Als Manuskript gedruckt

Technische Universität Dresden

Herausgeber: Der Rektor

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MATH-AN-07-02

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Dedicated to the memory of H. H. Schaefer

Abstract

In this paper we study C_0-semigroups on $X \times L_p(-h,0;X)$ associated with linear differential equations with delay, where X is a Banach space. In the case that X is a Banach lattice with order continuous norm, we describe the associated modulus semigroup, under minimal assumptions on the delay operator. Moreover, we present a new class of delay operators for which the delay equation is well-posed for p in a subinterval of $[1,\infty)$.

MSC 2000: 47D06, 34K06, 47B60

Keywords: functional differential equation, delay equation, modulus semigroup, perturbation theory, domination, Banach lattice

Introduction

We treat two topics arising in connection with the Cauchy problem for the linear delay equation

$$\begin{cases} u'(t) = Au(t) + Lu_t \quad (t \geqslant 0), \\ u(0) = x, \quad u_0 = f, \end{cases} \tag{DE}$$

with initial values $x \in X,\ f \in L_p(-h,0;X)$, where X is a Banach space, $1 \leqslant p < \infty$, and $0 < h \leqslant \infty$. (For a function $u\colon (-h,\infty) \to X$, we recall the notation

$$u_t(\theta) := u(t+\theta) \qquad (-h < \theta < 0),$$

for $t \geqslant 0$.) The foundations for treating this problem in the context of C_0-semigroups on $X \times L_p(-h,0;X)$ have been presented in [2]; we also refer to [3].

One of the topics concerns the question of the kind of operators L that are allowed in (DE). In the previous papers it was assumed that L is associated with a function

$\eta\colon [-h,0] \to \mathcal{L}(X)$ of bounded variation (cf. Example 5.1). Then the problem (DE) could be treated in $X \times L_p(-h,0;X)$ for any $p \in [1,\infty)$. We present a class of operators L that allows this treatment only for p in a proper subset of $[1,\infty)$: We only require $L\colon W_p^1(-h,0;X) \to X$ to be continuous as an operator from $L_r(\mu_L;X)$ to X, for some $r \in [1,p]$ and a suitable measure μ_L on $[-h,0]$.

For the other topic we additionally assume that X is a Banach lattice. Then we determine the modulus semigroup in a rather general context. This generalises the results of [5], [13], [10].

In order to put the results of the paper into the proper context, we now describe the C_0-semigroup setting of (DE). Let X be a Banach space, A the generator of a C_0-semigroup T on X. Let $\mathcal{T}_0$ be the C_0-semigroup on $X_p := X \times L_p(-h,0;X)$ given by

$$\mathcal{T}_0(t) = \begin{pmatrix} T(t) & 0 \\ T_t & S(t) \end{pmatrix} \qquad (t \geqslant 0),$$

where $T_t \in \mathcal{L}(X, L_p(-h,0;X))$ denotes the operator given by

$$T_t x(\theta) := \begin{cases} 0 & \text{for } -h < \theta \leqslant -t, \\ T(t+\theta)x & \text{for } -t < \theta < 0, \end{cases}$$

and S is the C_0-semigroup of left translation on $L_p(-h,0;X)$, i.e., $S(t)f = f_t$, where we assume that $f \in L_p(-h,0;X)$ is extended by 0 to a function on $(-h,\infty)$. It is known (see [2; Prop. 3.1]) that the generator of $\mathcal{T}_0$ is given by

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\theta} \end{pmatrix}, \quad D(\mathcal{A}_0) = \big\{ \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(A) \times W_p^1(-h,0;X);\ x = \varphi(0) \big\}.$$

Let now $L \in \mathcal{L}(W_p^1(-h,0;X),X)$. Then $\mathcal{B} := \left(\begin{smallmatrix} 0 & L \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{L}(D(\mathcal{A}_0),X_p)$, and we define

$$\mathcal{A} := \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A & L \\ 0 & \frac{d}{d\theta} \end{pmatrix}, \quad D(\mathcal{A}) = D(\mathcal{A}_0).$$

Assuming that $\mathcal{A}$ is the generator of a C_0-semigroup $\mathcal{T}$ one knows that, for $\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(\mathcal{A})$, the first component of the function $t \mapsto \mathcal{T}(t)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)$ is the unique solution of (DE); cf. [2].

Next, assume that X is a Banach lattice with order continuous norm and that T possesses a modulus semigroup (which is the smallest semigroup dominating T), whose generator will be denoted by $A^\#$. Assume that L possesses a modulus $|L|$, that L is massless at 0 (cf. Section 1), and that $\widetilde{\mathcal{A}} := \left(\begin{smallmatrix} A^\# & |L| \\ 0 & \frac{d}{d\theta} \end{smallmatrix}\right)$, with domain $D(\widetilde{\mathcal{A}}) = \big\{ \left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(A^\#) \times W_p^1(-h,0;X);\ x = \varphi(0) \big\}$, is a generator. Then we show that $\widetilde{\mathcal{A}}$ generates the modulus semigroup of $\mathcal{T}$ (Theorem 2.7).

The paper is organized as follows. In Section 1 we investigate operators from $W_p^1(-h,0;X)$ to Y, where X, Y are Banach lattices. The main objective is establishing relations between masslessness at 0 of L and $|L|$. In Section 2 we determine the modulus semigroup of $\mathcal{T}$; see above.

In Section 3 we discuss an inequality needed for L in order to make $\mathcal{B}$ a small Miyadera perturbation of $\mathcal{A}$. In Section 4 we present our class of operators L mentioned above; the conditions are such that the inequality singled out in Section 3 is satisfied. Section 5 presents an example illustrating that the class of operators from Section 4 contains more operators than those associated with functions $\eta\colon [-h,0] \to \mathcal{L}(X)$ of bounded variation.

Throughout this paper, let $1 \leqslant p < \infty,\ 0 < h \leqslant \infty$.

1 Operators defined on $W_p^1(-h,0;X)$ that are massless at 0

In this section we investigate how to describe that operators $L\colon W_p^1(-h,0;X) \to Y$ attribute no mass to the point 0, and we present relations between different notions of this kind. We start with a result in a more abstract setting.

Proposition 1.1. *Let X be a (real or complex) vector lattice, Y a Banach lattice with order continuous norm. Let $L\colon X \to Y$ be a linear operator, and assume that L has a modulus $|L|$,*

$$|L|x = \sup\bigl\{|Lz|;\, z \in X,\ |z| \leqslant x\bigr\}$$

for $x \in X_+$. Let $Q_n\colon X \to X$ be linear operators, $0 \leqslant Q_n \leqslant I$ $(n \in \mathbb{N})$.

(a) *Assume that the strong limit $L_Q := \text{s-lim}_{n\to\infty} LQ_n$ exists. Then the linear operators L_Q and $L - L_Q$ both have a modulus, and $|L|Q_n \to |L_Q|,\ |L|(I-Q_n) \to |L-L_Q|$ strongly.*

(b) *If the sequence (Q_n) is monotone then* $\text{s-lim}_{n\to\infty} LQ_n$ *exists.*

Proof. (a) For $x \in X$ we have $|L_Qx| = \lim_{n\to\infty} |LQ_nx| \leqslant |L||x|$ since $0 \leqslant Q_n \leqslant I$. Thus L_Q has a modulus, by the order completeness of Y (which, in turn, follows from the order continuity of the norm); cf. [9]. Similarly, $L - L_Q$ has a modulus.

