L_1-estimates for eigenfunctions and heat kernel estimates for semigroups dominated by the free heat semigroup

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Abstract

We investigate selfadjoint positivity preserving C_0-semigroups that are dominated by the free heat semigroup on $\mathbb{R}^d$. Major examples are semigroups generated by Dirichlet Laplacians on open subsets or by Schrödinger operators with absorption potentials. We show explicit global Gaussian upper bounds for the kernel that correctly reflect the exponential decay of the semigroup. For eigenfunctions of the generator that correspond to eigenvalues below the essential spectrum we prove estimates of their L_1-norm in terms of the L_2-norm and the eigenvalue counting function. This estimate is applied to a comparison of the heat content with the heat trace of the semigroup.

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1 Introduction and main results

In the recent paper [BHV13], the authors studied Dirichlet Laplacians on open subsets Ω of $\mathbb{R}^d$. They proved an estimate for the L_1-norm of eigenfunctions in terms of their L_2-norm and spectral data, and they used this to estimate the heat content of Ω by its heat trace. The aim of the present paper is to provide sharper estimates in the following more general setting.

Let $\Omega \subseteq \mathbb{R}^d$ be measurable, where $d \in \mathbb{N}$, and let T be a selfadjoint positivity preserving C_0-semigroup on $L_2(\Omega)$ that is dominated by the free heat semigroup, i.e.,

$$0 \leqslant T(t)f \leqslant e^{t\Delta} f \qquad \bigl(t \geqslant 0,\ 0 \leqslant f \in L_2(\Omega)\bigr).$$

Let $-H$ denote the generator of T.

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An important example for the operator $-H$ is the Dirichlet Laplacian with a locally integrable absorption potential on an open set $\Omega \subseteq \mathbb{R}^d$. For more general absorption potentials the space of strong continuity of the semigroup will be $L_2(\Omega')$ for some measurable $\Omega' \subseteq \Omega$.

In our first main result we estimate the L_1-norm of eigenfunctions of H in terms of their L_2-norm and the eigenvalue counting function $N_t(H)$, which for $t < \inf \sigma_{\mathrm{ess}}(H)$ denotes the number of eigenvalues of H that are $\leqslant t$, counted with multiplicity.

1.1 Theorem. *Let φ be an eigenfunction of H with eigenvalue $\lambda < \inf \sigma_{\mathrm{ess}}(H)$. Then*

$$\|\varphi\|_1^2 \leqslant c_d (t-\lambda)^{-d/2} \bigl(\ln \tfrac{3t}{t-\lambda}\bigr)^d N_t(H) \|\varphi\|_2^2 \qquad \bigl(\lambda < t < \inf \sigma_{\mathrm{ess}}(H)\bigr),$$

with $c_d = 35^{d+1} d^{d/2}$.

1.2 Remarks. (a) We point out that as in [BHV13; Thm. 1.6] one has the lower bound

$$\|\varphi\|_1^2 \geqslant \bigl(\tfrac{2\pi d}{e}\bigr)^{d/2} \lambda^{-d/2} \|\varphi\|_2^2\,.$$

Thus, the factor $d^{d/2}$ in the constant c_d is of the correct order. The factor $(t-\lambda)^{-d/2}$ matches the factor $\lambda^{-d/2}$; cf. Corollary 1.3 below. See [BHV13; Example 1.8(3)] for an explanation why the estimate should contain a factor like $N_t(H)$ with some $t > \lambda$.

(b) In [BHV13; Thm. 1.3], in the framework of Dirichlet Laplacians on open subsets of $\mathbb{R}^d$, the estimate

$$\|\varphi\|_1^2 \leqslant C_d E_0^{-d/2} \left(\left(\frac{\lambda}{E_0} \frac{\lambda}{t-\lambda} \right)^d \bigl(\ln N_t(H)\bigr)^d N_t(H) + \left(\frac{\lambda}{E_0}\right)^{4d-3} \left(\frac{\lambda}{t-\lambda}\right)^{4d} \right) \|\varphi\|_2^2$$

was shown under the additional assumption $t \leqslant 3\lambda$, where $E_0 = \inf \sigma(H)$. Our estimate $\|\varphi\|_1^2 \leqslant c_d \lambda^{-d/2} \bigl(\frac{\lambda}{t-\lambda}\bigr)^{d/2} \bigl(\ln \frac{3t}{t-\lambda}\bigr)^d N_t(H) \|\varphi\|_2^2$ improves on this in several regards; most notably, the factors $\frac{\lambda}{E_0}$, $\bigl(\ln N_t(H)\bigr)^d$ and the second summand are removed altogether.

(c) In [BHV13], a partition of $\mathbb{R}^d$ into cubes was used in the proof. We will work with a "continuous partition" into balls instead; see the proof of Lemma 4.2. Working with balls leads to a better constant c_d in the estimate.

(d) In the case $d = 1$ and H the Dirichlet Laplacian on an open subset of $\mathbb{R}$, an improved estimate is given in [BHV13; Rem. 1.5]. For that estimate it is crucial that H is a direct sum of Dirichlet Laplacians on intervals. The improvement is not possible for general H in dimension $d = 1$; this can be seen similarly as in [BHV13; Example 1.8(3)].

If H has compact resolvent, then one can apply Theorem 1.1 with $t = (1+\varepsilon)\lambda$ for any $\varepsilon > 0$ to obtain the following estimate. Note that it contains the same factor $\lambda^{-d/2}$ as the lower bound of Remark 1.2(a).

1.3 Corollary. *Assume that H has compact resolvent. Let φ be an eigenfunction of H with eigenvalue λ. Then*

$$\|\varphi\|_1^2 \leqslant c_d C_\varepsilon^d \lambda^{-d/2} N_{(1+\varepsilon)\lambda}(H)\|\varphi\|_2^2 \qquad (\varepsilon > 0),$$

with c_d as in Theorem 1.1 and $C_\varepsilon = \varepsilon^{-1/2}\ln(3+\frac{3}{\varepsilon})$.

1.4 Remark. The assumption that H has compact resolvent is in particular satisfied if Ω has finite volume. Note that then the trivial estimate $\|\varphi\|_1^2 \leqslant \mathrm{vol}(\Omega)\|\varphi\|_2^2$ holds. We point out that, up to a dimension dependent constant, the estimate of Corollary 1.3 is never worse since one has the bound $N_t(H) \leqslant K_d\,\mathrm{vol}(\Omega)t^{d/2}$ for all $t > 0$. (To obtain this bound, apply [LiYa83; Cor. 1] to open sets $\tilde{\Omega} \supseteq \Omega$ and note that $e^{-tH} \leqslant e^{-t\Delta_{\tilde{\Omega}}}$, where $\Delta_{\tilde{\Omega}}$ denotes the Dirichlet Laplacian on $\tilde{\Omega}$.)

Our second main result is the following heat kernel estimate for semigroups dominated by the free heat semigroup. This estimate is obtained as a by-product of the preparations for the proof of Theorem 1.1.

