Existence of Strict Transfer Operator Approaches for Non-Compact Developable Hyperbolic Orbisurfaces

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You will never arrive,
Unless you make peace with your dead ends.

Radical Face

Abstract

For geometrically finite non-compact hyperbolic orbisurfaces fulfilling mild assumptions, we provide transfer operator families whose Fredholm determinant functions are identical to the respective Selberg zeta function. Our proof yields an algorithmic and uniform construction. By application of the cusp expansion algorithm by Pohl [54] and introduction of a similar algorithmic procedure for orbisurfaces without cusps, we establish cross sections for the geodesic flow on the considered orbisurfaces, that yield highly faithful, but, in general, non-uniformly expanding discrete dynamical systems modeling the geodesic flow. The central object for this pursuit is the set of branches, which encapsulates the structure guaranteeing the cross section to be suitable for its purpose. These sets of branches are introduced and extensively studied. Through a number of algorithmic steps of reduction, elimination, and acceleration on a set of branches, we turn the associated cross section into one that yields a still highly faithful, but now uniformly expanding discrete dynamical system. By virtue of the strict transfer operator approach in the sense of Fedosova and Pohl [22], this gives rise to a family of transfer operators nuclear of order zero on a well-chosen Banach space, and the Fredholm determinant function is seen to admit a meromorphic continuation to the whole complex plane and to equal the Selberg zeta function. All statements allow for the inclusion of finite-dimensional representations with non-expanding cusp monodromy, in the sense that a twisted version of the Selberg zeta function as well as twisted transfer operators may be considered. A comprehensive overview of the required background knowledge in hyperbolic geometry precedes the investigations.

Zusammenfassung

Wir entwickeln Familien von Transferoperatoren für eine große Klasse von geometrisch endlichen, nichtkompakten, hyperbolischen Orbiflächen, deren Fredholmsche Determinantenfunktionen mit der jeweiligen Selbergschen Zetafunktion übereinstimmen. Unser Beweis stellt eine algorithmische und uniforme Konstruktion dieser Operatoren bereit. Durch Anwendung von Pohls *cusp expansion algorithm* [54] und der Einführung eines darauf basierenden alorithmischen Ansatzes in Situationen ohne Spitzen, geben wir Schnitte für den geodätischen Fluss auf den betrachteten Orbiflächen an, welche hochgradig treue, jedoch im Allgemeinen nicht uniform expandierende diskrete Dynamiken liefern, die den geodätischen Fluss modellieren. Das zentrale Objekt für dieses Unterfangen stellt das *set of branches* dar, welches die für die Eignung des betreffenden Schnittes wesentlichen Strukturinformationen enthält. Diese *sets of branches* werden eingeführt und umfassend untersucht. Mittels diverser algorithmischer Arbeitsschritte betreffend Reduktion, Elimination und Beschleunigung auf diesem *set of branches*, gelingt es, den zugehörigen Schnitt so umzuwandeln, dass nach wie vor ein hochgradig treues, nun jedoch ebenfalls uniform expandierendes diskretes dynamisches System induziert wird. Auf Grundlage des *strict transfer operator approach* von Fedosova und Pohl [54], ermöglicht dies die Konstruktion einer Familie von nuklearen Operatoren der Ordnung Null auf einem geeigneten Banachraum, deren Fredholm-Determinantenfunktion eine meromorphe Fortsetzung auf die gesamte komplexe Ebene ermöglicht und dort mit der Selbergschen Zetafunktion übereinstimmt. Alle Aussagen gestatten zudem die Betrachtung von endlich-dimensionalen Darstellungen mit nicht-expandierender Spitzenmonodromie, in dem Sinne, dass eine getwistete Version der Selbergschen Zetafunktion sowie Familien getwisteter Transferoperatoren betrachtet werden können. Den Untersuchungen ist ein umfänglicher Überblick über das benötigte Hintergrundwissen in hyperbolischer Geometrie vorangestellt.