Since $X = \operatorname{lin} X_+$ it now suffices to show that $|L|Q_nx \to |L_Q|x$ and $|L|(I - Q_n)x \to |L - L_Q|x$ for all $x \in X_+$. Let $\varepsilon > 0$. Since Y has order continuous norm, there exist $x_j \in X,\ |x_j| \leqslant x$ $(j = 1,\dots,m)$ such that $y := \sup_j |L_Qx_j|$ and $z := \sup_j |(L-L_Q)x_j|$ satisfy

$$\bigl\||L_Q|x - y\bigr\| \leqslant \varepsilon, \quad \bigl\||L-L_Q|x - z\bigr\| \leqslant \varepsilon. \tag{1.1}$$

The lattice operations in Y are continuous, so we have

$$\begin{aligned} y_n &:= \sup_j |LQ_nx_j| \to \sup_j |L_Qx_j| = y,\\ z_n &:= \sup_j |L(I-Q_n)x_j| \to \sup_j |(L-L_Q)x_j| = z. \end{aligned}$$

Let now $n_0 \in \mathbb{N}$ such that $\|y_n - y\| \leqslant \varepsilon,\ \|z_n - z\| \leqslant \varepsilon$ for all $n \geqslant n_0$. By the definition of y_n, z_n we obtain, using the estimate $|L|x \leqslant |L_Q|x + |L-L_Q|x$,

$$z_n - |L-L_Q|x \leqslant |L|(I-Q_n)x - |L-L_Q|x \leqslant |L_Q|x - |L|Q_nx \leqslant |L_Q|x - y_n.$$

For $n \geqslant n_0$ the left and right hand sides of this chain of inequalities have norm $\leqslant 2\varepsilon$ by (1.1), so $|L|Q_n x \to |L_Q|x$ and $|L|(I - Q_n)x \to |L - L_Q|x$ as $n \to \infty$.

(b) It clearly suffices to treat the case that (Q_n) is monotone increasing. Let $x \in X$. For $n \leqslant m$ we estimate

$$|LQ_m x - LQ_n x| \leqslant |L||Q_m x - Q_n x| \leqslant |L|(Q_m - Q_n)|x|.$$

The order continuity of the norm in Y implies that the increasing sequence $(|L|Q_n|x|)$ in $[0, |L||x|]$ is convergent, and therefore is a Cauchy sequence. The previous estimate implies that $(LQ_n x)$ is a Cauchy sequence as well. □

Definition. Let X, Y be Banach spaces, and let $L\colon W_p^1(-h, 0; X) \to Y$ be a bounded linear operator. We say that L is *massless at* 0 if for all $x \in X$ and for all sequences (χ_n) in $W_\infty^1(-h, 0)$, $0 \leqslant \chi_n \leqslant 1$, $\chi_n(t) = 0$ $(t \leqslant -1/n)$ for all $n \in \mathbb{N}$, one has

$$L(\chi_n x) \to 0 \qquad (n \to \infty).$$

We say that L is *strongly massless at* 0 if for all $\varphi \in W_p^1(-h, 0; X)$ and for all sequences (χ_n) as above, one has

$$L(\chi_n \varphi) \to 0 \qquad (n \to \infty) \tag{1.2}$$

(note that $\chi_n \varphi \in W_p^1(-h, 0; X)$).

Remark 1.2. Obviously, L is massless at 0 if and only if for all $x \in X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|L(\chi x)\| \leqslant \varepsilon$ for all $\chi \in W_\infty^1(-h, 0)$, $0 \leqslant \chi \leqslant 1$, $\chi(t) = 0$ $(t \leqslant -\delta)$.

Remarks 1.3. Let X, Y be Banach lattices with order continuous norm.

(a) We recall that $W_p^1(-h, 0; X)$ is a vector lattice, and that $\big\||\varphi|\big\|_{p,1} \leqslant \|\varphi\|_{p,1}$ for all $\varphi \in W_p^1(-h, 0; X)$; cf. [13; Thm. 1 and Rem. 5].

(b) Assume that $L\colon W_p^1(-h, 0; X) \to Y$ is linear and possesses a modulus. We show that then L and $|L|$ are continuous. Because of (a) it suffices to prove the assertion for the case that L is positive.

We show that $\varphi_n \to 0$ in $W_p^1(-h, 0; X)$ implies $L\varphi_n \to 0$ in Y. Without loss of generality suppose $\sum_{n=1}^\infty n\|\varphi_n\|_{p,1} < \infty$. Defining $\varphi := \sum_{n=1}^\infty n|\varphi_n|$ we obtain $|L\varphi_n| \leqslant L|\varphi_n| \leqslant \frac{1}{n} L\varphi$ for all $n \in \mathbb{N}$.

We note that the continuity of positive operators between ordered Banach spaces can be obtained in more general situations; cf. [7; Thm. 2.1].

(c) Assume that $L\colon W_p^1(-h, 0; X) \to Y$ is linear and possesses a modulus, and that L is strongly massless at 0. Proposition 1.1 implies that then $|L|$ is strongly massless at 0 as well. Indeed, if (χ_n) is a sequence as in the definition above then with Q_n defined by $Q_n \varphi := \chi_n \varphi$ we obtain $L_Q = 0$, and therefore $|L_Q| = 0$.

The following lemma shows that it suffices to require property (1.2) for a special sequence in the definition of 'strongly massless at 0' if L possesses a modulus.

Lemma 1.4. *Let X, Y be Banach lattices with order continuous norm, $L\colon W_p^1(-h,0;X)\to Y$ a linear operator. Assume that L possesses a modulus and that there exists a sequence (χ_n) in $W_\infty^1(-h,0)$, $0\leqslant\chi_n\leqslant 1$, $\chi_n(0)=1$ for all $n\in\mathbb{N}$, such that*

$$L(\chi_n\varphi)\to 0 \qquad (n\to\infty) \tag{1.3}$$

for all $\varphi\in W_p^1(-h,0;X)$. Then L is strongly massless at 0.

Proof. Since Proposition 1.1 implies $|L|(\chi_n\varphi)\to 0$ $(n\to\infty)$, it is sufficient to treat the case $L\geqslant 0$. Let $(\widetilde{\chi}_n)$ be a sequence as in the definition of 'strongly massless at 0', $\varphi\in W_p^1(-h,0;X)$, $\varphi\geqslant 0$. Let $n\in\mathbb{N}$. Then there exists $m_0\in\mathbb{N}$ such that for all $m\geqslant m_0$ one has $\widetilde{\chi}_m\leqslant 2\chi_n$, and therefore

$$0\leqslant L(\widetilde{\chi}_m\varphi)\leqslant 2L(\chi_n\varphi).$$

By (1.3) this implies that $\lim_{m\to\infty}L(\widetilde{\chi}_m\varphi)=0$. □

We are going to show that in a rather general context L can be decomposed as the sum of two operators, the first only depending on the value of the function at 0, the second being strongly massless at 0.

Proposition 1.5. *Let X, Y be Banach lattices with order continuous norm. Assume that $L\colon W_p^1(-h,0;X)\to Y$ is linear and possesses a modulus.*

(a) *Let (χ_n) be a sequence in $W_\infty^1(-h,0)$, $0\leqslant\chi_n\leqslant 1$, $\chi_n(t)=0$ $(t\leqslant -1/n)$, $\chi_n(0)=1$ for all $n\in\mathbb{N}$. Then*

$$L_0\varphi:=\lim_{n\to\infty}L(\chi_n\varphi)$$

exists for all $\varphi\in W_p^1(-h,0;X)$, and the limit does not depend on the sequence (χ_n). The linear operator L_0 thus defined satisfies $L_0\varphi=0$ if $\varphi\in W_p^1(-h,0;X)$, $\varphi(0)=0$; the operator $L-L_0$ is strongly massless at 0.