1.5 Theorem. *For all $t > 0$ the semigroup operator e^{-tH} has an integral kernel p_t. If $E_0 := \inf\sigma(H) > 0$ then*

$$0 \leqslant p_t(x,y) \leqslant \left(\frac{eE_0}{2\pi d}\right)^{d/2} \exp\left(-E_0 t - \frac{|x-y|^2}{4t}\right) \qquad \left(t \geqslant \tfrac{d}{2E_0},\ x,y \in \mathbb{R}^d\right).$$

1.6 Remark. (a) For $0 < t < \frac{d}{2E_0}$ one just has the estimate with respect to the free heat kernel,

$$0 \leqslant p_t(x,y) \leqslant (4\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right).$$

In combination with Theorem 1.5 this gives

$$0 \leqslant p_t(x,y) \leqslant (4\pi t)^{-d/2}\left(1 + \frac{2e}{d}E_0 t\right)^{d/2} \exp\left(-E_0 t - \frac{|x-y|^2}{4t}\right) \qquad (t > 0).$$

(In the case $E_0 = 0$ this estimate is true but inconsequential.)

(b) In [Ouh06; formula (22)], the following estimate was proved in the framework of Dirichlet Laplacians with absorption potentials on open subsets of $\mathbb{R}^d$:

$$p_t(x,y) \leqslant c_\varepsilon (4\pi t)^{-d/2}\left(1 + \frac{1}{2}E_0 t + \varepsilon\frac{|x-y|^2}{8t}\right)^{d/2} \exp\left(-E_0 t - \frac{|x-y|^2}{4t}\right),$$

where $\varepsilon > 0$ and $c_\varepsilon = e^2(1+\frac{1}{\varepsilon})^{d/2}$. Part (a) shows that the summand $\varepsilon\frac{|x-y|^2}{8t}$ is actually not needed, which may come as a surprise.

(c) In the generality of our setting, the estimate provided in Theorem 1.5 is probably the best one can hope for. Suppose, for example, that the semigroup T is irreducible and that E_0 is an isolated eigenvalue of H. Then the large time behaviour of p_t is known:

$$e^{E_0 t}p_t(x,y) \to \varphi(x)\varphi(y) \qquad (t\to\infty),$$

where φ is the non-negative normalized ground state of H; see, e.g., [KLVW13; Thm. 3.1]. Moreover, if $\inf\sigma(H) = 1$ then $E^{d/4}\varphi(E^{1/2}\cdot)$ is the ground state of an appropriately scaled operator H_E with $\inf\sigma(H_E) = E$. This explains the factor $E_0^{d/2}$ in our estimate.

Note, however, that better estimates are known for Dirichlet Laplacians under suitable geometric assumptions on the domain Ω. Then a boundary term like $\varphi(x)\varphi(y)$ can be included in the estimate. This can be shown via intrinsic ultracontractivity as in [OuWa07].

An important application of Corollary 1.3 is that it allows us to compare the "heat content" of H with its "heat trace". We assume that H has compact resolvent, with (λ_k) the increasing sequence of all the eigenvalues of H, repeated according to their multiplicity. For $t > 0$ we denote by $Q_H(t) := \|e^{-tH}1_\Omega\|_1$ the *heat content*, by $Z_H(t) := \sum_{k=1}^\infty e^{-t\lambda_k}$ the *heat trace* of H.

Note that Q_H, Z_H are decreasing functions. It may well occur that $Q_H(t) = \infty$ and/or $Z_H(t) = \infty$ for some but not all $t > 0$ if Ω has infinite Lebesgue measure, see [BeDa89; Thm. 5.5].

1.7 Theorem. *Assume that H has compact resolvent and that $Z_H(t_0) < \infty$ for some $t_0 > 0$. Then $Q_H(t) < \infty$ for all $t > 2t_0$,*

$$Q_H(t) \leqslant c_{\varepsilon,d}\lambda_1^{-d/2}Z_H\big(\tfrac{t}{2+\varepsilon}\big)^2 \qquad \big(0<\varepsilon<\tfrac{t}{t_0}-2\big),$$

with $c_{\varepsilon,d} = c_d C_\varepsilon^d$ as in Corollary 1.3.

The proof is rather short, so we give it right here. We will use the following simple estimate.

1.8 Lemma. *(cf. [BHV13; Lemma 5.2]) For $T,\lambda > 0$ one has $N_\lambda(H) \leqslant Z_H(T)e^{T\lambda}$.*

Proof. If $k\in\mathbb{N}$ is such that $\lambda_k \leqslant \lambda$, then $k \leqslant e^{T\lambda_k}\sum_{j=1}^k e^{-T\lambda_j} \leqslant e^{T\lambda}Z_H(T)$. Thus, $N_\lambda(H) = \#\{k;\, \lambda_k\leqslant\lambda\} \leqslant e^{T\lambda}Z_H(T)$. $\square$

Proof of Theorem 1.7. Let $T := \frac{t}{2+\varepsilon}$. Let (φ_k) be an orthonormal basis of $L_2(\Omega)$ such that $H\varphi_k = \lambda_k\varphi_k$ for all $k \in \mathbb{N}$. By Corollary 1.3 and Lemma 1.8 we obtain

$$\|\varphi_k\|_1^2 \leqslant c_{\varepsilon,d}\lambda_k^{-d/2}N_{(1+\varepsilon)\lambda_k}(H)\|\varphi_k\|_2^2 \leqslant c_{\varepsilon,d}\lambda_1^{-d/2}Z_H(T)e^{T(1+\varepsilon)\lambda_k}$$

for all $k \in \mathbb{N}$. For $f \in L_2(\Omega) \cap L_\infty(\Omega)$ one has $e^{-tH}f = \sum_{k=1}^\infty \langle f, \varphi_k\rangle e^{-t\lambda_k}\varphi_k$ and hence

$$\|e^{-tH}f\|_1 \leqslant \sum_{k=1}^\infty \|f\|_\infty e^{-t\lambda_k}\|\varphi_k\|_1^2.$$

Using a sequence (f_k) in $L_2(\Omega)$ with $0 \leqslant f_k \uparrow 1_\Omega$ and recalling $T(1+\varepsilon) = t - T$, we conclude that

$$\|e^{-tH}1_\Omega\|_1 \leqslant \sum_{k=1}^\infty e^{-t\lambda_k}\|\varphi_k\|_1^2 \leqslant c_{\varepsilon,d}\lambda_1^{-d/2}Z_H(T)\sum_{k=1}^\infty e^{-T\lambda_k} = c_{\varepsilon,d}\lambda_1^{-d/2}Z_H(T)^2. \quad \square$$

The paper is organized as follows. In Section 2 we investigate properties of selfadjoint positivity preserving semigroups dominated by the free heat semigroup. In Section 3 we prove Theorem 1.5, and we show off-diagonal resolvent estimates needed in the proof of Theorem 1.1, which in turn is given in Section 4.

2 Semigroups dominated by the free heat semigroup

Throughout this section let $\Omega \subseteq \mathbb{R}^d$ be measurable, and let T be a selfadjoint positivity preserving C_0-semigroup on $L_2(\Omega)$ that is dominated by the free heat semigroup, with generator $-H$. Let τ be the closed symmetric form associated with H. The purpose of this section is to collect some basic properties of τ and H.

