Holomorphic families of forms, operators and C_0-semigroups

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Abstract

If $z \mapsto a_z$ is a holomorphic function with values in the sectorial forms in a Hilbert space, then the associated operator valued function $z \mapsto A_z$ is resolvent holomorphic. We give a proof of this result of Kato, on the basis of the Lax-Milgram lemma. We also show that the C_0-semigroups T_z generated by $-A_z$ depend holomorphically on z.

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Introduction

The main objective of this note is to present a proof of the following theorem connecting holomorphic dependence of forms in a Hilbert space with holomorphy of the associated operator function.

0.1 Theorem. *Let H be a complex Hilbert space, $V \subseteq H$ a dense subspace, and let $\Omega \subseteq \mathbb{C}$ be open. For each $z \in \Omega$ let a_z be a closed sectorial form in H with domain $\operatorname{dom}(a_z) = V$, and let A_z denote the (m-sectorial) operator associated with a_z. Assume that for all $x, y \in V$ the function $\Omega \ni z \mapsto a_z(x, y) \in \mathbb{C}$ is holomorphic.*

Then the function $\Omega \ni z \mapsto A_z$ is resolvent holomorphic, and the sectoriality of $(A_z)_{z\in\Omega}$ is locally uniform.

This theorem is due to Kato [4; Theorem VII.4.2] and is proved there via a representation of m-sectorial operators involving the square roots of their real parts. We will present a proof that might be regarded as more natural; our crucial observation is a formula expressing the operator associated with a form in terms of the 'Lax-Milgram operator'; see Proposition 1.1 below.

A rather striking application of Theorem 0.1, due to B. Simon, has been given in Kato [3; Addendum]. In this application a Trotter product formula for sectorial forms is derived from the validity of the corresponding Trotter product formula for symmetric forms.

Concerning notation, we recall that a form a is called *sectorial* if there exist $\gamma \in \mathbb{R}$ and $C \geqslant 0$ such that

$$|\operatorname{Im} a(u)| \leqslant C(\operatorname{Re} a(u) - \gamma\|u\|^2) \qquad (u \in \operatorname{dom}(a)),$$

and similarly for an operator in H. This means that the numerical range of the form (or the operator) is contained in a sector with vertex γ and semi-angle $\arctan C$.

In Kato [4; Section VII.4.2], a function $z \mapsto a_z$ as in Theorem 0.1 is called *holomorphic of type* (a). We call a function $\Omega \ni z \mapsto A_z$ with values in the closed operators in H *resolvent holomorphic* if the following condition is satisfied. For all $z_0 \in \Omega$ and some (and then all) $\lambda \in \rho(A_{z_0})$ there exists an open neighbourhood Ω_{z_0} such that $\lambda \in \rho(A_z)$ for all $z \in \Omega_{z_0}$ and the function $\Omega_{z_0} \ni z \mapsto (\lambda - A_z)^{-1} \in \mathcal{L}(H)$ is holomorphic; see Kato [4; Theorem VII.1.3].

In Section 1 we recall the Lax-Milgram lemma and present the resulting formula mentioned above. Section 2 contains the proof of Theorem 0.1. In Section 3 we sketch a result that, in the particular context of Theorem 0.1, implies that the associated C_0-semigroups depend holomorphically on z.

1 The Lax–Milgram lemma

Let V be a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $a\colon V \times V \to \mathbb{K}$ be a coercive bounded sesquilinear form, where *coercive* means that there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u) \geqslant \alpha\|u\|_V^2 \qquad (u \in V).$$

Let V^* denote the anti-dual space of V, with the V^*-V-pairing denoted by $\langle\cdot,\cdot\rangle$. Then

$$\langle \mathcal{A}u, v\rangle := a(u,v) \qquad (u, v \in V)$$

defines a bounded operator $\mathcal{A}\colon V \to V^*$. The Lax–Milgram lemma states that $\mathcal{A}$ is an isomorphism, and $\|\mathcal{A}^{-1}\| \leqslant 1/\alpha$; see [5; Theorem 2.1], [1; Satz 4.9] (for the complex case).

Let H be a Hilbert space over $\mathbb{K}$, and let $j \in \mathcal{L}(V, H)$ be an injective operator with dense range. Then

$$A := \bigl\{(x, y) \in H \times H;\ \exists u \in V\colon ju = x,\ a(u,v) = (y\,|\,jv)_H\ \ (v \in V)\bigr\}$$

defines the operator A *associated with* (a, j). There exists $c > 0$ such that $\|ju\|_H \leqslant c\|u\|_V$ for all $v \in V$. If $x \in \operatorname{dom}(A)$, then there exists $u \in V$ such that $ju = x$ and $a(u,u) = (Ax\,|\,x)$; hence

$$\operatorname{Re}(Ax\,|\,x) = \operatorname{Re} a(u,u) \geqslant \alpha\|u\|_V^2 \geqslant \frac{\alpha}{c^2}\|x\|_H^2.$$

This inequality means that A is *strictly accretive*; see Kato [3; Chapter V, §3.11].