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Contents

Introduction

Historical Background and Motivation

Selberg's zeta function, introduced in his groundbreaking work [74] in 1956, is a mediator between the worlds of differential geometry and spectral theory on Riemannian orbifolds. Despite being defined in terms of the primitive geodesic length spectrum—and hence by purely geometric means—its zeros nevertheless encode profound spectral information in form of the L^2-eigenvalues and resonances of the Laplacian on the surface. That makes the Selberg zeta function a highly valued object in the study of resonances (or vice versa, in the investigation of periodic geodesics using knowledge on resonances). Selberg zeta functions are now available for all geometrically finite hyperbolic orbisurfaces, and their theory is already quite extensive. Nevertheless, new contributions are made regularly, as we will do with this thesis.

Let $\mathbb{X}$ be a geometrically finite hyperbolic orbisurface. By that we mean a two-dimensional hyperbolic connected Riemannian orbifold (see, e. g., [77]). Further denote by $\mathscr{L}_{\mathbb{X}}$ the multiset of lengths of prime periodic geodesics on $\mathbb{X}$—the primitive length spectrum of $\mathbb{X}$. Then the infinite Euler product

$$Z_{\mathbb{X}}(s) \coloneqq \prod_{\ell \in \mathscr{L}_{\mathbb{X}}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right) \tag{1}$$

converges for $\operatorname{Re} s \gg 1$ and admits an analytic continuation to a meromorphic function on $\mathbb{C}$ [74, 27, 78, 10], which we continue to denote by $Z_{\mathbb{X}}(s)$. The set of zeros of $Z_{\mathbb{X}}(s)$ is known to contain the resonances of the Laplacian. For instance, for various orbisurfaces (and also in combination with non-trivial finite-dimensional unitary representations) a factorization

$$Z_{\mathbb{X}}(s) = G(s) \cdot \mathcal{P}_{\mathbb{X}}$$

is known [11, 52, 19], where $G(s)$ is a meromorphic function and $\mathcal{P}_{\mathbb{X}}$ is the Weierstraß product of resonances of $\mathbb{X}$ (including multiplicities), that are the poles of the resolvent of the Laplacian on $\mathbb{X}$. Hence, every resonance re-appears as a zero of $Z_{\mathbb{X}}$ with (almost) matching multiplicities. This relationship has also been shown in further settings and by other means (see [74, 78]). We provide a more extensive survey of these results in Section 1.12.

The link to the spectral theory of hyperbolic orbisurfaces motivates extended interest in the study of the zeros of the Selberg zeta function. This undertaking has gained a lot of traction with the introduction of transfer operator techniques from statistical mechanics, whose use goes back to work of Ruelle [69, 70, 71], Mayer [39, 40, 41], Fried [23], and Pollicott [65] and has made a great leap forward in recent years due to the efforts of many further researchers (see the list of references below). In the most general setting, the transfer operator associated with a given dynamical system

$$T\colon X \longrightarrow X\,,$$

where X is an arbitrary set, is an operator on the space $\mathrm{Fct}(X;\mathbb{C})$ of functions $f\colon X \to \mathbb{C}$ defined as

$$\mathcal{L}f(x) \coloneqq \sum_{y\in T^{-1}(x)} \varrho(y)f(y)\,,$$

for $x \in X$, where $T^{-1}(x)$ denotes the complete preimage of x under T and ϱ is some auxiliary valuation function. In settings where T has a non-zero Jacobian determinant, a common choice for ϱ is $|T'|^{-1}$ (*Perron–Frobenius operator*). While the study of discrete-time dynamics on X naturally leads to considerations regarding the orbits of the points of X under iteration of T, the transfer operator encapsulates how functions of various regularity evolve under iteration. It is designed to disclose the action of the dynamical system on mass densities of initial conditions. Consider, for example, the case that the Lebesgue measure λ on X is non-singular with respect to T. Then, for every measure μ on X absolutely continuous with respect to λ, its push-forward $\mu \circ T^{-1}$ is also absolutely continuous with respect to λ, and both their densities are λ-almost unique by the Radon–Nikodym theorem. Then a transfer operator can be chosen which transforms the density of μ into the density of its pushforward, for every such measure μ [73].

