

# Algebra, Coalgebra, and Dualization

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## Introduction

While varieties of algebras (at least finitary ones) are the object of mathematical studies at least since Birkhoff's seminal paper [8], the concept of covariety of coalgebras is a very recent one: the term "covariety" probably first appeared in [13] while its precise relation to the notion of variety was only established in [3].

However, coalgebraic structures themselves are around for a much longer time, originating from topology and geometry (see [14] for a systematic account of this earlier work). The present interest comes from computer science. By restricting itself to the category **Set** of sets as a base category (though being aware of the possibility of choosing more general bases) these recent studies could, however, neither take into account the earlier (module based) work, nor could it make precise to what extent the "co" in coalgebra really was justified and thus would allow for obtaining relevant results for coalgebra from known algebraic ones by formal (categorical) dualization.

Both can be achieved when starting from a (nearly) arbitrary base category for doing algebra, as is to be shown in these notes. Emphasis is therefore given to a presentation of general algebraic structures which immediately allows for getting most of the general theory of coalgebra as duals of essentially well known algebraic results. This should not be misinterpreted: we do *not* say that coalgebra simply is the dual of algebra. In fact, covarieties (over **Set**)—though being the dual of varieties—for example have a strong algebraic flavour, too; moreover, the important coalge-

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braic concept of “bisimulation”—not covered in these notes at all—seems not to have any algebraically interesting (dual) counterpart.

There is a word in place concerning the (categorical) methods used in order to obtain algebraic (and, thus, coalgebraic) results. For the sake of simplicity—in particular w.r.t. (co)algebra over **Set**—we refrain from only employing the methods of the theory of locally presentable categories, which could be done throughout. However, some arguments, particularly those leading to cocompleteness and reflectivity results, are obtained more easily when using methods of the theory of concrete categories, which therefore are used whenever appropriate.

This report should be seen as “report on work in progress” which is presently undertaken by the author jointly with Jiří Adámek. It is hoped that it might help to stimulate others to become engaged in this exciting subject.

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# 1 Algebras and coalgebras with respect to a functor

Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be an endofunctor of some category  $\mathbf{C}$ . Categories  $\mathbf{Alg}(F)$  and  $\mathbf{Coalg}(F)$  are defined as follows.

Objects of  $\mathbf{Alg}(F)$ , called *F-algebras (over  $\mathbf{C}$ )*, are pairs  $(C, \alpha_C)$  with  $C$  a  $\mathbf{C}$ -object and  $\alpha_C: FC \rightarrow C$  a  $\mathbf{C}$ -morphism. Morphisms  $f: (C, \alpha_C) \rightarrow (D, \alpha_D)$  of  $\mathbf{Alg}(F)$ , called *F-algebra homomorphisms*, are  $\mathbf{C}$ -morphisms  $f: C \rightarrow D$  making the diagram

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FD \\ \alpha_C \downarrow & & \downarrow \alpha_D \\ C & \xrightarrow{f} & D \end{array}$$

commute.

Objects of  $\mathbf{Coalg}(F)$ , called *F-coalgebras (over  $\mathbf{C}$ )*, are pairs  $(C, \alpha_C)$  where  $C$  is a  $\mathbf{C}$ -object and  $\alpha_C: C \rightarrow FC$  is a  $\mathbf{C}$ -morphism. Morphisms  $f: (C, \alpha_C) \rightarrow (D, \alpha_D)$  of  $\mathbf{Coalg}(F)$ , called *F-coalgebra homomorphisms*, are  $\mathbf{C}$ -morphisms  $f: C \rightarrow D$  making the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \alpha_C \downarrow & & \downarrow \alpha_D \\ FC & \xrightarrow{Ff} & FD \end{array}$$

commute.

Composition and identities in  $\mathbf{Alg}(F)$  and  $\mathbf{Coalg}(F)$  respectively are those of  $\mathbf{C}$ .

$\mathbf{Alg}(F)$  and  $\mathbf{Coalg}(F)$  are concrete categories over  $\mathbf{C}$  in that they are equipped with canonical underlying functors

$${}_F U: \mathbf{Alg}(F) \rightarrow \mathbf{C} \quad \text{and} \quad U_F: \mathbf{Coalg}(F) \rightarrow \mathbf{C}$$

respectively<sup>1</sup>.

Recall that the dual of a functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  is the functor  $F^{op}: \mathbf{C}^{op} \rightarrow \mathbf{D}^{op}$  acting on objects and morphisms as  $F$ .

With these notations one has

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<sup>1</sup>Whenever confusion is unlikely to arise we will omit the subscript  $F$ .

**1.1 Lemma** For any functor  $F: \mathcal{C} \rightarrow \mathcal{C}$  the following hold:

1.  $\text{Coalg}(F) = (\text{Alg}(F^{op}))^{op}$
2.  $U_F = (F^{op}U)^{op}$

**1.2 Examples** 1. Let  $\Omega$  be a signature in Birkhoff's sense, i.e.,  $\Omega = (\Omega_n)_{n \in \mathbb{N}}$  is a countable family of sets  $\Omega_n$  (of so-called “ $n$ -ary operational symbols”). Then the category  $\text{Alg}\Omega$  of  $\Omega$ -algebras is the following category: objects are pairs  $(A, (\delta_n^A)_{n \in \mathbb{N}})$  with a set  $A$  and a family of maps

$$\delta_n^A: \Omega_n \rightarrow A^{A^n}$$

where often  $\omega^A: A^n \rightarrow A$  is written instead of  $\delta_n^A(\omega)$ . Morphisms  $f: (A, (\delta_n^A)_n) \rightarrow (B, (\delta_n^B)_n)$  of  $\text{Alg}\Omega$  are maps  $f: A \rightarrow B$  such that, for each  $n \in \mathbb{N}$ ,  $\omega \in \Omega_n$  the diagram

$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & B^n \\ \omega^A \downarrow & & \downarrow \omega^B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

Note that  $\text{Alg}\Omega$  is a concrete category by means of its canonical underlying functor  $|-|: \text{Alg}(F) \rightarrow \text{Set}$ .

We describe  $(\text{Alg}\Omega, |-|)$ —up to a concrete isomorphism—as  $(\text{Alg}(F), U)$  for a functor  $F = F_\Omega: \text{Set} \rightarrow \text{Set}$  as follows:

$F_\Omega$  assigns to a set  $X$  the set<sup>2</sup>  $\sum_{n \in \mathbb{N}} \Omega_n \times X^n$ . Correspondingly,  $F_\Omega$  assigns to a map  $f: X \rightarrow Y$  the map  $\sum_{n \in \mathbb{N}} \Omega_n \times f^n$ , i.e., the map  $\sum_{n \in \mathbb{N}} \Omega_n \times X^n \rightarrow \sum_{n \in \mathbb{N}} \Omega_n \times Y^n$  mapping a pair  $(\omega, (x_1, \dots, x_n))$  to the pair  $(\omega, (fx_1, \dots, fx_n))$ .

A concrete isomorphism  $\Phi: (\text{Alg}\Omega, |-|) \rightarrow (\text{Alg}(F), U)$  then is defined by the assignment:

$$\Phi(A, (\delta_n^A)_{n \in \mathbb{N}}) = (A, \delta_A) \quad \text{and} \quad \Phi f = f$$

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<sup>2</sup>Here  $\times$ - denotes the (cartesian) product of sets, while  $\sum$  denotes the coproduct (disjoint union) of sets.

where  $\delta^A = [\bar{\delta}_n^A]: \sum_{n \in \mathbb{N}} \Omega_n \times A^n \rightarrow A$  is the map acting on the  $n$ -th summand as  $\bar{\delta}_n^A: \Omega_n \times A^n \rightarrow A$ , which is the map corresponding to  $\delta_n^A$  by adjointness, i.e.,  $\bar{\delta}_n^A(\omega, (a_1, \dots, a_n)) = \omega^A(a_1, \dots, a_n)$ .

Functors of the form  $F_\Omega$  are called *type* or *polynomial functors*.

2. Let  $\mathbf{csLat}$  denote the category of complete semilattices and sup preserving maps.  $\mathbf{csLat}$  can be fully and concretely embedded into the category  $\mathbf{Alg}(F)$  for a Set-functor  $F$  as follows: chose  $F$  to be the covariant powerset functor  $\mathcal{P}$ . For a complete semilattice  $L$  let  $\Phi(L) = (L, \alpha_L)$  the  $\mathcal{P}$ -algebra with the map  $\alpha_L: \mathcal{P}(L) \rightarrow L$  sending  $U \subset L$  to  $\sup U$ . Then a map  $f: L \rightarrow L'$  between complete semilattices is a  $\mathbf{csLat}$ -morphism iff  $f: \Phi(L) \rightarrow \Phi(L')$  is an  $\mathcal{P}$ -homomorphism.
3. Let  $\mathbf{Alg}_k$  be the category of  $k$ -algebras over some field  $k$ .  $\mathbf{Alg}_k$  can be fully embedded into the category  $\mathbf{Alg}(F)$  (over  $\mathbf{Vect}_k$ , the category of  $k$ -vector spaces) for the functor  $F$  with  $F(V) = k + (V \otimes V)$ .
4. Let  $\mathbf{Part}$  be the category of partial endomaps of sets, i.e., objects of  $\mathbf{Part}$  are triples  $(U, f, X)$  where  $f: U \rightarrow X$  is a map with  $\text{dom} f = U \subset X$ ;  $\mathbf{Part}$ -morphisms  $\phi: (U, f, X) \rightarrow (V, g, Y)$  are maps  $\phi: X \rightarrow Y$  satisfying the conditions (1)  $x \in U \iff \phi(x) \in V$  and (2)  $g(\phi(x)) = \phi(f(x))$  for each  $x \in U$ .  $\mathbf{Part}$  is a concrete category by means of the underlying functor with  $|(U, f, X)| = X$ . A concrete isomorphism  $\Phi: (\mathbf{Part}, | - |) \rightarrow (\mathbf{Coalg}(F), U)$  for the functor  $F = 1 + Id$  is given by  $\Phi(U, f, X) = (X, \alpha_X)$  where  $\alpha_X: X \rightarrow 1 + X$  is the map with  $\alpha_X(x) = f(x)$  if  $x \in U$  and  $\alpha_X(x) = \star$  otherwise<sup>3</sup>.

The *extended natural numbers*  $(\bar{\mathbb{N}}, \alpha)$  with

$$\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \quad \text{and} \quad \alpha(x) = \begin{cases} \infty & \text{if } x = \infty, \\ x - 1 & \text{if } x \in \mathbb{N} \text{ and } x \neq 0, \\ \star & \text{if } x = 0. \end{cases}$$

provide an example of such a coalgebra.

5. So-called *labelled transition systems* (with a fixed set  $L$  of labels) are of importance in computer science. They form a category  $\mathbf{LTS}$  as follows. A labelled transition system consists of a set  $S$  (of states) and, for each state  $s \in S$ , a set of (possible) transitions from  $s$  to states  $s' \in S$ , each labelled by some  $l \in L$ . In other words, for each

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<sup>3</sup>Here 1 denotes a one-element set  $\{\star\}$ .

$s \in S$  there is given a set  $T_s \subset L \times S$  of possible (labelled) transitions such that the system can be viewed as an  $F$ -coalgebra  $(S, \alpha_S)$  (with  $\alpha_S(s) = T_s$ ) for the **Set**-functor  $F$  with  $FX = \mathcal{P}(L \times X)$ . The notion of homomorphism of LTS's coincides with the notion of  $F$ -coalgebra-homomorphism.

6. An *automaton* with *input set*  $I$ , *output set*  $O$ , and *set of states*  $S$  is a quintuple  $(I, O, S, \delta, \gamma)$  with sets  $I, O, S$  and maps

$$\gamma: S \rightarrow O \quad \text{and} \quad \delta: S \times I \rightarrow S.$$

$\gamma$  describes the output accruing from a given state while the “transition map”  $\delta$  gives the change of state according to input. A *homomorphism of automata* with the same input and output sets,  $\phi: (I, O, S, \delta, \gamma) \rightarrow (I, O, S', \delta', \gamma')$ , is a map  $\phi: S \rightarrow S'$  such that, for all  $s \in S$ ,  $i \in I$

$$\phi(\gamma(s)) = \gamma'(\phi(s)) \quad \text{and} \quad \phi(\delta(s, i)) = \delta'(\phi(s), i)$$

It is easily seen that the category **Aut** of automata obtained this way is concretely isomorphic to the category of  $F$ -coalgebras for the **Set**-functor  $F = O \times (-)^I$ : simply assign to an automaton  $(I, O, S, \delta, \gamma)$  the  $F$ -coalgebra  $(S, \alpha_S)$  where  $\alpha_S$  has components  $\gamma$  and  $\bar{\delta}$ , the map corresponding to  $\delta$  by adjointness.

7. Sets of so called *streams* over some alphabet  $A$ , i.e., sets  $A^\omega$  of countable sequences in  $A$ , are often modelled by their *head* (the first element) and *tail* (the remainder). They thus form a coalgebra  $(A^\omega, \langle h, t \rangle)$  of the **Set**-functor  $F$  with  $FX = A \times X$ , where  $h(a_0, a_1, \dots) = a_0$  and  $t(a_0, a_1, \dots) = (a_1, a_2, \dots)$  (see also Example 2.18).
8. *Nondeterministic systems* and *bounded nondeterministic systems* can be modelled as coalgebras of the powerset functor  $\mathcal{P}$  and the finite powerset functor  $\mathcal{P}_{fin}$ <sup>4</sup> respectively.
9. Denote by  $F$  the endofunctor on  $\mathbf{Vect}_k$  with  $FV = k \times (V \otimes V)$  and by  $\mathbf{Coalg}(F)$  its category of coalgebras. Then the category  $\mathbf{Coalg}_k$  of  $k$ -coalgebras is a full subcategory of  $\mathbf{Coalg}(F)$ . Similarly, for a fixed  $k$ -coalgebra  $M$ , the category of (left)  $M$ -modules is a full subcategory of  $\mathbf{Coalg}(F_M)$  for the endofunctor  $F_M$  on  $\mathbf{Vect}_k$  with  $F_M(V) = M \otimes V$  (see Section 3.B and [14] for details of these notions).

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<sup>4</sup>I.e., the functor sending a set to its collection of finite subsets.

**1.3 Remark** By a result of the Prague school (see e.g. [2]), for every **Set**-functor  $F$  there is a functor  $G$  which preserves monomorphisms and finite intersections and coincides with  $F$  on all nonempty sets and all nonempty functions. The resulting categories  $\mathbf{Coalg}(F)$  and  $\mathbf{Coalg}(G)$  clearly are (concretely) isomorphic. Moreover, if  $F(\emptyset) \neq \emptyset$ , then  $G(\emptyset) \neq \emptyset$ ; thus,  $\mathbf{Alg}(F) = \mathbf{Alg}(G)$  in this case. Consequently we may assume that, when dealing with a category  $\mathbf{Alg}(F)$  over **Set**,  $F$  preserves monomorphisms.

## 1.A Elementary properties of $\mathbf{Alg}(F)$

Many elementary properties in algebra are related to the following basic facts:

- **Set** has a regular factorization structure, i.e., every source  $(X, (X \xrightarrow{f_i} X_i)_I)$  of maps can be factored (uniquely) as

$$X \xrightarrow{f_i} X_i = X \xrightarrow{e} Y \xrightarrow{m_i} X_i$$

with a surjection (regular epimorphism)  $e$  and a point-separating family (monosource)  $(Y, (m_i))$ ;

- every **Set**-functor preserves the (regular) epimorphisms (by the axiom of choice).

In order to generalize these observations to arbitrary base categories  $\mathbf{C}$  (which we are doing for good reasons as will become clear later and which hardly requires an argument different from those to be used in the **Set**-based case) we will often work under the following assumption, referred to as *basic situation* in the sequel.

$\mathbf{C}$  has kernel pairs and regular factorizations of sources and  $F: \mathbf{C} \rightarrow \mathbf{C}$  is a functor which preserves regular epimorphisms.

**1.4 Examples** Each of the following is a basic situation:

1. **Set** and  $F$  any **Set**-functor;
2.  $\mathbf{Set}^{op}$  and  $F$  any endofunctor of  $\mathbf{Set}^{op}$ <sup>5</sup>.
3. Any endofunctor on  $\mathbf{Set}_{fin}$ , the category of finite sets, and on  $\mathbf{Set}_{fin}^{op}$ .

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<sup>5</sup>The regular factorization here “is” the factorization of a family of maps  $X_i \xrightarrow{f_i} X = X_i \rightarrow \bigcup f_i[X_i] \hookrightarrow X$ .

4.  $\text{Set}^I$  and  $F$  any endofunctor of  $\text{Set}^I$ .
5.  $(\text{Set}^{op})^I$  and  $F$  any endofunctor of  $(\text{Set}^{op})^I$  whose components are polynomial functors<sup>6</sup>, in particular, thus, if  $F$  is a polynomial functor.
6. Any concrete or polynomial functor on a regular epi-reflexive subcategory of  $\text{Alg}(F)$  where  $F$  is in the basic situation<sup>7</sup>.
7. Any generalized polynomial functor on  $\text{Comp}_2$ , the category of compact Hausdorff spaces, i.e., any functor  $F$  of the form  $FY = \sum_{i \in I} Y^{C_i}$ , and any polynomial functor on each of the categories  $\text{POS}$  of partially ordered sets,  $\text{kTop}$  of compactly generated topological spaces,  $\text{Unif}$  of uniform spaces, or  $\text{Top}_2\text{Grp}$  of Hausdorff topological groups.
8. The functors  $V \mapsto M \otimes V$  with some fixed  $k$ -vector space  $M$ , both on  $\text{Vect}_k$  and  $\text{Vect}_k^{op}$ .
9. The functor  $V \mapsto k \times (V \otimes V)$ , both on  $\text{Vect}_k$  and  $\text{Vect}_k^{op}$ .