1.1 Proposition. *In the situation described above the operator A is strictly m-accretive, and*

$$A^{-1} = j\mathcal{A}^{-1}k, \tag{1.1}$$

with the canonical injection $k \in \mathcal{L}(H, V^*)$ defined by $H \ni y \mapsto (y\,|\,j(\cdot))_H \in V^*$ (the 'anti-dual operator' of j).

Proof. Let $y \in H$. Then $(y\,|\,j(\cdot))_H \in V^*$, so by the Lax-Milgram lemma there exists $u \in V$ such that

$$a(u,v) = (y\,|\,jv)_H \qquad (v \in V),$$

i.e., $\mathcal{A}u = ky$. By the definition of A, this implies that $x := ju \in \operatorname{dom}(A)$ and $Ax = y$. This shows that $y \in \operatorname{ran}(A)$ and $A^{-1}y = x = j\mathcal{A}^{-1}ky$. We conclude that A is strictly m-accretive and that (1.1) holds. □

2 Proof of the main theorem

We start with a preliminary step of the proof of Theorem 0.1; this also serves to fix some notation. We note that for each $z \in \Omega$ there exist $\gamma_z \in \mathbb{R}$ and $C_z \geqslant 0$ such that

$$|\operatorname{Im} a_z(u)| \leqslant C_z(\operatorname{Re} a_z(u) - \gamma_z\|u\|_H^2) \qquad (u \in V).$$

The closedness of a_z means that the space $(V, \|\cdot\|_{a_z})$, with the norm

$$\|u\|_{a_z} = \big(\operatorname{Re} a_z(u) + (1-\gamma_z)\|u\|_H^2\big)^{1/2} \qquad (u \in V),$$

is complete. Using that the embedding $(V, \|\cdot\|_{a_z}) \hookrightarrow (H, \|\cdot\|_H)$ is continuous and applying the closed graph theorem we conclude that the norms $\|\cdot\|_{a_z}$ are pairwise equivalent. For notational convenience we can therefore assume that $(V, (\cdot\,|\,\cdot)_V)$ is a Hilbert space with a norm equivalent to all norms $\|\cdot\|_{a_z}$.

Proof of Theorem 0.1. For $z \in \Omega$ we define $\mathcal{A}_z \in \mathcal{L}(V, V^*)$ by

$$\langle \mathcal{A}_z u, v\rangle := a_z(u,v) \qquad (u, v \in V)$$

and note that the hypotheses together with Kato [4; Theorem III.3.12] yield the holomorphy of $\Omega \ni z \mapsto \mathcal{A}_z \in \mathcal{L}(V, V^*)$.

Let $z_0 \in \Omega$. Without loss of generality we assume that $z_0 = 0$ and that a_0 is sectorial with vertex $\gamma_0 = 1$; then there exists $C > 0$ such that

$$\|u\|_V^2 \leqslant C\|u\|_{a_0}^2 = C\operatorname{Re} a_0(u) \qquad (u \in V).$$

There exists $r > 0$ such that $B[0,r] \subseteq \Omega$ and $\|\mathcal{A}_z - \mathcal{A}_0\| \leqslant \frac{1}{2C}$ for all $z \in B[0,r]$. This implies that for all $z \in B[0,r]$, $u \in V$ one has

$$|a_z(u) - a_0(u)| \leqslant \frac{1}{2C}\|u\|_V^2 \leqslant \frac{1}{2}\operatorname{Re} a_0(u), \tag{2.1}$$

in particular

$$\operatorname{Re} a_z(u) \geqslant \frac{1}{2} \operatorname{Re} a_0(u) \geqslant \frac{1}{2C} \|u\|_V^2. \tag{2.2}$$

This inequality shows that a_z is coercive for all $z \in B[0, r]$. Therefore Proposition 1.1 implies that A_z is strictly m-accretive, and

$$A_z^{-1} = j\mathcal{A}_z^{-1}k, \tag{2.3}$$

where $j\colon V \hookrightarrow H$ denotes the embedding and k is as in Proposition 1.1. The holomorphy of $z \mapsto \mathcal{A}_z$ and the existence of the inverse $\mathcal{A}_z^{-1} \in \mathcal{L}(V^*, V)$ for all $z \in B(0, r)$ imply that $B(0, r) \ni z \mapsto \mathcal{A}_z^{-1} \in \mathcal{L}(V^*, V)$ is holomorphic; cf. [4; bottom of p. 365]. By (2.3), this implies the holomorphy of $B(0, r) \ni z \mapsto A_z^{-1} \in \mathcal{L}(H)$.