(b) *The operator L_0 has the modulus $|L|_0$, and $L-L_0$ has the modulus $|L|-|L|_0$.*

Proof. (a) We first suppose that (χ_n) is a monotone decreasing sequence satisfying the assumptions. Then $L_0\varphi:=\lim_{n\to\infty}L(\chi_n\varphi)$ exists for all $\varphi\in W_p^1(-h,0;X)$, and L_0 as well as $L-L_0$ possess a modulus, by Proposition 1.1. Obviously, $L_0\varphi=0$ if $\varphi=0$ in a neighbourhood of 0. Since the set of those φ is dense in $\{\varphi\in W_p^1(-h,0;X);\ \varphi(0)=0\}$ and L_0 is continuous (cf. Remark 1.3(b)), we obtain that $L_0\varphi=0$ if $\varphi(0)=0$. From this we conclude that

$$(L-L_0)(\chi_n\varphi)=L(\chi_n\varphi)-L_0(\chi_n\varphi)=L(\chi_n\varphi)-L_0\varphi\to 0 \qquad (n\to\infty).$$

Thus, by Lemma 1.4, $L-L_0$ is strongly massless at 0.

Let now (χ_n) be any sequence satisfying the assumptions. Then with L_0 obtained as above we have

$$L(\chi_n\varphi)=(L-L_0)(\chi_n\varphi)+L_0\varphi\to L_0\varphi \qquad (n\to\infty)$$

by the definition of 'strongly massless at 0'. This completes the proof of (a).

(b) follows from Proposition 1.1. □

The following result is another version of Lemma 1.4. It contains as a consequence that, for operators possessing a modulus, 'massless at 0' implies 'strongly massless at 0'.

Proposition 1.6. *Let X, Y be Banach lattices with order continuous norm, $L\colon W_p^1(-h,0;X) \to Y$ a linear operator. Assume that L possesses a modulus and that there exists a sequence (χ_n) in $W_\infty^1(-h,0)$, $0 \leqslant \chi_n \leqslant 1$, $\chi_n(t) = 0$ ($t \leqslant -1/n$), $\chi_n(0) = 1$ for all $n \in \mathbb{N}$ such that $L(\chi_n x) \to 0$ ($n \to \infty$) for all $x \in X$. Then L is strongly massless at 0.*

Proof. We use the decomposition obtained in Proposition 1.5. Let $\chi \in W_p^1(-h,0)$, $\chi = 1$ in a neighbourhood of 0. Then

$$L_0\varphi = L_0(\chi\varphi(0)) = \lim_{n\to\infty} L(\chi_n\chi\varphi(0)) = 0$$

for all $\varphi \in W_p^1(-h,0;X)$, i.e., $L_0 = 0$. Therefore $L = L - L_0$ is strongly massless at 0. □

2 The modulus semigroup

The following two lemmas are technical results which will be needed later. We assume that X is a Banach space, that A is the generator of a C_0-semigroup T on X, and that $L \in \mathcal{L}(W_p^1(-h,0;X),X)$. Let $\mathcal{A}_0$, $\mathcal{T}_0$ and $\mathcal{B}$ be as defined in the introduction. For later use we note that

$$\mathcal{B}\mathcal{T}_0(s)\binom{x}{\varphi} = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T(s)x \\ T_s x + S(s)\varphi \end{pmatrix} = \begin{pmatrix} L(T_s x + S(s)\varphi) \\ 0 \end{pmatrix} \tag{2.1}$$

for $\binom{x}{\varphi} \in D(\mathcal{A})_0$.

Lemma 2.1. *The spectral radius of $\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}$ tends to 0 as $\lambda \to \infty$.*

Proof. For $c > 0$ we endow $X_p = X \times L_p(-h,0;X)$ with the norm $\|\binom{x}{f}\|_c := \|x\| + c\|f\|_p$ and also denote the corresponding norm on $\mathcal{L}(X_p)$ by $\|\cdot\|_c$.

Let $\binom{x}{f} \in X_p$, λ larger than the type of $\mathcal{T}_0$. Denoting by P_2 the projection of X_p onto its second component, we obtain that

$$\left\|\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}\binom{x}{f}\right\|_c = \|LP_2(\lambda - \mathcal{A}_0)^{-1}\binom{x}{f}\| \leqslant \|L\|\|P_2(\lambda - \mathcal{A}_0)^{-1}\binom{x}{f}\|_{p,1}$$

and

$$P_2(\lambda - \mathcal{A}_0)^{-1}\binom{x}{f} = \int_0^\infty e^{-\lambda t}P_2\mathcal{T}_0(t)\binom{x}{f}\,dt = \int_0^\infty e^{-\lambda t}\big(T_t x + S(t)f\big)\,dt.$$

A straightforward computation shows that $\int_0^\infty e^{-\lambda t}T_t x\,dt = \varepsilon_\lambda(\lambda - A)^{-1}x$, where $\varepsilon_\lambda(\theta) := e^{\lambda\theta}$ for $\theta \in (-h,0)$ (cf. [6; Prop. VI.6.7]). This implies

$$P_2(\lambda - \mathcal{A}_0)^{-1}\binom{x}{f} = \varepsilon_\lambda(\lambda - A)^{-1}x + (\lambda - \tfrac{d}{d\theta})^{-1}f,$$

where $\frac{d}{d\theta}$ denotes the generator of S. There exist $\lambda_0 > 0$, $M \geqslant 1$ such that $(\lambda + 1)\|(\lambda - A)^{-1}\| \leqslant M$ and $\|(\lambda - \frac{d}{d\theta})^{-1}\colon L_p(-h,0;X) \to W_p^1(-h,0;X)\| \leqslant M$ for all $\lambda \geqslant \lambda_0$.

In the following let $\lambda \geqslant \lambda_0$. Then

$$\begin{aligned}\|P_2(\lambda - \mathcal{A}_0)^{-1}\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_{p,1} &\leqslant (\lambda+1)\|\varepsilon_\lambda\|_{L_p(-h,0)}\|(\lambda - A)^{-1}x\| + M\|f\|_p \\ &\leqslant M\delta_\lambda\|x\| + M\|f\|_p,\end{aligned}$$

with $\delta_\lambda := \|\varepsilon_\lambda\|_{L_p(-h,0)} \to 0$ as $\lambda \to \infty$. For $c := \frac{1}{\delta_\lambda}$ we conclude that

$$\left\|\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\|_c \leqslant \|L\|M\big(\delta_\lambda\|x\| + \|f\|_p\big) = \delta_\lambda\|L\|M\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_c,$$

so $\|\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}\|_c \leqslant \delta_\lambda\|L\|M$, and this implies that the spectral radius of $\mathcal{B}(\lambda - \mathcal{A}_0)^{-1}$ is less or equal $\delta_\lambda\|L\|M$. □

Lemma 2.2. *Assume that L is massless at 0 and that $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ is the generator of a C_0-semigroup $\mathcal{T}$ on X_p. Then*

$$\tfrac{1}{t}\big(\mathcal{T}(t) - \mathcal{T}_0(t)\big)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \to 0 \qquad (t \to 0)$$

for all $x \in D(A)$.