**1.5 Proposition** *For every functor  $F: \mathcal{C} \rightarrow \mathcal{C}$  the following hold:*

1. A sink  $((C_i, \alpha_i) \xrightarrow{e_i} (C, \alpha_C))_I$  in  $\text{Alg}(F)$  is  $U$ -final<sup>8</sup> provided that  $(FC_i \xrightarrow{Fe_i} FC)_I$  is an episink in  $\mathcal{C}$ .
2. A source  $((C, \alpha_C) \xrightarrow{m_i} (C_i, \alpha_i))_I$  in  $\text{Alg}(F)$  is  $U$ -initial<sup>9</sup> provided that  $(m_i)_I$  is a monosource in  $\mathcal{C}$ .

**Proof** In the following diagram (with notations as in the footnote) only commutativity of the right hand square is to be shown. In case  $(Ff_i)$  is an episink this follows by diagram chasing.

<sup>6</sup>I.e.,  $F(X_i)$  is a coproduct of products of the  $X_i$ .

<sup>7</sup>Compare Section 3.A.

<sup>8</sup>This is: a  $\mathcal{C}$ -morphism  $h: C \rightarrow D$  with  $(D, \alpha_D)$  in  $\text{Alg}(F)$  lifts to an  $\text{Alg}(F)$ -homomorphism  $(C, \alpha_C) \rightarrow (D, \alpha_D)$  provided that  $g_i = h \circ e_i$  is a homomorphism  $(C_i, \alpha_i) \rightarrow (D, \alpha_D)$  for each  $i \in I$ .

<sup>9</sup>This is the dual: a  $\mathcal{C}$ -morphism  $h: D \rightarrow C$  with  $(D, \alpha_D)$  in  $\text{Alg}(F)$  lifts to an  $\text{Alg}(F)$ -homomorphism provided that  $g_i = m_i \circ h$  is a homomorphism for each  $i \in I$ .

$$\begin{array}{ccccc}
& & & & Fg_i \\
& & & & \curvearrowright \\
FC_i & \xrightarrow{Fe_i} & FC & \xrightarrow{Fh} & FD \\
\alpha_i \downarrow & & \downarrow \alpha_C & & \downarrow \alpha_D \\
C_i & \xrightarrow{e_i} & C & \xrightarrow{h} & D \\
& & & & \curvearrowleft g_i
\end{array}$$

The proof of 2. is done analogously by a diagram chase.  $\diamond$

The following then is a trivial consequence of each of the statements above:

**1.6 Corollary** For every functor  $F: \mathbf{C} \rightarrow \mathbf{C}$ , an  $\mathbf{Alg}(F)$ -morphism  $f$  is an isomorphism in  $\mathbf{Alg}(F)$  iff  $f$  is an isomorphism in  $\mathbf{C}$ .

**1.7 Proposition** For every functor  $F: \mathbf{C} \rightarrow \mathbf{C}$  the functor  $U: \mathbf{Alg}(F) \rightarrow \mathbf{C}$  creates limits<sup>10</sup>.

**Proof** Let, for  $D: \mathbf{I} \rightarrow \mathbf{C}$ , with  $D(i) = (D_i, \alpha_i)$ ,  $(L, L \xrightarrow{l_i} (D(i))_I)$  be a limit of  $UD$  in  $\mathbf{C}$ . Observe first that the source  $FL \xrightarrow{Fl_i} FD_i \xrightarrow{\alpha_i} D_i$  is natural for  $UD$ , i.e., for each morphism  $t: i \rightarrow j$  in  $\mathbf{I}$  one has

$$\begin{aligned}
Dt \circ \alpha_i \circ Fl_i &= \alpha_j \circ FDt \circ Fl_i && \text{(since } Dt \text{ is a homomorphism)} \\
&= \alpha_j \circ Fl_j && \text{(since } (l_i) \text{ is natural for } UD).
\end{aligned}$$

By the limit property of  $(L, (l_i))$  then there exists a unique  $\mathbf{C}$ -morphism  $\alpha_L: FL \rightarrow L$  making the diagram

$$\begin{array}{ccc}
FL & \xrightarrow{\alpha_L} & L \\
Fl_i \downarrow & & \downarrow l_i \\
FD_i & \xrightarrow{\alpha_i} & D_i
\end{array}$$

<sup>10</sup>This is: given a (small) diagram  $D: \mathbf{I} \rightarrow \mathbf{Alg}(F)$  and a limit  $(L, L \xrightarrow{l_i} U(D(i)))$  of  $U \circ D$  in  $\mathbf{C}$ , there exists a unique  $F$ -algebra structure  $\alpha_L: FL \rightarrow L$  such that all  $l_i: (L, \alpha_L) \rightarrow D(i)$  are homomorphisms; moreover this source then is a limit of  $D$  in  $\mathbf{Alg}(F)$ .

commute. Next, for any source  $((K, \alpha_K) \xrightarrow{f_i} D(i))_I$  in  $\mathbf{Alg}(F)$ , natural for  $D$ , there exists a unique  $\mathbf{C}$ -morphism  $f: K \rightarrow L$  with  $l_i \circ f = f_i$  since, clearly,  $(f_i)$  is natural for  $UD$ .  $f$  is a homomorphism by Proposition 1.5.  $\diamond$

**1.8 Proposition** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a functor which preserves colimits of a given class  $\mathcal{K}$  of diagrams. Then  $U: \mathbf{Alg}(F) \rightarrow \mathbf{C}$  creates colimits of diagrams from  $\mathcal{K}$ .*

**Proof** Let, for  $D: \mathbf{I} \rightarrow \mathbf{C}$  in  $\mathcal{K}$ ,  $((D(i) \xrightarrow{l_i} L)_I, L)$  be a colimit of  $UD$  in  $\mathbf{C}$ . Observe first that, with  $D(i) = (D_i, \alpha_i)$ , the sink  $FD_i \xrightarrow{\alpha_i} D_i \xrightarrow{l_i} L$  is natural for  $FD$ , i.e., for each morphism  $t: i \rightarrow j$  in  $\mathbf{I}$  one has

$$\begin{aligned} l_j \circ \alpha_j \circ FDt &= l_j \circ Dt \circ \alpha_i && \text{(since } Dt \text{ is a homomorphism)} \\ &= l_i \circ \alpha_i && \text{(since } (l_i) \text{ is natural for } UD). \end{aligned}$$

Since  $(Fl_i, FL)$  is a colimit of  $FD$  by hypothesis there exists a unique  $\mathbf{C}$ -morphism  $\alpha_L: FL \rightarrow L$  making the diagram

$$\begin{array}{ccc} FD_i & \xrightarrow{\alpha_i} & D_i \\ Fl_i \downarrow & & \downarrow l_i \\ FL & \xrightarrow{\alpha_L} & L \end{array}$$

commute. Next, for any sink  $(D(i) \xrightarrow{f_i} (K, \alpha_K))_I$  in  $\mathbf{Alg}(F)$ , natural for  $D$ , there exists a unique  $\mathbf{C}$ -morphism  $f: L \rightarrow K$  with  $f \circ l_i = f_i$  since, clearly,  $(f_i)$  is natural for  $UD$ . Since  $(Fl_i)$ , being a colimit sink by hypothesis, is an episink in particular,  $f$  is a homomorphism by Proposition 1.5.  $\diamond$

The following is an important application of this result.

**1.9 Corollary** *For every polynomial functor on  $\mathbf{Set}$  the underlying functor  $U: \mathbf{Alg}(F) \rightarrow \mathbf{Set}$  creates directed colimits.*

**Proof** Multiplication with a constant set preserves directed colimits in  $\mathbf{Set}$  (by cartesian closedness); also sums of functors preserving directed colimits will do so. Thus the result follows from the next well known lemma (the proof of which might savely be left to the reader—see e.g. [5]), which also will be of use lateron.  $\diamond$

**1.10 Lemma** *The Set-functor  $X \mapsto X^A$  preserves directed colimits iff  $A$  is finite.*

**1.11 Proposition** *For  $F: \mathbf{C} \rightarrow \mathbf{C}$  in the basic situation<sup>11</sup>  $\mathbf{Alg}(F)$  has a  $(U^{-1}[\mathbf{RegEpi}], U^{-1}[\mathbf{MonoSource}])$  factorization structure, and the factorizations are created by  $U$ .*

**Proof** Let  $((C, \alpha_C), (f_i: (C, \alpha_C) \rightarrow (C_i, \alpha_i))_I)$  be a source in  $\mathbf{Alg}(F)$  and

$$C \xrightarrow{e} D \xrightarrow{m_i} C_i$$

be the regular factorization of its underlying source in  $\mathbf{C}$ . Then, in the following commutative diagram, there exists a unique dotted arrow  $\alpha_D$  making the cells (I) and (II) commutative provided  $Fe \in \mathbf{E}$ . Hence  $(C, \alpha_C) \xrightarrow{e} (D, \alpha_D) \xrightarrow{m_i} (C_i, \alpha_i)$  is the required factorization of the given source.

$$\begin{array}{ccc}
 FC & \xrightarrow{Ff_i} & FC_i \\
 \downarrow \alpha_C & \begin{array}{c} \searrow Fe \\ \downarrow \alpha_D \\ \swarrow Fm_i \end{array} & \downarrow \alpha_i \\
 & FD & \\
 & \downarrow \alpha_D & \\
 & D & \\
 \downarrow \alpha_C & \begin{array}{c} \swarrow e \\ \downarrow \alpha_D \\ \searrow m_i \end{array} & \downarrow \alpha_i \\
 C & \xrightarrow{f_i} & C_i
 \end{array}$$

That  $(U^{-1}[\mathbf{RegEpi}], U^{-1}[\mathbf{MonoSource}])$  has the diagonalization property is consequence of this property for the regular factorization structure on  $\mathbf{C}$  and of Proposition 1.5.  $\diamond$

The factorization structure obtained above can be identified to be the regular one again under a mild additional hypothesis.

**1.12 Proposition** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be in the basic situation. Then the following holds.*

<sup>11</sup>In fact a somewhat relaxed notion of basic situation would suffice here: the factorization structure needn't be the regular one, and we don't need kernel pairs.

1. A homomorphism  $e: (C, \alpha_C) \rightarrow (D, \alpha_D)$  is a regular epimorphism in  $\text{Alg}(F)$  iff  $e$  is a regular epimorphism in  $\mathbf{C}$ .
2. A source  $(C, ((C, \alpha_C) \xrightarrow{m_i} (C_i, \alpha_i))_I)$  is a monosource in  $\text{Alg}(F)$  iff  $(C, (C \xrightarrow{m_i} C_i))_I$  is a monosource in  $\mathbf{C}$ .
3.  $\text{Alg}(F)$  has regular factorizations and therefore coequalizers in particular.

**Proof** 1. Let  $e: (C, \alpha_C) \rightarrow (D, \alpha_D)$  be a morphism such that  $e$  is a regular epimorphism in  $\mathbf{C}$ . Since  $U$  creates kernel pairs there is a kernel pair  $r, s: (K, \alpha_K) \rightarrow (C, \alpha_C)$  of  $e$  where  $r, s: K \rightarrow C$  is a kernel pair of  $e$  in  $\mathbf{C}$ . Then  $e: C \rightarrow D$  is a coequalizer of  $(r, s)$  in  $\mathbf{C}$  and it is easy to see that  $e$  is also a coequalizer of  $(r, s)$  in  $\text{Alg}(F)$ .

For the converse let  $e$  be a regular epimorphism in  $\text{Alg}(F)$  and  $e = m \circ q$  its factorization according to Proposition 1.11. Then  $m$  is a monomorphism (since  $U$  is faithful), hence an isomorphism (since regular epimorphisms are extremal). Thus, with  $Uq$ , also  $Ue$  is a regular epimorphism.

2. By faithfulness of  $U$  we only have to prove that  $U$  preserves monosources. Let now  $(C, ((C, \alpha_C) \xrightarrow{m_i} (C_i, \alpha_i))_I)$  be a monosource in  $\text{Alg}(F)$  and  $m_i = n_i \circ e$  its factorization according to Proposition 1.11. By part 1,  $e$  is a regular epimorphism, thus an isomorphism (since  $(m_i)$  is a monosource). Thus, with  $U(n_i)$ , also  $U(m_i)$  will be a monosource.

3. That  $\text{Alg}(F)$  has regular factorizations is clear. In any such category the coequalizer of a pair  $r, s$  of parallel morphisms is well known (and easily seen) to be the morphism  $e$  appearing in the regular factorization of the source of all morphisms  $f$  with  $fr = fs$ .  $\diamond$

**1.13 Remark** From the description of coequalizers in the proof above and the fact that the regular factorizations are created by  $U$  the following characterization of coequalizers is easily seen to hold under the assumptions of Proposition 1.12: For any pair of homomorphisms  $r, s: (C, \alpha_C) \rightarrow (D, \alpha_D)$  in  $\text{Alg}(F)$  the coequalizer  $e: (D, \alpha_D) \rightarrow (E, \alpha_E)$  is the smallest regular quotient of  $(D, \alpha_D)$  with the property that, in the category  $\mathbf{C}$ , the coequalizer of  $r, s$  is a regular quotient of  $D$ <sup>12</sup>.

<sup>12</sup>This is: if  $(A, \alpha_A)$  is a regular quotient of  $(E, \alpha_E)$  and the coequalizer of  $r, s$  is a regular quotient of  $A$  in  $\mathbf{C}$ , then  $(A, \alpha_A)$  and  $(E, \alpha_E)$  are isomorphic.

Applying the previous results to the interesting case of endofunctors of (powers) of  $\mathbf{Set}$  one thus has<sup>13</sup>

**1.14 Theorem** *For every endofunctor  $F$  of (a power of)  $\mathbf{Set}$  the following hold:*

1.  $\mathbf{Alg}(F)$  has all limits and these are created by  $U$ .
2.  $\mathbf{Alg}(F)$  has all colimits which are preserved by  $F$ , and these are created by  $U$ .  
*In particular  $U$  creates directed colimits in  $\mathbf{Alg}(F)$  for every polynomial functor.*
3.  $\mathbf{Alg}(F)$  has coequalizers and regular factorizations; the latter are created by  $U$ .
4. The monosources in  $\mathbf{Alg}(F)$  are precisely those sources which are point-separating in  $\mathbf{Set}$ , and these are  $U$ -initial.
5. The regular epimorphisms in  $\mathbf{Alg}(F)$  are precisely those homomorphisms which are surjective in  $\mathbf{Set}$ , and these are  $U$ -final.
6. The isomorphisms in  $\mathbf{Alg}(F)$  are precisely the bijective homomorphisms.

The following important corollary—being an immediate consequence of the facts that intersections (as limits) are created by  $U$  and monomorphisms in  $\mathbf{Alg}(F)$  and  $\mathbf{Set}$  coincide—is worth being stated explicitly.

**1.15 Corollary** *For every endofunctor  $F$  of  $\mathbf{Set}$  the subalgebras<sup>14</sup> of an  $F$ -algebra  $(C, \alpha_C)$  form a complete lattice with  $\bigcap = \bigwedge$ . In particular, for each subset  $X$  of an  $F$ -algebra  $(C, \alpha_C)$  there exists a smallest subalgebra  $\langle X \rangle$ <sup>15</sup> of  $(C, \alpha_C)$  containing  $X$ .*

A further simple corollary is the following:

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<sup>13</sup>Note that none of the proofs given above would have been simpler if only performed in the case of  $\mathbf{C} = \mathbf{Set}$ .

<sup>14</sup> $(D, \alpha_D)$  is called *subalgebra* of  $(C, \alpha_C)$  iff  $D \subset C$  and the embedding is a homomorphism.

<sup>15</sup> $\langle X \rangle$  is called the *subalgebra generated by  $X$* .

**1.16 Proposition** *Let  $F$  be an endofunctor of  $\mathbf{Set}$ , which preserves  $\lambda$ -directed colimits for some regular cardinal  $\lambda$ . Then any  $F$ -algebra  $(C, \alpha_C)$  is a directed colimit of its subalgebras  $\langle X \rangle$  generated by subsets  $X$  of  $C$  of cardinality  $< \lambda$ .*

## 1.B Elementary properties of $\mathbf{Coalg}(F)$

By means of Lemma 1.1 one can obtain results on categories  $\mathbf{Coalg}(F)$  by simple dualization of the corresponding results on categories of  $F$ -algebras. When applying these results to a specific functor one only has to make sure that this particular base category and this functor respectively satisfy the dual assumptions if there were made any. So for example the results of Section 1.A can not only be applied to  $\mathbf{C} = \mathbf{Set}$  (as in Theorem 1.14—since one here is in a basic situation for every  $F$ ) but also to  $\mathbf{Set}^{op}$ , since here one is in a basic situation as well for every  $F$  (see Section 1.A).

For the sake of the reader not familiar with categorical dualization we formulate the dual of the formally most involved result of Section 1.A explicitly as an example of this process:

**Proposition 1.11<sup>op</sup>** *Let  $\mathbf{C}$  be a category with an  $(\mathcal{E}, \mathbf{M})$ -factorization structure for sinks and  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a functor such that, for each  $m \in \mathbf{M}$ ,  $Fm \in \mathbf{M}$ . Then  $(U^{-1}[\mathcal{E}], U^{-1}[\mathbf{M}])$  is a factorization structure for sinks in  $\mathbf{Coalg}(F)$  and these factorizations are created by  $U (= (F^{op_U})^{op})$ .*

In this section we simply collect the elementary properties of categories of coalgebras, usually proved from scratch, as duals of the corresponding (known) algebraic results as presented in Section 1.A.