It is crucial that $D(\tau)$ is a subset of $H^1(\mathbb{R}^d)$ (in fact an ideal; see, e.g., [MVV05; Cor. 4.3]). Thus we can define a symmetric form σ by

$$\sigma(u,v) := \tau(u,v) - \langle\nabla u, \nabla v\rangle \qquad \big(u, v \in D(\sigma) := D(\tau)\big). \tag{2.1}$$

This gives a decomposition of the form τ as the standard Dirichlet form plus a form σ that is positive and local in the sense of the following lemma. If $-H$ is the Dirichlet Laplacian with an absorption potential $V \geqslant 0$ on an open set $\Omega \subseteq \mathbb{R}^d$, then $\sigma(u,v) = \int Vu\bar{v}$. In this case the next three results are trivial.

2.1 Lemma. *Let $0 \leqslant u, v \in D(\tau)$. Then $\sigma(u,v) \geqslant 0$, and $\sigma(u,v) = 0$ if $u \wedge v = 0$.*

Proof. By [MVV05; Cor. 4.3], the first assertion follows from the assumption that T is a positive semigroup dominated by the free heat semigroup. For the second assertion let $w := u - v$. Then $\tau(u,v) = \tau(w^+, w^-) \leqslant 0$ since T is a positive semigroup (see, e.g., [MVV05; Cor. 2.6]). Since $\langle\nabla u, \nabla v\rangle = 0$, this implies $\sigma(u,v) \leqslant 0$ and hence $\sigma(u,v) = 0$. $\square$

2.2 Lemma. *If $\xi \in W^1_\infty(\mathbb{R}^d)$ and $u \in D(\tau)$, then $\xi u \in D(\tau)$. Moreover, $f\colon \mathbb{R}^d \to D(\tau)$, $f(x) := \xi(\cdot - x)u$ is continuous.*

Proof. By [MVV05; Cor. 4.3], $D(\tau)$ is an ideal of $H^1(\mathbb{R}^d)$. This implies the first assertion $\xi u \in D(\tau)$ since $\xi u \in H^1(\mathbb{R}^d)$ and $|\xi u| \leqslant \|\xi\|_\infty |u| \in D(\tau)$.

For the second assertion it suffices to show continuity at 0, and we can assume without loss of generality that ξ, u are real-valued. From the identity

$$f(x) - f(0) = \xi(\cdot - x)\big(u - u(\cdot - x)\big) + (\xi u)(\cdot - x) - \xi u$$

one deduces that $f\colon \mathbb{R}^d \to H^1(\mathbb{R}^d)$ is continuous at 0. By Lemma 2.1 we obtain

$$\sigma\big(f(x) - f(0)\big) = \sigma\big(|f(x) - f(0)|\big) \leqslant \sigma\big(\|\xi(\cdot - x) - \xi\|_\infty |u|\big) \leqslant \|\nabla\xi\|_\infty^2 |x|^2 \sigma(|u|).$$

Due to the decomposition (2.1) this yields continuity of $f\colon \mathbb{R}^d \to D(\tau)$ at 0. ☐

2.3 Lemma. *Let $u, v \in D(\tau)$. Then $\sigma(\xi u, v) = \sigma(u, \xi v)$ for all $\xi \in W^1_\infty(\mathbb{R}^d)$.*

Proof. Since $D(\tau)$ is a lattice, it suffices to show the assertion for $u, v \geqslant 0$ and real-valued ξ. Throughout the proof we consider only real-valued function spaces. We define a bilinear form b by

$$b(\varphi, \psi) := \sigma(\varphi u, \psi v) \qquad \big(\varphi, \psi \in D(b) := W^1_{\infty,0}(\mathbb{R}^d)\big).$$

Then $b(\varphi, \psi) \geqslant 0$ for $\varphi, \psi \geqslant 0$ by Lemma 2.1. Now one can proceed similarly as in [ArWa03; proof of Thm. 4.1] to show that

$$\sigma(\varphi u, \psi v) = \int \varphi\psi \, d\mu \qquad \big(\varphi, \psi \in W^1_{\infty,0}(\mathbb{R}^d)\big) \tag{2.2}$$

for some finite positive Borel measure μ on $\mathbb{R}^d$ (depending of course on u, v). We only sketch the argument: first one can extend b to a continuous bilinear form on $C_0(\mathbb{R}^d)$, by positivity. Then one uses the linearisation of b in $C_0(\mathbb{R}^d \times \mathbb{R}^d)'$ to obtain a finite Borel measure ν on $\mathbb{R}^d \times \mathbb{R}^d$ such that $b(\varphi, \psi) = \int \varphi(x)\psi(y) \, d\nu(x, y)$ for all $\varphi, \psi \in W^1_{\infty,0}(\mathbb{R}^d)$. Finally, $\operatorname{spt} \nu \subseteq \big\{(x, x);\, x \in \mathbb{R}^d\big\}$ since $b(\varphi, \psi) = 0$ in the case $\operatorname{spt} \varphi \cap \operatorname{spt} \psi = \varnothing$, by Lemma 2.1, and this leads to the asserted measure μ.

To complete the proof, we show that the representation (2.2) is valid for all $\varphi, \psi \in W^1_\infty(\mathbb{R}^d)$. Let $\chi \in C^1_{\mathrm{c}}(\mathbb{R}^d)$ such that $0 \leqslant \chi \leqslant 1$ and $\chi|_{B(0,1)} = 1$. Then $u_n := \chi(\frac{\cdot}{n})u \to u$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$, and $\sigma(u_n) \leqslant \sigma(u)$ for all $n \in \mathbb{N}$ by Lemma 2.1. Therefore, $\limsup \tau(u_n) \leqslant \tau(u)$, and this implies $u_n \to u$ in $D(\tau)$. Applying (2.2) to $\sigma\big(\chi(\frac{\cdot}{n})\varphi u, \chi(\frac{\cdot}{n})\psi v\big)$ and letting $n \to \infty$ we derive (2.2) for any $\varphi, \psi \in W^1_\infty(\mathbb{R}^d)$. For real-valued $\xi \in W^1_\infty(\mathbb{R}^d)$ we now obtain

$$\sigma(\xi u, v) = \int \xi \, d\mu = \sigma(u, \xi v).$$ ☐

In the proof of Theorem 1.1 we will work with operators that are subordinated to H as follows. For an open set $U \subseteq \mathbb{R}^d$ let H_U denote the selfadjoint operator in $L_2(\Omega \cap U)$ associated with the form τ restricted to $D(\tau) \cap H_0^1(U)$. (Observe that this form domain is dense in $L_2(\Omega \cap U)$.)

