ such that*

$$\|T_s(t)\|_{p_0\to q_0, t^{-1/m}} \leqslant Ct^{-\frac{N}{m}(\frac{1}{p_0}-\frac{1}{q_0})} \quad (0 < t \leqslant t_0).$$

Then T_s extrapolates to a consistent family of C_0-semigroups $T_p(t) = e^{-tA_p}$ on $L_p(\Omega)$, $p \in [p_0, q_0] \setminus \{\infty\}$, with angle of analyticity and spectrum $\sigma(A_p)$ not depending on p, and the operators A_p have consistent resolvents.

The weight functions $\rho_{\gamma,y}$ were not used in the context of weighted norm estimates before [KuVo00]. In [ScVo94] the functions ρ_ξ defined by $\rho_\xi(x) := e^{\xi x}$ $(x, \xi \in \mathbb{R}^N)$ were used to prove L_p-spectral independence, in [Sch96] also to prove analyticity of angle $\frac{\pi}{2}$. E. B. Davies demonstrated in [Dav95b] that (approximations of) these weight functions ρ_ξ are suitable for studying all three problems of interest in the present paper: extrapolation, analyticity, and L_p-spectral independence.

For the proofs of our results, the crucial advantage of the weights $\rho_{\gamma,y}$ is that they are integrable for $\gamma > 0$ whereas the weights ρ_ξ grow exponentially in direction ξ. We point out that in the case of higher order elliptic operators on $\mathbb{R}^N$, it is hard to estimate $\|B\|_{p\to q,\gamma}$ directly: the functions $\rho_{\gamma,y} = e^{-\gamma|\cdot-y|_\infty}$ that are involved in the estimate have only one bounded weak derivative. This difficulty is overcome by the following result, which also shows that Corollary 3 covers the analyticity results of [Sch96], [Dav95b], [Sem00].

Remark 4. ([KuVo00; end of Sec. 2.2]) Let $\Omega \subseteq \mathbb{R}^N$ be open, $B\colon L_{\infty,c}(\Omega) \to L_{1,loc}(\Omega)$ a linear operator, $\gamma > 0$, $1 \leqslant p < q \leqslant \infty$. Let $\|\cdot\|_{p\to q,\gamma}$ be the weighted operator norm corresponding to the supremum metric. Then

$$\|B\|_{p\to q,\gamma} \leqslant 2N \sup_{|\xi|=\gamma} \|e^{\xi x} B e^{-\xi x}\|_{p\to q}.$$

Finally we show how our main abstract result can be applied to Schrödinger semigroups on Riemannian manifolds, and more generally in the context of perturbation of Dirichlet forms by potentials. This application is based on the following perturbation result that is essentially due to V. Liskevich and Yu. Semenov, and which is valid in the context of an arbitrary measure space (Ω, μ).

Theorem 5. *(cf. [LiSe93; Thm. 2], [LiSe96; Thm. 3.2]) Let τ be a symmetric Dirichlet form in $L_2(\mu)$, $V\colon \Omega \to \mathbb{R}$ measurable. Assume that $\tau + V^+$ is densely defined, and $V^- \leqslant \beta\tau + V^+ + c_\beta$ in the form sense for some $\beta < 1$, $c_\beta \in \mathbb{R}$. Let $p_\pm := \frac{2}{1\mp\sqrt{1-\beta}}$ (the roots of the equation $\frac{4}{pp'} = \beta$).*

Then $\tau + V$ is a densely defined closable symmetric form, and for all $p \in [p_-, p_+]$ the analytic semigroup $T_{V,2}$ on $L_2(\mu)$ associated with $\overline{\tau + V}$ extrapolates to a positive C_0-semigroup $T_{V,p}$ on $L_p(\mu)$, with $\|T_{V,p}(t)\| \leqslant e^{c_\beta t}$ $(t \geqslant 0)$.

