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Abstract. It is shown that the function $x \mapsto 1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$ is strictly completely monotone on $(-1, \infty)$ and tends to one as $x \to -1$, to zero as $x \to \infty$. This property is derived from a suitable integral representation of $\ln \Gamma(x+1)$.

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The starting point of this note was an inequality,

$$1 \leqslant \frac{\Gamma(\frac{n}{2}+1)^{\frac{n-d}{n}}}{\Gamma(\frac{n-d}{2}+1)} \leqslant e^{\frac{d}{2}}, \tag{1}$$

for all pairs of integers $0 \leqslant d \leqslant n$, in [5; Lemma 2.1]. Note that the left hand side of this inequality is an immediate consequence of the logarithmic convexity of the Γ-function; see [5]. Looking for a stream-lined proof of inequality (1), we first found a proof of the more general inequality

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \leqslant e^{\frac{p}{q}-1}, \tag{2}$$

valid for all $0 < q \leqslant p$, and finally showed

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \leqslant \frac{p+1}{q+1}, \tag{3}$$

for all $-1 < q \leqslant p$. These inequalities will be immediate consequences of the following result.

Theorem 1. *The function* $f(x) := 1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$ *is strictly completely monotone on* $(-1, \infty)$*,*

$$\lim_{x \to -1} f(x) = 1, \quad \lim_{x \to \infty} f(x) = 0,$$
$$f(0) = \lim_{x \to 0} f(x) = 1 - \gamma.$$

(Here, γ *is the Euler-Mascheroni constant, and* strictly completely monotone *means* $(-1)^n f^{(n)}(x) > 0$ *for all* $x \in (-1, \infty)$*,* $n \in \mathbb{N}_0$*.)*

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Proof. The main ingredient of the proof is the integral representation

$$\ln\Gamma(x+1) = x\ln(x+1) - x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-t}\frac{1}{t}(1-e^{-xt})\,dt,$$

which is an immediate consequence of [6; formula 1.9 (2) (p. 21)] and [6; formula 1.7.2 (18) (p. 17)]. We obtain

$$f(x) = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-t}\frac{1}{xt}(1-e^{-xt})\,dt.$$

The function

$$g(y) := \frac{1}{y}(1-e^{-y}) = \int_0^1 e^{-sy}\,ds$$

is strictly completely monotone on $\mathbb{R}$. Since $\frac{1}{t}-\frac{1}{e^t-1} > 0$ for all $t>0$, we conclude that f is strictly completely monotone. As $y \to \infty$, $g(y)$ tends to zero, and hence $\lim_{x\to\infty} f(x) = 0$. The definition of f shows $\lim_{x\to 0} f(x) = 1+\psi(1) = 1-\gamma$; cf. [6; formula 1.7 (4) (p. 15)]. Finally,

$$\lim_{x\to -1} f(x) = 1 - \lim_{x\to -1}\left(\frac{1}{x}\big(\ln\Gamma(x+2) - \ln(x+1)\big) - \ln(x+1)\right) = 1. \qquad \square$$

Corollary 2. *Inequalities* (3), (2), (1) *are valid for the indicated ranges.*

Proof. Inequality (3) is just a reformulation of the monotonicity of the function f from Theorem 1. Continuing (3) to the right,

$$\frac{p+1}{q+1} \leqslant \frac{p}{q} \leqslant e^{\frac{p}{q}-1} \quad (0<q\leqslant p),$$

we obtain (2). Setting $q = \frac{n-d}{2}$, $p = \frac{n}{2}$ we get (1). $\square$

Remark 3. (a) In [4] it was shown that the function $\xi \mapsto \xi\big(\Gamma(1+\frac{1}{\xi})\big)^\xi$ is increasing on $(0,\infty)$. This fact follows immediately from our Theorem 1, because of

$$\ln\left(\frac{1}{x}\Gamma(x+1)^{\frac{1}{x}}\right) + 1 = -\ln x + \frac{1}{x}\Gamma(x+1) + 1 = \ln(x+1) - \ln x + f(x).$$

(In fact, the latter function even is strictly completely monotone as well.)

(b) For other recent results on (complete) monotonicity properties of the Γ-function we refer to [1], [2], [3].

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