2.4 Lemma. *Let φ be an eigenfunction of H with eigenvalue λ. Let U be an open subset of $\mathbb{R}^d$, and let $\xi \in W_\infty^2(\mathbb{R}^d)$, $\xi = 0$ on $\mathbb{R}^d \setminus U$. Then $\xi\varphi \in D(H_U)$ and*

$$(H_U - \lambda)(\xi\varphi) = -2\nabla\xi \cdot \nabla\varphi - (\Delta\xi)\varphi.$$

Proof. By Lemma 2.2 we have $\xi\varphi \in D(\tau)$. Moreover, $\xi\varphi \in H_0^1(U)$ due to the assumption $\xi = 0$ on $\mathbb{R}^d \setminus U$. For $v \in D(\tau) \cap H_0^1(U)$ we have $\xi v \in D(\tau) \cap H_0^1(U)$ and

$$(\tau - \lambda)(\varphi, \xi v) = \langle (H - \lambda)\varphi, \xi v\rangle = 0.$$

Since $\sigma(\xi\varphi, v) = \sigma(\varphi, \xi v)$ by Lemma 2.3, the decomposition (2.1) yields

$$\begin{aligned}(\tau - \lambda)(\xi\varphi, v) &= (\tau - \lambda)(\xi\varphi, v) - (\tau - \lambda)(\varphi, \xi v)\\ &= \langle \nabla(\xi\varphi), \nabla v\rangle - \langle \nabla\varphi, \nabla(\xi v)\rangle = \langle \varphi\nabla\xi, \nabla v\rangle - \langle \nabla\varphi, v\nabla\xi\rangle.\end{aligned}$$

Now $\varphi\nabla\xi$ is in $H^1(\mathbb{R}^d)^d$ and $\nabla \cdot (\varphi\nabla\xi) = \nabla\varphi \cdot \nabla\xi + \varphi\Delta\xi$, so we conclude that

$$(\tau - \lambda)(\xi\varphi, v) = -\langle 2\nabla\varphi \cdot \nabla\xi + \varphi\Delta\xi, v\rangle$$

for all $v \in D(\tau) \cap H_0^1(U)$, which proves the assertion. □

3 Heat kernel estimates

In this section we prove Theorem 1.5, and we provide resolvent estimates needed in the proof of Theorem 1.1. Throughout we denote

$$\mathbb{C}_+ := \{z \in \mathbb{C};\ \operatorname{Re} z > 0\}.$$

We point out that in the following result T is not required to be a semigroup.

3.1 Proposition. *Let (Ω, μ) be a measure space, and let $\rho\colon \Omega \to \mathbb{R}$ be measurable. Let $\lambda \in \mathbb{R}$, and let $T\colon \mathbb{C}_+ \to \mathcal{L}(L_2(\mu))$ be analytic, $\|T(z)\| \leqslant e^{-\lambda \operatorname{Re} z}$ for all $z \in \mathbb{C}_+$. Assume that there exists $C \geqslant 0$ such that*

$$\|e^{\alpha\rho}T(t)e^{-\alpha\rho}\| \leqslant Ce^{\alpha^2 t} \qquad (\alpha, t > 0).$$

Then

$$\|e^{\alpha\rho}T(z)e^{-\alpha\rho}\| \leqslant \exp\bigl(\alpha^2/\operatorname{Re}\tfrac{1}{z} - \lambda \operatorname{Re} z\bigr) \qquad (\alpha > 0,\ z \in \mathbb{C}_+),$$

in particular, $\|e^{\alpha\rho}T(t)e^{-\alpha\rho}\| \leqslant e^{\alpha^2 t - \lambda t}$ for all $\alpha, t > 0$.

Here and in the following we denote

$$\|wBw^{-1}\| := \sup\bigl\{\|wBw^{-1}f\|_2;\ f \in L_2(\mu),\ \|f\|_2 \leqslant 1,\ w^{-1}f \in L_2(\mu)\bigr\}$$

for an operator $B \in \mathcal{L}(L_2(\mu))$ and a measurable function $w\colon \Omega \to (0,\infty)$.

Proof of Proposition 3.1. Observe that

$$M := \bigl\{f \in L_2(\mu);\ \rho \text{ bounded on } [f \neq 0]\bigr\}$$

is dense in $L_2(\mu)$. Let $\alpha > 0$, and let $f, g \in M$ with $\|f\|_2 = \|g\|_2 = 1$. Define the analytic function $F\colon \mathbb{C}_+ \to \mathbb{C}$ by

$$F(z) := e^{\lambda z - \alpha^2/z}\langle e^{\alpha\rho/z}T(z)e^{-\alpha\rho/z}f, g\rangle.$$

Let $c > 0$ such that $|\rho| \leqslant c$ on $[f \neq 0] \cup [g \neq 0]$. Then

$$|F(z)| \leqslant \|e^{\lambda z}T(z)\|\|e^{-\alpha\rho/z}f\|_2\|e^{\alpha\rho/\bar{z}}g\|_2 \leqslant \exp(2\alpha c \operatorname{Re}\tfrac{1}{z}) \qquad (z \in \mathbb{C}_+),$$

in particular $|F(t)| \leqslant e^{2\alpha c}$ for all $t \geqslant 1$. Moreover,

$$|F(t)| \leqslant e^{\lambda t - \alpha^2/t}\|e^{\alpha\rho/t}T(t)e^{-\alpha\rho/t}\| \leqslant e^{\lambda t - \alpha^2/t} \cdot Ce^{(\alpha/t)^2 t} \leqslant Ce^{|\lambda|}$$

for all $0 < t < 1$. Thus, $|F(z)| \leqslant 1$ for all $z \in \mathbb{C}_+$ by the next lemma, and this yields

$$\|e^{\alpha\rho/z}T(z)e^{-\alpha\rho/z}\| \leqslant \exp\bigl(\alpha^2 \operatorname{Re}\tfrac{1}{z} - \lambda \operatorname{Re} z\bigr) \qquad (\alpha > 0,\ z \in \mathbb{C}_+).$$

The assertion follows by replacing α with $\alpha/\operatorname{Re}\frac{1}{z}$. ☐

The following Phragmén-Lindelöf type result is similar to [CoSi08; Prop. 2.2].

3.2 Lemma. *Let $F\colon \mathbb{C}_+ \to \mathbb{C}$ be analytic. Assume that there exist $c_1, c_2 > 0$ such that*

$$|F(z)| \leqslant \exp(c_1 \operatorname{Re}\tfrac{1}{z}) \quad (z \in \mathbb{C}_+), \qquad\qquad |F(t)| \leqslant c_2 \quad (t > 0).$$

Then $|F(z)| \leqslant 1$ for all $z \in \mathbb{C}_+$.

Proof. Note that $\limsup_{z \to iy}|F(z)| \leqslant 1$ for all $y \in \mathbb{R} \setminus \{0\}$. Thus, $|F(z)| \leqslant c_2 \vee 1$ for all $z \in \mathbb{C}_+$ by the Phragmén-Lindelöf principle applied to the sectors $\bigl\{z \in \mathbb{C};\ \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\bigr\}$ and $\bigl\{z \in \mathbb{C};\ \operatorname{Re} z > 0,\ \operatorname{Im} z < 0\bigr\}$. Then an application of the Phragmén-Lindelöf principle to the sector $\mathbb{C}_+$ implies $|F| \leqslant 1$ on $\mathbb{C}_+$. ☐

In the next lemma we state a version of the well-known Davies' trick; cf. [Dav95; proof of Lemma 19]. For the proof note that $\inf_{\xi \in \mathbb{R}^d} \exp\bigl(|\xi|^2 t - \xi \cdot x\bigr) = \exp\bigl(-\frac{|x|^2}{4t}\bigr)$ for all $t > 0,\ x \in \mathbb{R}^d$.