The (discrete-time or Ruelle-type) transfer operators, which are of interest for the thesis in hand, are associated to a discretizations of the (continuous-time) geodesic flow on a hyperbolic orbisurface $\mathbb{X}$. This is realized via a *cross section* sufficiently well structured to admit a *first return map*, hence yielding a time-discrete dynamical system on the unit tangent bundle of the orbisurface semi-conjugate to a Hadamard-type symbolic dynamics on subsets of the geodesic boundary of the hyperbolic plane. The transfer operators associated to this symbolic dynamics (with well-chosen valuation functions; see Section 1.11 for the details) then bear profound geometric information in terms of their eigenvalues and eigenfunctions. Via a complexification procedure on said subsets and an application of Grothendieck theory [26, 25], a Banach space of holomorphic functions can be obtained, on which every member of a certain one-parameter family of transfer operators associated to the symbolic dynamics (and hence, by proxy, to the geodesic flow) is seen to be nuclear of sufficiently small order, i. e., of (generalized) trace-class. This makes it possible to assign a *Fredholm determinant*. See Section 3.2 for a brief survey on nuclear operators and their traces and determinants.

Coming back to the Selberg zeta function, the result most seminal for the study of transfer operators in this setting is Mayer's thermodynamic formalism approach on the modular surface $\mathbb{M}$ [41]. Based on a Farey tessellation of $\mathbb{H}$ provided by Series [75], Mayer showed that a certain family of trace-class operators $\{\mathcal{L}_s^{\mathbb{M}}\}_{\operatorname{Re} s>\frac{1}{2}}$ related to the Gauß map (and including the famous Gauß–Kuzmin–Wirsing operator as a member) represents the Selberg zeta function by means of their Fredholm determinants:

$$Z_{\mathbb{X}}(s) = \det\bigl(1 - \mathcal{L}_s^{\mathbb{X}}\bigr)\,, \tag{2}$$

for $\mathbb{X} = \mathbb{M}$. The identity (2) converts the search for zeros of $Z_{\mathbb{X}}$ (and hence for the resonances of $\mathbb{X}$) to a question on the existence of eigenfunctions with eigenvalue 1 of the transfer operator $\mathcal{L}_s^{\mathbb{X}}$ with parameter s. *A priori*, the latter seems to be the more involved problem. However, this conversion allows to take advantage of, e. g., functional-analytic properties of $\mathcal{L}_s^{\mathbb{X}}$ and the spectral theory for compact operators for investigations. This explains why a relation as in (2) is so powerful and thus desired to have, not only in the case of the modular surface, but for as many hyperbolic orbisurfaces as feasible. Accordingly, in recent years much progress has been made for several (classes of) hyperbolic orbisurfaces by various researchers, and thus representations of the form (2) are now available in many settings (see the references listed below).

In such studies of resonances, sometimes also a *twisted variant* of the Selberg zeta function, $Z_{\mathbb{X},\chi}$, features, which enjoys quite similar properties as $Z_{\mathbb{X}}$ for χ a finite-dimensional representation of the fundamental group of $\mathbb{X}$ of appropriate regularity (see below). We refer the reader to Section 1.12 for exact definitions. For some hyperbolic orbisurfaces, also a family of *twisted transfer operators* $\{\mathcal{L}_{s,\chi}^{\mathbb{X}}\}_s$ is available, yielding a twisted analogue of (2):

$$Z_{\mathbb{X},\chi}(s) = \det\bigl(1 - \mathcal{L}_{s,\chi}^{\mathbb{X}}\bigr)\,. \tag{3}$$

While the (twisted or untwisted) Selberg zeta function focuses on the *static geometry* of hyperbolic orbisurfaces (namely, the lengths of periodic geodesics), transfer operators take advantage of the *dynamics* of the geodesic flow (namely, the paths of periodic geodesics). Accordingly, investigations of spectral properties of hyperbolic surfaces by means of transfer operators on the one hand and by means of Selberg zeta functions on the other hand are complementary. A number of particularly good results have been achieved by combining both approaches and making crucial use of (2) or (3), of which we can list here but a few examples:

- prime geodesic theorems, including error terms [50, 51, 66, 67, 47, 49] (some of these works use a variant of (2) or (3) or are for other spaces, but are nevertheless good examples),
- relations between Patterson–Sullivan distributions and Wigner distributions [3],
- numerical investigations of resonances [9, 5],
- distribution and counting results for resonances [28, 32, 48, 62],
- meromorphic continuation of Selberg zeta functions [23, 70, 71, 69, 40, 41, 65, 46, 17, 42, 44, 59, 61, 22] (some of these results use variants of (2) in order to compensate for non-exact codings).