Proof. Let $t > 0$, $x \in D(A)$. For $n \in \mathbb{N}$ define $\varphi_n \in W_p^1(-h,0;X)$ by $\varphi_n(\theta) := (1 + n\theta)^+ x$. Then $\varphi_n \to 0$ in $L_p(-h,0;X)$ as $n \to \infty$ and hence

$$\left\|\big(\mathcal{T}(t) - \mathcal{T}_0(t)\big)\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)\right\| = \lim_{n\to\infty}\left\|\big(\mathcal{T}(t) - \mathcal{T}_0(t)\big)\left(\begin{smallmatrix} x \\ \varphi_n \end{smallmatrix}\right)\right\|. \tag{2.2}$$

For $s \in [0,t]$ let now $\chi_{n,s} \in W_\infty^1(-h,0)$, $\chi_{n,s}(\theta) := \big(1 + n(\theta + s)\big)^+ \wedge 1$ and $\psi_s \in W_p^1(-h,0;X)$, $\psi_s(\theta) := 0$ for $\theta \leqslant -s$, $\psi_s(\theta) := T(s+\theta)x - x$ for $\theta > -s$. Then $T_s x + S(s)\varphi_n = \chi_{n,s}x + \psi_s$. Defining $M := \sup_{0 \leqslant s \leqslant t}\|\mathcal{T}(s)\|$ and using the Duhamel formula and (2.1), we obtain

$$\begin{aligned}\left\|\big(\mathcal{T}(t) - \mathcal{T}_0(t)\big)\left(\begin{smallmatrix} x \\ \varphi_n \end{smallmatrix}\right)\right\| &= \left\|\int_0^t \mathcal{T}(t-s)\mathcal{B}\mathcal{T}_0(s)\left(\begin{smallmatrix} x \\ \varphi_n \end{smallmatrix}\right) ds\right\| \\ &\leqslant M\int_0^t \|L(T_s x + S(s)\varphi_n)\|\, ds \leqslant M\int_0^t \big(\|L(\chi_{n,s}x)\| + \|L\|\|\psi_s\|_{p,1}\big)\, ds.\end{aligned}$$

Together with (2.2) this implies the assertion: We have

$$\|\psi_s\|_{p,1}^p = \|\psi_s\|_p^p + \int_{-s}^0 \|T(s+\theta)Ax\|^p\, d\theta \to 0 \qquad (s \to 0),$$

and $L(\chi_{n,s}x) \to 0$ as $n \to \infty$ and $s \to 0$ since L is massless at 0. □

The following proposition is an abstract result on generation, positivity and domination of perturbed semigroups. It will be needed for the application to delay semigroups.

We recall that for operators $B, \widetilde{B} \in \mathcal{L}(X)$, where X is a Banach lattice, B is *dominated by* $\widetilde{B}$ if $|Bx| \leqslant \widetilde{B}|x|$ for all $x \in X$. If $T, \widetilde{T}$ are C_0-semigroups on X then T is *dominated by* $\widetilde{T}$ if $T(t)$ is dominated by $\widetilde{T}(t)$ for all $t \geqslant 0$.

Proposition 2.3. *Let X be a Banach lattice. Let T, $\widetilde{T}$ be C_0-semigroups on X, with generators A, $\widetilde{A}$ respectively, and assume that T is dominated by $\widetilde{T}$. Let $B\colon D(A) \to X$, $\widetilde{B}\colon D(\widetilde{A}) \to X$ be linear operators, $\widetilde{B} \geqslant 0$,*

$$B(\lambda - A)^{-1} \text{ dominated by } \widetilde{B}(\lambda - \widetilde{A})^{-1} \qquad (\lambda \geqslant \lambda_0)$$

for some $\lambda_0 \in \mathbb{R}$. Assume that $\widetilde{A} + \widetilde{B}$ generates a C_0-semigroup $\widetilde{T}_{\widetilde{B}}$ on X and that the spectral radius of $\widetilde{B}(\lambda - \widetilde{A})^{-1}$ is less that 1 for all $\lambda \geqslant \lambda_0$. Then $\widetilde{T}_{\widetilde{B}}$ is positive, and $A + B$ generates a C_0-semigroup T_B on X that is dominated by $\widetilde{T}_{\widetilde{B}}$.

Proof. Let $\lambda \geqslant \lambda_0$. The assumptions imply that

$$(\lambda - \widetilde{A} - \widetilde{B})^{-1} = (\lambda - \widetilde{A})^{-1} \sum_{k=0}^{\infty} \big(\widetilde{B}(\lambda - \widetilde{A})^{-1}\big)^k \geqslant 0$$

(cf. [1; Thm. 3.1]). Therefore, the C_0-semigroup $\widetilde{T}_{\widetilde{B}}$ is positive.

Since $B(\lambda - A)^{-1}$ is dominated by $\widetilde{B}(\lambda - \widetilde{A})^{-1}$, we obtain in the same way that

$$(\lambda - A - B)^{-1} = (\lambda - A)^{-1} \sum_{k=0}^{\infty} \big(B(\lambda - A)^{-1}\big)^k.$$

We conclude that $(\lambda - A - B)^{-1}$ is dominated by $(\lambda - \widetilde{A} - \widetilde{B})^{-1}$. Hence $\|(\lambda - A - B)^{-n}\| \leqslant \|(\lambda - \widetilde{A} - \widetilde{B})^{-n}\|$ for all $n \in \mathbb{N}$, and the assertion follows from the Hille-Yosida generation theorem. □

The following 'domination lemma' is a generalisation of [10; Lemma 1.1]; a more restricted form has already been used in [11; proof of Prop. 1.2].

Lemma 2.4. *Let X be a Banach lattice. Let T, S be C_0-semigroups on X, S positive, and let A be the generator of T. Let $k \in \mathbb{N}_0$, $R\colon [0,1] \to \mathcal{L}(D_{A^k}, X)$ (where D_{A^k} is the domain of A^k endowed with the graph norm of A^k), and assume that*

$$\tfrac{1}{t} R(t)x \to 0 \qquad (t \to 0)$$

for all $x \in D(A^k)$. Further assume that

$$|T(t)x| \leqslant S(t)|x| + |R(t)x|$$

for all $x \in D(A^k)$, $0 \leqslant t \leqslant 1$. Then T is dominated by S.

Proof. Let $x \in D(A^k)$, $t > 0$. As in [10; proof of Lemma 1.1, eqn. (1.2)] we obtain

$$|T(t)x| \leqslant S(t)|x| + \sum_{m=1}^{n} S(\tfrac{n-m}{n}t)\, \big|R(\tfrac{t}{n})T(\tfrac{m-1}{n}t)x\big| \tag{2.3}$$

for all $n \in \mathbb{N},\ n \geqslant t$. With $c_t := \sup_{0\leqslant s\leqslant t} \|S(s)\|$ we estimate

$$\Big\| \sum_{m=1}^{n} S(\tfrac{n-m}{n}t)\, \big|R(\tfrac{t}{n})T(\tfrac{m-1}{n}t)x\big| \Big\| \leqslant c_t \sum_{m=1}^{n} \big\|R(\tfrac{t}{n})T(\tfrac{m-1}{n}t)x\big\|$$
$$\leqslant tc_t \max\big\{ \big\| \tfrac{n}{t} R(\tfrac{t}{n})T(\tfrac{m-1}{n}t)x \big\|;\ 1 \leqslant m \leqslant n \big\}.$$

The hypothesis on R implies that $\frac{n}{t}R(\frac{t}{n}) \to 0$ strongly in $\mathcal{L}(D_{A^k}, X)$ as $n \to \infty$. We recall that strong convergence is uniform on compact sets and that $T(\cdot)x\colon [0,\infty) \to D_{A^k}$ is continuous. Therefore the right hand side of the last inequality converges to 0 as $n \to \infty$. Letting $n \to \infty$ in (2.3) we obtain $|T(t)x| \leqslant S(t)|x|$ for all $x \in D(A^k)$. This implies the assertion since $D(A^k)$ is dense in X. □

The following proposition generalises [10; Prop. 2.1].

Proposition 2.5. *Let X be a Banach lattice. Assume that L is massless at 0, and that $\mathcal{A}$ is the generator of a C_0-semigroup $\mathcal{T}$ on X_p.*

If $\mathcal{T}$ is dominated by a C_0-semigroup $\mathcal{S}$ on X_p then $\mathcal{T}_0$ is dominated by $\mathcal{S}$ as well.