3.3 Lemma. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable, and let B be a positive operator on $L_2(\Omega)$. For $\xi, x \in \mathbb{R}^d$ let $\rho_\xi(x) := e^{\xi x}$. Then for $t > 0$ the following are equivalent:*

(i) $B \leqslant e^{t\Delta}$,

(ii) $\|\rho_\xi B \rho_\xi^{-1}\|_{1\to\infty} \leqslant (4\pi t)^{-d/2} e^{|\xi|^2 t}$ *for all* $\xi \in \mathbb{R}^d$.

In (i), the inequality $B \leqslant e^{t\Delta}$ is meant in the sense of positivity preserving operators, i.e.,

$$Bf \leqslant e^{t\Delta} f \qquad \bigl(0 \leqslant f \in L_2(\Omega)\bigr).$$

The following result provides an estimate of the resolvent of H by the free resolvent. Together with Proposition 3.5 below this will be an important stepping stone in the proof of Theorem 1.1.

3.4 Theorem. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable, and let T be a selfadjoint positive C_0-semigroup on $L_2(\Omega)$ that is dominated by the free heat semigroup. Let $-H$ be the generator of T, and let $E_0 := \inf \sigma(H)$. Then for all $\varepsilon \in (0,1]$ one has*

$$T(t) \leqslant \varepsilon^{-d/2} e^{-(1-\varepsilon)E_0 t} e^{t\Delta} \qquad (t \geqslant 0),$$

$$(H-\lambda)^{-1} \leqslant \varepsilon^{-d/2}\bigl((1-\varepsilon)E_0 - \lambda - \Delta\bigr)^{-1} \qquad (\lambda < (1-\varepsilon)E_0).$$

Proof. As above let $\rho_\xi(x) := e^{\xi x}$. The assumptions imply $\|T(z)\|_{2\to2} \leqslant e^{-E_0 \operatorname{Re} z}$ for all $z \in \mathbb{C}_+$ and

$$\|\rho_\xi T(t) \rho_\xi^{-1}\|_{2\to2} \leqslant \|\rho_\xi e^{t\Delta} \rho_\xi^{-1}\|_{2\to2} = e^{t|\xi|^2} \qquad (\xi \in \mathbb{R}^d,\ t \geqslant 0).$$

By Proposition 3.1 it follows that

$$\|\rho_\xi T(t) \rho_\xi^{-1}\|_{2\to2} \leqslant e^{t|\xi|^2 - E_0 t} \qquad (\xi \in \mathbb{R}^d,\ t \geqslant 0). \tag{3.1}$$

Let $t > 0$, and let k_t be the convolution kernel of $e^{t\Delta}$. Then for $\xi \in \mathbb{R}^d$ the kernel of $e^{-t|\xi|^2}\rho_\xi e^{t\Delta}\rho_\xi^{-1}$ is given by

$$e^{-t|\xi|^2 + \xi\cdot(x-y)} k_t(x-y) = k_t(x - 2t\xi - y) \qquad (x, y \in \mathbb{R}^d)$$

since $-t|\xi|^2 + \xi \cdot (x-y) - \frac{|x-y|^2}{4t} = -\frac{|x-y-2t\xi|^2}{4t}$. (The above identity is the key point in the proof; this is why we need unbounded weights in Proposition 3.1.) Therefore,

$$e^{-t|\xi|^2}\|\rho_\xi T(t)\rho_\xi^{-1}\|_{2\to\infty} \leqslant \|e^{t\Delta}\|_{2\to\infty} = \|e^{2t\Delta}\|_{1\to\infty}^{1/2} = (8\pi t)^{-d/4}.$$

By duality we also have $e^{-t|\xi|^2}\|\rho_\xi T(t)\rho_\xi^{-1}\|_{1\to2} \leqslant (8\pi t)^{-d/4}$. Using the semigroup property and (3.1), we conclude for $\varepsilon \in (0,1]$ that

$$\begin{aligned}\|\rho_\xi T(t)\rho_\xi^{-1}\|_{1\to\infty} &\leqslant \|\rho_\xi T(\tfrac{\varepsilon}{2}t)\rho_\xi^{-1}\|_{2\to\infty}\|\rho_\xi T((1-\varepsilon)t)\rho_\xi^{-1}\|_{2\to2}\|\rho_\xi T(\tfrac{\varepsilon}{2}t)\rho_\xi^{-1}\|_{1\to2} \\ &\leqslant e^{t|\xi|^2}(8\pi\tfrac{\varepsilon}{2}t)^{-d/4} e^{-E_0(1-\varepsilon)t}(8\pi\tfrac{\varepsilon}{2}t)^{-d/4} = (4\pi\varepsilon t)^{-d/2} e^{t|\xi|^2 - E_0(1-\varepsilon)t}.\end{aligned}$$

Now the first assertion follows from Lemma 3.3, and this gives the second assertion by the resolvent formula. $\square$

Proof of Theorem 1.5. The existence of the kernel p_t follows from the Dunford-Pettis theorem, and Theorem 3.4 implies

$$p_t(x,y) \leqslant (4\pi\varepsilon t)^{-d/2} e^{\varepsilon E_0 t} \exp\left(-E_0 t - \frac{|x-y|^2}{4t}\right)$$

for all $t > 0$. Then for $t \geqslant \frac{d}{2E_0}$ the assertion follows by setting $\varepsilon := \frac{d}{2E_0 t}$. □

We conclude this section with an off-diagonal L_1-estimate for the free resolvent.

3.5 Proposition. *Let $A, B \subseteq \mathbb{R}^d$ be measurable, and let $d(A,B)$ denote the distance between A and B. Then*

$$\|1_A(\mu-\Delta)^{-1}1_B\|_{1\to 1} \leqslant (1-\theta^2)^{-d/2}\tfrac{1}{\mu}\exp\bigl(-\theta\sqrt{\mu}\, d(A,B)\bigr)$$

for all $\mu > 0$, $0 < \theta < 1$.

Proof. Let $r := d(A,B)$. By duality we have to show

$$\|1_B(\mu-\Delta)^{-1}1_A\|_{\infty\to\infty} \leqslant (1-\theta^2)^{-d/2}\tfrac{1}{\mu}\exp\bigl(-\theta r\sqrt{\mu}\bigr) =: C,$$

or equivalently, $(\mu-\Delta)^{-1}1_A \leqslant C$ on B. Let $x \in B$. By the resolvent formula we obtain

$$(\mu-\Delta)^{-1}1_A(x) = \int_0^\infty e^{-\mu t}\int_{\mathbb{R}^d} k_t(y)1_A(x-y)\,dy\,dt \leqslant \int_0^\infty e^{-\mu t}\int_{|y|\geqslant r} k_t(y)\,dy\,dt,$$

where $k_t(y) = (4\pi t)^{-d/2}\exp\bigl(-\frac{|y|^2}{4t}\bigr)$. We substitute $y = (4t)^{1/2}z$ and note that $|y| \geqslant r$ if and only if $t \geqslant \bigl(\frac{r}{2|z|}\bigr)^2$; then by Fubini's theorem we infer that

$$\begin{aligned}(\mu-\Delta)^{-1}1_A(x) &\leqslant \pi^{-d/2}\int_{\mathbb{R}^d}\int_{\left(\frac{r}{2|z|}\right)^2}^\infty e^{-\mu t}\,dt\, e^{-|z|^2}\,dz \\ &= \pi^{-d/2}\frac{1}{\mu}\int_{\mathbb{R}^d}\exp\left(-\frac{\mu r^2}{4|z|^2} - |z|^2\right)dz.\end{aligned}$$

