Bands in L_p-spaces

Hendrik Vogt and Jürgen Voigt

Abstract

For a general measure space (Ω, μ), it is shown that for every band M in $L_p(\mu)$ there exists a decomposition $\mu = \mu' + \mu''$ such that $M = L_p(\mu') = \{f \in L_p(\mu);\ f = 0\ \mu''\text{-a.e.}\}$. The theory is illustrated by an example, with an application to absorption semigroups.

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Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space (in particular, $\mathcal{A}$ is assumed to be a σ-algebra), and let $1 \leqslant p < \infty$. If Ω' is a locally measurable subset of Ω, and M is the set of all functions in $L_p(\mu)$ vanishing a.e. outside Ω', then M is a band in $L_p(\mu)$ (see the beginning of Section 2 for the definition). This description holds for all bands if the measure μ is σ-finite (or, more generally, weakly localisable). Looking at the case of general measures one realises that such a property is true 'locally', but the local sets cannot be composed to a locally measurable set, in general. Our main result, Theorem 2.2, is that this composition can be achieved on the level of measures.

The issue sketched above came up in connection with absorption semigroups; see [8]. More specifically, for an absorption semigroup T_V on $L_p(\mu)$ with a 'very singular' absorption rate V it is known that $P := \text{s-lim}_{t\to 0+} T_V(t)$ is a band projection, and we demonstrate by an example that also in this specific context the band $\operatorname{ran}(P)$ can be of the above 'strange type'.

In Section 1 we discuss some properties of measure spaces. In particular, for a general measure space we recall the definitions of locally measurable functions and sets, and we study relations between weak localisability and localisability.

In Section 2 we recall properties of bands in weakly localisable measure spaces, and we show our main result on bands for general measure spaces.

Section 3 is devoted to an example illustrating the main theorem in the case of a measure space that is not weakly localisable and to an example of an absorption semigroup as mentioned above.

Throughout we assume our L_p-spaces to consist of real-valued functions.

1 Preliminaries

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $1 \leqslant p < \infty$. Then for every function $f \in L_p(\mu)$ the set $[f \neq 0]$ is σ-finite, and this implies that for the definition and properties of $L_p(\mu)$ only the values of μ on the σ-ring

$$\mathcal{A}_{\sigma\text{-fin}} := \bigl\{A \in \mathcal{A};\ A\ \sigma\text{-finite}\bigr\}$$

play a role. We also define

$$\mathcal{A}_{\text{fin}} := \bigl\{A \in \mathcal{A};\ \mu(A) < \infty\bigr\}.$$

We recall that a function $f\colon \Omega \to \mathbb{R}$ is *locally measurable* if $\mathbf{1}_A f$ is measurable for all $A \in \mathcal{A}_{\text{fin}}$. A set $B \subseteq \Omega$ is *locally measurable* if $B \cap A \in \mathcal{A}$ for all $A \in \mathcal{A}_{\text{fin}}$, and B is a *local null set* if additionally $\mu(B \cap A) = 0$ for all $A \in \mathcal{A}_{\text{fin}}$. We say that the measure space (or the measure μ) is *weakly localisable* if for any consistent family $(f_A)_{A\in\mathcal{A}_{\text{fin}}}$ of measurable functions $f_A\colon A \to \mathbb{R}$ there exists a locally measurable function $f\colon \Omega \to \mathbb{R}$ such that $f|_A = f_A$ μ-a.e. for all $A \in \mathcal{A}_{\text{fin}}$. The property just introduced is called 'localisable' in [7; Sec. 14, M]. It appears that, more recently, 'localisable' is associated with a stronger property; see our discussion below.

It will be convenient to use the following notation. Analogously to the family of functions $(f_A)_{A\in\mathcal{A}_{\text{fin}}}$ used above we will need a family of sets with index set $\mathcal{A}_{\text{fin}}$, and for each $A \in \mathcal{A}_{\text{fin}}$ the associated set will be denoted by A', where A' belongs to $\mathcal{A} \cap A$, the trace σ-algebra of $\mathcal{A}$ on A. In other words, the family $(A')_{A\in\mathcal{A}_{\text{fin}}}$ is a mapping $\ '\colon \mathcal{A}_{\text{fin}} \to \mathcal{A}_{\text{fin}}$, satisfying $A' \in \mathcal{A} \cap A$ for all $A \in \mathcal{A}_{\text{fin}}$.