The inequalities (2.1) and (2.2) imply

$$|\operatorname{Im} a_z(u)| \leqslant |\operatorname{Im} a_0(u)| + \frac{1}{2} \operatorname{Re} a_0(u) \leqslant \Big(C_0 + \frac{1}{2}\Big) \operatorname{Re} a_0(u) \leqslant (2C_0 + 1) \operatorname{Re} a_z(u).$$

This estimate shows that the form a_z is sectorial with semi-angle $\arctan(2C_0 + 1)$ and vertex 0, for all $z \in B[0, r]$. □

2.1 Remark. We will show here that the equivalence of the norms $\|\cdot\|_{a_z}$ is locally uniform. Note that this was not needed explicitly in the proof of Theorem 0.1.

Putting ourselves into the context of the proof of Theorem 0.1 we show the uniform equivalence of the norms on $B(0, r)$. For $z \in B(0, r)$ the form a_z is sectorial with vertex 0; so we will use the norm

$$\|u\|_{a_z} = \big(\operatorname{Re} a_z(u) + \|u\|_H^2\big)^{1/2} \qquad (u \in V).$$

From (2.2) and (2.1) we know that

$$\frac{1}{2} \operatorname{Re} a_0(u) \leqslant \operatorname{Re} a_z(u) \leqslant \frac{3}{2} \operatorname{Re} a_0(u) \qquad (u \in V)$$

for all $z \in B(0, r)$, and this implies

$$\frac{1}{2} \|u\|_{a_0}^2 \leqslant \|u\|_{a_z}^2 \leqslant \frac{3}{2} \|u\|_{a_0}^2 \qquad (u \in V).$$

3 Holomorphic dependence of C_0-semigroups

In the context of Theorem 0.1, every operator $-A_z$ is the generator of a holomorphic C_0-semigroup T_z. The following theorem shows that the function $z \mapsto T_z$ is also holomorphic, in a suitable sense. Note, however, that in this result no holomorphy of the semigroups is required.

3.1 Theorem. *Let X be a complex Banach space, and let $\Omega \subseteq \mathbb{C}$ be open. For $z \in \Omega$ let T_z be a C_0-semigroup on X, with generator A_z, and assume that there exists $\omega \in \mathbb{R}$ such that*

$$M := \sup\bigl\{\mathrm{e}^{-\omega t}\|T_z(t)\|;\, t \geqslant 0,\ z \in \Omega\bigr\} < \infty.$$

Assume further that $\Omega \ni z \mapsto (\lambda - A_z)^{-1} \in \mathcal{L}(X)$ is holomorphic, for some $\lambda > \omega$. Then

(a) *the function $\Omega \ni z \mapsto T_z(\cdot)x \in C([0,t_1];X)$ is holomorphic for all $t_1 > 0$, $x \in X$,*

(b) *the function $\Omega \ni z \mapsto T_z(t) \in \mathcal{L}(X)$ is holomorphic for all $t \geqslant 0$.*

Sketch of the proof. Without loss of generality we assume $\omega = 0$; then

$$\bigl\|(\lambda - A_z)^{-n}\bigr\| \leqslant \frac{M}{\lambda^n} \qquad (n \in \mathbb{N},\ \lambda > 0). \tag{3.1}$$

The holomorphy hypothesis implies that $\Omega \ni z \mapsto (\lambda - A_z)^{-1} \in \mathcal{L}(X)$ is holomorphic for all $\lambda > 0$; see Kato [4; Theorem VII.1.3]. The exponential formula shows that

$$T_z(t) = \operatorname*{s-lim}_{n\to\infty}\Bigl(I - \frac{t}{n}A_z\Bigr)^{-n} \qquad (t \geqslant 0), \tag{3.2}$$

and the strong convergence is uniform for t in bounded subsets of $[0,\infty)$; see Pazy [6; Theorem I.8.3].

From (3.1) and (3.2) one obtains the assertions, using standard facts of the theory of Banach space valued holomorphic functions; see [2; Proposition A.3]. □

3.2 Remarks. (a) In Theorem 3.1, assume additionally that all the semigroups T_z are holomorphic on a common sector $\Sigma_\theta := \bigl\{\tau \in \mathbb{C};\, |\operatorname{Arg}\tau| < \theta\bigr\}$, with some $\theta \in (0,\pi/2]$, and that

$$M := \sup\bigl\{\mathrm{e}^{-\omega \operatorname{Re}\tau}\|T_z(\tau)\|;\, \tau \in \Sigma_\theta,\ z \in \Omega\bigr\} < \infty,$$

for some $\omega \in \mathbb{R}$. Then as above one can show that

$$\Omega \ni z \mapsto T_z(\cdot)x \in C\bigl((\Sigma_{\theta'} \cup \{0\}) \cap B_{\mathbb{C}}(0,r);X\bigr)$$

is holomorphic for all $x \in X$, $\theta' \in (0,\theta)$, $r > 0$.

(b) The statement presented in part (a) above is a well-established result; see [3; Theorem IX.2.6]. The proof given in this reference uses the representation of the semigroups expressed by contour integrals. The authors are not aware of a source in the literature for the result stated in Theorem 3.1, for non-holomorphic semigroups.

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