**1.17 Proposition** *For every functor  $F: \mathbf{C} \rightarrow \mathbf{C}$  the following hold:*

1. *A source  $((C, \alpha_C) \xrightarrow{m_i} (C_i, \alpha_i))_I$  in  $\mathbf{Coalg}(F)$  is  $U$ -initial provided that  $(Fm_i)_I$  is a monosource in  $\mathbf{C}$ .*
2. *A sink  $((C_i, \alpha_i) \xrightarrow{e_i} (C, \alpha_C))_I$  in  $\mathbf{Coalg}(F)$  is  $U$ -final provided  $(C_i \xrightarrow{e_i} C)_I$  is an episink in  $\mathbf{C}$ .*
3. *A homomorphism  $f$  is an isomorphism in  $\mathbf{Coalg}(F)$  iff  $f$  is an isomorphism in  $\mathbf{C}$ .*

**Proof** This is the dual of Proposition 1.5 and Corollary 1.6. ◇

**1.18 Proposition** *For every functor  $F: \mathbf{C} \rightarrow \mathbf{C}$  the functor  $U: \mathbf{Coalg}(F) \rightarrow \mathbf{C}$  creates colimits.*

**Proof** This is the dual of Proposition 1.7. ◇

**1.19 Proposition** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a functor which preserves limits of a given class  $\mathcal{K}$  of diagrams. Then  $U: \mathbf{Coalg}(F) \rightarrow \mathbf{C}$  creates limits of diagrams from  $\mathcal{K}$ .*

**Proof** This is the dual of Proposition 1.8. ◇

Since polynomial functors on **Set** preserve limits along countable chains (see [7]) one gets as an application:

**1.20 Corollary** *The functor  $U: \mathbf{Coalg}(F) \rightarrow \mathbf{Set}$  creates limits along countable chains for any polynomial functor  $F$ .*

By Remark 1.3 we may assume that **Set**-functors preserve finite intersections. Since the empty set obviously carries a subcoalgebra structure of any coalgebra one thus gets, together with the dualization of Theorem 1.14, the following properties of categories of coalgebras over **Set**:

**1.21 Theorem** *Let  $F$  be an endofunctor of **Set**. Then the following hold:*

1.  $\mathbf{Coalg}(F)$  has all colimits and these are created by  $U$ .
2.  $\mathbf{Coalg}(F)$  has all limits which are preserved by  $F$  and these are created by  $U$ .

*In particular*

- (a)  $U$  creates finite intersections.
- (b)  $U$  creates limits along countable chains in  $\mathbf{Coalg}(F)$  if  $F$  is a polynomial functor.
3.  $\mathbf{Coalg}(F)$  has equalizers and regular factorizations for sinks; the latter are created by  $U$ .
4. The episinks in  $\mathbf{Coalg}(F)$  are precisely those sinks which are covering in **Set**, and these are  $U$ -final.
5. The regular monomorphisms in  $\mathbf{Coalg}(F)$  are precisely those homomorphisms which are injective in **Set**, and these are  $U$ -initial.

6. *The isomorphisms in  $\text{Coalg}(F)$  are precisely the bijective homomorphisms.*

The following important corollary—being an immediate consequence of the fact that the regular factorization of a sink of embeddings is given by the union and that these factorizations are created by  $U$ —is worth being stated explicitly.

**1.22 Corollary** *For every endofunctor  $F$  of  $\text{Set}$  the subcoalgebras<sup>16</sup> of an  $F$ -coalgebra  $(C, \alpha_C)$  form a complete lattice with  $\bigcup = \bigvee$ . In particular, for each subset  $X$  of an  $F$ -coalgebra  $(C, \alpha_C)$  there exists a largest subalgebra  $[X]$ <sup>17</sup> of  $(C, \alpha_C)$  contained in  $X$ .*

**1.23 Remark** For a number of  $\text{Set}$ -functors  $F$  the meet in the subcoalgebra lattice is in fact  $\bigwedge$ . To ensure this one only has to know that the intersection of subcoalgebras is a subcoalgebra again. And this clearly is the case provided that  $F$  preserves intersections. This is the case for each of the following functors:

1. Every polynomial functor  $F_\Omega$ ;
2. The powerset functor  $\mathcal{P}$  and the finite powerset functor  $\mathcal{P}_{fin}$ .

Thus, for a coalgebra  $(C, \alpha_C)$  w.r.t. any such functor and an element  $a \in C$ , one has, in particular, a subcoalgebra  $\langle a \rangle$  of  $(C, \alpha_C)$  *generated by*  $a$ , i.e., a smallest coalgebra of  $(C, \alpha_C)$  containing  $a$ .

With notation as in Corollary 1.22, equalizers in  $\text{Coalg}(F)$  can be described as follows using (the dual of) Remark 1.13.

**1.24 Corollary** *The equalizer of a pair of homomorphisms  $f, g: (C, \alpha_C) \rightarrow (D, \alpha_D)$  in  $\text{Coalg}(F)$  is the subalgebra of  $(C, \alpha_C)$  cogenerated by the equalizer of  $f$  and  $g$  in  $\text{Set}$ .*

**1.25 Remark** Note that, if specializing 4. of Theorem 1.21 to the factorization of single morphisms only, one obtains the existence of *images*, only seemingly the same result as in algebra; in fact images arise in algebra and in coalgebra respectively as specializations of dual constructions!

<sup>16</sup> $(D, \alpha_D)$  is called *subcoalgebra* of  $(C, \alpha_C)$  iff  $D \subset C$  and the embedding is a homomorphism. More properly they should be called *regular subalgebras* in view of Theorem 1.21.

<sup>17</sup> $[X]$  is called the *subcoalgebra cogenerated by  $X$* .

However, it is an immediate consequence of creation of images that, for any  $\text{Coalg}(F)$ -homomorphism  $f: (C, \alpha_C) \rightarrow (D, \alpha_D)$ , the quotient  $C/\ker f$  of  $C$  modulo the kernel of the map  $f$  carries a unique coalgebra structure such that the quotient map becomes a homomorphism, and this coalgebra is isomorphic to  $f$ 's image (Isomorphism Theorem).

## 2 Free algebras and co-free coalgebras

**2.1 Example** For polynomial endofunctors  $F_\Omega$  on  $\text{Set}$ , the concept of free algebra  $X^\sharp$  on a set  $X$  of generators is well known. We can describe it recursively as  $X^\sharp = \cup_{i < \omega} X_i^\sharp$  where

$$\begin{aligned} X_0^\sharp &= X + \Omega_0 \\ &= X + F_\Omega \emptyset \quad \text{terms of depths 0 are variables and nullary operations} \\ X_{i+1}^\sharp &= X + \{(\omega, t_0, \dots, t_{n-1}) \mid \omega \in \Omega_n, t_0, \dots, t_{n-1} \in X_i^\sharp\} \\ &= X + F_\Omega X_i^\sharp \quad \text{terms of depths } i+1 \end{aligned}$$

**2.2 Remark** Free  $F$ -algebras on  $X$  for an object  $X$  (of “variables”) of  $\mathbf{C}$  can be defined for all functors  $F: \mathbf{C} \rightarrow \mathbf{C}$  as pairs consisting of an  $F$ -algebra

$$FX^\sharp \xrightarrow{\varphi_X} X^\sharp \quad \text{and a morphism} \quad \eta_X: X \rightarrow X^\sharp$$

with the universal property that given an  $F$ -algebra  $(C, \alpha_C)$  and a morphism  $f: X \rightarrow C$  of  $\mathbf{C}$  there exists a unique  $F$ -homomorphism  $f^\sharp$  extending  $f$ , i.e., such that the following diagram commutes.

$$\begin{array}{ccc} FX^\sharp & \xrightarrow{\varphi_X} & X^\sharp & \xleftarrow{\eta_X} & X \\ Ff^\sharp \downarrow & & \downarrow f^\sharp & \swarrow f & \\ FC & \xrightarrow{\alpha_C} & C & & \end{array}$$

In other words,  $\eta_X$  is a universal arrow for and  $(-)^{\sharp}$  is a left adjoint of the forgetful functor  $U: \text{Alg}(F) \rightarrow \mathbf{C}$ .

Our first observation is the following generalization of the familiar fact that, in the above construction, the  $\Omega$ -term algebra over a set  $X$  is the same as the  $\Omega_X$ -term algebra over the empty set where  $\Omega_X$  is obtained from  $\Omega$  by adding  $X$  as an additional set of constants.

**2.3 Lemma** Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a functor where  $\mathbf{C}$  has finite coproducts and  $X$  a  $\mathbf{C}$ -object. The following are equivalent for a morphism  $\iota_X: X + FI_X \rightarrow I_X$  with components  $\eta_X: X \rightarrow I_X$  and  $\alpha_X: FI_X \rightarrow I_X$ .

- (i)  $(I_X, \iota_X)$  is the initial algebra in  $\mathbf{Alg}(X + F)$ .
- (ii)  $(I_X, \alpha_X)$  is the free algebra in  $\mathbf{Alg}(F)$  over  $X$  with universal morphism  $\eta_X$ .

**Proof** The proof is based on the observation that the structure morphism  $X + FC \xrightarrow{[f, \alpha_C]} C$  of an  $X + F$ -algebra on  $C$  is induced by a pair  $(X \xrightarrow{f} C, (C, \alpha_C))$  with an  $F$ -algebra  $(C, \alpha_C)$ . For a morphism  $f^\sharp: I_X \rightarrow C$  the following then are equivalent:

- $f^\sharp$  is the unique  $X + F$ -homomorphism  $(I_X, \iota_X) \rightarrow (C, [f, \alpha_C])$ ;
- $f^\sharp$  is the unique  $F$ -homomorphism  $(I_X, \alpha_X) \rightarrow (C, \alpha_C)$  with  $f^\sharp \circ \eta_X = f$ .

as can be read off the following diagram (where the  $\mu$ 's and  $\nu$ 's denote the coproduct injections).

$$\begin{array}{ccc}
 FI_X & \xrightarrow{Ff^\sharp} & FC \\
 \downarrow \mu_2 & & \downarrow \nu_2 \\
 X + FI_X & \xrightarrow{X + Ff^\sharp} & X + FC \\
 \downarrow \iota_X & \swarrow \mu_1 & \searrow \nu_1 \\
 & X = X & \\
 \downarrow \eta_X & & \downarrow f \\
 I_X & \xrightarrow{f^\sharp} & C \\
 \downarrow \alpha_X & & \downarrow \alpha_C
 \end{array}$$

◇

In view of the above result the following construction is of particular interest:

**2.4 Construction (of initial algebras)** Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a functor which preserves colimits of chains. Then the *initial*  $F$ -algebra (the free  $F$ -algebra on 0, an initial object of  $\mathbf{C}$ ) can be constructed as a colimit of the chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \dots$$

In fact, if  $l_i: F^i 0 \rightarrow L$  is this colimit,  $Fl_i: F^{i+1} 0 \rightarrow FL$  again is a colimit of (essentially) the same diagram. Thus there exists a unique morphism  $\alpha: FL \rightarrow L$  with  $\alpha \circ Fl_i = l_{i+1}$  for all  $i$ . The  $F$ -algebra  $(L, \alpha)$  then is initial: indeed, let  $(C, \alpha_C)$  be any  $F$ -algebra, then the unique morphism  $0 \rightarrow C$  induces a cocone of the above chain and the morphism  $L \rightarrow C$  induced by this cocone is the unique homomorphism to be found. We leave a more detailed argument for the construction to follow.

Of particular interest when looking for initial or free algebras is the following result.

**2.5 Lemma (Lambek's Lemma)** *If  $(I, \alpha_I)$  is an initial  $F$ -algebra then  $\alpha_I$  is an isomorphism.*

**Proof** Let  $f: (I, \alpha_I) \rightarrow (FI, F\alpha_I)$  be the unique  $F$ -homomorphism. Since, trivially,  $\alpha_I: (FI, F\alpha_I) \rightarrow (I, \alpha_I)$  is an  $F$ -homomorphism the composition  $\alpha_I \circ f$  is an endomorphism of  $(I, \alpha_I)$ , thus the identity, by initiality of this algebra. This implies, since  $f$  is a homomorphism,  $f \circ \alpha_I = F(\alpha_I) \circ F(f) = id_{FI}$ .  $\diamond$

**2.6 Corollary** *There exist no free algebras for the powerset functor  $\mathcal{P}$ .*

**2.7 Construction (of free finitary algebras)** (see [1]): This is an application of the above construction of an initial  $F$ -algebra to the functor  $F_X = X + F(-)$  (see Lemma 2.3 above). Let  $\mathbf{C}$  have countable colimits. Given an object  $X$  in  $\mathbf{C}$  we define an  $\omega$ -chain  $X_i^\sharp$  ( $i < \omega$ ) as follows:

$$0 \xrightarrow{\cdot} X + F0 \xrightarrow{X + F!} X + F(X + F0) \xrightarrow{X + F(X + F!)} (X + F(X + F(X + F0))) \cdots$$

That is:

- $X_0^\sharp = 0$ ,  $X_1^\sharp = X + F0 = X + FX_0^\sharp$  and  $x_{0,1}^\sharp = 0 \xrightarrow{\cdot} X + F0$  is the unique morphism
- $X_{i+1}^\sharp = X + FX_i^\sharp$  and  $x_{i+1,j+1}^\sharp = X + Fx_{i,j}^\sharp$  for all  $i \leq j$

**Claim:** If  $F$ —and thus  $X + F$ —preserve a colimit  $X^\sharp = \text{colim}_{i < \omega} X_i^\sharp$  of the above chain, then  $X^\sharp$  is a free  $F$ -algebra on  $X$ . More detailed: suppose  $(X_i^\sharp \xrightarrow{x_i} X^\sharp)$  is a colimit cocone. If  $X + F$  preserves that colimit we have a unique morphism

$$\varphi_X: X + FX^\sharp \rightarrow X^\sharp \quad \text{with} \quad \varphi_X \circ (X + Fx_i) = x_{i+1}$$

The two components  $\eta_X: X \rightarrow X^\sharp$  and  $\alpha_X: FX^\sharp \rightarrow X^\sharp$  of  $\varphi_X$  form a free  $F$ -algebra on  $X$ .

**Proof** For every  $F$ -algebra  $(C, \alpha_C)$  and any morphism (“assignment to variables”)  $f: X \rightarrow C$  define a cocone of the above chain (*computation of terms*) recursively as follows:

$$f_0^\sharp =! \quad \text{and} \quad f_{i+1}^\sharp = [f, \alpha_C \circ Ff_i^\sharp]$$

Then the (unique) factorization  $X_i^\sharp \xrightarrow{x_i} X^\sharp \xrightarrow{f^\sharp} C = f_i^\sharp$  gives the (unique) homomorphism  $f^\sharp: (X^\sharp, \alpha_X) \rightarrow (C, \alpha_C)$  with  $f = f^\sharp \circ \eta_X$ .  $\diamond$

**2.8 Construction (of free algebras)** (see [1] or [6, IV.3.2]): Let  $\mathbf{C}$  be a cocomplete category. For every endofunctor  $F$  on  $\mathbf{C}$  and every object  $X$  (“of variables”) in  $\mathbf{C}$  define a transfinite chain of objects  $X_i^\sharp$  ( $i$  any ordinal) and connecting morphisms

$$x_{i,j}^\sharp: X_i^\sharp \rightarrow X_j^\sharp \quad (i \leq j)$$

by the following transfinite induction:

- $- X_0^\sharp = 0, X_1^\sharp = X + F0$   
 $- x_{0,1}^\sharp: 0 \xrightarrow{!} X + F0$  the unique morphism
- $- X_{i+1}^\sharp = X + FX_i^\sharp$  for all ordinals  $i$ ,  
 $- x_{i+1,j+1}^\sharp = X + Fx_{i,j}^\sharp$  for all  $i \leq j$
- $- X_j^\sharp = \text{colim}_{i < j} X_i^\sharp$  for all limit ordinals  $j$   
 $- x_{i,j}^\sharp, i < j$  the colimit cocone .

**Claim:** If the above chain construction *stops after  $k$  steps*, i.e, if  $k$  is an ordinal such that  $x_{k,k+1}^\sharp: X_k^\sharp \rightarrow X + FX_k^\sharp$  is an isomorphism, then  $X_k^\sharp$  is a free  $F$ -algebra on  $X$ . More detailed: Denoting the inverse of  $x_{k,k+1}^\sharp$  by  $\varphi_X$  with components

$$\alpha_X: FX_k^\sharp \rightarrow X_k^\sharp \quad \text{and} \quad \eta_X: X \rightarrow X_k^\sharp$$

these form a free  $F$ -algebra on  $X$ .

**Proof** Given an  $F$ -algebra  $(C, \alpha_C)$  and a morphism  $f: X \rightarrow C$  we define a cocone  $f_i^\sharp: X_i^\sharp \rightarrow C$  ( $i$  an ordinal) by transfinite induction as above

(leaving out the limit steps; compatibility  $f_j \circ x_{i,j}^\# = f_i$  ( $i < j$ ) implies that the  $f_i$  ( $i < j$ ) determine  $f_j$  for limit ordinals  $j$ ):

$$f_0^\# = ! \quad \text{and} \quad f_{i+1}^\# = [f, \alpha_C \circ Ff_i^\#]$$

Now  $f_k^\#: X_k^\# \rightarrow C$  is the unique homomorphism with, trivially,  $f = f_k^\# \circ \eta_X$ : By our definitions we have the following commutative diagram (with  $\mu$  the coproduct injection), which proves that  $f_k^\#$  is a homomorphism.

$$\begin{array}{ccc} FX_k^\# & \xrightarrow{\mu} & X + FX_k^\# & \xrightarrow{\varphi_X} & X_k^\# \\ Ff_k^\# \downarrow & & & \searrow f_{k+1}^\# & \downarrow f_k^\# \\ FC & \xrightarrow{\alpha_C} & & & C \end{array}$$

To prove uniqueness of  $f^\#$  assume that  $h: (X_k^\#, \alpha_X) \rightarrow (C, \alpha_C)$  is any homomorphism with  $f = h \circ \eta_X$ . For  $h_n = h \circ x_{1,n}^\#$  for  $n \leq k$  one proves by transfinite induction  $h_n = f_n^\#$  and, thus,  $h = f_k^\#$ .  $\diamond$

We will make use of the following definition:

**2.9 Definition** 1. A functor  $F: \mathbf{C} \rightarrow \mathbf{C}$  is called *variator* provided that the functor  $U: \mathbf{Alg}(F) \rightarrow \mathbf{C}$  has a left adjoint.