Proof. For $t > 0$ we define $\mathcal{R}_1(t) := \mathcal{T}(t) - \mathcal{T}_0(t),\ \mathcal{R}(t) := \mathcal{R}_1(t) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Let $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$. Estimating as in [10; proof of Prop. 2.1] we obtain that

$$|\mathcal{T}_0(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)| \leqslant \mathcal{S}(t)|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)| + |\mathcal{R}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)|.$$

Lemma 2.2 implies that $\frac{1}{t}\mathcal{R}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \to 0$ as $t \to 0$. Now applying Lemma 2.4, with $k = 1$, we obtain the assertion. □

Remark 2.6. Assume that X is a Banach lattice with order continuous norm and that L possesses a modulus. Then one may rearrange the setting in such a way that L is (strongly) massless at 0.

Indeed, let $\chi \in W_p^1(-h,0),\ \chi(0) = 1$. Define $E \in \mathcal{L}(X, W_p^1(-h,0;X))$ by $Ex := \chi x$. Let $L_0 \in \mathcal{L}(W_p^1(-h,0;X), X)$ be as in Proposition 1.5. Since $\varphi(0) = x$ for $\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(\mathcal{A})$, we obtain

$$\mathcal{A} = \begin{pmatrix} A & L \\ 0 & \frac{d}{d\theta} \end{pmatrix} = \begin{pmatrix} A + L_0E & L - L_0 \\ 0 & \frac{d}{d\theta} \end{pmatrix}.$$

Thus, we have written $\mathcal{A}$ with a generater $A + L_0E$ and an off-diagonal term $L - L_0$ that is (strongly) massless at 0.

For the remainder of this section we assume that X is a Banach lattice with order continuous norm and that T possesses a modulus semigroup, $T^{\#}$, with generator $A^{\#}$. (We recall that the modulus semigroup of T is the smallest C_0-semigroup dominating T; see [4] for results on modulus semigroups.) It was shown in [10; Prop. 3.2(b)] that then the C_0-semigroup $\mathcal{T}_0$ possesses a modulus semigroup, $\mathcal{T}_0^{\#}$, whose generator is given by $\mathcal{A}_0^{\#} = \left(\begin{smallmatrix} A^{\#} & 0 \\ 0 & \frac{d}{d\theta} \end{smallmatrix}\right)$, with domain $D(\mathcal{A}_0^{\#}) = \{\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(A^{\#}) \times W_p^1(-h,0;X);\ x = \varphi(0)\}$.

Assume additionally that L is massless at 0, that L possesses a modulus and that $\widetilde{\mathcal{A}} := \mathcal{A}_0^{\#} + \left(\begin{smallmatrix} 0 & |L| \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A^{\#} & |L| \\ 0 & \frac{d}{d\theta} \end{smallmatrix}\right)$, with domain $D(\widetilde{\mathcal{A}}) = D(\mathcal{A}_0^{\#})$, is the generator of a C_0-semigroup $\widetilde{\mathcal{T}}$.

Taking into account Lemma 2.1, we then obtain by Proposition 2.3 that $\mathcal{A}$ is the generator of a C_0-semigroup $\mathcal{T}$ on X_p, and that $\mathcal{T}$ is dominated by $\widetilde{\mathcal{T}}$. From the order completeness of X_p we conclude that $\mathcal{T}$ possesses a modulus semigroup, $\mathcal{T}^{\#}$, with generator $\mathcal{A}^{\#}$, and it is a consequence of Proposition 2.5 that then $\mathcal{T}_0^{\#}(t) \leqslant \mathcal{T}^{\#}(t) \leqslant \widetilde{\mathcal{T}}(t)$ for all $t \geqslant 0$; cf. [10; Prop. 3.2(a)].

Theorem 2.7. *Let X, $\mathcal{T}$, and $\widetilde{\mathcal{T}}$ be as introduced above. Then $\mathcal{T}^{\#} = \widetilde{\mathcal{T}}$.*

The proof of this result is the same as for [10; Theorem 3.1]. Therefore we are not going to reproduce it; cf. [10; pp. 397–398]. We only mention that in [10; bottom of p. 397], for $0 \leqslant \varphi \in W_p^1(-h, 0; X)$, the existence of a sequence (φ_k) in $W_p^1(-h, 0; X)$, $\varphi_k(0) = 0$, $0 \leqslant \varphi_k \leqslant \varphi$ $(k \in \mathbb{N})$ such that $|L|\varphi_k \to |L|\varphi$ $(k \to \infty)$ was needed. In the present context, the existence of such a sequence follows from the fact that $|L|$ is strongly massless at 0. (Note that the hypotheses and Proposition 1.6 imply that L is strongly massless at 0, and recall Remark 1.3(c).)

3 Miyadera perturbations of delay semigroups

We assume that X is a Banach space, T a C_0-semigroup on X, $L \in \mathcal{L}(W_p^1(-h, 0; X), X)$, and we use the notation of the introduction.

The following theorem is a slight generalisation of [2; Thm. 3.2]; see also [3; Thm. 3.26]. Our contribution—already used in [10; Sec. 1.2] as well as in the proof of Lemma 2.1 above—is the use of a suitable norm on the product space $X_p = X \times L_p(-h, 0; X)$.

Theorem 3.1. *Assume that there exist $t, c > 0$ and $\gamma < 1$ such that*

$$\int_0^t \|L(T_s x + S(s)\varphi)\|\, ds \leqslant \gamma\|x\| + c\|\varphi\|_p \qquad \left(\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(\mathcal{A}_0)\right). \tag{3.1}$$

Then $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ is the generator of a C_0-semigroup $\mathcal{T}$ on X_p.

More precisely, if X_p is endowed with the norm $\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\|_{c/\gamma} := \|x\| + \frac{c}{\gamma}\|f\|_p$ then $\mathcal{B}$ is a small Miyadera perturbation of $\mathcal{A}_0$, i.e.,

$$\int_0^t \|\mathcal{B}\mathcal{T}_0(s)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\|_{c/\gamma}\, ds \leqslant \gamma\|\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\|_{c/\gamma} \qquad \left(\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right) \in D(\mathcal{A}_0)\right).$$

(For Miyadera perturbations and the Miyadera perturbation theorem we refer to [12], [6; III.3.c].)

Proof. By (2.1) we obtain that

$$\int_0^t \|\mathcal{B}\mathcal{T}_0(s)\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\|_{c/\gamma}\, ds = \int_0^t \|L(T_s x + S(s)\varphi)\|\, ds \leqslant \gamma\|x\| + c\|\varphi\|_p = \gamma\|\left(\begin{smallmatrix} x \\ \varphi \end{smallmatrix}\right)\|_{c/\gamma}. \qquad \square$$

Remarks 3.2. (a) The only difference between condition (3.1) in Theorem 3.1 and [2; Thm. 3.2, condition (M)] is that in the latter $c = \gamma < 1$ is assumed, whereas we allow arbitrary $c > 0$. This refinement is important for the application to more general perturbations than studied in [2], and it avoids the separate study of the case $p = 1$ given in [8; Sec. 2]; see also [10; Sec. 1.2].

(b) Condition (3.1) is in particular satisfied if there exist $t, c > 0$ such that

$$\int_0^t \|L\psi_s\|\, ds \leqslant c\|\psi\|_p \qquad \big(\psi \in W_p^1(-h,t;X)\big) \tag{3.2}$$

(recall that $\psi_s(\theta) = \psi(\theta+s)$). Indeed, let $\binom{x}{\varphi} \in D(\mathcal{A}_0)$ and define $\psi \in W_p^1(-h,t;X)$ by $\psi := \varphi$ on $(-h,0)$, $\psi(\theta) := T(\theta)x$ for $0 \leqslant \theta \leqslant t$. Then

$$\|\psi\|_p^p = \|\varphi\|_p^p + \int_0^t \|T(\theta)x\|^p\, d\theta \leqslant \|\varphi\|_p^p + t(c_t\|x\|)^p,$$

where $c_t := \sup_{0\leqslant\theta\leqslant t} \|T(\theta)\|$. Hence (3.2) implies

$$\int_0^t \|L(T_s x + S(s)\varphi)\|\, ds = \int_0^t \|L\psi_s\|\, ds \leqslant c(t^{1/p}c_t\|x\| + \|\varphi\|_p).$$

It remains to choose $t > 0$ such that $\gamma := ct^{1/p}c_t < 1$.