Note that $\theta r\sqrt{\mu} \leqslant \frac{\mu r^2}{4|z|^2} + \theta^2|z|^2$ and hence $\exp\bigl(-\frac{\mu r^2}{4|z|^2} - |z|^2\bigr) \leqslant e^{-\theta r\sqrt{\mu}}e^{-(1-\theta^2)|z|^2}$ for all $z \in \mathbb{R}^d$. We conclude that

$$(\mu-\Delta)^{-1}1_A(x) \leqslant \tfrac{1}{\mu}e^{-\theta r\sqrt{\mu}}\,\pi^{-d/2}\int_{\mathbb{R}^d} e^{-(1-\theta^2)|z|^2}dz = \tfrac{1}{\mu}e^{-\theta r\sqrt{\mu}}(1-\theta^2)^{-d/2},$$

which proves the assertion. □

3.6 Remark. For $\mu > \bigl(\frac{d}{r}\bigr)^2$ (where $r = d(A,B)$), optimizing the estimate of Proposition 3.5 with respect to θ leads to the choice $\theta = \bigl(1 - \frac{d}{r\sqrt{\mu}}\bigr)^{1/2}$. For $\mu > \bigl(\frac{d}{2r}\bigr)^2$, the choice $\theta = 1 - \frac{d}{2r\sqrt{\mu}}$ yields

$$\|1_A(\mu-\Delta)^{-1}1_B\|_{1\to 1} \leqslant \bigl(\tfrac{2e}{d} + r\sqrt{\mu}\bigr)^{d/2}\tfrac{1}{\mu}e^{-r\sqrt{\mu}}.$$

4 Proof of Theorem 1.1

Throughout this section we assume the setting of Section 2, i.e., $\Omega \subseteq \mathbb{R}^d$ is measurable, T a selfadjoint positivity preserving C_0-semigroup on $L_2(\Omega)$ dominated by the free heat semigroup, with generator $-H$, and τ the closed symmetric form associated with H. We denote

$$E_0(H) := \inf \sigma(H).$$

Recall that, for an open set $U \subseteq \mathbb{R}^d$, H_U is the selfadjoint operator in $L_2(\Omega \cap U)$ associated with the form τ restricted to $D(\tau) \cap H_0^1(U)$.

For $A \subseteq \mathbb{R}^d$ we denote by $U_\varepsilon(A) = \bigcup_{x\in A} B(x,\varepsilon)$ the ε-neighborhood of A. If A is measurable, then we write $|A|$ for the Lebesgue measure of A. For $r > 0$ and $E_0(H) < t < \inf \sigma_{\mathrm{ess}}(H)$ we define the sets

$$\begin{aligned} F_r(t) &:= \bigl\{x \in \mathbb{R}^d;\, E_0(H_{B(x,r)}) < t\bigr\}, \\ G_r(t) &:= \mathbb{R}^d \setminus \overline{U_r(F_r(t))}. \end{aligned} \tag{4.1}$$

For the proof of Theorem 1.1 the following two facts will be crucial. On the one hand, the set $F_r(t)$ is "small" in the sense that the Lebesgue measure of $U_{3r}(F_r(t))$ is not too large, as is expressed in the next lemma. On the other hand, the set $G_r(t)$ is "spectrally small" in the sense that the ground state energy of $H_{G_r(t)}$ is not much smaller than t; see Lemma 4.2 below.

4.1 Lemma. *Let $r > 0$ and $E_0(H) < t < \inf \sigma_{\mathrm{ess}}(H)$. Then*

$$|U_s(F_r(t))| \leqslant \omega_d(2r+s)^d N_t(H) \qquad (s > 0),$$

where $\omega_d := |B(0,1)|$.

Proof. Let $M \subseteq F_r(t)$ be a maximal subset with the property that the balls $B(x,r)$, $x \in M$ are pairwise disjoint. Then by the min-max principle and the definition of $F_r(t)$ one sees that M has at most $N_t(H)$ elements. Moreover, $F_r(t) \subseteq \bigcup_{x\in M} B(x,2r)$ by the maximality of M. Therefore,

$$|U_s(F_r(t))| \leqslant \sum_{x\in M} |B(x,2r+s)| \leqslant N_t(H)\cdot \omega_d(2r+s)^d. \qquad \square$$

4.2 Lemma. *Let $E_{0,d}$ denote the ground state energy of the Dirichlet Laplacian on $B(0,1)$. Then $E_{0,d} \leqslant \frac{1}{2}(d+1)(d+2) \leqslant \frac{3}{4}(d+1)^2$, and*

$$E_0(H_{G_r(t)}) \geqslant t - E_{0,d}/r^2 \qquad \bigl(r > 0,\ E_0(H) < t < \inf \sigma_{\mathrm{ess}}(H)\bigr).$$

Proof. For $\psi \in W^1_{2,0}\big(B(0,1)\big)$ defined by $\psi(x) = 1 - |x|$ one easily computes $\|\nabla\psi\|_2^2/\|\psi\|_2^2 = \frac{1}{2}(d+1)(d+2)$, thus proving the first assertion. Let now ψ denote the normalized ground state of the Dirichlet Laplacian on $B(0,1)$. For $r > 0$ let $\psi_r := r^{-d/2}\psi(\frac{\cdot}{r})$; note that $\|\psi_r\|_2 = 1$ and $\psi_r \in W^1_\infty(\mathbb{R}^d)$.

To prove the second assertion, we need to show that

$$\tau(u) \geqslant \big(t - E_{0,d}/r^2\big)\|u\|_2^2 \tag{4.2}$$

for all $u \in D(\tau) \cap H^1_0(G_r(t))$, without loss of generality u real-valued. We will use $\big(\psi_r(\cdot - x)^2\big)_{x\in\mathbb{R}^d}$ as a continuous partition of the identity. By Lemma 2.2 we have $\psi_r(\cdot - x)u \in D(\tau)$ for all $x \in \mathbb{R}^d$. Using (2.1) and Lemma 2.3 we obtain

$$\begin{aligned}
\tau\big(\psi_r(\cdot - x)u\big) &= \big\|\psi_r(\cdot - x)\nabla u + u\nabla\psi_r(\cdot - x)\big\|_2^2 + \sigma\big(\psi_r(\cdot - x)u\big) \\
&= \int \Big(\nabla\big(\psi_r(\cdot - x)^2 u\big)\cdot\nabla u + u^2|\nabla\psi_r(\cdot - x)|^2\Big) + \sigma\big(\psi_r(\cdot - x)^2 u, u\big) \\
&= \tau\big(\psi_r(\cdot - x)^2 u, u\big) + \int u^2|\nabla\psi_r(\cdot - x)|^2.
\end{aligned}$$