2. A functor  $F: \mathbf{C} \rightarrow \mathbf{C}$  is called *constructive variator* provided that, for each  $\mathbf{C}$ -object  $X$ , the free algebra construction stops.

**2.10 Examples** 1. Clearly, every constructive variator is a variator.

2. An endofunctor  $F$  on a cocomplete category which preserves colimits of  $\lambda$ -chains for some infinite cardinal  $\lambda$  is a variator.

In fact, if  $F$  preserves  $X_\lambda^\# = \text{colim}_{j < \lambda} X_j^\#$ , then the free algebra construction stops after  $\lambda$  steps.

3. Every polynomial functor on  $\mathbf{Set}$  is a variator (since  $X \mapsto X^n$  preserves colimits of chains).

4.  $M \otimes -$  ( $M$  a fixed module) is a variator on  $\mathbf{Mod}_R$ , the category of  $R$ -modules for some ring  $R$  (since  $M \otimes -$  preserves colimits, being a left adjoint).

5. The functor  $-\otimes-$  sending a module  $X$  to  $X\otimes X$  is a variator on  $\text{Mod}_R$  since it preserves colimits of chains (see also Example 5.2).
6. The (covariant) powerset functor  $\mathcal{P}$  fails to be a variator on  $\text{Set}$  by Lemmas 2.3 and 2.5.

In important cases each variator is constructive as is shown by the next result.

**2.11 Proposition** 1. *Functors  $F: \text{Set}^{op} \rightarrow \text{Set}^{op}$  and  $F: \text{Set} \rightarrow \text{Set}$ <sup>18</sup> are constructive variators provided that they are variators.*

2. *Every variator on  $\text{Vect}_k$  and on  $\text{Vect}_k^{op}$  is a constructive variator.*

**Proof** Ad 1: It suffices, using the facts that both,  $\text{Set}$  and  $\text{Set}^{op}$ , are wellpowered categories and that, in the cases under consideration, the connecting morphisms obviously are monomorphisms, to prove that, for each set  $X$ , there exists a family  $f_i^\sharp: X_i^\sharp \rightarrow I_X$  of monomorphisms compatible with the connection morphisms.

Choose first an infinite set  $Y_X$  of cardinality  $\text{card}Y_X$  at least the cardinality  $\text{card}X$  of  $X$  and form then the free  $F$ -algebra  $(I_X, \alpha_X)$  over  $Y_X$ . By Lambek's Lemma 2.5 in connection with Lemma 2.3 there is an isomorphism  $\alpha'_X: X + FI_X \rightarrow I_X$  such that, in particular,  $\text{card}FI_X \leq \text{card}I_X$ . According to the choice of  $Y_X$  and by Lemma 3.12 we have moreover  $\text{card}X \leq \text{card}Y_X \leq \text{card}I_X$ . Now the equality  $\text{card}I_X = \text{card}(I_X + FI_X)$  follows, since  $\text{card}Y_X$  is infinite. We thus might chose an isomorphism  $\psi: I_X + FI_X \rightarrow I_X$  (in  $\text{Set}$  and in  $\text{Set}^{op}$  respectively) as well as a monomorphism  $f: X \rightarrow I_X$  (always in  $\text{Set}$ ; in  $\text{Set}^{op}$  provided that  $X \neq \emptyset$ ; but this case is trivial since  $\emptyset_i^\sharp = \emptyset$  for all  $i$ ). Consider the  $F$ -algebra  $(I_X, \psi \circ i_X)$  (where  $i_X: FX \rightarrow X + FX$  denotes the coproduct injection) and  $f$  as an assignment of variables herein. Then the computation of terms  $f_i^\sharp$  will be monomorphic for each ordinal  $i$  (this follows immediately from the definition of the  $f_i^\sharp$ ; for  $\text{Set}^{op}$  because  $g \times Fg$  is surjective whenever  $g$  is, and for  $\text{Set}$  because  $g + Fg$  is injective whenever  $g$ —see Remark 1.3).

The same arguments prove 2., if  $\text{card}$  is replaced by  $\text{dim}$ , the dimension of a vector space.  $\diamond$

Knowing that  $F$  is a variator (or even a special variator like a functor preserving directed colimits—thus colimits of  $\lambda$ -chains) provides further properties of the category  $\text{Alg}(F)$ . We here give two examples.

<sup>18</sup>In fact we will prove a somewhat weaker statement here: we will assume that  $F$  preserves monomorphisms—but see Remark 1.3

The first one concerns the existence of coproducts. Since this construction often is considered to be difficult (see e.g. [11]) we give a somewhat more detailed proof.

**2.12 Proposition** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a varietor in the basic situation where, in addition,  $\mathbf{C}$  is cocomplete. Then  $\mathbf{Alg}(F)$  has all colimits.*

**Proof** By Proposition 1.12  $\mathbf{Alg}(F)$  has coequalizers such that only existence of coproducts has to be proved. The obvious strategy to construct a coproduct of a family  $(C_i, \alpha_i)_I$  in  $\mathbf{Alg}(F)$  is to obtain this as a quotient of the free  $F$ -algebra  $(F, \alpha_F)$  over the  $\mathbf{C}$ -coproduct  $\coprod_I C_i$ . For getting this quotient there are essentially two options: either to provide a suitable pair of homomorphisms  $r, s: (D, \alpha_D) \rightarrow (F, \alpha_F)$  such that its coequalizer is the desired coproduct, or to provide a source  $(F, \alpha_F) \xrightarrow{f_i} (D_j, \alpha_{D_j})$  whose regular factorization  $f_j = (F, \alpha_F) \xrightarrow{e} (C, \alpha_C) \xrightarrow{m_j} (D_j, \alpha_{D_j})$  has  $(C, \alpha_C)$  as the coproduct one is looking for. We are to make use of the second option here (for applying the first one which wouldn't need any further assumptions on  $F$  see e.g. [9]).

To fix notation let  $\mu_i: C_i \rightarrow \coprod C_i$  be the coproduct injections in  $\mathbf{C}$  and  $\eta: \coprod C_i \rightarrow F$  the universal arrow. Let  $g_j: \coprod C_i \rightarrow U(D_j, \alpha_{D_j})$  the source of all those  $\mathbf{C}$ -morphisms such that, for each  $i \in I$ , there is an  $F$ -algebra homomorphism  $h_{ij}: (C_i, \alpha_i) \rightarrow (D_j, \alpha_{D_j})$  with  $h_{ij} = g_j \circ \mu_i$ .

Now each  $g_j$  determines an  $F$ -algebra homomorphism  $f_j: (F, \alpha_F) \rightarrow (D_j, \alpha_{D_j})$  with  $f_j \circ \eta = g_j$  by the universal property of  $\eta$ , and the source of these homomorphisms has a regular factorization  $(F, \alpha_F) \xrightarrow{e} (C, \alpha_C) \xrightarrow{m_j} (D_j, \alpha_{D_j})$  by Proposition 1.12. The situation is visualized by the following commutative diagram.

$$\begin{array}{ccccc}
 C_i & \xrightarrow{\mu_i} & \coprod C_i & \xrightarrow{\eta} & F & \xrightarrow{e} & D \\
 & \searrow & \downarrow g_j & & \downarrow f_j & & \downarrow m_j \\
 & & & & & & D_j \\
 & \searrow h_{ij} & & & & & \\
 & & & & & & 
 \end{array}$$

Observe now that, for each  $i \in I$ , the  $\mathbf{C}$ -morphism  $e \circ \eta \circ \mu_i$  lifts to an  $F$ -algebra homomorphism  $\nu_i: (C_i, \alpha_i) \rightarrow (D, \alpha_D)$  since the source  $(m_j)$ —being a monosource—is initial (see Proposition 1.5 in conjunction with Proposition 1.12).

To prove that the sink  $\nu_i: (C_i, \alpha_i) \rightarrow (C, \alpha_C)$  is in fact a coproduct now is trivial: given any sink  $l_i: (C_i, \alpha_i) \rightarrow (E, \alpha_E)$ , it induces, considered

as a sink of  $\mathbf{C}$ -morphisms, a  $\mathbf{C}$ -morphism  $[l_i]: \coprod_I C_i \rightarrow E$  which then is  $g_j$  for some  $j$ . Thus,  $l_i = m_j \circ \nu_i$  for each  $i \in I$  as required. Obviously then  $m_j$  is unique with this property.  $\diamond$

For possibly unfamiliar notions appearing in the next theorem we refer to the appendix.

**2.13 Theorem** *Let  $F$  be an accessible endofunctor on  $\mathbf{Set}$ <sup>19</sup>. Then  $\mathbf{Alg}(F)$  is a locally presentable category. Each  $\lambda$ -presentable  $F$ -algebra is a homomorphic image of a free  $F$ -algebra on a set  $X$  with  $\text{card}X < \lambda$ .*

**Proof**  $\mathbf{Alg}(F)$  is cocomplete by the previous theorem. Since  $F$  preserves  $\lambda$ -directed colimits, so does the underlying functor of  $\mathbf{Alg}(F)$  by Proposition 1.8; by Example 2.10  $U$  has a left adjoint. By Lemma 5.5 the free  $F$ -algebras on sets of cardinality less than  $\lambda$  are finitely presentable. They clearly form a strong generator.

Let now  $(C, \alpha_C)$  be  $\lambda$ -presentable. Consider  $(C, \alpha_C)$  as a  $\lambda$ -directed colimit of all  $(\langle X \rangle, \alpha_{\langle X \rangle})$  with  $X \subset C$  and  $\text{card}X < \lambda$  (see Proposition 1.16). Then, since  $(C, \alpha_C)$  is  $\lambda$ -presentable, the identity  $id_C$  factorizes over one such  $(\langle X \rangle, \alpha_{\langle X \rangle})$ ; thus,  $(C, \alpha_C) = (\langle X \rangle, \alpha_{\langle X \rangle})$ . Trivially,  $(\langle X \rangle, \alpha_{\langle X \rangle})$  is a quotient of the free  $F$ -algebra over  $X$ .  $\diamond$

**Cofree coalgebras** are the corresponding dualization of free algebras. A cofree  $F$ -coalgebra (with respect to a functor  $F: \mathbf{C} \rightarrow \mathbf{C}$ ) on a  $\mathbf{C}$ -object  $X$  (“of colours”) is a coalgebra  $\psi_X: X_{\#} \rightarrow FX_{\#}$  together with a (“colouring”) morphism  $\rho_X: X_{\#} \rightarrow X$  having the universal property that given an  $F$ -coalgebra  $(C, \alpha_C)$  and a morphism  $f: C \rightarrow X$  of  $\mathbf{C}$  there exists a unique  $F$ -coalgebra homomorphism  $f_{\#}$  extending  $f$ , i.e., such that the diagram

$$\begin{array}{ccc}
 C & \xrightarrow{\alpha_C} & FC \\
 \swarrow f & \downarrow f_{\#} & \downarrow Ff_{\#} \\
 X & \xleftarrow{\rho_X} X_{\#} & \xrightarrow{\psi_X} FX_{\#}
 \end{array}$$

commutes. In other words,  $\rho_X$  is a couniversal arrow of the forgetful functor  $U: \mathbf{Coalg}(F) \rightarrow \mathbf{C}$ .

<sup>19</sup>Set here could be replaced by an arbitrary locally presentable category: simply replace in the proof “sets of cardinality less than  $\lambda$ ” by “objects of the respective generator”

**2.14 Definition** An endofunctor  $F: \mathbf{C} \rightarrow \mathbf{C}$  is called a *covariator* provided that a cofree  $F$ -coalgebra exists on every  $\mathbf{C}$ -object.

This terminology is justified by the following remark based on Lemma 1.1 and Lemma 2.3.

**2.15 Remark** The following are equivalent for any  $F: \mathbf{C} \rightarrow \mathbf{C}$ :

- $F$  is a covariator.
- $U: \mathbf{Coalg}(F) \rightarrow \mathbf{C}$  has a right adjoint.
- $U: \mathbf{Alg}(F^{op}) \rightarrow \mathbf{C}^{op}$  has a left adjoint.
- $F^{op}$  is a variator.

In case  $\mathbf{C}$  has finite products, another equivalent condition is:

- For every object  $X$  in  $\mathbf{C}$  the functor  $F^X = X \times F$  has a terminal (= final) coalgebra.

Dualization of the free-algebra construction above gives the following

**2.16 Construction (of cofree coalgebras)** Let  $\mathbf{C}$  be a complete category. For every endofunctor  $F$  on  $\mathbf{C}$  and every object  $X$  (“of colours”) in  $\mathbf{C}$  define a transfinite cochain of objects  $X_{\#}^i$  ( $i$  any ordinal) and connecting morphisms  $x_{\#}^{i,j}: X_{\#}^i \rightarrow X_{\#}^j$  ( $i \geq j$ ) as follows (where  $1$  denotes a terminal object of  $\mathbf{C}$ ):

- –  $X_{\#}^0 = 1$ ,  $X_{\#}^i = X \times F1$   
–  $x_{\#}^{1,0}: X \times F1 \xrightarrow{!} 1$  the unique morphism
- –  $X_{\#}^{i+1} = X \times F X_{\#}^i$  for all ordinals  $i$ ,  
–  $x_{\#}^{i+1,j+1} = X \times F x_{\#}^{i,j}$  for all  $i \geq j$
- –  $X_{\#}^j = \lim_{i < j} X_{\#}^i$  for all limit ordinals  $j$   
–  $x_{\#}^{j,i}$ ,  $i < j$  the limit cone.

If this cochain construction *stops after  $k$  steps*, i.e., if  $k$  is an ordinal such that  $x_{\sharp}^{k,k+1}: X \times FX_{\sharp}^k \rightarrow X_{\sharp}^k$  is an isomorphism, then  $X_{\sharp}^k$  is a cofree  $F$ -coalgebra on  $X$ . More detailed: Denoting the inverse of  $x_{\sharp}^{k,k+1}$  by  $\varphi_X: X_{\sharp}^k \rightarrow X \times FX_{\sharp}^k$  with components

$$\alpha_X: X_{\sharp}^k \rightarrow FX_{\sharp}^k \quad \text{and} \quad \rho_X: X_{\sharp}^k \rightarrow X$$

these form a cofree  $F$ -coalgebra on  $X$ . For an  $F$ -coalgebra  $(C, \alpha_C)$  and a morphism  $f: C \rightarrow X$  the extension  $f_{\sharp}$  of  $f$  is the  $k$ -th member of the cocone  $f_{\sharp}^i: C \rightarrow X_{\sharp}^i$  which is defined by transfinite induction (leaving out the limit steps) as follows:

$$f_{\sharp}^0 = ! \quad \text{and} \quad f_{\sharp}^{i+1} = \langle f, Ff_{\sharp}^i \circ \alpha_C \rangle.$$

By the dual of Examples 2.10.1 the following holds:

**2.17 Corollary** *An endofunctor  $F$  on a complete category which preserves limits of  $\lambda$ -cochains for some infinite cardinal  $\lambda$  is a covariator.*

**2.18 Examples** 1. Polynomial functors on **Set** are covariators (here  $\lambda = \omega$ ).

The polynomial functor  $F$  with  $FX = A \times X$  has the coalgebra of streams over  $A$  as its final (= cofree over 1) coalgebra (see Exxample 1.2).

2. *Generalized polynomial functors on **Set***, i.e., functors  $FY = \sum_{i \in I} Y^{C_i}$  for a given family  $(C_i)_{i \in I}$  of (not necessarily finite) sets are covariators (again,  $\lambda = \omega$ ).

**2.19 Remark** A criterion of quite a different kind for a functor to be a covariator will be given in Section 4. This will imply, e.g., that tensoring by a fixed module and "tensor squaring" are covariators on  $\mathbf{Mod}_R$ .

## 3 Varieties and covarieties

### 3.A Equations and varieties

The motivation for the following definitions stems from classical universal algebra; it will be shown in due course how the classical notions fit into the general concept. We will use the notation  $X_i^{\sharp}$  and  $f_i^{\sharp}$  as in 2.8.

**3.1 Definition** Let  $F$  be an endofunctor of a cocomplete category  $\mathbf{C}$ .

1. An *equation arrow over  $X$*  is a regular epimorphism  $e: X_i^\sharp \rightarrow E$  for some ordinal  $i$ . An  $F$ -algebra  $(C, \alpha_C)$  is said to *satisfy  $e$*  provided that for every morphism  $f: X \rightarrow C$  the morphism  $f_i^\sharp$  factorizes through  $e$ :

$$\begin{array}{ccc} X_i^\sharp & \xrightarrow{e} & E \\ & \searrow f_i^\sharp & \downarrow \\ & & C \end{array}$$

2. For any class  $\mathcal{E}$  of equation arrows,  $\mathbf{Alg}(F, \mathcal{E})$  denotes the full subcategory of  $\mathbf{Alg}(F)$  spanned by all  $F$ -algebras satisfying every  $e \in \mathcal{E}$ . Such categories will be called *equational categories (of  $F$ -algebras)* over  $\mathbf{C}$ .
3. An equational category  $\mathbf{Alg}(F, \mathcal{E})$  over  $\mathbf{C}$  will be called a *variety (of  $F$ -algebras)* over  $\mathbf{C}$  provided that the underlying functor  $U^\mathcal{E} = U|_{\mathbf{Alg}(F, \mathcal{E})}: \mathbf{Alg}(F, \mathcal{E}) \rightarrow \mathbf{C}$  has a left adjoint.

As mentioned above the guiding example for these concepts is the case  $\mathbf{C} = \mathbf{Set}$ . Here the term objects are the usual sets of terms in view of their traditional recursive definition, while for each assignment of variables  $f: X \rightarrow C$  the map  $f_i^\sharp$  calculates the  $(C, \alpha_C)$ -interpretations of terms correspondingly.