1.1 Remark. It is a standard fact that μ is weakly localisable if and only if for each consistent family $(A')_{A\in\mathcal{A}_{\text{fin}}}$ of sets $A' \in \mathcal{A} \cap A$ there exists a locally measurable set $\Omega' \subseteq \Omega$ such that $A' = \Omega' \cap A$ μ-a.e. for all $A \in \mathcal{A}_{\text{fin}}$. (For $B, C \in \mathcal{A}$ we use the notation '$B = C$ μ-a.e.' if $\mathbf{1}_B = \mathbf{1}_C$ μ-a.e.)

For the necessity one considers the consistent family $(\mathbf{1}_{A'})_{A\in\mathcal{A}_{\text{fin}}}$. If $f\colon \Omega \to \mathbb{R}$ is the corresponding locally measurable function, then the set $\Omega' := [f = 1]$ satisfies $A' = \Omega' \cap A$ μ-a.e. for all $A \in \mathcal{A}_{\text{fin}}$.

For the less trivial implication, the sufficiency, we refer to [6; 213N], [10; Exercise 4.59]. Even though the context in these references differs from ours, the proofs can be adapted to our situation.

For the information of the reader we are going to put the concept of weak localisability in relation to other closely related notions of measure theory. We emphasise that the results presented in the remainder of this section will not be used in the following Sections 2 and 3.

We recall that the measure space $(\Omega, \mathcal{A}, \mu)$ is *semi-finite* if for all $B \in \mathcal{A}$ with $\mu(B) = \infty$ there exists $A \in \mathcal{A} \cap B$ such that $0 < \mu(A) < \infty$. This property is equivalent to the requirement that every local null set belonging to $\mathcal{A}$ is already a null set. (This holds because $\mu(A) \in \{0, \infty\}$ for all local null sets $A \in \mathcal{A}$.)

From [6; Def. 211G] we adopt the terminology that $(\Omega, \mathcal{A}, \mu)$ is *localisable* if μ is semi-finite, and for each system $\mathcal{E} \subseteq \mathcal{A}$ there exists $A = \operatorname{ess\,sup} \mathcal{E} \in \mathcal{A}$ (that is, A is the smallest set, up to μ-null sets, containing all $B \in \mathcal{E}$, again up to μ-null sets). We also refer to [10]; the concept of localisability was first introduced by Segal [11] (in a context where 'measure space' is defined slightly differently).

We recall that σ-finite measure spaces are (weakly) localisable. More information on localisable measure spaces can be found in [6] and [10].

The following proposition shows, in particular, that localisability implies weak localisability.

1.2 Proposition. *A measure space $(\Omega, \mathcal{A}, \mu)$ is localisable if and only if it is semi-finite, and for all consistent families $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$ (as in Remark 1.1) there exists $\Omega' \in \mathcal{A}$ such that $A' = \Omega' \cap A$ μ-a.e. for all $A \in \mathcal{A}_{\mathrm{fin}}$.*

Proof. If the measure μ is localisable, then by definition it is semi-finite. For a consistent family $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$ the set $\Omega' := \operatorname{ess\,sup}\{A';\ A \in \mathcal{A}_{\mathrm{fin}}\}$ is as required. This proves the necessity.

For the sufficiency let $\mathcal{E} \subseteq \mathcal{A}$. For all $A \in \mathcal{A}_{\mathrm{fin}}$ there exists $A' := \operatorname{ess\,sup}(\mathcal{E} \cap A)$. (Here we use that $\sup\{\mathbf{1}_{B \cap A}\,;\ B \in \mathcal{E}\}$ exists in the order complete space $L_1(\mu)$ and is the indicator function of a set $A' \in \mathcal{A} \cap A$.) Then $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$ is a consistent family, and by hypothesis there exists $\Omega' \in \mathcal{A}$ such that $A' = \Omega' \cap A$ for all $A \in \mathcal{A}_{\mathrm{fin}}$. The semi-finiteness of μ then implies that $\Omega' = \operatorname{ess\,sup} \mathcal{E}$. □

1.3 Remark. Clearly, in the 'localisation condition' formulated in Proposition 1.2 one does not need the whole index set $\mathcal{A}_{\mathrm{fin}}$, but it is sufficient to use an index set $\mathcal{A}' \subseteq \mathcal{A}_{\mathrm{fin}}$ such that for every set $A \in \mathcal{A}_{\mathrm{fin}}$ there exists $B \in \mathcal{A}'$ such that $B = A$ μ-a.e. (The same comment applies to the definition of weak localisability and to Remark 1.1.)