Remark 6. In [LiSe93], [LiSe96], the above theorem is proved in the more general setting of perturbation by sub-Markovian generators, not only by potentials. But the assumption on the perturbation is slightly more restrictive, namely (expressed for perturbation by a potential) $V_- \leqslant \beta(\tau + V_+) + c_\beta$, with $V_-, V_+\colon \Omega \to [0,\infty)$ such that $V = V_+ - V_-$. In the above form, the theorem is proved in [Vog01; Thm. 1.32].

Under additional conditions on the measure space and the semigroup, now with M, Ω, μ, d as before, the L_p-scale $[p_-, p_+]$ of existence of the semigroup can be extended.

Theorem 7. *([Vog01; Thm. 2.10]) Assume that* (2) *and* (4) *hold for $v_r(x) = \mu(B(x,r))$. Let τ be a symmetric Dirichlet form in $L_2(\Omega)$, and assume that the associated symmetric sub-Markovian semigroup T on $L_2(\Omega)$ satisfies the Gaussian upper bound* (8) *with $m = 2$. Let $V\colon \Omega \to \mathbb{R}$ be measurable such that $\tau + V^+$ is densely defined and $V^- \leqslant \beta\tau + V^+ + c_\beta$ for some $\beta < 1$, $c_\beta \in \mathbb{R}$. Assume $N > 2$ in* (4) *and let $p_+ := \frac{2}{1-\sqrt{1-\beta}}$, $p_{\max} := \frac{N}{N-2}p_+$, $p_{\min} := p'_{\max}$.*

(a) *Then T_V, the analytic semigroup on $L_2(\Omega)$ associated with $\overline{\tau + V}$, extrapolates to an analytic semigroup $T_{V,p}$ on $L_p(\Omega)$ of angle $\frac{\pi}{2}$, for all $p \in (p_{\min}, p_{\max})$.*

(b) *If* (5) *holds instead of* (2), *then the spectrum of the generators of the semigroups $T_{V,p}$ is independent of $p \in (p_{\min}, p_{\max})$.*

Remark 8. (a) It was first observed by Yu. Semenov that the L_p-scale $[p_-, p_+]$ given in Theorem 5 can be extended: in [Sem00] he studied the form τ corresponding to a selfadjoint second order uniformly elliptic operator on $\mathbb{R}^N$. He showed that T_V extrapolates to an analytic semigroup on $L_p(\mathbb{R}^N)$, for all $p \in (p_{\min}, p_{\max})$, but he did not obtain the (optimal) angle $\frac{\pi}{2}$.

More generally, the above theorem can be applied to the following situation. Let M be a complete Riemannian manifold with Ricci curvature bounded below, d the Riemannian distance and μ the Riemannian volume. Then (2) and (4) hold for $v_r(x) = \mu(B(x,r))$ and N the dimension of M, by Bishop's comparison principle. Let $\Omega \subseteq M$ be open, τ the form corresponding to a selfadjoint second order uniformly elliptic operator on Ω subject to Dirichlet boundary conditions. Then the associated semigroup on $L_2(\Omega)$ satisfies a Gaussian upper bound of order $m = 2$ (see [Sal92; Thm. 6.3]). So Theorem 7(a) can in particular be applied to Schrödinger semigroups with form small negative part of the potential on Riemannian manifolds with Ricci curvature bounded below.

(b) An interesting point about Theorem 7 is the following. If

$$V^- \leqslant \beta\tau + V^+ + c \qquad \text{for some } c \in \mathbb{R} \tag{9}$$

holds for some $\beta \leqslant 1$ then $\tau + V$ is closable (see, e.g., [SoVo02; Prop. 3.4(b)]), and $\overline{\tau + V}$ is associated with an analytic semigroup T_V on $L_2(\Omega)$. If $\beta < 1$ then Theorem 7 shows that T_V extrapolates to an analytic semigroup on $L_p(\Omega)$ for all p in $[\frac{2N}{N+2}, \frac{2N}{N-2}]$, an interval not depending on β.

If (9) holds for $\beta = 1$, but not for any $\beta < 1$, then it is not clear whether T_V extrapolates to a C_0-semigroup on $L_p(\Omega)$ for any $p \neq 2$.

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