Calling now a pair of elements  $s, t \in X_i^\sharp$  (equivalently a pair of maps  $\bar{s}, \bar{t}: 1 \rightarrow X_i^\sharp$ ) an *equation*, this equation determines an equation arrow by taking a coequalizer  $e: X_i^\sharp \rightarrow E$  of  $(\bar{s}, \bar{t})$ . An algebra  $(C, \alpha_C)$  then satisfies the equation  $(s, t)$  in the usual sense iff  $f_i^\sharp(s) = f_i^\sharp(t)$  for all assignments of variables  $f$ ; but this is equivalent to the factorization of  $f_i^\sharp$  over  $e$ . Conversely, every surjection  $e: X_i^\sharp \rightarrow E$  determines the set of equations *ker* $e$  and  $(C, \alpha_C)$  satisfies these equations in the usual sense iff  $(C, \alpha_C)$  satisfies  $e$ . Thus, equational categories over  $\mathbf{Set}$  are precisely the equationally defined categories of algebras.

By transfinite induction one proves the following

**3.2 Lemma** *Homomorphisms of  $F$ -algebras satisfy computation of terms, i.e., given a homomorphism  $h: (C, \alpha_C) \rightarrow (D, \alpha_D)$  and an assignment of*

variables  $f: X \rightarrow C$  then the following diagram commutes for all ordinals  $i$ :

$$\begin{array}{ccc} X_i^\sharp & \xrightarrow{f_i^\sharp} & C \\ & \searrow (hf)_i^\sharp & \downarrow h \\ & & D \end{array}$$

We provide a non-trivial example of an equational category (in fact a variety) as follows:

**3.3 Example** The power-set functor  $\mathcal{P}$  on **Set** is not a variator. However, we can consider equational categories of  $\mathcal{P}$ -algebras. Complete semilattices are an example. In fact, the join-operation of a complete (upper) semilattice  $C$  is an arrow  $\alpha_C: \mathcal{P}C \rightarrow C$  satisfying (i)  $\alpha_C\{x\} = x$ , and (ii)  $\alpha_C \bigcup M_i = \alpha_C\{\alpha_C M_i \mid i \in I\}$  for any collection  $M_i$  in  $\mathcal{P}C$ . Conversely, every  $\mathcal{P}$ -algebra satisfying (i) and (ii) is a (join operation of a unique) complete semilattice. Now (i) can be expressed by the equation arrow  $e: X_2^\sharp \rightarrow E$  where  $X = \{x\}$  and  $e$  just merges  $x$  and  $\{x\}$ , whereas (ii) corresponds to the equation arrows  $f: X_3^\sharp \rightarrow F$  where  $X$  is an arbitrary set and, for a given collection  $M_i$  in  $\mathcal{P}X$ ,  $f$  merges  $\bigcup M_i$  with  $\{M_i \mid i \in I\}$ . The homomorphisms are precisely the functions preserving all joins (see Examples 1.2).

In case  $F$  is a variator varieties of  $F$ -algebras often can be described as *orthogonality classes* as follows. Denote, for some class  $\mathcal{M}$  of morphisms in a category  $\mathbf{C}$ , by  $\mathcal{M}^\perp$  the full subcategory of  $\mathbf{C}$  spanned by all objects  $C$  *orthogonal to  $\mathcal{M}$*  in the sense that, for each  $m: M \rightarrow M' \in \mathcal{M}$ , each morphism  $f: M \rightarrow C$  factorizes uniquely over  $m$ :

$$\begin{array}{ccc} M & \xrightarrow{m} & M' \\ & \searrow f & \downarrow \text{---} \\ & & C \end{array}$$

**3.4 Definition** Let  $F$  be a variator. Regular epimorphisms  $q: (F, \alpha_F) \rightarrow (D, \alpha_D)$  in  $\text{Alg}(F)$  with a free algebra  $(F, \alpha_F)$  are called *equational homomorphisms*. An  $F$ -algebra  $(C, \alpha_C)$  is said to *satisfy* the equational homomorphism  $q$  provided  $(C, \alpha_C)$  is orthogonal to  $q$ .

Orthogonality classes of collections of equational homomorphisms will be called *weak varieties* over  $\mathbf{C}$  provided that they have free algebras.

**3.5 Lemma** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a varietor in the basic situation.*

1. *Every variety is a weak variety, i.e., if  $\mathcal{E}$  is class of equation arrows in  $\mathbf{C}$ , then there exists a class  $\mathcal{E}_h$  of equational homomorphisms in  $\text{Alg}(F)$  such that  $\text{Alg}(F, \mathcal{E}) = \mathcal{E}_h^\perp$ .*
2. *If  $F$  is even a constructive varietor, then every weak variety is a variety, i.e., if  $\mathcal{E}_h$  is class of equational homomorphisms in  $\text{Alg}(F)$ , then there exists a class  $\mathcal{E}$  of equation arrows in  $\mathbf{C}$  such that  $\mathcal{E}_h^\perp = \text{Alg}(F, \mathcal{E})$ .*

**Proof** 1. Let  $e: X_i^\sharp \rightarrow E \in \mathcal{E}$  be a coequalizer of  $r, s: Y \rightarrow X_i^\sharp$  and  $(F, \alpha_F)$  be free over  $X$  with universal arrow  $\eta$ . Denote by  $r', s': F' \rightarrow F$  the  $F$ -homomorphisms with  $r' \circ \eta' = \eta_i^\sharp \circ r$  and  $s' \circ \eta' = \eta_i^\sharp \circ s$  respectively, where  $\eta': Y \rightarrow F'$  is the  $U$ -universal arrow for  $Y$ , and let  $q_e: (F, \alpha_F) \rightarrow (D, \alpha_D)$  be a coequalizer of  $r', s'$ . Then an  $F$ -algebra  $(C, \alpha_C)$  satisfies  $e$  (in the sense of Definition 3.1) iff it satisfies  $q_e$  (in the sense of Definition 2.9): one only has to make use of the fact that, for any assignment of variables  $f$  from  $X$  in the algebra  $(C, \alpha_C)$  with homomorphic extension  $f^\sharp$  to the free algebra  $(F, \alpha_F)$ , one has  $f^\sharp \circ \eta_i^\sharp = f_i^\sharp$  by Lemma 3.2.

2. is trivial by the proof of Lemma 2.11.  $\diamond$

### Closure properties of varieties

**3.6 Proposition** *Let  $\mathbf{K}$  be a subcategory of  $\text{Alg}(F)$  of the form  $\text{Alg}(F, \mathcal{E})$  or  $\mathcal{E}_h^\perp$ . Then  $\mathbf{K}$  is closed in  $\text{Alg}(F)$  under*

1. *monosources (in particular, thus, under subalgebras and all limits which might exist);*
2. *homomorphic images carried by split epimorphisms in  $\mathbf{C}$ .*

**Proof** We provide the arguments for the case  $\mathbf{K} = \text{Alg}(F, \mathcal{E})$ ; the second case is proved analogously. 1. is trivial by an obvious diagonal fill-in argument. 2. Let  $r: (C, \alpha_C) \rightarrow (D, \alpha_D)$  be a homomorphism with coretraction  $s$  in  $\mathbf{C}$ , where  $(C, \alpha_C)$  satisfies the equation arrow  $e: X_i^\sharp \rightarrow E$ . Given  $f: X \rightarrow D$ , one has  $r \circ (sf)_i^\sharp = (r \circ s \circ f)_i^\sharp = f_i^\sharp$ . Thus, since  $(s \circ f)_i^\sharp$  factorizes through  $e$  so does  $f_i^\sharp$ .  $\diamond$

As a corollary one obtains

**3.7 Proposition** *Let  $F: \mathbb{C} \rightarrow \mathbb{C}$  be in the basic situation and let  $\mathcal{E}$  be a class of equations. Then*

1.  $\text{Alg}(F, \mathcal{E})$  is regular epi-reflexive in  $\text{Alg}(F)$ ;
2.  $\text{Alg}(F, \mathcal{E})$  has coequalizers;
3.  $\text{Alg}(F, \mathcal{E})$  is closed in  $\text{Alg}(F)$  under regular factorizations<sup>20</sup>;
4.  $\text{Alg}(F, \mathcal{E})$  has a regular factorization structure, created by its underlying functor and both, regular epimorphisms and monosources, are preserved and reflected by the embedding  $\text{Alg}(F, \mathcal{E}) \hookrightarrow \text{Alg}(F)$ .

*The same statements hold for weak varieties.*

**Proof** The reflection arrow for an  $F$ -algebra  $(C, \alpha_C)$  in  $\text{Alg}(F)$  is given by the regular factorization of the source of all  $F$ -homomorphisms from  $(C, \alpha_C)$  into algebras in  $\text{Alg}_{\mathbb{C}}(F, \mathcal{E})$ . As a faithful right adjoint the embedding clearly preserves and reflects monosources; it preserves regular epimorphisms since coequalizers in  $\text{Alg}(F, \mathcal{E})$  are reflections of those in  $\text{Alg}(F)$ . It follows that the regular factorization in  $\text{Alg}(F)$  of a regular epimorphism from  $\text{Alg}(F, \mathcal{E})$  is trivial. Altogether this proves 2. and 3.

The proof for weak varieties is analogous.  $\diamond$

**3.8 Proposition** *Let  $F$  be an endofunctor of  $\text{Set}$ . Then every equationally defined category  $\text{Alg}(F, \mathcal{E})$  is regular epi-reflexive in  $\text{Alg}(F)$  and closed under homomorphic images. It has regular factorizations of sources and these are created by its underlying functor.*

**3.9 Corollary** *Let  $F$  be a  $\text{Set}$ -functor preserving  $\lambda$ -directed colimits (for some regular cardinal  $\lambda$ ). Then  $\text{Alg}(F, \mathcal{E})$  is closed in  $\text{Alg}(F)$  under  $\lambda$ -directed colimits.*

**Proof** If we could assume that the equations in  $\mathcal{E}$  were all over sets of variables of cardinality less than  $\lambda$ , we might proceed as follows (which allows for a straightforward generalization to more general base categories): Given a directed colimit  $l_k: (C_k, \alpha_k) \rightarrow (C, \alpha_C)$  with each  $(C_k, \alpha_k)$  in  $\text{Alg}(F, \mathcal{E})$  and an assignment to variables  $f: X \rightarrow C$ , where  $e: X_i^{\sharp} \rightarrow E \in \mathcal{E}$  and  $\text{card}X < \lambda$ , the map  $f$  factorizes as  $f = l_k \circ g$  for some  $k$  since,

<sup>20</sup>I.e., if  $f_i = (C, \alpha_C) \xrightarrow{m_i} (D, \alpha_D) \xrightarrow{e} (C_i, \alpha_i)$  is a regular factorization in  $\text{Alg}(F)$  of an  $\text{Alg}(F, \mathcal{E})$ -source  $(f_i)$  then  $(D, \alpha_D)$  belongs to  $\text{Alg}(F, \mathcal{E})$ .

in the given situation, directed colimits are preserved by the underlying functor (see Corollary 1.9). Then, by Lemma 3.2, also each  $f_i^\sharp$  factorizes as  $f_i^\sharp = l_k g_i^\sharp$ . Since  $g_i^\sharp$  factorizes through  $e$  so does  $f_i^\sharp$ .

As for the general case we resort to the concrete construction of directed colimits in **Set** as quotients of disjoint sums: If  $(d_{i,j} : (D_i, \alpha_i) \rightarrow (D_j, \alpha_j))_{i \leq j}$  is a  $\lambda$ -directed diagram of  $F$ -algebras, its colimit  $(D, \alpha_D)$  is constructed as in **Set**: elements of  $D$  are equivalence classes  $[x]$  with  $x \in D_i$  where  $x$  is equivalent to  $y \in D_j$  iff, for some upper bound  $k$  of  $i$  and  $j$ ,  $d_{i,k}(x) = d_{j,k}(y)$ . We are going to present  $(D, \alpha_D)$  as a quotient of a subalgebra  $(C, \alpha_C)$  of  $\prod_{k \in I} (D_k, \alpha_k)$  which, by the closure properties established above, will prove the claim.

1. Construct, for each  $i \in I$ , the multiple equalizer (intersection of equalizers)  $m_i : E_i \rightarrow \prod_I D_k$  of the family of pairs

$$\left( \prod D_k \xrightarrow{\pi_i} D_i \xrightarrow{d_{i,j}} D_j, \prod D_k \xrightarrow{p_j} D_j \right)_{j \geq i}$$

These then are subalgebras of the product  $\prod_I (D_k, \alpha_k)$ .

2. By construction, the  $E_i$  form a directed (over  $I$ ) diagram (of monomorphisms) such that the homomorphisms

$$p_i : E_i \xrightarrow{m_i} \prod_I D_k \xrightarrow{\pi_i} D_i$$

are a natural transformation and, thus, induce a homomorphism (!)  
 $f : \text{colim}_I E_i \rightarrow \text{colim}_I D_i$ .

3. One can identify  $\text{colim}_I E_i$  as consisting of those tuples  $(x_k) \in \prod D_k$  for which there exists  $i \in I$  such that  $x_j = d_{i,j}(x_i)$  for all  $j \geq i$  and the map  $f$  as sending  $(x_k)$  to  $[x_i]$  whenever  $x_j = d_{i,j}(x_i)$  for all  $j \geq i$ . This shows that  $f$  is surjective.  $\diamond$

Concerning coproducts in a variety  $\text{Alg}(F, \mathcal{E})$ , one clearly knows they exist, provided that  $F$  is a variator in a basic basic situation over a cocomplete category (since then  $\text{Alg}(F, \mathcal{E})$  is reflective in  $\text{Alg}(F)$ , and  $\text{Alg}(F)$  is cocomplete by Proposition 2.12). However, one even can prove—by simply referring to Proposition 3.7 instead of Proposition 1.12—in the proof of Proposition 2.12

**3.10 Proposition** *Let  $\text{Alg}(F, \mathcal{E})$  be a variety where  $F : \mathbf{C} \rightarrow \mathbf{C}$  is in a basic situation and  $\mathbf{C}$  is cocomplete. Then  $\text{Alg}(F, \mathcal{E})$  is cocomplete.*

Essentially as a corollary to these results we obtain (see Theorem 5.3):

**3.11 Theorem** *Every variety  $\text{Alg}(F, \mathcal{E})$  over  $\text{Set}$  is cocomplete. If  $F$  is even accessible,  $\text{Alg}(F, \mathcal{E})$  is a locally presentable category.*

The following lemma supports the intuition to think of the universal map of a free algebra as an “embedding” of generators.

**3.12 Lemma** *Let  $\text{Alg}(F, \mathcal{E})$  be a variety over  $\text{Set}$  or  $\text{Set}^{op}$ . For any set  $X$  the  $U^{\mathcal{E}}$ -universal map  $\eta_X$  from  $X$  into the free algebra over  $X$  is a monomorphism provided that  $\text{Alg}(F, \mathcal{E})$  has an algebra  $(C, \alpha_C)$  where  $C$  is a set with at least two elements.*

**Proof** Since  $\text{Alg}(F, \mathcal{E})$  has products which are created by  $U^{\mathcal{E}}$ , the algebra  $(C, \alpha_C)^X$  has  $C^X$  as its underlying set. Now choose any monomorphism  $f: X \rightarrow C^{X^{21}}$ . Since  $f$  factorizes as  $f = f^\# \circ \eta_X$ ,  $\eta_X$  is a monomorphism.  $\diamond$

### How to characterize varieties

For weak varieties we have the following characterization which, for the case  $\mathbf{C} = \text{Set}$ , gives (in connection with Proposition 3.14) the famous Birkhoff Variety Theorem.

**3.13 Theorem ([3])** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be a variator in the basic situation. The following are equivalent for any full isomorphism-closed subcategory  $\mathbf{K}$  of  $\text{Alg}(F)$ :*

- (i)  $\mathbf{K}$  is closed under
  - (a) monosources (in particular, thus, under subalgebras and all limits which might exist);
  - (b) homomorphic images carried by split epimorphisms in  $\mathbf{C}$ .
- (ii)  $\mathbf{K}$  is a weak variety.

*If the variator  $F$  is a constructive variator, (i) is even equivalent to*

- (ii')  $\mathbf{K}$  is a variety.

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<sup>21</sup>Note that this is, for  $\text{Set}$ , an injection  $X \rightarrow C^X$ , while it is, for  $\text{Set}^{op}$ , a surjection  $X \cdot C \rightarrow X$ .

**Proof** (ii) implies (i) by Proposition 3.6. To show the converse observe first that  $\mathbf{K}$  is regularly epi-reflexiv in  $\mathbf{Alg}(F)$  by Proposition 3.7. Let now  $\mathcal{E}_h$  be the class of all reflection-arrows of free algebras. We claim  $\mathbf{K} = \mathbf{Alg}(F, \mathcal{E})$ . Trivially each algebra in  $\mathbf{K}$  satisfies all reflection arrows. Conversely, if  $(C, \alpha_C)$  satisfies all reflection arrows, it does so in particular for the reflection  $r$  of the free algebra  $(C^\#, \alpha^\#)$  over  $C$ . But then the homomorphic extension  $id_C^\#$  of the identity of  $C$  factorizes as  $id_C^\# = g \circ r$ . This shows that  $g$  is—as a  $\mathbf{C}$ -morphism— a retraction. Thus,  $(C, \alpha_C)$  is a split epi-carried quotient of the  $\mathbf{K}$ -reflection of  $(C^\#, \alpha^\#)$ , hence belonging to  $\mathbf{K}$  by hypotheses.  $\diamond$

The following result complements the above in that it shows that **Set**-functors often only need a *set* of equations in order to describe their varieties (the above proof provided us only with a *proper class* of equation arrows).