Let us define the σ-algebra

$$\mathcal{A}_{\mathrm{loc}} := \bigl\{B \subseteq \Omega;\ B \text{ locally measurable}\bigr\},$$

and define

$$\mu_{\mathrm{loc}}(B) := \sup\bigl\{\mu(B \cap A);\ A \in \mathcal{A}_{\mathrm{fin}}\bigr\} \qquad (B \in \mathcal{A}_{\mathrm{loc}}).$$

Then μ_{loc} is a measure, $\mu_{\mathrm{loc}}|_{\mathcal{A}_{\sigma\text{-fin}}} = \mu|_{\mathcal{A}_{\sigma\text{-fin}}}$, and $\mu_{\mathrm{loc}}(B) = 0$ for all local null sets. Clearly, the measure μ_{loc} is semi-finite, and μ is semi-finite if and only if $\mu_{\mathrm{loc}}|_{\mathcal{A}} = \mu$.

We note that for any $A \in \mathcal{A}_{\mathrm{loc,fin}}$ there exist a set $A' \in \mathcal{A}_{\mathrm{fin}}$ and a local null set A'' such that $A = A' \cup A''$. Indeed, $\mu_{\mathrm{loc}}(A)$ can be obtained as $\lim_{n\to\infty} \mu(A_n)$, with an increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_{\mathrm{fin}}$ of subsets of A; therefore $A' := \bigcup_{n\in\mathbb{N}} A_n \in$

$\mathcal{A}_{\mathrm{fin}}$, and $A'' := A \setminus A'$ is a local null set. As a consequence one also concludes that every set $A \in \mathcal{A}_{\mathrm{loc},\sigma\text{-fin}}$ can be written as $A = A' \cup A''$, with $A' \in \mathcal{A}_{\sigma\text{-fin}}$ and a local null set A''.

We further observe that for all $p \in [1, \infty)$ one obtains $L_p(\mu) = L_p(\mu_{\mathrm{loc}})$, with the following interpretation. It is clear that every function $f \in L_p(\mu)$ is also measurable with respect to the σ-Algebra $\mathcal{A}_{\mathrm{loc}}$ and belongs to $L_p(\mu_{\mathrm{loc}})$. On the other hand, if $f \in L_p(\mu_{\mathrm{loc}})$, then $[f \neq 0] \in \mathcal{A}_{\mathrm{loc},\sigma\text{-fin}}$, and by the previous paragraph one has $[f \neq 0] = A' \cup A''$, with $A' \in \mathcal{A}_{\sigma\text{-fin}}$ and a local null set A''. This implies that $\mathbf{1}_{A'} f = f$ in $L_p(\mu_{\mathrm{loc}})$ and that $\mathbf{1}_{A'} f \in L_p(\mu)$. One then concludes that the mapping $L_p(\mu_{\mathrm{loc}}) \ni f \mapsto \mathbf{1}_{A'} f \in L_p(\mu)$ is an isomorphism.

1.4 Proposition. *The measure space* $(\Omega, \mathcal{A}, \mu)$ *is weakly localisable if and only if* $(\Omega, \mathcal{A}_{\mathrm{loc}}, \mu_{\mathrm{loc}})$ *is localisable.*

Proof. Recall that $\mu|_{\mathcal{A}_{\mathrm{fin}}} = \mu_{\mathrm{loc}}|_{\mathcal{A}_{\mathrm{fin}}}$ and that for every $A \in \mathcal{A}_{\mathrm{loc,fin}}$ there exists $A' \in \mathcal{A}_{\mathrm{fin}}$ such that $A' = A$ μ_{loc}-a.e. This implies that, in order to verify the 'localisation condition' of Proposition 1.2 for μ_{loc}, it is sufficient to use μ_{loc}-consistent families $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$; cf. Remark 1.3. In fact, such a family is μ_{loc}-consistent if and only if it is μ-consistent.

If $\Omega' \in \mathcal{A}_{\mathrm{loc}}$, $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$ is a consistent family, and $A \in \mathcal{A}_{\mathrm{fin}}$, then $A' = \Omega' \cap A$ μ-a.e. if and only if $A' = \Omega' \cap A$ μ_{loc}-a.e. This shows that μ is weakly localisable if and only if μ_{loc} satisfies the 'localisation condition' of Proposition 1.2. As μ_{loc} is semi-finite the latter is equivalent to the localisability of μ_{loc}. $\square$

The measure space $(\Omega, \mathcal{A}, \mu)$ is *locally determined* if it is semi-finite and every locally measurable set is measurable. One easily sees that $(\Omega, \mathcal{A}, \mu)$ is locally determined if and only if $\mathcal{A}_{\mathrm{loc}} = \mathcal{A}$ and $\mu_{\mathrm{loc}} = \mu$. This implies that the following statement is immediate from Proposition 1.4.