4 Perturbation classes

In this section we present a condition on L sufficient for (3.2). Here, as in Section 1, we assume more generally that X, Y are Banach spaces and that $L \in \mathcal{L}(W_p^1(-h,0;X),Y)$. For a Borel measure μ on $\mathbb{R}$ and $t \geqslant 0$ we define $v_{\mu,t}\colon \mathbb{R} \to [0,\infty)$ by

$$v_{\mu,t}(s) := \mu((s-t,s]) \qquad (s \in \mathbb{R}).$$

Theorem 4.1. *Assume that there exist $r \in [1,p]$ and a Borel measure μ_L on $[-h,0]$ ($(-\infty,0]$ in case $h = \infty$) such that $v_{\mu_L,1} \in L_{\frac{p}{p-r}}(-h,1)$ and*

$$\|L\varphi\| \leqslant \|\varphi\|_{L_r(\mu_L;X)} \qquad \big(\varphi \in W_p^1(-h,0;X)\big). \tag{4.1}$$

Then

$$\int_0^t \|L\psi_s\|\, ds \leqslant t^{1/p'} \|v_{\mu_L,1}\|_{\frac{p}{p-r}}^{1/r} \|\psi\|_p$$

for all $0 < t \leqslant 1$, $\psi \in W_p^1(-h,t;X)$.

This theorem is an immediate consequence of Proposition 4.3 below.

Remarks 4.2. (a) Assume that $X = Y$. If (4.1) holds then L satisfies condition (3.2) and hence condition (3.1). Thus, (4.1) is a condition for $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ to be a generator. Observe that this condition does not depend on the generator A.

(b) If μ_L is finite (in particular if $h < \infty$) then the condition $v_{\mu_L,1} \in L_{\frac{p}{p-r}}(-h, 1)$ in Theorem 4.1 is automatically satisfied. If μ_L is not finite then this condition is responsible for the inclusion $W_p^1(-h, 0; X) \subseteq L_r(\mu_L; X)$.

Indeed, for $\varphi \in W_p^1(-h, 0; X)$ we compute

$$\|\varphi\|_{L_r(\mu_L;X)}^r = \int_{-\infty}^{0} \int_{\theta}^{\theta+1} dt\, \|\varphi(\theta)\|^r \, d\mu_L(\theta) = \int_{-\infty}^{1} \int_{t-1}^{t\wedge 0} \|\varphi(\theta)\|^r \, d\mu_L(\theta)\, dt.$$

Now $\int_{t-1}^{t\wedge 0} \|\varphi(\theta)\|^r \, d\mu_L(\theta) \leqslant v_{\mu_L,1}(t) \|\varphi|_{(t-1,t\wedge 1)}\|_\infty^r$, so by Hölder's inequality we conclude that

$$\|\varphi\|_{L_r(\mu_L;X)}^r \leqslant \|v_{\mu_L,1}\|_{\frac{p}{p-r}} \Big\| t \mapsto \|\varphi|_{(t-1,t\wedge 0)}\|_\infty^r \Big\|_{\frac{p}{r}} = \|v_{\mu_L,1}\|_{\frac{p}{p-r}} \Big\| t \mapsto \|\varphi|_{(t-1,t\wedge 0)}\|_\infty \Big\|_p^r.$$

Using the Sobolev embedding $W_p^1(0, 1; X) \subseteq L_\infty(0, 1; X)$, we deduce that the latter can be estimated by $c\|\varphi\|_{p,1}^r$, with $c \geqslant 0$ not depending on φ, so the asserted inclusion follows.

(c) Assume that (4.1) holds. Then the operator L extends to a bounded operator $\widehat{L}\colon L_r(\mu_L; X) \to Y$.

(d) Assume additionally that $\mu_L(\{0\}) = 0$. Then it is obvious that L is strongly massless at 0, in the sense of Section 1.

Conversely, assume that L is massless at 0. We show that then μ_L can be chosen such that $\mu_L(\{0\}) = 0$. Indeed, define $\widetilde{\mu}_L := \mu_L - \mu_L(\{0\})\delta_0$, where δ_0 is the Dirac measure at 0. Let $\varphi \in W_p^1(-h, 0; X)$, φ constant in a neighbourhood of 0. Let (χ_n) be a sequence as in the definition of 'massless at 0', $\chi_n(0) = 1$ $(n \in \mathbb{N})$. Then

$$\|L((1-\chi_n)\varphi)\| \leqslant \|(1-\chi_n)\varphi\|_{L_r(\mu_L;X)} = \|(1-\chi_n)\varphi\|_{L_r(\widetilde{\mu}_L;X)}$$

for all $n \in \mathbb{N}$. From $L((1-\chi_n)\varphi) \to L\varphi$ in Y, $(1-\chi_n)\varphi \to \varphi$ in $L_r(\widetilde{\mu}_L; X)$ $(n \to \infty)$ we obtain that $\|L\varphi\| \leqslant \|\varphi\|_{L_r(\widetilde{\mu}_L;X)}$. Thus, (4.1) holds with μ_L replaced by $\widetilde{\mu}_L$ since the set of φ under consideration in dense in $L_r(\mu_L; X)$.

Proposition 4.3. *Let $r \in [1, p]$, and let μ be a Borel measure on $\mathbb{R}$ with $v_{\mu,1} \in L_{\frac{p}{p-r}}(\mathbb{R})$. Let $0 < t \leqslant 1$, $\psi \in L_p(\mathbb{R})$. Then*

$$\int_0^t \|\psi_s\|_{L_r(\mu)}\, ds \leqslant t^{1/r'} \|v_{\mu,t}|_{\operatorname{spt}\psi}\|_{\frac{p}{p-r}}^{1/r} \|\psi\|_p \leqslant t^{1/p'} \|v_{\mu,1}\|_{\frac{p}{p-r}}^{1/r} \|\psi\|_p.$$

Proof. We only show the first inequality, the second one being a consequence of

Lemma 4.4 below. We have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} |\psi(\theta+s)|^r \, d\mu(\theta)\, ds &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{[0,t)}(s)\, |\psi(\theta+s)|^r \, ds\, d\mu(\theta) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{[0,t)}(s-\theta)\, d\mu(\theta)\, |\psi(s)|^r \, ds \\ &= \int_{\mathbb{R}} v_{\mu,t}(s)\, |\psi(s)|^r \, ds \leqslant \|v_{\mu,t}|_{\operatorname{spt}\psi}\|_{\frac{p}{p-r}} \| |\psi|^r \|_{\frac{p}{r}}.\end{aligned}$$

Since $\| |\psi|^r \|_{\frac{p}{r}} = \|\psi\|_p^r$, we obtain by Hölder's inequality that

$$\int_0^t \|\psi_s\|_{L_r(\mu)}\, ds \leqslant t^{1/r'} \Big(\int_0^t \|\psi_s\|_{L_r(\mu)}^r \, ds \Big)^{1/r} \leqslant t^{1/r'} \|v_{\mu,t}|_{\operatorname{spt}\psi}\|_{\frac{p}{p-r}}^{1/r} \|\psi\|_p. \qquad \square$$

In view of Remark 4.2(a) one could be content with the first inequality given in Proposition 4.3. Note, however, that the second inequality yields a better t-exponent if $p > r$.