Presentation of the Main Results

In this thesis we greatly expand the realm of hyperbolic orbisurfaces for which a representation as in (3) can be obtained. With respect to this objective, our main result reads as follows.

Theorem A. *For admissible developable hyperbolic orbisurfaces $\mathbb{X}$ and good representations χ there exists a Banach space $\mathcal{B}$ of functions and a family of operators $\{\mathcal{L}_{s,\chi}^{\mathbb{X}}\}_s$ which are nuclear of order zero on $\mathcal{B}$ and such that (3) holds true for $\operatorname{Re} s$ sufficiently large. Both $Z_{\mathbb{X},\chi}(s)$ and $s \mapsto \mathcal{L}_{s,\chi}$ extend to meromorphic functions in $s \in \mathbb{C}$, the latter with values in nuclear operators of order zero.*

A hyperbolic orbisurface $\mathbb{X}$ is called *developable*, if it has a fundamental group, that is, if there exists a group Γ of isometries on $\mathbb{H}$ such that $\mathbb{X} = \Gamma\backslash\mathbb{H}$. By a *good representation* we mean a linear representation of Γ on a finite-dimensional Hermitian vector space that has *non-expanding cusp monodromy* (see Section 1.12). This includes, in particular, all finite-dimensional unitary representations. In what follows we discuss the strategy of proof for Theorem A and, by doing so, elaborate on which orbisurfaces we shall call *admissible*.

We start with a briefly survey of a result by Möller and Pohl [44], where they established (2) (i. e., (3) for χ the trivial one-dimensional representation) for cofinite Hecke triangle groups. These groups Γ_q are generated by the two isometries

$$s\colon z \longmapsto -\frac{1}{z} \qquad \text{and} \qquad t_\lambda\colon z \longmapsto z + \lambda\,,$$

for $z \in \mathbb{H}$ and $\lambda \in \{2\cos(\pi/q) \mid q \geq 3\}$. The orbisurface $\mathbb{X}_q$ associated with Γ_q has a single cusp and two conical singularities, one of order 2 and one of order q (the modular surface $\mathbb{M}$ is the Hecke triangle surface to the parameter $q = 3$). The cusp of $\mathbb{X}_q$ makes it prone to the *cusp expansion algorithm* developed by Pohl in her dissertation thesis [54]. For every geometrically finite Fuchsian group with cusps which fulfills a certain technical condition (see below), this algorithm establishes a cross section for the geodesic flow that gives rise to a symbolic dynamics as described above. In the case of a cofinite Hecke triangle surface, the representatives of this cross section in $\mathbb{H}$ are of particularly convenient structure: one obtains a subset of the unit tangent vectors based on the imaginary axis and pointing into the half-space $\{\operatorname{Re} z > 0\}$ (see Figure 1). The arising symbolic dynamics

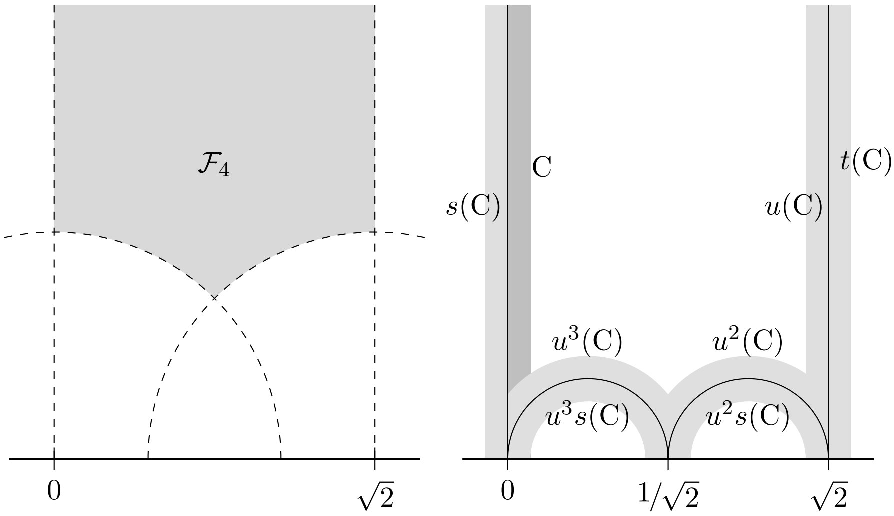


Figure 1: A fundamental domain $\mathcal{F}_4$ for Γ_4 and a representative C of the cross section constructed by Pohl [54] together with the relevant translates. We abbreviate here $t := t_{\sqrt{2}}$ and $u := t \cdot s$. The gray stripes indicate that the respective set consists of unit tangent vectors which are based on the adjacent geodesic arc and point into the indicated half-space relative to that arc.