1.5 Corollary. *If the measure space* $(\Omega, \mathcal{A}, \mu)$ *is locally determined and weakly localisable, then it is localisable.*

2 Bands in $L_p(\mu)$

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $1 \leqslant p < \infty$. A *band* M in a vector lattice E is a lattice ideal that additionally is stable under forming suprema of subsets, i.e., if $A \subseteq M$ is a set for which $\sup A$ exists in E, then $\sup A \in M$. We recall from [9; Chap. II, §2] that every band M in $L_p(\mu)$ is automatically a *projection band*, i.e.,

$$L_p(\mu) = M \oplus M^{\perp}$$

is the topological direct sum of M and its disjoint complement

$$M^{\perp} := \big\{ f \in L_p(\mu);\ |f| \wedge |g| = 0\ (g \in M) \big\},$$

and then one has $M = M^{\perp\perp}$. The projection $P \in \mathcal{L}(L_p(\mu))$ onto M satisfies $0 \leqslant P \leqslant I$, and conversely, every projection P satisfying $0 \leqslant P \leqslant I$ is a band projection ([1; Thm. 1.44]).

In the following we recall that in the case of 'nice' measure spaces, band projections are multiplication operators by indicator functions of locally measurable sets. If P is a band projection and there exists a locally measurable function $h\colon \Omega \to \mathbb{R}$ such that $Pf = hf$ for all $f \in L_p(\mu)$, then it is easy to see that $\Omega \setminus \big([h = 0] \cup [h = 1]\big)$ is a local null set. This implies that P is the multiplication operator by the indicator function of the set $\Omega' := [h = 1]$, and the band $M := \operatorname{ran}(P)$ is given by

$$M = \big\{f \in L_p(\mu);\ f = 0\ \mu\text{-a.e. on } \Omega \setminus \Omega'\big\}. \tag{2.1}$$

It is a consequence of [15; Theorem 7] that in the case of σ-finite measures, a band is always of the kind described in the previous paragraph. This fact is part of the assertion in the following proposition.

2.1 Proposition. *The measure μ is weakly localisable if and only if for every band M in $L_p(\mu)$ there exists a locally measurable set Ω' such that the band projection P onto M is given by $Pf = \mathbf{1}_{\Omega'} f$ $(f \in L_p(\mu))$.*

Proof. In order to show the necessity we first treat the case that μ is finite. Then from $P1 + (I-P)1 = 1$ and $P1 \wedge (I-P)1 = 0$ it is immediate that $h := P1 = \mathbf{1}_{\Omega'}$ for a measurable set $\Omega' \subseteq \Omega$. If $f \in L_p(\mu)$, $0 \leqslant f \leqslant 1$, then $0 \leqslant Pf \leqslant P1$ and $0 \leqslant (I-P)f \leqslant (I-P)1$, and using also $f = Pf + (I-P)f$ one obtains $Pf = \mathbf{1}_{\Omega'} f = hf$. By denseness of $\operatorname{lin}\{f \in L_p(\mu);\ 0 \leqslant f \leqslant 1\}$ in $L_p(\mu)$ and continuity the equality $Pf = hf$ carries over to all $f \in L_p(\mu)$.

Now assume that μ is weakly localisable. For $A \in \mathcal{A}_{\mathrm{fin}}$ let μ_A denote the restriction of μ to $\mathcal{A} \cap A$, and consider $L_p(A, \mu_A)$ as as a subspace of $L_p(\mu)$. Then it is immediate that the restriction P_A of P to $L_p(A, \mu_A)$ is the band projection onto $M \cap L_p(A, \mu_A)$; hence – by the first part of the proof – there exists $A' \in \mathcal{A} \cap A$ such that $P_A f = \mathbf{1}_{A'} f$ for all $f \in L_p(A, \mu_A)$. It is not difficult to see that the family $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$ is consistent. The weak localisability of μ – together with Remark 1.1 – implies that there exists a locally measurable set Ω' such that $Pf = \mathbf{1}_{\Omega'} f$ for all $f \in L_p(\mu)$ with $\mu([f \neq 0]) < \infty$. By denseness of $\big\{f \in L_p(\mu);\ \mu([f \neq 0]) < \infty\big\}$ in $L_p(\mu)$ and continuity, the equality $Pf = \mathbf{1}_{\Omega'} f$ carries over to all $f \in L_p(\mu)$.