**3.14 Proposition** *Let  $F$  be a **Set**-functor preserving  $\lambda$ -directed colimits. Then the following hold for any variety  $\mathbf{Alg}(F, \mathcal{E})$ .*

1. *There exists a set of equations  $\mathcal{E}'$ , each in less than  $\lambda$  variables only<sup>22</sup>, such that  $\mathbf{Alg}(F, \mathcal{E}) = \mathbf{Alg}(F, \mathcal{E}')$ .*
2. *There exists an equational homomorphism  $e: \Lambda^\# \rightarrow E$  on the free  $F$ -algebra  $\Lambda^\#$  on a set  $\Lambda$  with  $\text{card}\Lambda = \lambda$  such that  $\mathbf{Alg}(F, \mathcal{E}) = \{e\}^\perp$ .*
3.  *$\mathbf{Alg}(F, \mathcal{E})$  is locally  $\lambda$ -presentable.*

**Proof** Ad 1. While by the previous proof one has (with reflection morphisms denoted by  $r$ )

$$\mathbf{Alg}(F, \mathcal{E}) = \{r_{(C, \alpha_C)} \mid (C, \alpha_C) \in \mathbf{Alg}(F)\}^\perp$$

Since  $F$  preserves  $\lambda$ -directed colimits we already know better by Proposition 3.8 and Corollary 3.9 in view of Remark 5.4:

$$\mathbf{Alg}(F, \mathcal{E}) = \{r_{(C, \alpha_C)} \mid (C, \alpha_C) \text{ } \lambda\text{-presentable}\}^\perp$$

Thus  $\mathbf{Alg}(F, \mathcal{E})$  already is presented as an orthogonality class of a *set* of regular epimorphisms, whose domains, however, aren't free in general. This

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<sup>22</sup>I.e., the equations from  $\mathcal{E}'$  all belong to squares  $(X_i^\#)^2$  with sets  $X$  of cardinality less than  $\lambda$ .

can be achieved by showing that an  $F$ -algebra is orthogonal to the reflection morphisms of all  $\lambda$ -presentable  $F$ -algebras, provided that it is orthogonal to the reflection morphisms of all free  $F$ -algebras on sets  $X$  of cardinality less than  $\lambda$ <sup>23</sup>, i.e., by proving:

$$\text{Alg}(F, \mathcal{E}) = \{r_{X^\sharp} \mid \text{card}X < \lambda\}^\perp$$

Thus, let  $(D, \alpha_D)$  be orthogonal to all  $r_{X^\sharp}$  with  $\text{card}X < \lambda$  and  $(P, \alpha_P)$   $\lambda$ -presentable. By Theorem 2.13 there exists a surjective homomorphism  $e: X^\sharp \rightarrow (P, \alpha_P)$ . In the pushout

$$\begin{array}{ccc} X^\sharp & \xrightarrow{r_{X^\sharp}} & RX^\sharp \\ e \downarrow & & \downarrow e' \\ (P, \alpha_P) & \xrightarrow{r'} & (P', \alpha_{P'}) \end{array}$$

$e'$  is surjective since  $e$  is. Thus,  $(P', \alpha_{P'})$  belongs to  $\text{Alg}(F, \mathcal{E})$  since this subcategory is closed under homomorphic images. It is now easy to check that (1)  $r'$  is an  $\text{Alg}(F, \mathcal{E})$ -reflection arrow for  $(P, \alpha_P)$  and (2) an  $F$ -algebra is orthogonal to  $r'$  iff it is orthogonal to  $r_{X^\sharp}$ .

Ad 2. This is an easy consequence of 1: represent  $\Lambda$  as a directed colimit of its subsets  $U$  of cardinality  $< \lambda$ . There results a representation of  $\Lambda^\sharp$  as a directed colimits of free  $F$ -algebras  $U^\sharp$  on these subsets of  $\Lambda$ , and  $\text{Alg}(F, \mathcal{E})$  is the orthogonality class of their reflection morphisms  $r_{U^\sharp}$ . The directed colimit of these reflection morphisms  $r_{U^\sharp}$  is a regular epimorphism  $e$  with domain  $\Lambda^\sharp$  and orthogonality to  $e$  is equivalent to orthogonality to every  $r_{U^\sharp}$ .

3. follows from Theorem 5.4.  $\diamond$

**Problem.** We do not know whether the above theorem can be generalized to any locally presentable category (instead of **Set**).

We end this section by relating (weak) varieties to monadic categories.

**3.15 Proposition** *Let  $F: \mathbf{C} \rightarrow \mathbf{C}$  be an endofunctor on a cocomplete category  $\mathbf{C}$  and let  $\mathbf{K}$  be a subcategory of  $\text{Alg}(F)$  admitting free algebras and being closed under monosources and quotients carried by split epimorphisms. Then  $\mathbf{K}$  is monadic over  $\mathbf{C}$  (by means of the restriction of  $U$ ).*

<sup>23</sup>In fact, we here only need  $X$  to vary over a representative collection of these sets.

In particular, varieties and weak varieties over a cocomplete category are monadic.

**Proof** By Beck's Theorem we have to verify that the underlying functor of  $\mathbf{K}$  creates split coequalizers. Since  $\mathbf{K}$  is closed under quotients splitting in  $\mathbf{C}$ , it suffices to prove that  $U$  creates absolute coequalizers.

Given  $f_1, f_2: (C, \alpha_C) \rightarrow (D, \alpha_D)$  and an absolute coequalizer  $q: D \rightarrow Q$  in  $\mathbf{C}$ , the left hand square of the following diagram commutes (since the  $f_i$  are homomorphisms).

$$\begin{array}{ccccc}
 FC & \xrightarrow{Ff_i} & FD & \xrightarrow{Fq} & FQ \\
 \alpha_C \downarrow & & \downarrow \alpha_D & & \downarrow \alpha_Q \\
 C & \xrightarrow{f_i} & D & \xrightarrow{q} & Q
 \end{array}$$

Then there exists a unique (dotted) arrow  $\alpha_Q$  making the right hand square commute, since the top row is a coequalizer diagram (the coequalizer  $(q, Q)$  is absolute). That  $(c, (Q, \alpha_Q))$  then is a coequalizer of  $f_1, f_2$  in  $\mathbf{Alg}(F)$  is checked easily.  $\diamond$

**3.16 Theorem ([3])** *Monadic categories over a cocomplete category  $\mathbf{C}$  are precisely the categories concretely equivalent to varieties over  $\mathbf{C}$ .*

**Proof** In view of the results above it suffices to show that, for any monad  $\mathbb{T} = (T, \eta, \mu)$  on a cocomplete category  $\mathbf{C}$ , the Eilenberg-Moore category  $\mathbf{C}^{\mathbb{T}}$  of  $\mathbb{T}$ -algebras coincides with the subcategory  $\mathbf{Alg}(T, \mathcal{E})$  of  $\mathbf{Alg}(T)$  for a suitable class  $\mathcal{E}$  of equation arrows. For doing so consider, for every  $\mathbf{C}$ -object  $X$ , the coproduct

$$X \xrightarrow{m_{i+1}} X + TX_i^{\sharp} = X_{i+1}^{\sharp} \xleftarrow{n_{i+1}} TX_i^{\sharp}.$$

A class  $\mathcal{E}_1$  of equation arrows now is defined as follows: for every  $\mathbf{C}$ -object  $X$  let  $e_X: X_2^{\sharp} \rightarrow E_X$  be a coequalizer of the pair  $m_2, n_2 \circ \eta_{X_1^{\sharp}} \circ m_1$ . A  $T$ -algebra  $(C, \alpha_C)$  satisfies  $e_X$  iff, for every morphism  $f: X \rightarrow C$ , the morphism

$$f_2^{\sharp} = [f, \alpha_C \circ Tf_1^{\sharp}]: X + TX_1^{\sharp} \rightarrow C$$

satisfies  $f_2^{\sharp} \circ m_2 = f_2^{\sharp} \circ n_2 \circ \eta_{X_1^{\sharp}} \circ m_1$  or, equivalently,  $f = \alpha_C \circ Tf_1^{\sharp} \circ \eta_{X_1^{\sharp}} \circ m_1$ . Since  $\eta$  is natural, this is equivalent to  $f = \alpha_C \circ \eta_C \circ f$  which, for  $X = C$

and  $f = 1_C$  yields satisfaction of the  $\mathbb{T}$ -algebra axiom  $\alpha_C \circ \eta_C = 1_C$ . Conversely,  $\alpha_C \circ \eta_C = 1_C$  yields  $f = \alpha_C \circ \eta_C \circ f$  by composition with  $f$ . Thus, the satisfaction of  $\mathcal{E}_1$  is equivalent  $\alpha_C \circ \eta_C = 1_C$ .

Next we define a class  $\mathcal{E}_2$  of equation arrows as follows: for every  $\mathbb{C}$ -object  $X$  let  $d_X: X + TX_2^\# = X_3^\# \rightarrow D_X$  be a coequalizer of the pair  $n_3 \circ Tm_2 \circ \mu_X, n_3 \circ Tn_2 \circ T^2m_1$ .

$$\begin{array}{ccccc}
 & & TX & & \\
 & \nearrow \mu_X & & \searrow Tm_2 & \\
 T^2X & & & & TX_2^\# \xrightarrow{n_3} X + TX_2^\# \xrightarrow{d_X} D_X \\
 & \searrow T^2m_1 & & \nearrow Tn_2 & \\
 & & T^2X_1^\# & & 
 \end{array}$$

A  $T$ -algebra  $(C, \alpha_C)$  satisfies  $d_X$  iff, for every morphism  $f: X \rightarrow C$ , the morphism

$$f_3^\# = [f, \alpha_C \circ T[f, \alpha_C \circ Tf_1^\#]]: X + TX_2^\# \rightarrow C$$

satisfies  $f_3^\# \circ n_3 \circ Tn_2 \circ T^2m_1 = f_3^\# \circ n_3 \circ Tm_2 \circ \mu_X$ . This is equivalent to  $\alpha_C \circ T\alpha_C \circ T^2f = \alpha_C \circ Tf \circ \mu_X$  or, since  $\mu$  is natural, to  $\alpha_C \circ T\alpha_C \circ T^2f = \alpha_C \circ \mu_C \circ T^2f$ . Choosing  $f = 1_C$  this yields satisfaction of the  $\mathbb{T}$ -algebra axiom  $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$ . Conversely,  $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$  yields  $\alpha_C \circ T\alpha_C \circ T^2f = \alpha_C \circ \mu_C \circ T^2f$  by composition with  $T^2f$ . Thus, the satisfaction of  $\mathcal{E}_2$  is equivalent  $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$ .

Chosing  $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$  one thus gets  $\mathbb{C}^\mathbb{T} = \text{Alg}(T, \mathcal{E})$ .  $\diamond$

### 3.B Covarieties and Co-equations

By formally dualizing Definition 3.1, see Lemma 1.1, we obtain the following

**3.17 Definition** Let  $F$  be an endofunctor of a complete category  $\mathbb{C}$ .

1. An *coequation arrow over  $X$*  is a regular monomorphism  $m: M \rightarrow X_\#^i$  for some ordinal  $i$ . An  $F$ -coalgebra  $(C, \alpha_C)$  is said to *satisfy  $m$*  provided that for every morphism  $f: C \rightarrow X$  the morphism  $f_\#^i$  factorizes through  $m$ .

2. For any class  $\mathcal{M}$  of coequation arrows  $\text{Coalg}(F, \mathcal{M})$  is the full subcategory of  $\text{Coalg}(F)$  spanned by all  $F$ -coalgebras satisfying every  $m \in \mathcal{M}$  (that is,  $\text{Coalg}(F, \mathcal{M})$  is the dual of  $\text{Alg}(F^{op}, \mathcal{M})$ ). Such categories are called *coequational categories (of  $F$ -coalgebras)* over  $\mathbf{C}$ .
3. A coequational category  $\text{Coalg}(F, \mathcal{M})$  will be called a *covariety (of  $F$ -coalgebras)* over  $\mathbf{C}$  provided that the underlying functor

$$U^{\mathcal{M}} = U|_{\text{Coalg}(F, \mathcal{M})} : \text{Coalg}(F, \mathcal{M}) \rightarrow \mathbf{C}$$

has a right adjoint (that is, if  $\text{Alg}(F^{op}, \mathcal{M})$  is a variety over  $\mathbf{C}^{op}$ ).

By dualizing the respective results on equational categories and varieties we immediately obtain the following results.

**3.18 Theorem ([3])** *Comonadic categories over a complete category  $\mathbf{C}$  are precisely the categories concretely equivalent to covarieties over  $\mathbf{C}$ .*

**3.19 Proposition** *For every covariety  $\text{Coalg}(F, \mathcal{M})$  over  $\text{Set}$  or over  $\text{Vect}_k$  the following hold:*

1.  $\text{Coalg}(F, \mathcal{M})$  is regular monoreflexiv in  $\text{Coalg}(F)$ ;
2.  $\text{Coalg}(F, \mathcal{M})$  has all limits;
3.  $\text{Coalg}(F, \mathcal{M})$  has a regular factorization structure of sinks, created by its underlying functor and both, regular monomorphisms and episinks, are preserved and reflected by the embedding  $\text{Coalg}(F, \mathcal{M}) \hookrightarrow \text{Coalg}(F)$ .

**3.20 Theorem** *Let  $F$  be a covariator on  $\text{Set}$  or on  $\text{Vect}_k$ . Then the following are equivalent for a full and isomorphism closed subcategory  $\mathbf{K}$  of  $\text{Coalg}(F)$ :*

- (i)  $\mathbf{K}$  is a covariety.
- (ii)  $\mathbf{K}$  is a coorthogonality class<sup>24</sup>  $M_{\perp}$  of some class of regular monomorphisms  $\mathbf{M}$  with cofree codomains<sup>25</sup>.

<sup>24</sup>This is the dual of an orthogonality class: a coalgebra  $(C, \alpha_C)$  belongs to that class provided that, for each  $m: M \rightarrow M'$  in  $\mathbf{M}$  each homomorphism  $f: C \rightarrow M'$  factorizes over  $m$ .

<sup>25</sup>See Theorem 4.13 for an strengthening of this statement.

(iii)  $\mathbf{K}$  is closed under homomorphic images, coproducts and subcoalgebras<sup>26</sup>.

While Theorem 3.18 shows that the dual of a covariety over  $\mathbf{Set}$  is a variety over  $\mathbf{Set}^{op}$  it moreover implies the following additional dualization principle:

**3.21 Proposition ([3])** *The dual of a covariety over  $\mathbf{Set}$  is equivalent to a variety over  $\mathbf{Set}$ .*

**Proof** By means of the contravariant power-set functor  $\mathcal{P}'$  the category  $\mathbf{Set}^{op}$  is monadic over  $\mathbf{Set}$ . Let  $V: \mathbf{Coalg}(F, \mathcal{M}) \rightarrow \mathbf{Set}$  be the composite of  $(U^{\mathcal{M}})^{op}$  and  $\mathcal{P}'$ . We need to show that  $V$  is monadic. Since  $V$  has a left adjoint and creates limits it suffices to prove that  $V$  creates coequalizers of congruence relations (= kernel pairs). Hence let  $r, s: (C, \alpha_C) \rightarrow (D, \alpha_D)$  be a pair of  $\mathbf{Coalg}(F, \mathcal{M})$ -morphisms such that  $Vr, Vs$  is a congruence relation and let  $q: \mathcal{P}'(D) \rightarrow X$  be its coequalizer. Since  $\mathcal{P}'$  reflects congruence relations and creates their coequalizers there is a unique  $\mathbf{Set}^{op}$ -morphism  $q': D \rightarrow X'$  with  $\mathcal{P}'(q') = q$  and this is a coequalizer of the congruence relation  $U^{\mathcal{M}}r, U^{\mathcal{M}}s$ . If  $X' \neq \emptyset$  this will even be a split coequalizer such that  $U^{\mathcal{M}}$  creates from it a coequalizer of  $r, s$ . The remaining case  $X' = \emptyset$  is trivial: the unique  $F$ -coalgebra structure on  $\emptyset$  does the job.  $\diamond$

**3.22 Remark** Coequations and their satisfaction have the following simple interpretation in the case of coalgebras over  $\mathbf{Set}$ : define, for every ‘‘coterm’’  $x \in X_{\#}^i$ , the coequation  $[x]$  as the following embedding

$$X_{\#}^i \setminus \{x\} \hookrightarrow X_{\#}^i.$$

A coalgebra  $(C, \alpha_C)$  satisfies  $[x]$  iff  $x$  does not lie in the image of  $f_{\#}^i: C \rightarrow X_{\#}^i$  for any colouring  $f: C \rightarrow X$ . These are all the coequations needed: we can substitute an arbitrary coequation

$$m: M \rightarrow X_{\#}^i$$

by the set of coequations  $\{[x] \mid x \in X_{\#}^i \setminus m[M]\}$ .

The final part of this section will be used to discuss examples of (weak) covarieties.

<sup>26</sup>Observe that, in  $\mathbf{Coalg}(F)$ , the homomorphic images are given by (plain) epimorphisms while the embeddings of subalgebras are the regular monomorphisms.

### Comonoids and Coacts

Let  $(\mathbf{C}, \otimes, I)$  be a (symmetric) monoidal category. Given  $C$  in  $\mathbf{C}$ ,  $s_C$  will denote the symmetry on  $C \otimes C$  (in the symmetric case) while  $a_C: (C \otimes C) \otimes C \rightarrow C \otimes (C \otimes C)$  denotes the associativity.  $l_C$  and  $r_C$  are the left and right unit-isomorphisms.