Lemma 4.4. *Let μ be a Borel measure on $\mathbb{R}$, $t \in [0,1]$, $q \in [1,\infty]$. Then $\|v_{\mu,t}\|_q \leqslant t^{1/q} \|v_{\mu,1}\|_q$.*

Proof. For $q = \infty$ the assertion is clear since $v_{\mu,t} \leqslant v_{\mu,1}$, so let $q < \infty$. Below we will show: If $\alpha > 0$, $t \in [0,1)$ and

$$\|v_{\mu,s\alpha}\|_q^q \leqslant s \|v_{\mu,\alpha}\|_q^q \tag{4.2}$$

holds for $s = t$ then (4.2) holds for $s = \frac{1+t}{2}$, too. Since (4.2) trivially holds for $s = 0$, by induction we obtain (4.2) for $s = t_n := 1 - 2^{-n}$ ($n \in \mathbb{N}$) and $\alpha > 0$. Again by induction, this shows the assertion of the lemma for $t \in D := \{t_n^k;\, k, n \in \mathbb{N}\}$. Since D is dense in $[0,1]$, and $t \mapsto \|v_{\mu,t}\|_q$ is increasing, the asserted inequality follows for all $t \in [0,1]$.

Now let $\alpha > 0$, $t \in [0,1)$, and assume that (4.2) holds for $s = t$. Let $t' := \frac{1+t}{2}$. Observe that, since $t' \in [\frac{1}{2}, 1)$, we have $\theta - \alpha < \theta - t'\alpha \leqslant \theta - \alpha + t'\alpha < \theta$ and hence

$$\mathbf{1}_{(\theta-\alpha,\theta-\alpha+t'\alpha]} + \mathbf{1}_{(\theta-t'\alpha,\theta]} = \mathbf{1}_{(\theta-\alpha,\theta]} + \mathbf{1}_{(\theta-t'\alpha,\theta-\alpha+t'\alpha]} \qquad (\theta \in \mathbb{R}).$$

Since $t = 2t' - 1$, this implies

$$v_{\mu,t'\alpha}(\cdot - \alpha + t'\alpha) + v_{\mu,t'\alpha} = v_{\mu,\alpha} + v_{\mu,t\alpha}(\cdot - \alpha + t'\alpha), \tag{4.3}$$

where both terms on the left hand side are bounded by $v_{\mu,\alpha}$ since $(\theta - \alpha, \theta - \alpha + t'\alpha]$, $(\theta - t'\alpha, \theta]$ are subsets of $(\theta - \alpha, \theta]$.

We now need the following inequality for numbers $a, b, c, d \geqslant 0$: If $a + b = c + d =: \sigma$ and $a, b \leqslant c$ then $a^q + b^q \leqslant c^q + d^q$. This is a direct consequence of the fact that the function $[\frac{\sigma}{2}, \sigma] \ni t \mapsto t^q + (\sigma - t)^q$ is monotone increasing. Together with (4.3) we obtain

$$v_{\mu,t'\alpha}(\cdot - \alpha + t'\alpha)^q + v_{\mu,t'\alpha}^q \leqslant v_{\mu,\alpha}^q + v_{\mu,t\alpha}(\cdot - \alpha + t'\alpha)^q.$$

Integrating both sides we infer, using (4.2) for $s = t$, that

$$2\|v_{\mu,t'\alpha}\|_q^q \leqslant \|v_{\mu,\alpha}\|_q^q + \|v_{\mu,t\alpha}\|_q^q \leqslant (1+t)\|v_{\mu,\alpha}\|_q^q,$$

i.e., (4.2) holds for $s = t'$. □

Remarks 4.5. (a) It is easy to see that for $q = 1$ the inequality in Lemma 4.4 is in fact an equality. Since for $q = \infty$ the inequality is clear, one is tempted to use interpolation to prove Lemma 4.4. The authors did not succeed in realising this idea.

(b) Applying Lemma 4.4 to $\mu = f\lambda$ with $0 \leqslant f \in L_q(\mathbb{R})$ and Lebesgue measure λ on $\mathbb{R}$, we obtain

$$\|\mathbf{1}_{[0,t]} * f\|_q \leqslant t^{1/q}\|\mathbf{1}_{[0,1]} * f\|_q \quad (0 \leqslant t \leqslant 1).$$

We did not find a straightforward proof for this inequality.

We close this section by some observations concerning the modulus of operators L admitting a measure μ_L as in Theorem 4.1.

Let X, Y be Banach lattices with order continuous norm. Assume that $L \in \mathcal{L}(W_p^1(-h,0;X),Y)$ possesses a modulus, and that for $|L|$ there exists a measure $\mu_{|L|}$ such that the hypotheses of Theorem 4.1 are satisfies for $|L|$ and $\mu_{|L|}$. Then clearly the hypotheses of Theorem 4.1 are also satisfied for L, with $\mu_L := \mu_{|L|}$.

Proposition 4.6. *Let X, Y be Banach lattices with order continuous norm. Let $L \in \mathcal{L}(W_p^1(-h,0;X),Y)$, and assume that there exist $r \in [1,p]$ and μ_L such that the hypotheses of Theorem 4.1 are satisfied. Assume further that the operator $\widehat{L}$ defined in Remark 4.2(c) possesses a modulus. Then L possesses a modulus, $|L|$ is the restriction of $|\widehat{L}|$ to $W_p^1(-h,0;X)$, and $|L|$ satisfies the hypotheses of Theorem 4.1, with $\mu_{|L|} := \big\||\widehat{L}|\big\|\mu_L$.*

Proof. It is sufficient to show that the restriction of $|\widehat{L}|$ to $W_p^1(-h,0;X)$ is the modulus of L. This, however, is a consequence of [13; Thm. 1, Rem. 2]; see also Remark 4.7(a) below. □

Remark 4.7. (a) In order to apply [13; Rem. 2]) in the present context we have to convince ourselves of the following fact: If $(\Omega, \mathcal{A}, \mu)$ is a measure space, X a Banach lattice, and $f, g \in L_p(\mu;X),\ g \geqslant 0$ then

$$\tau_g f(t) = \tau_{g(t)} f(t) \qquad (t \in \Omega). \tag{4.4}$$

(We refer to [13; Sec. 2] for the truncation τ). If f, g are simple functions then the right hand side of (4.4) defines a measurable function enjoying the properties of the truncation; therefore (4.4) holds. Approximating general f, g by simple functions and applying

$$\big|\tau_{y_1}x_1 - \tau_{y_2}x_2\big| \leqslant |x_1 - x_2| + |y_1 - y_2|$$

(cf. [13; Eqn. (1)]) simultaneously to elements x_1, y_1, x_2, y_2 in X and $L_p(\mu;X)$, one obtains (4.4) for $f, g \in L_p(\mu;X),\ g \geqslant 0$.

(b) We point out that, in Proposition 4.7, it is much more restrictive to require the existence of a modulus for $\widehat{L}$ than for L.

5 Examples

Example 5.1. (a) Let X, Y be Banach spaces. Let $\eta\colon [-h,0] \to \mathcal{L}(X,Y)$ be a function of bounded variation. Then a continuous linear operator $L_\eta\colon C([-h,0];X) \to Y$ is defined by

$$L_\eta\varphi := \int d\eta(\theta)\varphi(\theta);$$

cf. [13; Sec. 3]. Due to the embedding $W_p^1(-h,0;X) \subseteq C([-h,0];X)$, the operator L_η can be restricted to $W_p^1(-h,0;X)$; the restriction will be denoted by $L_{\eta,p}$.

It it easy to see that $L = L_{\eta,p}$ satisfies (4.1) for $r = 1$ if the variation $|\eta|$ (a measure; cf. [13; Sec. 3, p. 198]) of η is used as μ_L. Therefore, $L_{\eta,p}$ satisfies the hypotheses of Theorem 4.1; cf. Remark 4.2(b).