For the proof of the sufficiency let $(A')_{A \in \mathcal{A}_{\mathrm{fin}}}$ be a consistent family of sets $A' \in \mathcal{A} \cap A$. Then

$$M := \big\{f \in L_p(\mu);\ f = 0\ \mu\text{-a.e. on } A \setminus A'\ (A \in \mathcal{A}_{\mathrm{fin}})\big\}$$

is a band in $L_p(\mu)$. By hypothesis, there exists a locally measurable set $\Omega' \subseteq \Omega$ such that the band projection P onto M is given by $Pf = \mathbf{1}_{\Omega'} f$ for all $f \in L_p(\mu)$. In particular it follows that $\mathbf{1}_{A'} = P\,\mathbf{1}_A = \mathbf{1}_{\Omega'} \mathbf{1}_A$, i.e., $A' = \Omega' \cap A$ μ-a.e., for all $A \in \mathcal{A}_{\mathrm{fin}}$. $\square$

As a consequence of Proposition 2.1 one concludes that for measure spaces that are not weakly localisable, there will always exist a band which is not of the form described in (2.1). This will be illustrated by an example in Section 3. In order to motivate our replacement for the description (2.1) we mention that this description can also be formulated by stating that there exists a locally measurable set $\Omega' \subseteq \Omega$ such that for the measures $\mu' := \mathbf{1}_{\Omega'}\,\mu$, $\mu'' := \mathbf{1}_{\Omega\setminus\Omega'}\,\mu$ one has $M = L_p(\mu')$, $M^\perp = L_p(\mu'')$ – in a sense made precise in Remark 2.4(c).

If μ', μ'' are measures on $\mathcal{A}$ such that $\mu' \leqslant \mu$, $\mu'' \leqslant \mu$, then μ', μ'' will be called *locally disjoint*, denoted by $\mu' \perp_{\rm loc} \mu''$, if for all $A \in \mathcal{A}_{\rm fin}$ there exist $A', A'' \in \mathcal{A}$, $A = A' \cup A''$, $A' \cap A'' = \varnothing$, such that $\mu'(A'') = \mu''(A') = 0$.

The following theorem is the main result of this paper.

2.2 Theorem. (a) *Let $\mu = \mu' + \mu''$ be a locally disjoint decomposition of μ. Then*

$$M := \bigl\{f \in L_p(\mu);\ f = 0\ \ \mu''\text{-a.e.}\bigr\} \tag{2.2}$$

is a band in $L_p(\mu)$, with

$$M^\perp = \bigl\{f \in L_p(\mu);\ f = 0\ \ \mu'\text{-a.e.}\bigr\}. \tag{2.3}$$

(b) *Conversely, for each band M in $L_p(\mu)$ there exists a locally disjoint decomposition $\mu = \mu' + \mu''$ of μ such that M and $M^\perp$ are given by* (2.2) *and* (2.3).

2.3 Remark. We note that the definition of M in Theorem 2.2(a) is meaningful: if $f, g\colon \Omega \to \mathbb{R}$ are measurable functions, $f = 0$ μ''-a.e. and $f = g$ μ-a.e., then $\mu''([f \neq g]) \leqslant \mu([f \neq g]) = 0$, hence $g = f = 0$ μ''-a.e.

Proof of Theorem 2.2. (a) Denote the right hand side of (2.3) by N. We show that $L_p(\mu) = M \oplus N$ is a direct sum and that the projection P onto M satisfies $0 \leqslant P \leqslant I$; this implies the assertions.

If $f \in L_p(\mu)$ satisfies $f = 0$ μ''-a.e. as well as $f = 0$ μ'-a.e., then $\mu([f \neq 0]) = \mu'([f \neq 0]) + \mu''([f \neq 0]) = 0$, i.e., $f = 0$ μ-a.e. This shows $M \cap N = \{0\}$. Now let $f \in L_p(\mu)$. Then $A := [f \neq 0]$ is σ-finite, and from $\mu' \perp_{\rm loc} \mu''$ we infer that A is the disjoint union of two sets $A', A'' \in \mathcal{A}$ with $\mu'(A'') = \mu''(A') = 0$. This shows that $f = \mathbf{1}_{A'} f + \mathbf{1}_{A''} f \in M + N$. In particular one obtains $Pf = \mathbf{1}_{A'} f$, and this implies $0 \leqslant P \leqslant I$.

(b) Let P denote the band projection onto M.

First let $A \in \mathcal{A}_{\sigma\text{-fin}}$. The restriction of P to $L_p(A, \mu_A)$ is a band projection, and from Proposition 2.1 we know that A is the disjoint union of two sets $A', A'' \in \mathcal{A}$ such that $Pf = \mathbf{1}_{A'} f$ $(f \in L_p(A, \mu_A))$. We define

$$\mu'(A) := \mu(A'), \quad \mu''(A) := \mu(A'').$$

For $A \in \mathcal{A} \setminus \mathcal{A}_{\sigma\text{-fin}}$ we define $\mu'(A) = \mu''(A) := \infty$. From these definitions it is now clear that $\mu = \mu' + \mu''$, and that $\mu' \perp_{\rm loc} \mu''$ (provided we know that μ' and μ'' are measures).