Note that $\int \psi_r(y - x)^2\,dx = \|\psi_r\|_2^2 = 1$ and

$$\int |\nabla\psi_r(y - x)|^2\,dx = \|\nabla\psi_r\|_2^2 = \|\nabla\psi\|_2^2/r^2 = E_{0,d}/r^2$$

for all $y \in \mathbb{R}^d$. Taking into account Lemma 2.2 (with $\xi = \psi_r^2$) we thus obtain $\int \tau\big(\psi_r(\cdot - x)^2 u, u\big)\,dx = \tau(u,u)$ and hence

$$\int \tau\big(\psi_r(\cdot - x)u\big)\,dx = \tau(u) + \|u\|_2^2 \cdot E_{0,d}/r^2.$$

To conclude the proof of (4.2), we show that the left hand side of this identity is greater or equal $t\|u\|_2^2$: note that $\psi_r(\cdot - x)u \in H^1_0(B(x,r))$. For $x \in \mathbb{R}^d \setminus F_r(t)$ we have $\tau\big(\psi_r(\cdot - x)u\big) \geqslant t\|\psi_r(\cdot - x)u\|_2^2$ by the definition of $F_r(t)$; for $x \in F_r(t)$ we have $\psi_r(\cdot - x)u = 0$ since $u \in H^1_0(G_r(t))$. Therefore,

$$\int \tau\big(\psi_r(\cdot - x)u\big)\,dx \geqslant t\int \|\psi_r(\cdot - x)u\|_2^2\,dx = t\|u\|_2^2\,. \qquad \square$$

4.3 Remark. It is known that $E_{0,d}$ behaves like $\frac{1}{4}d^2$ for large d. For $d = 3$, however, the estimate $E_{0,d} \leqslant \frac{1}{2}(d+1)(d+2) = 10$ from Lemma 4.2 is quite sharp since $E_{0,3} = \pi^2 > 9.86$.

4.4 Lemma. *There exists* $0 \leqslant \rho \in C^2(\mathbb{R}^d)$ *such that* $\operatorname{spt}\rho \subseteq B(0,1)$, $\int \rho = 1$ *and*

$$\|\nabla\rho\|_1 \leqslant d+1, \qquad \|\Delta\rho\|_1 \leqslant 2(d+1)^2. \tag{4.3}$$

Proof. Let $\rho_0 \in W_1^1(\mathbb{R}^d)$, $\rho_0(x) := \frac{d(d+2)}{2\sigma_{d-1}}(1-|x|^2)1_{B(0,1)}(x)$, where σ_{d-1} denotes the surface measure of the unit sphere $\partial B(0,1)$. Then one easily computes

$$\int \rho_0 = 1, \qquad \|\nabla\rho_0\|_1 = \frac{d(d+2)}{d+1} < d+1$$

and

$$\Delta\rho_0 = \frac{d(d+2)}{\sigma_{d-1}}\big(-d\,1_{B(0,1)} + \delta_{\partial B(0,1)}\big)$$

in the distributional sense, so $\Delta\rho_0$ is a measure with $\|\Delta\rho_0\| = 2d(d+2) < 2(d+1)^2$. Using a suitable mollifier and scaling, one obtains ρ as asserted. $\square$

Proof of Theorem 1.1. (i) Let $r > \big(\frac{E_{0,d}}{t-\lambda}\big)^{1/2}$, and let $F_r := F_r(t)$, $G_r := G_r(t)$ be as in (4.1). Then $E_0(H_{G_r}) > \lambda$ by Lemma 4.2. We define $\xi \in C^2(\mathbb{R}^d)$ satisfying

$$\operatorname{spt}\xi \subseteq G_r, \qquad \operatorname{spt}(1_{\mathbb{R}^d} - \xi) \subseteq U_{2r}(F_r)$$

as follows: let $\rho_r := r^{-d}\rho(\frac{\cdot}{r})$, where ρ is as in Lemma 4.4. Then

$$\xi := 1_{\mathbb{R}^d} - \rho_{r/2} * 1_{U_{3r/2}(F_r)} = \tfrac{1}{2}1_{\mathbb{R}^d} + \rho_{r/2} * \big(\tfrac{1}{2}1_{\mathbb{R}^d} - 1_{U_{3r/2}(F_r)}\big)$$

has the above properties, and

$$\|\nabla\xi\|_\infty \leqslant \tfrac{1}{2}\|\nabla\rho_{r/2}\|_1 = \tfrac{1}{r}\|\nabla\rho\|_1, \qquad \|\Delta\xi\|_\infty \leqslant \tfrac{1}{2}\|\Delta\rho_{r/2}\|_1 = \tfrac{2}{r^2}\|\Delta\rho\|_1. \tag{4.4}$$

By Lemma 2.4 we obtain $\xi\varphi \in D(H_{G_r})$ and

$$f_r := (H_{G_r} - \lambda)(\xi\varphi) = -2\nabla\xi\cdot\nabla\varphi - (\Delta\xi)\varphi, \qquad \operatorname{spt} f_r \subseteq \operatorname{spt}\nabla\xi \subseteq U_{2r}(F_r).$$

Then $\xi\varphi = (H_{G_r} - \lambda)^{-1}f_r = (H_{G_r} - \lambda)^{-1}1_{U_{2r}(F_r)}f_r$. Since $\xi = 1$ on $\Omega\setminus U_{3r}(F_r)$, we can now estimate

$$\begin{aligned} \|\varphi\|_1 &= \|1_{U_{3r}(F_r)}\varphi\|_1 + \|1_{\Omega\setminus U_{3r}(F_r)}\xi\varphi\|_1 \\ &\leqslant \|1_{U_{3r}(F_r)}\varphi\|_1 + \|1_{\Omega\setminus U_{3r}(F_r)}(H_{G_r} - \lambda)^{-1}1_{U_{2r}(F_r)}\|_{1\to 1}\|f_r\|_1. \end{aligned} \tag{4.5}$$

The remainder of the proof consists of estimating the terms in this pivotal inequality.

Lemma 4.1 implies

$$\|1_{U_{3r}(F_r)}\varphi\|_1 \leqslant |U_{3r}(F_r)|^{1/2}\|\varphi\|_2 \leqslant \big(\omega_d(5r)^d N_t(H)\big)^{1/2}\|\varphi\|_2 \tag{4.6}$$

and

$$\|f_r\|_1 \leqslant |U_{2r}(F_r)|^{1/2}\|f_r\|_2 \leqslant \big(\omega_d(4r)^d N_t(H)\big)^{1/2}\|f_r\|_2\,, \tag{4.7}$$

$$\|f_r\|_2 \leqslant 2\|\nabla\xi\|_\infty\|\nabla\varphi\|_2 + \|\Delta\xi\|_\infty\|\varphi\|_2 \leqslant \frac{2}{r}\|\nabla\rho\|_1\sqrt{\lambda}\,\|\varphi\|_2 + \frac{2}{r^2}\|\Delta\rho\|_1\|\varphi\|_2\,, \tag{4.8}$$

where in (4.8) we used (4.4) and $\|\nabla\varphi\|_2^2 = \lambda\|\varphi\|_2^2$.

