Stability of uniformly eventually positive C_0-semigroups on L_p-spaces

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Abstract

We give a short and elementary proof of the theorem of Lutz Weis that the growth bound of a positive C_0-semigroup on $L_p(\mu)$ equals the spectral bound of its generator. In addition, we generalise the result to the case of uniformly eventually positive semigroups.

MSC 2020: 47D06, 47B65, 46G10

Keywords: C_0-semigroups, eventually positive semigroups

Throughout let (Ω, μ) be a measure space, and let $1 < p < \infty$; we do not assume μ to be σ-finite. Let T be a C_0-semigroup on $L_p(\mu)$. A basic question of stability theory is when the growth bound

$$\omega_0(T) = \inf\bigl\{\omega \in \mathbb{R};\, \exists M \geqslant 1\, \forall t \geqslant 0\colon \|T(t)\| \leqslant Me^{\omega t}\bigr\}$$

of the semigroup T coincides with the spectral bound

$$s(A) = \sup\bigl\{\operatorname{Re}\lambda;\, \lambda \in \sigma(A)\bigr\}$$

of its generator A. In [Wei95], Weis proved that this is the case if the semigroup T is positive; in [Wei98] he gave a shorter proof of the same fact.

The aim of this paper is to give an even shorter and more elementary proof. Also, we show that the result generalises to the more general case of uniformly eventually positive semigroups. As in [DGK16; Def. 5.1] we say that T is *uniformly eventually positive* if there exists $t_0 \geqslant 0$ such that $T(t) \geqslant 0$ for all $t \geqslant t_0$. There are several important examples of operators that generate uniformly eventually positive semigroups, such as certain Dirichlet-to-Neumann operators and bi-Laplace operators, see [DaGl18; Sec. 4].

We denote by $L_p(\mu)_+$ the set of positive functions in $L_p(\mu)$.

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Lemma 1. *Let $I \subseteq \mathbb{R}$ be a compact interval, and let $g \in C(I; L_p(\mu)_+)$. Then*

$$\Big(\int_I g(t)^p \, dt\Big)^{1/p} = \sup\big\{\textstyle\int_I h(t)g(t)\,dt;\ h \in L_{p'}(I;\mathbb{R})_+,\ \|h\|_{p'} \leqslant 1\big\},$$

where the integrals are Bochner integrals and the supremum is taken in the Banach lattice $L_p(\mu)$.

Proof. Let $D \subseteq I$ be a countable dense subset. Then the set $\Omega_0 := \bigcup_{t\in D}\big\{x \in \Omega;\ g(t)(x) \neq 0\big\}$ is σ-finite, and $g(t)|_{\Omega\setminus\Omega_0} = 0$ a.e. for all $t \in I$. Now by Fubini's theorem there exists a function $\tilde g \in L_p(I \times \Omega_0)_+$ such that $g(t)|_{\Omega_0} = \tilde g(t,\cdot)$ for a.e. $t \in I$ (see [HNVW16; Prop. 1.2.24]). Then $\big(\int_I g(t)^p\,dt\big)(x) = \int_I \tilde g(t,x)^p\,dt$ and $\big(\int_I h(t)g(t)\,dt\big)(x) = \int_I h(t)\tilde g(t,x)\,dt$ for a.e. $x \in \Omega_0$. Thus the assertion follows from the separability of $L_{p'}(I;\mathbb{R})$ and the validity of the asserted identity for scalar-valued g. □

Theorem 2. *Let T be a uniformly eventually positive C_0-semigroup on $L_p(\mu)$. Then the growth bound of T equals the spectral bound of its generator.*

Proof. Let A be the generator of T, and assume that $s(A) < 0$. We show $\omega_0(T) < 0$, using Datko's theorem; then the assertion follows since $s(A) \leqslant \omega_0(T)$ is always true.

Let $t_0 \geqslant 0$ be such that $T(t) \geqslant 0$ for all $t \geqslant t_0$. Let $f \in L_p(\mu)_+$ and $t_1 > 2t_0 + 1$. Let $h \in L_{p'}(2t_0,\infty)_+$ satisfy $h|_{(2t_0,2t_0+1)} = 0$, $h|_{(t_1,\infty)} = 0$ and $\|h\|_{p'} \leqslant 1$. Then

$$\int_{2t_0+1}^{t_1} h(t)T(t)f\,dt = \int_{2t_0+1}^{\infty}\int_{t-t_0-1}^{t-t_0} ds\, h(t)T(t)f\,dt = \int_{t_0}^{\infty}\int_{s+t_0}^{s+t_0+1} h(t)T(t)f\,dt\,ds$$

by Fubini's theorem. Lemma 1 shows that

$$\int_{t_0}^{t_0+1} h(t+s)T(t)f\,dt \leqslant \Big(\int_{t_0}^{t_0+1} \big(T(t)f\big)^p dt\Big)^{1/p} =: g \in L_p(\mu),$$

so

$$\int_{s+t_0}^{s+t_0+1} h(t)T(t)f\,dt = T(s)\int_{t_0}^{t_0+1} h(t+s)T(t)f\,dt \leqslant T(s)g$$

for all $s \geqslant t_0$, by the positivity of $T(s)$. It follows that

$$\int_{2t_0+1}^{t_1} h(t)T(t)f\,dt \leqslant \int_{t_0}^{\infty} T(s)g\,ds =: g_1 \in L_p(\mu),$$

where the latter integral is convergent as an improper Riemann integral by our assumption $s(A) < 0$; see [DGK16; Prop. 7.1]. Using Lemma 1 again we conclude that $\int_{2t_0+1}^{t_1}\big(T(t)f\big)^p dt \leqslant g_1^p$ for all $t_1 > 2t_0 + 1$ and hence

$$\int_{2t_0+1}^{\infty} \|T(t)f\|_p^p\,dt \leqslant \|g_1\|_p^p < \infty.$$

Now Datko's theorem (see [EnNa00; Thm. V.1.8]) implies that $\omega_0(T) < 0$. □

Remarks 3. (a) It remains an open question whether Theorem 2 is also true if T is merely an *individually* eventually positive C_0-semigroup, i.e., if for all $f \in L_p(\mu)_+$ there exists $t_0 \geqslant 0$ such that $T(t)f \geqslant 0$ for all $t \geqslant t_0$. Note that our proof breaks down in this case because it relies on the positivity of the operator $T(s)$ for $s \geqslant t_0$. If $p = 1$ or $p = 2$, then the answer to the above question is positive; see [DGK16; Thm. 7.8].

(b) In [RoVe18; Thm. 3.8], sharp estimates are given for the asymptotic behaviour of positive C_0-semigroups on $L_p(\mu)$ in terms of resolvent bounds. This result can be viewed as a quantitative version of Theorem 2 for the case of positive semigroups.

Acknowledgment. I would like to thank Jochen Glück for valuable discussions.

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