As a preparation for the proof of the σ-additivity of μ', μ'' we show: if $A \subseteq B$ belong to $\mathcal{A}_{\sigma\text{-fin}}$, and A', A'', B', B'' are as defined above, then $A' = B' \cap A$ and $A'' = B'' \cap A$ μ-a.e. Indeed, let $f \in L_p(\mu)$ satisfy $[f \neq 0] \subseteq A$. Then the previous definitions imply that $\mathbf{1}_{A'} f = Pf = \mathbf{1}_{B'} f = \mathbf{1}_{B' \cap A} f$ μ-a.e. As this holds for all $f \in L_p(\mu)$ with $[f \neq 0] \subseteq A$ one concludes that $A' = B' \cap A$ μ-a.e. This implies also the analogous property for A'' and B''.

Now we show that μ', μ'' are σ-additive. Recall that $\mu' = \mu'' = \infty$ on $\mathcal{A} \setminus \mathcal{A}_{\sigma\text{-fin}}$, and note that sets in $\mathcal{A} \setminus \mathcal{A}_{\sigma\text{-fin}}$ cannot be obtained as a countable union of sets in $\mathcal{A}_{\sigma\text{-fin}}$. Thus it is sufficient to show the σ-additivity on $\mathcal{A}_{\sigma\text{-fin}}$. Let $(A_n)_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{A}_{\sigma\text{-fin}}$, and let $A := \bigcup_{n\in\mathbb{N}} A_n$. Let A', A'', A'_n, A''_n $(n \in \mathbb{N})$ be the corresponding sets defined above. From the previous paragraph we know that $A'_n = A' \cap A_n$ and $A''_n = A'' \cap A_n$ μ-a.e. Therefore the σ-additivity of μ shows that

$$\sum_{n=1}^{\infty} \mu'(A_n) = \sum_{n=1}^{\infty} \mu(A'_n) = \sum_{n=1}^{\infty} \mu(A' \cap A_n) = \mu(A') = \mu'(A),$$

and similarly for μ''. This completes the proof that μ', μ'' are (locally disjoint) measures on $\mathcal{A}$.

From the definition of μ', μ'' it is clear that $Pf = 0$ μ''-a.e. and $(I - P)f = 0$ μ'-a.e. for all $f \in L_p(\mu)$. This shows that μ', μ'' are as asserted. □

2.4 Remark. (a) The description of bands given in Theorem 2.2 implies the description in Proposition 2.1. Indeed, if μ', μ'' as in Theorem 2.2 are locally disjoint and the measure μ is weakly localisable, then it follows that there exists a locally measurable set Ω' such that $\mu' = \mathbf{1}_{\Omega'}\, \mu$ and $\mu'' = \mathbf{1}_{\Omega\setminus\Omega'}\, \mu$.

(b) In the construction of μ' and μ'' in the proof of part (b) of Theorem 2.2 we obtained μ', μ'' with

$$\mathcal{A}_{\sigma\text{-fin},\mu'} = \mathcal{A}_{\sigma\text{-fin},\mu''} = \mathcal{A}_{\sigma\text{-fin}},$$

in particular such that $\mu'(A) = \mu''(A) = \infty$ for all $A \in \mathcal{A} \setminus \mathcal{A}_{\sigma\text{-fin}}$. This might not be the case for the decomposition supposed in part (a) of Theorem 2.2.

(c) We want to rephrase the assertion of Theorem 2.2(b) in the form that every band M of $L_p(\mu)$ is of the form $M = L_p(\mu')$, with μ' as in Theorem 2.2(b).

In order to interpret this statement we embed $L_p(\mu')$ into $L_p(\mu)$ as follows. For all $f \in L_p(\mu')$ there exists an element $\tilde{f}$ in the μ'-equivalence class of f with $\tilde{f} = 0$ μ''-a.e. (Recall that $A := [f \neq 0] \in \mathcal{A}_{\sigma\text{-fin}}$, $\mu'(A'') = 0$, hence $\tilde{f} := \mathbf{1}_{A'} f = f$ as elements of $L_p(\mu')$.) The embedding of $L_p(\mu')$ into $L_p(\mu)$ is then given by $f \mapsto \tilde{f}$.