A (commutative) comonoid  $(M, m, u)$  consists of a  $\mathbf{C}$ -object  $M$  and morphisms  $m: M \rightarrow M \otimes M$  and  $u: M \rightarrow I$  such that the following diagrams commute:

$$\begin{array}{ccc}
 & M \otimes M & \xrightarrow{m \otimes M} (M \otimes M) \otimes M \\
 & \nearrow m & \parallel a_M \\
 M & & \\
 & \searrow m & \\
 & M \otimes M & \xrightarrow{M \otimes m} M \otimes (M \otimes M)
 \end{array}$$

This condition will be called *co-associativity*.

$$\begin{array}{ccccc}
 M \otimes M & \xleftarrow{m} & M & \xrightarrow{m} & M \otimes M & & M & \xrightarrow{m} & M \otimes M \\
 \downarrow u \otimes M & & \downarrow M & & \downarrow M \otimes u & & \searrow m & & \downarrow s_M \\
 I \otimes M & \xrightarrow{l_M} & M & \xleftarrow{r_M} & M \otimes I & & & & M \otimes M
 \end{array}$$

These are the *co-neutral law* and *co-commutativity* respectively.

By  $F$  we denote the endofunctor on  $\mathbf{C}$  with  $FC = I \times (C \otimes C)$  and by  $\text{Coalg}(F)$  its category of coalgebras. Thus, an  $F$ -coalgebra  $(C, \alpha_C)$  has  $\alpha_C = \langle u_C, m_C \rangle$  with

$$u_C: C \rightarrow I \quad \text{and} \quad m_C: C \rightarrow C \otimes C$$

and, for  $F$ -coalgebras  $(C, \alpha_C), (D, \alpha_D)$ , a morphism  $f: C \rightarrow D$  is an  $F$ -coalgebra homomorphism iff the following diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{u_C} & I \\
 \downarrow f & & \downarrow I \\
 D & \xrightarrow{u_D} & I
 \end{array}
 \quad
 \begin{array}{ccc}
 C & \xrightarrow{m_C} & C \otimes C \\
 \downarrow f & & \downarrow f \otimes f \\
 D & \xrightarrow{m_D} & D \otimes D
 \end{array}$$

**3.23 Proposition** Let  $(C, \otimes, I)$  be a monoidally closed category where  $C$  is locally presentable. Let  $F$  be as above. Then the underlying functor  $\text{Coalg}(F) \rightarrow C$  has a right adjoint.

**Proof** Since  $C \otimes -$  preserves colimits for each  $C$ , the functor  $C \otimes C$  preserves colimits of chains. Thus,  $F$  is accessible since  $I$  is presentable and  $C$  is locally presentable. Now the claim follows by Proposition 4.3.  $\diamond$

**3.24 Proposition** Let  $C$  be a (symmetric) monoidal category and  $F$  the endofunctor on  $C$  with  $FC = C \otimes C$ . Then the category of (commutative) comonoids is closed in  $\text{Coalg}(F)$  under

1. episinks
2. subobjects carried by split monomorphisms.

**Proof**

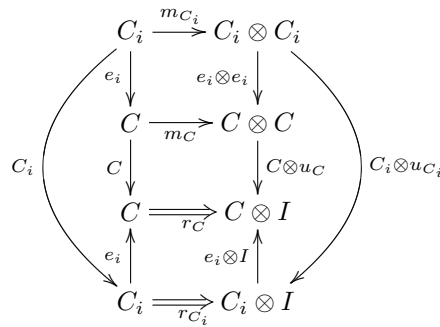
Ad 1: Let  $e_i: (C_i, \alpha_i) \rightarrow (C, \alpha_C)$  be an episink.

CO-ASSOCIATIVITY: Let every  $(C_i, \alpha_i)$  be co-associative. Consider the following diagram: Here the left hand cells commute since the  $e_i$  are homomorphisms; the right hand cell commutes by naturality of  $a$ ; the top and bottom cells commute by (bi)functoriality of  $- \otimes -$ ; the outer frame commutes since the  $(C_i, \alpha_i)$  are co-associative. Thus, the inner cell commutes as required, since the  $e_i$  are jointly cancellable.

$$\begin{array}{ccccc}
 & & C_i \otimes C_i & \xrightarrow{m_{C_i \otimes C_i}} & (C_i \otimes C_i) \otimes C_i \\
 & & \downarrow e_i \otimes e_i & & \downarrow (e_i \otimes e_i) \otimes e_i \\
 & & C \otimes C & \xrightarrow{m_{C \otimes C}} & (C \otimes C) \otimes C \\
 & & \downarrow m_C & & \downarrow a_C \\
 C_i & \xrightarrow{e_i} & C & & C \\
 & & \downarrow m_C & & \downarrow C \otimes m_C \\
 & & C \otimes C & \xrightarrow{C \otimes m_C} & C \otimes (C \otimes C) \\
 & & \downarrow e_i \otimes e_i & & \downarrow e_i \otimes (e_i \otimes e_i) \\
 & & C_i \otimes C_i & \xrightarrow{C_i \otimes m_{C_i}} & C_i \otimes (C_i \otimes C_i)
 \end{array}$$

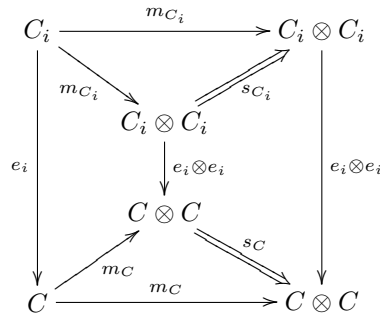
$a_{C_i}$  (curved arrow from  $(C_i \otimes C_i) \otimes C_i$  to  $C_i \otimes (C_i \otimes C_i)$ )

CO-NEUTRAL LAW: Let every  $(C_i, \alpha_i)$  satisfy the co-neutral law. Consider the following diagram:



Here the top cell commutes since the  $e_i$  are homomorphisms; the bottom cell commutes by naturality of  $r$ ; the right hand cell commutes by functoriality of  $- \otimes -$  and since the  $e_i$  are homomorphisms; the left hand cell commutes trivially and the outer frame commutes since the  $(C_i, \alpha_i)$  satisfy the co-neutral law. Thus, the inner cell commutes as required, since the  $e_i$  are jointly cancellable.

CO-COMMUTATIVITY: Let every  $(C_i, \alpha_i)$  be co-commutative. Consider the following diagram:

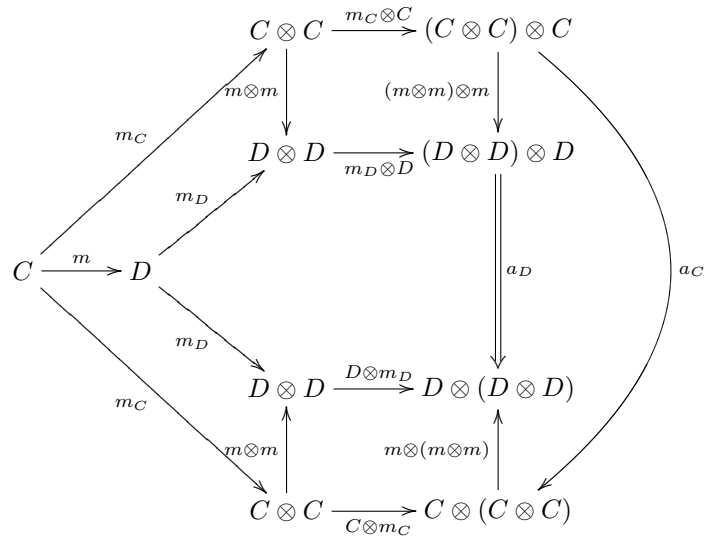


The left hand cell as well as the outer frame commute since the  $e_i$  are homomorphisms; the right hand cell commutes by naturality of  $s$ ; the top cell commutes since the  $(C_i, \alpha_i)$  are co-commutative. Thus, the bottom cell commutes as required, since the  $e_i$  are jointly cancellable.

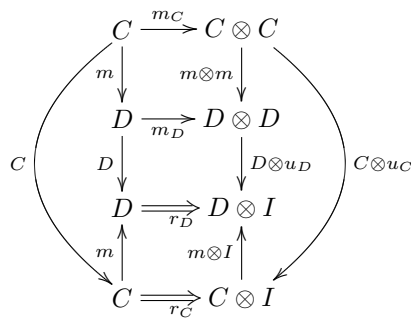
Ad 2: Let  $m: (C, \alpha_C) \rightarrow (D, \alpha_D)$  be a monomorphism splitting in  $\mathbf{C}$ .

CO-ASSOCIATIVITY: Let  $(D, \alpha_D)$  be co-associative. In the following dia-

gram the left hand cells commute since  $m$  is a homomorphism; the right hand cell commutes by naturality of  $a$ ; the top and bottom cells commute by (bi)functoriality of  $-\otimes-$ ; the inner cell commutes since the  $(D, \alpha_D)$  is co-associative. Thus, the outer cell commutes as required, since with  $m$  the morphism  $m \otimes (m \otimes m)$  is a (split) monomorphism and therefore cancellable.



CO-NEUTRAL LAW: Let  $(D, \alpha_D)$  satisfy the co-neutral law. Consider the following diagram:



Here the top cell commutes since  $m$  is a homomorphism; the bottom cell commutes by naturality of  $r$ ; the right hand cell commutes by functoriality

of  $- \otimes -$  and since  $m$  is a homomorphism; the left hand cell commutes trivially and the inner cell commutes since the  $(D, \alpha_D)$  satisfies the co-neutral law. Thus, the outer frame commutes as required, since  $m \otimes I$  again is cancellable.

CO-COMMUTATIVITY: follows analogously.  $\diamond$

For a fixed comonoid  $M = (M, m, u)$  the category of  $M$ -coacts is the full subcategory of  $\text{Coalg}(F_M)$  for the endofunctor  $F_M$  of  $\mathbf{C}$  with  $F_M C = C \otimes M$  consisting of those  $F_M$ -coalgebras  $(C, \alpha_C)$  for which the following diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\alpha_C} & C \otimes M \\
 \alpha_C \downarrow & & \downarrow C \otimes m \\
 C \otimes M & \xrightarrow{\alpha_C \otimes M} & C \otimes M \otimes M
 \end{array}
 \qquad
 \begin{array}{ccc}
 C & \xrightarrow{\alpha_C} & C \otimes M \\
 \searrow r_M & & \downarrow C \otimes u \\
 & & C \otimes I
 \end{array}$$

By essentially the same argument used for the proof of Proposition 3.23 one gets

**3.25 Proposition** *Let  $(\mathbf{C}, \otimes, I)$  be a monoidally closed category where  $\mathbf{C}$  is locally presentable, and let  $M$  be a comonoid. Let  $F_M$  be as above. Then the underlying functor  $\text{Coalg}(F_M) \rightarrow \mathbf{C}$  has a right adjoint.*

In the same way as Proposition 3.24 was proved one gets

**3.26 Proposition** *Given a comonoid  $M = (M, m, u)$  The category of  $M$ -coacts is closed in  $\text{Coalg}(F_M)$  under*

1. *episinks*
2. *subobjects carried by split monomorphisms.*

### $k$ -coalgebras and comodules

If the monoidal category of the previous section is the category  $\text{Vect}_k$  of  $k$ -vector spaces for some field  $k$ , the comonoids are called  $k$ -coalgebras and the  $M$ -coacts are called  $M$ -comodules. We denote the category of  $k$ -coalgebras by  $\text{Coalg}_k$  and the category of  $M$ -comodules by  $\text{Comod}_M$ .

Note that  $F$  and  $F_M$  constitute, in this particular case, (the dual of) a basic situation (see Example 1.4). Since both functors are constructive varieties by Propositions 3.23 and 3.25 in connection with Proposition 2.11, the closure properties of Propositions 3.24 and 3.26 imply:

**3.27 Proposition**  $\text{Coalg}_k$  and  $\text{Comod}_M$  are covarieties over  $\text{Vect}_k$ ; in particular,  $\text{Coalg}_k$  and  $\text{Comod}_M$  are comonadic over  $\text{Vect}_k$ .

**3.28** As an illustration of how to describe these categories coequationally, we now give a collection of coequations whose satisfaction is equivalent to co-associativity.

Let  $V$  be any  $k$ -vector space and  $f: C \rightarrow V$  a linear map where  $(C, \alpha_C)$  is a comonoid and  $\alpha_C = \langle u, m \rangle$  with  $u_C: C \rightarrow k$  and  $m_C: C \rightarrow C \otimes C$ .

We first calculate  $f_{\#}^1: C \rightarrow V_{\#}^1$  and  $f_{\#}^2: C \rightarrow V_{\#}^2$ , using the simplified description, i.e., starting with  $V_{\#}^0 = V$ ; we make use of the fact that, for each  $x \in C$ , the element  $m(x) \in C \otimes C$  has a representation as  $m(x) = \sum_i a_i \otimes b_i$ .

$$V_{\#}^1 = V \times F(V) = V \times k \times (V \otimes V).$$

$$f_{\#}^1: C \rightarrow V \times k \times (V \otimes V) \text{ acts by}$$

$$f_{\#}^1(x) = (fx, ux, f \otimes f(mx)) \quad (1)$$

$$= (fx, ux, \sum_i f a_i \otimes f b_i) \quad (2)$$

$$V_{\#}^2 = V \times F(V_{\#}^1) = V \times k \times ((V \times k \times (V \otimes V)) \otimes (V \times k \times (V \otimes V)))$$

$$f_{\#}^2: C \rightarrow V \times k \times ((V \times k \times (V \otimes V)) \otimes (V \times k \times (V \otimes V))) \text{ acts by}$$

$$\begin{aligned} f_{\#}^2(x) &= (fx, F f_{\#}^1(\langle u, m \rangle(x))) \\ &= (fx, ux, f_{\#}^1 \otimes f_{\#}^1(mx)) \\ &= (fx, ux, f_{\#}^1 \otimes f_{\#}^1(\sum_i a_i \otimes b_i)) \\ &= (fx, ux, \sum_i f_{\#}^1 a_i \otimes f_{\#}^1 b_i) \\ &= (fx, ux, \sum_i (f a_i, u a_i, (f \otimes f)(m a_i)) \otimes (f b_i, u b_i, (f \otimes f)(m b_i))) \end{aligned}$$

Now consider the following subspaces of  $V_{\#}^2$ :

Let  $U'$  be the subspace of  $F(V_{\#}^1) = (V \times k \times (V \otimes V)) \otimes (V \times k \times (V \otimes V))$  generated by

$$\{(a, \kappa, b \otimes c) \otimes (a', \kappa', b' \otimes c') \mid a \otimes b' \otimes c' = a' \otimes b \otimes c\}$$

and let

$$i_{V,ass}: V \times k \times U' \hookrightarrow V_{\#}^2$$

be the embedding. This notation is symbolic; what is meant is following: if  $(V \times k \times (V \otimes V)) \otimes (V \times k \times (V \otimes V))$  is decomposed as

$$((V \times k) \otimes (V \times k)) \times ((V \times k) \otimes (V \otimes V)) \times ((V \otimes V) \otimes (V \times k)) \times ((V \otimes V) \otimes (V \otimes V))$$

and the two middle components further as

$$(V \otimes (V \otimes V)) \times (k \otimes (V \otimes V)) \times ((V \otimes V) \otimes V) \times ((V \otimes V) \otimes k)$$

then  $U'$  consists of those elements having identical components in  $(V \otimes V) \otimes V$  and  $V \otimes (V \otimes V)$ .

$f_{\sharp}^2$  then factorizes over  $i_{V,ass}$  iff

$$\sum_i f a_i \otimes (f \otimes f)(m b_i) = \sum_i (f \otimes f)(m a_i) \otimes f b_i \quad (3)$$

which is, for  $V = C$  and  $f = id$ , precisely the coassociativity condition for  $m$ . Clearly, if  $m$  satisfies this condition, the equation 3 follows for each  $f: C \rightarrow V$ . Thus  $(C, \alpha_C)$  is coassociative iff it satisfies the coequations  $i_{V,ass}$  for all  $V$ .

## 4 Coalgebra of accessible functors

The notion of locally presentable category for good reasons is considered to capture the idea of a category of (generalized) algebras (see e.g. [5]). It might therefore come as surprise that covarieties of coalgebras quite often are locally presentable.

We will, in this final section, collect a couple of results related to local presentability of covarieties of coalgebras. This then, in a sense, "embeds" coalgebra into "algebra". More important, however, these results imply some surprising results on the structure of covarieties and thus shed some light, too, on the structure of those varieties of which covarieties are duals.

Clearly, accessible functors are (constructive) variators (see example 2 of 2.10). It is, however, somewhat surprising that accessibility of an endofunctor has striking consequences for its categories of coalgebras, as we are going to prove next. Here, the following lemma is crucial:

**4.1 Lemma ([4])** . *Let  $\mathbf{C}$  be a  $\lambda$ -presentable category and  $F$  an  $\lambda$ -accessible endofunctor on  $\mathbf{C}$ . Then every  $F$ -coalgebra  $(C, \alpha_C)$  is a  $\lambda$ -filtered colimit of  $F$ -coalgebras  $(C_i, \alpha_i)$  where, for each  $i$ ,  $C_i$  is a directed colimit of  $\lambda$ -presentable objects in  $\mathbf{C}$ .*

As a first consequence we get, using Example 5.2.3 below:

**4.2 Theorem** *Let  $F$  be an accessible endofunctor on a locally presentable category  $\mathbf{C}$ . Then  $\mathbf{Coalg}(F)$  is a locally presentable category. If  $F$  is  $\lambda$ -accessible, then  $\mathbf{Coalg}(F)$  is a locally  $\lambda$ -presentable, provided that  $\lambda > \aleph_0$ .*

This implies, in view of Theorem 5.3, by (the dual of) the Special Adjoint Functor Theorem:

**4.3 Proposition** *Let  $F$  be an accessible endofunctor on a locally presentable category  $\mathbf{C}$ . Then  $F$  is a covariator.*

As a consequence of Theorem 4.2 one can prove moreover (see [4]):

**4.4 Proposition** *Every covariety of  $F$ -coalgebras over  $\mathbf{Set}$ , where  $F$  is an accessible  $\mathbf{Set}$ -functor, is a locally presentable category. If  $F$  is  $\lambda$ -accessible, then every covariety of  $F$ -coalgebras is a locally  $\lambda$ -presentable, provided that  $\lambda > \aleph_0$ .*

The condition  $\lambda > \aleph_0$  in Theorem 4.2 cannot be dropped as is shown by the following example.