(b) Let μ_L be a finite Borel measure on $[-h,0]$ ($(-\infty,0]$ in case $h = \infty$). Let $L_{\eta,p}$ be as above and observe that η is uniquely determined by $L_{\eta,p}$. It thus follows from part (a) and Proposition 5.2 below that $L = L_{\eta,p}$ satisfies (4.1) for $r = 1$ if and only if $|\eta| \leqslant \mu_L$.

(c) From part (b) and Remark 4.2(d) we obtain that $L_{\eta,p}$ is (strongly) massless at 0 if and only if $|\eta|(\{0\}) = 0$. The latter holds if and only if $\eta(0) = \eta(0-) := \lim_{\theta\to 0-}\eta(\theta)$ (i.e., η does not give rise to mass at 0).

Proposition 5.2. *Let X, Y be Banach spaces, $L \in \mathcal{L}(W_p^1(-h,0;X),Y)$, and assume that* (4.1) *holds for $r = 1$ and a finite Borel measure μ_L. Then there exists $\eta\colon [-h,0] \to \mathcal{L}(X,Y)$ of bounded variation such that $|\eta| \leqslant \mu_L$ and $L = L_{\eta,p}$.*

Proof. Let $\widehat{L}$ be as in Remark 4.2(c). For a bounded interval $I \subseteq [-h,0]$ we define

$$\widehat{\eta}(I)x := \widehat{L}(\mathbf{1}_I x) \qquad (x \in X);$$

moreover, $\eta(\theta) := \widehat{\eta}((\theta,0])$ $(\theta \in (-h,0])$, $\eta(-h) := \widehat{\eta}([-h,0])$ if $h \in (0,\infty)$. We show that the variation of η on I is bounded by $\mu_L(I)$; then $L = L_{\eta,p}$ follows from the definition of L_η.

Let $(I_1,\ldots,I_n)$ be a partition of I into subintervals. Let $\varepsilon > 0$. Then there exist $x_j \in X$, $\|x_j\| = 1$ $(j = 1,\ldots,n)$ such that

$$\sum_{j=1}^n \|\widehat{\eta}(I_j)\| \leqslant (1+\varepsilon)\sum_{j=1}^n \|\widehat{\eta}(I_j)x_j\|.$$

Since (4.1) implies $\|\widehat{\eta}(I_j)x_j\| = \|\widehat{L}(\mathbf{1}_{I_j}x_j)\| \leqslant \|\mathbf{1}_{I_j}x_j\|_{L_1(\mu_L;X)} = \mu_L(I_j)$ for $j = 1,\ldots,n$, we conclude that

$$\sum_{j=1}^n \|\widehat{\eta}(I_j)\| \leqslant (1+\varepsilon)\mu_L\big(\bigcup_{j=1}^n I_j\big) = (1+\varepsilon)\mu_L(I)$$

and hence $|\eta|(I) \leqslant \mu_L(I)$. □

Remark 5.3. In Example 5.1, let X, Y be Banach lattices, Y order complete. Assume that the regular variation $\widetilde{\eta}$ of η exists, and that $\widetilde{\eta}$ is of bounded variation; cf. [13; Sec. 3].

(a) It was shown in [13; Prop. 9] that then the modulus of L_η exists, and that $|L_\eta| = L_{\widetilde{\eta}}$.

(b) Assuming additionally that X, Y have order continuous norm we show that $\eta(0) = \eta(0-)$ implies $\widetilde{\eta}(0) = \widetilde{\eta}(0-)$. This reproduces the result of [13; Lemma 10].

Indeed, part (a) and [13; Thm. 1 and Rem. 2] show that the modulus of $L_{\eta,p}$ exists and is given by $L_{\widetilde{\eta},p}$ (the restriction of $L_{\widetilde{\eta}}$ to $W_p^1(-h, 0; X)$). Remark 5.1(c) shows that $L_{\eta,p}$ is massless at 0. Therefore $L_{\widetilde{\eta},p}$ $(= |L_{\eta,p}|)$ is massless at 0, by Proposition 1.1. Applying again Remark 5.1(c) we obtain $\widetilde{\eta}(0) = \widetilde{\eta}(0-)$.

Example 5.4. Let X be a Banach space, (Y_n) a sequence of Banach spaces, $q \in (1, \infty)$,

$$Y := \ell_q((Y_n)_{n\in\mathbb{N}}) := \Big\{(y_n) \in \prod_{n=1}^{\infty} Y_n;\ \|(y_n)\|^q := \sum_{n=1}^{\infty} \|y_n\|^q < \infty\Big\}.$$

(The above is an abstraction of the case that $Y = L_q(\Omega, \nu)$, where $(\Omega, \mathcal{A}, \nu)$ is a measure space, and $Y_n = L_q(\Omega_n, \nu)$, with a sequence (Ω_n) of pairwise disjoint measurable subsets of Ω such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$.) For $n \in \mathbb{N}$ let $\eta_n\colon [-h, 0] \to \mathcal{L}(X, Y_n)$ be a function of bounded variation. Then $\mu_n := |\eta_n|$ is a finite measure on $[-h, 0]$.

Let $r \in (1, q]$ and assume that $\sum_{n=1}^{\infty} \|\mu_n\|^r < \infty$. Then $\sum_{n=1}^{\infty} \|L_{\eta_n}\|^q < \infty$, so we can define $L\colon C([-h, 0]; X) \to Y$ by

$$Lf := (L_{\eta_n} f)_{n\in\mathbb{N}}.$$

Assume without loss of generality that $\eta_n(0) = 0$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} \|\eta_n(\theta)\|^q \leqslant \sum_{n=1}^{\infty} \|\mu_n\|^q < \infty$ for all $\theta \in [-h, 0]$, and hence we can define $\eta\colon [-h, 0] \to \mathcal{L}(X, Y)$ by $\eta(\theta)f := \big(\eta_n(\theta)f\big)_{n\in\mathbb{N}}$. Then formally $L = L_\eta$, but η is not of bounded variation if the μ_n have pairwise disjoint support and $\sum_{n=1}^{\infty} \|\mu_n\| = \infty$. Nevertheless, we will show that L satisfies the hypotheses of Theorem 4.1 with the finite measure $\mu_L := \sum_{n=1}^{\infty} \|\mu_n\|^{r-1} \mu_n$.

Let $\varphi \in C([-h, 0]; X)$. Recall from Example 5.1(a) that $\|L_{\eta_n}\varphi\| \leqslant \|\varphi\|_{L_1(\mu_n;X)}$ for all $n \in \mathbb{N}$. Since $r \leqslant q$, we infer that

$$\|L\varphi\|^r = \Big(\sum_{n=1}^{\infty} \|L_{\eta_n}\varphi\|^q\Big)^{\frac{r}{q}} \leqslant \Big(\sum_{n=1}^{\infty} \|\varphi\|_{L_1(\mu_n;X)}^q\Big)^{\frac{r}{q}} \leqslant \sum_{n=1}^{\infty} \|\varphi\|_{L_1(\mu_n;X)}^r.$$

By Hölder's inequality we have $\|\varphi\|_{L_1(\mu_n;X)} \leqslant \|\mu_n\|^{\frac{1}{r'}} \|\varphi\|_{L_r(\mu_n;X)}$ for all $n \in \mathbb{N}$. We conclude that

$$\|L\varphi\|^r \leqslant \sum_{n=1}^{\infty} \|\mu_n\|^{r-1} \int \|\varphi(\theta)\|^r \, d\mu_n(\theta) = \int \|\varphi(\theta)\|^r \, d\mu_L(\theta),$$

by the definition of μ_L. This proves $\|L\varphi\| \leqslant \|\varphi\|_{L_r(\mu_L;X)}$.

We note that the operator L defined above is massless at 0 if and only if the operators L_{η_n} are massless at 0.

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