(d) Theorem 2.2(b) is remindful of [12; Theorem 6], [3; Theorem 4.1], where it is shown that every band in $L_p(\mu)$ (and more generally every range of a contractive projection) is isomorphic to some $L_p(\nu)$. The point in our result is that μ' and μ'' are such that $\mu = \mu' + \mu''$, $M = L_p(\mu')$, and $M^{\perp} = L_p(\mu'')$.

3 Examples

The first issue of this section is to give an example of a measure space $(\Omega, \mathcal{A}, \mu)$ with a band in $L_p(\mu)$ that cannot be described as $L_p(\Omega', \mu_{\Omega'})$ for a locally measurable set $\Omega' \subseteq \Omega$. The example is a version of an example given by Fremlin [5; Example 5(c)].

In view of Proposition 2.1 it obviously is redundant to present this example. However, our modification of Fremlin's example consists in the feature that one of the factors of the cartesian product Ω is the interval $(0,1)$, with Lebesgue measure, and it is this modification that makes it possible to present the second example treated in this section, an interesting non-trivial absorption semigroup on $L_p(\mu)$.

Let $\mathfrak{c}$ denote the cardinality of the continuum. Let Γ be a set of cardinality greater than $\mathfrak{c}$, and let

$$\Omega := (0,1) \times \Gamma.$$

On $(0,1)$ we use the Lebesgue measure λ. On Γ we use the σ-algebra consisting of the countable and the co-countable subsets of Γ and the measure

$$\nu(G) := \begin{cases} 0 & \text{if } G \subseteq \Gamma \text{ is countable,} \\ 1 & \text{if } G \subseteq \Gamma \text{ is co-countable.} \end{cases}$$

The σ-algebra $\mathcal{A}$ on Ω is defined by

$$\begin{aligned} \mathcal{A} := \big\{ A \subseteq \Omega;\ & A_x \text{ countable or co-countable } \big(x \in (0,1)\big), \\ & A^\gamma \text{ a Borel set in } (0,1)\ (\gamma \in \Gamma) \big\}, \end{aligned}$$

where

$$\begin{aligned} A_x &:= \big\{ \gamma \in \Gamma;\ (x,\gamma) \in A \big\}, \\ A^\gamma &:= \big\{ x \in (0,1);\ (x,\gamma) \in A \big\}. \end{aligned}$$

For $A \in \mathcal{A}$ we define

$$\mu'(A) := \sum_{x \in (0,1)} \nu(A_x), \quad \mu''(A) := \sum_{\gamma \in \Gamma} \lambda(A^\gamma),$$

and we define $\mu := \mu' + \mu''$. It is easy to check that μ', μ'' and μ are measures on $\mathcal{A}$.

If $A \in \mathcal{A}$, $\mu(A) < \infty$, then the set

$$I' := \big\{ x \in (0,1);\ \Gamma \setminus A_x \text{ countable} \big\}$$

is finite. Then for the set

$$A' := \big\{ (x,\gamma) \in A;\ x \in I' \big\} = A \cap (I' \times \Gamma) \in \mathcal{A} \cap A$$

one obtains

$$\mu''(A') \leqslant \mu''(I' \times \Gamma) = \sum_{\gamma\in\Gamma} \lambda(I') = 0$$

and

$$\mu'(A \setminus A') \leqslant \mu'\Big(A \cap \big(((0,1)\setminus I') \times \Gamma\big)\Big) = \sum_{x\in(0,1)\setminus I'} \nu(A_x) = 0.$$

This shows that $\mu' \perp_{\mathrm{loc}} \mu''$.

3.1 Proposition. *Let $1 \leqslant p < \infty$. With the measure space $(\Omega, \mathcal{A}, \mu)$ and the decomposition $\mu = \mu' + \mu''$ defined above let*

$$M := \big\{f \in L_p(\mu);\ f = 0\ \mu''\text{-a.e.}\big\}$$

be the band as in Theorem 2.2(a). Then the band projection P onto M is not a multiplication operator, i.e., there exists no locally measurable function $h\colon \Omega \to \mathbb{R}$ such that $Pf = hf$ $(f \in L_p(\mu))$.

Proof. Let $h\colon \Omega \to \mathbb{R}$ be locally measurable, with $hf = f$ for all $f \in M$. Applying this equality to functions $f = \mathbf{1}_{\{x\}\times\Gamma} \in M$, where $x \in I$, one obtains $h(x,\cdot) = 1$ ν-a.e. for all $x \in (0,1)$. This implies that the set $[h(x,\cdot) \neq 1]$ is countable for all $x \in (0,1)$; hence the set $[h \neq 1]$ has cardinality less or equal $\mathfrak{c}$.