(ii) Next we estimate $\|1_{\Omega\setminus U_{3r}(F_r)}(H_{G_r}-\lambda)^{-1}1_{U_{2r}(F_r)}\|_{1\to 1}$. Let $\delta,\theta\in(0,1)$ and

$$\varepsilon := \delta\,\frac{t-\lambda}{t}, \qquad \mu := (1-\varepsilon)E_0(H_{G_r})-\lambda.$$

Then $(H_{G_r}-\lambda)^{-1}\leqslant \varepsilon^{-d/2}(\mu-\Delta)^{-1}$ by Theorem 3.4, and hence Proposition 3.5 implies

$$\|1_{\Omega\setminus U_{3r}(F_r)}(H_{G_r}-\lambda)^{-1}1_{U_{2r}(F_r)}\|_{1\to 1}\leqslant \varepsilon^{-d/2}(1-\theta^2)^{-d/2}\tfrac{1}{\mu}e^{-\theta r\sqrt{\mu}}. \tag{4.9}$$

By Lemma 4.2 and the definition of ε we have

$$\mu\geqslant(1-\varepsilon)(t-E_{0,d}/r^2)-\lambda\geqslant t-\varepsilon t-E_{0,d}/r^2-\lambda=(1-\delta)(t-\lambda)-E_{0,d}/r^2.$$

We now choose r such that $r^2=\frac{c^2+E_{0,d}}{(1-\delta)(t-\lambda)}$, with $c\geqslant d+1$ to be determined later. Then

$$r^2\leqslant\frac{7/4}{(1-\delta)(t-\lambda)}\,c^2 \tag{4.10}$$

since $E_{0,d}\leqslant\frac{3}{4}(d+1)^2\leqslant\frac{3}{4}c^2$ by Lemma 4.2, and

$$\mu r^2\geqslant(1-\delta)(t-\lambda)r^2-E_{0,d}=c^2.$$

By (4.9) we thus obtain

$$\|1_{\Omega\setminus U_{3r}(F_r)}(H_{G_r}-\lambda)^{-1}1_{U_{2r}(F_r)}\|_{1\to 1}\leqslant \varepsilon^{-d/2}(1-\theta^2)^{-d/2}\frac{r^2}{c^2}\,e^{-\theta c}. \tag{4.11}$$

(iii) In this step we incorporate an estimate for $\|f_r\|_1$ into (4.11). By (4.3) we have $\|\nabla\rho\|_1\leqslant c$ and $\|\Delta\rho\|_1\leqslant 2c^2$. Thus, using (4.8), (4.10) and $\lambda<t$ we obtain

$$\begin{aligned}\frac{r^2}{2}\|f_r\|_2&\leqslant\|\nabla\rho\|_1 r\sqrt{\lambda}\,\|\varphi\|_2+\|\Delta\rho\|_1\|\varphi\|_2\\&\leqslant c^2\sqrt{\frac{7/4}{1-\delta}}\cdot\sqrt{\frac{\lambda}{t-\lambda}}\,\|\varphi\|_2+2c^2\|\varphi\|_2\leqslant c^2C_\delta\sqrt{\frac{t}{t-\lambda}}\,\|\varphi\|_2\,,\end{aligned}$$

with $C_\delta=\sqrt{2/(1-\delta)}+2$. Recalling $\varepsilon=\delta\frac{t-\lambda}{t}$, we infer by (4.11) that

$$\begin{aligned}\|1_{\Omega\setminus U_{3r}(F_r)}(H_{G_r}-\lambda)^{-1}1_{U_{2r}(F_r)}\|_{1\to 1}\|f_r\|_2&\leqslant\varepsilon^{-d/2}(1-\theta^2)^{-d/2}e^{-\theta c}\frac{r^2}{c^2}\|f_r\|_2\\&\leqslant\delta^{-d/2}\big(\tfrac{t}{t-\lambda}\big)^{(d+1)/2}(1-\theta^2)^{-d/2}e^{-\theta c}\cdot 2C_\delta\|\varphi\|_2\,.\end{aligned} \tag{4.12}$$

Now we set $K_{\delta,\theta}:=\frac{5}{4}\delta(1-\theta^2)$ and choose

$$c:=\frac{d+1}{2\theta}\ln\left(\frac{1}{K_{\delta,\theta}}\cdot\frac{t}{t-\lambda}\right).$$

Then
$$\delta^{-d/2}\big(\tfrac{t}{t-\lambda}\big)^{(d+1)/2}(1-\theta^2)^{-d/2}e^{-\theta c} = \big(\tfrac{5}{4}\big)^{d/2}K_{\delta,\theta}^{1/2},$$
so by (4.7) and (4.12) we obtain
$$\begin{aligned}\|1_{\Omega\setminus U_{3r}(F_r)}(H_{G_r}-\lambda)^{-1}1_{U_{2r}(F_r)}\|_{1\to1}\|f_r\|_1 \\ \leqslant \big(\omega_d(5r)^d N_t(H)\big)^{1/2}\cdot K_{\delta,\theta}^{1/2}\cdot 2C_\delta\|\varphi\|_2\,. \end{aligned}\tag{4.13}$$

(iv) We set $\theta := \frac{1}{2}$ and $\delta := \frac{16}{45}$, so that $K_{\delta,\theta} = \frac{1}{3}$ and hence
$$c = (d+1)\ln\tfrac{3t}{t-\lambda} \geqslant d+1 \tag{4.14}$$
as required above. Moreover, one easily verifies that $K_{\delta,\theta}^{1/2}\cdot 2C_\delta \leqslant \frac{9}{2}$. By (4.5), (4.6) and (4.13) we conclude that
$$\|\varphi\|_1^2 \leqslant \Big(\tfrac{11}{2}\big(\omega_d(5r)^d N_t(H)\big)^{1/2}\|\varphi\|_2\Big)^2 = \big(\tfrac{11}{2}\big)^2\omega_d(5r)^d N_t(H)\|\varphi\|_2^2\,. \tag{4.15}$$

Stirling's formula yields
$$\omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \leqslant \frac{\pi^{d/2}}{\sqrt{2\pi d/2}\,\big(\frac{d}{2e}\big)^{d/2}} = (\pi d)^{-1/2}(2\pi e)^{d/2}d^{-d/2},$$
so by (4.10) we obtain
$$\omega_d(5r)^d \leqslant (\pi d)^{-1/2}\big(2\pi e\cdot\tfrac{7/4}{1-\delta}\big)^{d/2}d^{-d/2}\cdot 5^d c^d(t-\lambda)^{-d/2}.$$
Using $2\pi e\cdot\frac{7/4}{1-\delta} \leqslant 7^2$, $(d+1)^d \leqslant 2d^{d+1/2}$ and (4.14) we finally derive
$$\omega_d(5r)^d \leqslant \pi^{-1/2}(7\cdot5)^d\cdot 2d^{d/2}\big(\ln\tfrac{3t}{t-\lambda}\big)^d(t-\lambda)^{-d/2}.$$
Together with (4.15) this proves the assertion since $\big(\frac{11}{2}\big)^2\pi^{-1/2}\cdot 2 \leqslant 35$. □

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