**4.5 Example ([4])** Let  $\mathbf{A} = \mathcal{P}(\omega)$  the algebraic lattice of all sets of natural numbers, ordered by inclusion. Then  $\mathbf{A}$ , considered as a category, is locally finitely presentable. Define  $F: \mathbf{A} \rightarrow \mathbf{A}$  by

$$F(M) = \begin{cases} M \setminus \{a\} & \text{if } a \text{ is a largest element of } M, \\ M & \text{if } M \text{ does not have a largest element.} \end{cases}$$

Then  $F$  is obviously order-preserving (i.e., a functor); in fact,  $F$  even preserves directed joins (i.e., is a finitary functor): if  $M = \bigcup_{i \in I} M_i$  is directed and  $M$  is finite, then  $M = M_j$  for some  $j$  and  $F(M) = \bigcup_{i \in I} F(M_i)$  trivially. If  $M$  is infinite, then  $F(M) = M$  and for each  $x \in M$  there is  $i \in I$  such that  $x$  is not the largest element of  $M_i$ ; thus  $x \in F(M_i)$ ; consequently,  $\bigcup_{i \in I} F(M_i) = M$ . An object  $M$  in  $\mathbf{A}$  carries an  $F$ -coalgebra structure iff  $M$  is infinite or empty. The subposet  $\mathbf{A}_0$  of  $\mathbf{A}$  formed by  $\emptyset$  and all infinite subsets of  $\omega$  is thus isomorphic to  $\mathbf{Coalg}(F)$ . and  $\mathbf{A}_0$  is not an algebraic lattice (i.e., not locally finitely presentable).

However, the following holds:

**4.6 Proposition ([4])** *For any finitary endofunctor  $F$  on  $\mathbf{Set}$  its category  $\mathbf{Coalg}(F)$  of coalgebras is finitely presentable, provided that  $F$  preserves arbitrary intersections and preimages.*

**Proof** We first prove that for any finite subset  $M$  of an  $F$ -coalgebra  $(C, \alpha_C)$  the subcoalgebra  $(M', \alpha') = \langle M \rangle$  generated by  $M$  (see Remark 1.23 for its existence) is finitely presentable as an  $F$ -coalgebra: let  $h: \langle M \rangle \rightarrow (D, \alpha_D)$  be a homomorphism where  $(D, \alpha_D)$  is a directed colimit of  $(D_i, \alpha_i)$ . Then the restriction of  $h$  to  $M$  factorizes over some  $(D_i, \alpha_i)$  and therefore maps  $M$  into some subcoalgebra  $(D_i, \alpha_i)'$  of  $(D, \alpha_D)$ . The preimage of this  $(D_i, \alpha_i)'$  thus contains  $M$  and then  $M'$  as well since  $F$  preserves preimages. Thus  $h$  factorizes (as a map) through  $(D_i, \alpha_i)$ . That this then actually is a factorization as a homomorphism and that it is essentially unique follows by categorical routine.

It remains to be mentioned that, in this case, every coalgebra is a directed colimit of all its finitely generated subcoalgebras.  $\diamond$

**4.7 Examples** The following  $\mathbf{Set}$ -functors preserve intersections and preimages.

1. Every polynomial functor  $F_\Omega$ ;
2. The finite powerset functor  $\mathcal{P}_{fin}$ .

Lemma 4.1 has, for the particular case  $\mathbf{C} = \mathbf{Set}$ , the following corollary:

**4.8 Proposition** *Let  $F$  be an accessible endofunctor on  $\mathbf{Set}$ . Then the category  $\mathbf{Coalg}(F)$  has the following property:*

- (\*) *There exists a cardinal  $c$  such that for each  $F$ -coalgebra  $(C, \alpha_C)$  and each  $x \in C$  there exists a subcoalgebra  $(D, \alpha_D)$  of  $(C, \alpha_C)$  with  $x \in D$  and  $\text{card}D \leq c$ .*

**4.9 Remark** Endofunctors  $F$  on  $\mathbf{Set}$ , for which  $\mathbf{Coalg}(F)$  has property (\*) have been called *bounded* ([13, 10, 12]). It is shown in [3] that boundedness is in fact equivalent to accessibility.

The boundedness condition of a  $\mathbf{Set}$ -functor can be related to a familiar algebraic property. For this purpose we recall the following notions from [10] (which are nothing but the dual versions of the familiar concepts of “subdirect representation”, “subdirect product” and “subdirectly irreducible”):

**4.10 Definition** Let  $\mathcal{S}$  a class of objects in a category  $\mathbf{C}$ .

1. An episink  $(C, (S_i \xrightarrow{m_i} C)_I)$  of regular monomorphisms  $m_i$  with  $S_i \in \mathcal{S}$  for each  $i$  is called *conjunct representation of  $C$  (by  $\mathcal{S}$ )*;  $C$  is then also called a *conjunct sum* of  $\mathcal{S}$ .
2. A  $\mathbf{C}$ -object  $C$  is called *conjunctly irreducible* if in each conjunct representation of  $C$  at least one  $m_i$  is an isomorphism.

**4.11 Remark** The following statements are clear:

1. If every  $\mathbf{C}$ -object has a conjunct representation by  $\mathcal{S}$ , and  $\mathcal{S}$  even is a set, then this set is a generator<sup>27</sup> of  $\mathbf{C}$ .
2. If  $\mathbf{C}$  has coproducts, a conjunct sum  $(C, (S_i \xrightarrow{m_i} C)_I)$  represents  $C$  as a (epimorphic) quotient of a coproduct of the family  $(S_i)_I$ . Thus, any subcategory of  $\mathbf{C}$ , which is closed under coproducts and (epimorphic) quotients, is closed under conjunct sums.

For coalgebras we have the following result.

**4.12 Proposition** *For an endofunctor  $F$  on  $\mathbf{Set}$  the following are equivalent:*

- (i)  $F$  is bounded.
- (ii) There exists a set  $\mathcal{S}$  of  $F$ -coalgebras such that every  $F$ -coalgebra has a conjunct representation by  $\mathcal{S}$ .
- (iii) There exists a set  $\mathcal{S}'$  of subcoalgebras of some cofree  $F$ -coalgebra such that every  $F$ -coalgebra has a conjunct representation by  $\mathcal{S}'$ .

**Proof** (i)  $\implies$  (ii): If  $F$  is bounded by  $\mathfrak{c}$  let  $\mathcal{S}$  be the collection of all  $F$ -coalgebras  $(D, \alpha_D)$  with  $\text{card}D \leq \mathfrak{c}$ . For each coalgebra the family of embeddings of its subcoalgebras from  $\mathcal{S}$  is a conjunct representation.

(ii)  $\implies$  (iii): Let  $\mathfrak{c}$  be a cardinal greater or equal  $\text{card}D$  for each  $(D, \alpha_D) \in \mathcal{S}$  and  $X$  a set of cardinality  $\mathfrak{c}$ . There exist a cofree coalgebra  $(X_{\sharp}, \alpha_{\sharp})$  on  $X$  and, for each  $(D, \alpha_D) \in \mathcal{S}$ , an embedding into  $(X_{\sharp}, \alpha_{\sharp})$ , induced by an embedding of  $D \hookrightarrow X$  and cofreeness of  $(X_{\sharp}, \alpha_{\sharp})$ . Let  $\mathcal{S}'$  be the collection of the images  $(D', \alpha')$  of these embeddings.

(iii)  $\implies$  (i): If the  $F$ -coalgebra  $(C, \alpha_C)$  has a conjunct representation by a collection  $\mathcal{S}'$  of subcoalgebras of some cofree coalgebra  $(X_{\sharp}, \alpha_{\sharp})$ , each

<sup>27</sup>Some authors might prefer *generating set*.

element of  $C$  is contained in a subcoalgebra  $(D, \alpha_D)$  of  $(C, \alpha_C)$  isomorphic to an element of  $\mathcal{S}'$ ; thus,  $\text{card}D \leq \text{card}X_\#$ .  $\diamond$

The following is an easy consequence of this result. Its second item (due to [10]) is remarkable in that it *looks* like the dual of Proposition 3.14 though doesn't seem to be obtainable by dualization: Proposition 3.14 wouldn't hold for varieties dual to covarieties in view of Remark 4.15.

**4.13 Theorem** *Let  $\text{Coalg}(F, \mathcal{M})$  be a covariety for an accessible Set-functor. Then there exists an embedding  $m$  of a subcoalgebra  $(S, \alpha_S)$  of some cofree coalgebra  $(X_\#, \alpha_\#)$  such that*

1. *Each coalgebra of  $\text{Coalg}(F, \mathcal{M})$  has a conjunct representation by subcoalgebras of  $(S, \alpha_S)$ .*
2.  $\text{Coalg}(F, \mathcal{M}) = \{m\}_\perp$ .

**Proof** Let

$$(f: (C, \alpha_C) \rightarrow (X_\#, \alpha_\#))_f$$

be the sink of all morphisms with codomain  $(X_\#, \alpha_\#)$ , the cofree coalgebra chosen as in Proposition 4.12 and domain  $(C, \alpha_C)$  in  $\text{Coalg}(F, \mathcal{M})$ . We now form the regular factorization of this sink according to Proposition 1.21

$$(C, \alpha_C) \xrightarrow{f} (X_\#, \alpha_\#) = (C, \alpha_C) \xrightarrow{e_f} (S, \alpha_S) \xrightarrow{m} (X_\#, \alpha_\#).$$

Let  $\mathcal{U}$  be the collection of all coalgebras in the set  $\mathcal{S}'$  as defined in the proof above, which are subcoalgebras of  $(S, \alpha_S)$ . Observe that, by construction,  $\text{Coalg}(F, \mathcal{M})$  is contained in  $\{m\}_\perp$ .

Ad 1. Let  $(C, \alpha_C) \in \text{Coalg}(F, \mathcal{M})$ .  $(C, \alpha_C)$  is a conjunct sum of coalgebras from  $\mathcal{S}'$ , which then—being subcoalgebras of  $(C, \alpha_C)$ —belong to  $\text{Coalg}(F, \mathcal{M})$ , too. Since  $\text{Coalg}(F, \mathcal{M}) \subset \{m\}_\perp$ , they are subcoalgebras of  $(S, \alpha_S)$ .

Ad 2. In order to prove  $\{m\}_\perp \subset \text{Coalg}(F, \mathcal{M})$  observe first that  $(S, \alpha_S)$  belongs to  $\text{Coalg}(F, \mathcal{M})$  since  $\text{Coalg}(F, \mathcal{M})$  is closed under episinks. Since  $\text{Coalg}(F, \mathcal{M})$  is closed under subcoalgebras and conjunct sums the result follows from 1.  $\diamond$

The conjunct representations in Proposition 4.12 can, in a number of interesting cases (see Example 4.7 above for examples of such functors), even chosen to be conjunctly irreducible as was also observed in [10]:

**4.14 Proposition** *Let  $F$  be a bounded **Set**-functor preserving intersections. Then every  $F$ -coalgebra has a conjunctly irreducible conjunct representation by (a set of) subcoalgebras of some cofree  $F$ -coalgebra.*

**Proof** By Remark 1.23 each element  $a$  of a coalgebra generates a subcoalgebra  $\langle a \rangle$ ; these subcoalgebras are conjunctly irreducible and isomorphic to subcoalgebras of some (fixed) cofree coalgebra (see the proof of Proposition 4.12). As in that proposition, every  $F$ -coalgebra has a conjunct representation by these subcoalgebras.  $\diamond$

**4.15 Remark** Let  $F$  be an accessible **Set**-functor. By Proposition 3.21 the dual of any covariety of  $F$ -coalgebras is a variety (over **Set**). The previous results give some further—partly surprising—information about the structure of those varieties:

1. They never are varieties “with rank”.
2. They always have a cogenerator.
3. They are residually small, provided that  $F$  preserves intersections.

## 5 Appendix

- 5.1 Definition**
1. An object  $C$  in a category  $\mathbf{C}$  is called  *$\lambda$ -presentable* (where  $\lambda$  is a regular cardinal) provided that its associated hom-functor  $\mathbf{hom}(C, -): \mathbf{C} \rightarrow \mathbf{Set}$  preserves  $\lambda$ -directed colimits.  $C$  is called *presentable* if  $C$  is  $\lambda$ -presentable for some  $\lambda$ .
  2. A category  $\mathbf{C}$  is called *locally  $\lambda$ -presentable* provided that  $\mathbf{C}$  is co-complete and admits a set  $\mathcal{S}$  of  $\lambda$ -presentable objects such that each  $\mathbf{C}$ -object is a  $\lambda$ -directed colimit of objects from  $\mathcal{S}$ .  $\mathbf{C}$  is called *locally presentable* if  $\mathbf{C}$  is locally  $\lambda$ -presentable for some  $\lambda$ .
  3. An endofunctor  $F$  on a locally  $\lambda$ -presentable category is called  *$\lambda$ -accessible* provided that  $F$  preserves  $\lambda$ -directed colimits.  $F$  is called *accessible* if  $F$  is  $\lambda$ -accessible for some  $\lambda$ .

- 5.2 Examples**
1. Each variety of  $F$ -algebras, where  $F$  is a polynomial **Set**-functor, is locally finitely (locally  $\aleph_0$ )-presentable. The finitely presentable algebras then are precisely those which can be represented by finitely many generators and finitely many equations (see e.g. [5]).

2. Algebraic lattices are (when considered as categories) locally finitely presentable. The finitely presentable objects are the compact elements.
3. Given a  $\lambda$ -accessible endofunctor on a  $\lambda$ -presentable category  $\mathbf{C}$ , an  $F$ -coalgebra  $(C, \alpha_C)$  is  $\lambda$ -presentable in  $\mathbf{Coalg}(F)$  provided that  $C$  is  $\lambda$ -presentable in  $\mathbf{C}$  (see [4]).
4. Each polynomial **Set**-functor is finitary (=finitely  $(\aleph_0)$ -accessible).
5. The functors  $V \mapsto M \otimes V$  and  $V \mapsto k \times (V \otimes V)$  are (finitely) accessible endofunctors on  $\mathbf{Vect}_k$ .

Locally presentable categories can be described in many equivalent ways some of which we collect as follows (see e.g [5]):

**5.3 Theorem** *The following are equivalent for a cocomplete category  $\mathbf{C}$ .*

- (i)  $\mathbf{C}$  is locally  $\lambda$ -presentable.
- (ii)  $\mathbf{C}$  has strong generator consisting of  $\lambda$ -presentables.
- (iii) The  $\lambda$ -presentables in  $\mathbf{C}$  have a representative set  $\mathbf{Pres}_\lambda \mathbf{C}$  and each object in  $\mathbf{C}$  is a  $\lambda$ -directed colimit of  $\lambda$ -presentables.
- (iv) The full subcategory of  $\mathbf{C}$  spanned by its  $\lambda$ -presentables  $\mathbf{Pres}_\lambda \mathbf{C}$  is essentially small and dense in  $\mathbf{C}$ .

They have—amongst others—the following properties (see e.g. [5]):

**5.4 Theorem** 1. *If  $\mathbf{C}$  and  $\mathbf{C}^{op}$  both are locally presentable then  $\mathbf{C}$  is equivalent to a complete lattice.*

2. *Let  $\mathbf{A}$  be a full, reflective subcategory of a locally  $\lambda$ -presentable category  $\mathbf{C}$ , closed under  $\lambda$ -directed colimits. Then*

$$\mathbf{A} = \{r_P \mid P \text{ } \lambda\text{-presentable}\}^\perp$$

*where  $r_P$  denotes the reflection morphism of  $P$  into  $\mathbf{A}$ . Moreover,  $\mathbf{A}$  is locally  $\lambda$ -presentable.*

Concerning preservation of  $\lambda$ -presentables by functors the following is obvious:

**5.5 Lemma** *Any left adjoint of a functor preserving  $\lambda$ -directed colimits preserves  $\lambda$ -presentables.*

## References

- [1] J. Adámek. Free algebras and automata realizations in the language of categories. *Comment. Math. Univ. Carolinae* **15** (1974), 589–602.
- [2] J. Adámek. Theory of Mathematical Structures. *D. Reidel Publ. Comp.*, 1983. Dordrecht.
- [3] J. Adámek and H.-E. Porst. From varieties of algebras to covarieties of coalgebras. *Electronic Notes in Theor. Comp. Sci.* **44.1** (2001).
- [4] J. Adámek and H.-E. Porst. On covarieties of coalgebras. *In preparation*.
- [5] J. Adámek and J. Rosický. Locally Presentable and Accessible Categories. *Cambridge University Press*, 1994. Cambridge. Corrections at [ftp.iti.cs.tu-bs.de](ftp:iti.cs.tu-bs.de)
- [6] J. Adámek and V. Trnková. Automata and Algebras in Categories. *Kluwer Acad. Publ.*, 1990. Dordrecht
- [7] M. Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science* **114** (1993), 299–315.
- [8] G. Birkhoff. On the structure of abstract algebras. *Proc. Cambridge Phil Soc.* **31** (1935), 433–454.
- [9] F. Borceux. Handbook of categorical algebra Vol. 2. *Cambridge University Press*, 1994. Cambridge.
- [10] H. P. Gumm. Elements of the general theory of coalgebras. *Preprint*, 1999
- [11] H. P. Gumm and T. Schröder. Products of coalgebras. *Preprint* 1999
- [12] Y. Kawahara and M. Mori. A small final coalgebra theorem. *Theoretical Computer Science* **233** (2000), 129–145.
- [13] J.J.M.M. Rutten. Universal Coalgebra: a theory of systems. Report CS-R9652. *CWI Amsterdam*, 1996
- [14] M. E. Sweedler. Hopf algebras. *Benjamin*, 1969. New York.

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