Since Γ has cardinality greater than $\mathfrak{c}$, it follows that there exists $\gamma \in \Gamma$ such that $h(\cdot,\gamma) = \mathbf{1}_{(0,1)}$. This means that $hg = g$ for the function $g = \mathbf{1}_{(0,1)\times\{\gamma\}} \in M^\perp$, and this shows that multiplication by h cannot be the band projection onto M. □

Taking into account that $L_p(\Gamma,\nu)$ is isomorphic to $\mathbb{R}$, one sees that the space $M = L_p(\mu')$ is isomorphic to $\ell_p(0,1)$. The space $M^\perp = L_p(\mu'')$ is isomorphic to $\ell_p\big(\Gamma; L_p(0,1)\big)$.

The second issue of this section is in connection with absorption semigroups. Starting with a positive C_0-semigroup $T = (T(t))_{t\geqslant 0}$ on $L_p(\mu)$, for some measure space (Ω,μ) and some $p \in [1,\infty)$, and with a locally measurable function $V\colon \Omega \to [0,\infty]$ (the 'absorption rate'), one defines the 'absorption semigroup' T_V by

$$T_V(t) := \operatorname*{s-lim}_{n\to\infty} \mathrm{e}^{t(A - V\wedge n)} \qquad (t \geqslant 0),$$

where A denotes the generator of T. (The limit exists by monotone convergence. We refer to [13], [14], [2], [8] for more information.) It was shown in [2; Corollary 3.3] that then

$$P := \operatorname*{s-lim}_{t\to 0+} T_V(t)$$

exists and is a band projection. One might hope that P is a multiplication operator in this particular context. However, we will present an example where the band corresponding to P is as in Proposition 3.1.

We return to the special setting of Proposition 3.1. We define a C_0-semigroup T on $L_p(\mu)$ as the direct sum of C_0-semigroups T' and T'' on M and $M^\perp$. On M we define $T'(t) = I'$ (the identity operator on M) for all $t \geqslant 0$. On $M^\perp$ we define T'' as the right translation semigroup on all the 'fibers' of $M^\perp = \ell_p\big(\Gamma; L_p(0,1)\big)$,

$$T''(t)f(x,\gamma) = f(x-t,\gamma) \qquad (t \geqslant 0,\ (x,\gamma) \in \Omega),$$

for $f \in M^\perp$ (with $f(x-t,\gamma) := 0$ if $x - t \leqslant 0$).

Putting T' and T'' together we obtain a positive C_0-semigroup $T = T' \oplus T''$ on $L_p(\mu) = M \oplus M^\perp$.

Let $W\colon (0,1) \to [0,\infty)$ be a Borel-measurable function with the property that $W \notin L_1(U)$ for any open subset $\varnothing \neq U \subseteq (0,1)$. Then $V(x,\gamma) := W(x)$ defines a locally measurable function $V\colon \Omega \to [0,\infty)$.

3.2 Proposition. *In the example described above, the absorption semigroup T_V is given by*

$$T_V(t)f = \mathrm{e}^{-tV}f$$

for $f \in M = L_p(\mu')$, and by

$$T_V(t)f = 0 \qquad (t > 0)$$

for $f \in M^\perp = L_p(\mu'')$. As a consequence,

$$P := \operatorname*{s-lim}_{t\to 0+} T_V(t)$$

is the band projection onto M.

Proof. The expression for T_V on $M = L_p(\mu') = \ell_p(0,1)$ is immediate.

In order to motivate $T_V(t) = 0$ on $M^\perp$ we mention that on each fiber the semigroup describes right translation with the absorption rate W. This is expressed by the formula

$$T_V(t)f(x,\gamma) = \exp\Big(-\int_{x-t}^{t} W(s)\,\mathrm{d}s\Big) f(x-t,\gamma) \tag{3.1}$$

for $f \in M^\perp = L_p(\mu'') = \ell_p\big(\Gamma; L_p\big(0,1\big)\big)$. (For bounded W, the expression (3.1) is a special case of [4; VI.2, (2.4)]. Approximating general W by $W \wedge n$, with $n \in \mathbb{N}$, one obtains (3.1) for general W.) The circumstance that

$$\int_{x-t}^{t} W(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_{x-t}^{t} (W(s) \wedge n)\,\mathrm{d}s = \infty$$

for all the integrals in the exponential yields the result $T_V(t)f = 0$.

The last assertion is then obvious. $\square$

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Hendrik Vogt
Fachbereich Mathematik
Universität Bremen
Postfach 330 440
28359 Bremen, Germany
hendrik.vogt@uni-bremen.de

Jürgen Voigt
Technische Universität Dresden
Fachrichtung Mathematik
01062 Dresden, Germany
juergen.voigt@tu-dresden.de