and family of transfer operators can be described as follows: for $k = 1, \ldots, q-1$ define the diffeomorphisms

$$g_k \colon x \longmapsto \frac{x \cdot \sin\left(\frac{k}{q}\pi\right) - \sin\left(\frac{k+1}{q}\pi\right)}{\sin\left(\frac{k}{q}\pi\right) - x \cdot \sin\left(\frac{k-1}{q}\pi\right)}$$

on $\widehat{\mathbb{R}}$, as well as the subsets[1]

$$D_k := \left(-\frac{\sin\left(\frac{k+1}{q}\pi\right)}{\sin\left(\frac{k}{q}\pi\right)}, -\frac{\sin\left(\frac{k}{q}\pi\right)}{\sin\left(\frac{k-1}{q}\pi\right)} \right) \setminus \Gamma_q.\infty\,,$$

where $\Gamma_q.\infty$ denotes the orbit of the ideal point ∞ under Γ_q, or equivalently, the set of representatives of the cusp of $\mathbb{X}_q$. Then the symbolic dynamical system is

[1] We consider the sets D_k as subsets of $\widehat{\mathbb{R}}$. This means we identify $+\infty$ and $-\infty$. Furthermore, we use the convention "$1/0 = \infty$".

given by

$$F\colon D \longrightarrow D\,, \qquad F|_{D_k} = g_k\,,\ k = 1, \ldots, q-1\,,$$

for $D \coloneqq \bigcup_{k=1}^{q-1} D_k$. Via the operators $\tau_s(g)\colon\ \mathrm{Fct}(D;\mathbb{C}) \to \mathrm{Fct}(D;\mathbb{C})$ given by

$$\tau_s(g^{-1})f(x) \coloneqq \left|g'(x)\right|^s \big(f \circ g\big)(x)\,,$$

for $g \in \Gamma_q$, $s \in \mathbb{C}$, and $x \in D$, an action of Γ_q on $\mathrm{Fct}(D;\mathbb{C})$ can be defined. With that the associated transfer operator family

$$\mathcal{L}_{F,s}\colon\ \mathrm{Fct}(D;\mathbb{C}) \longrightarrow \mathrm{Fct}(D;\mathbb{C})$$

for $s \in \mathbb{C}$ is given by

$$\mathcal{L}_{F,s} = \sum_{k=1}^{q-1} \tau_s(g_k)\,,$$

for $x \in D$. The family $\{\mathcal{L}_{F,s}\}_s$ is called the *family of slow transfer operators* and it will fail to be nuclear of order zero on any Banach space of holomorphic functions on open $\widehat{\mathbb{C}}$-neighborhoods of the sets D_k (*complexification*). The reason for that is the cusp of the orbisurface $\mathbb{X}_q$, or more precisely the local structure of the underlying cross section in proximity to the cusp. Periodic geodesics with extended sojourns into the cusp experience particularly slow coding, wherefore the symbolic dynamics (D, F) fails to be uniformly expanding on certain subsets of the geodesic boundary of $\mathbb{H}$.

To overcome this issue Möller and Pohl applied an *acceleration procedure*, also called *induction on parabolic elements.* This procedure gives rise to a second family, called the *family of fast transfer operators.* For its definition let

$$D_1^{(n)} \coloneqq \big(n\lambda, (n+1)\lambda\big) \setminus \Gamma_q.\infty \quad \text{and} \quad D_{q-1}^{(n)} \coloneqq \left(\frac{1}{(n+1)\lambda}, \frac{1}{n\lambda}\right) \setminus \Gamma_q.\infty\,,$$

for $n \in \mathbb{N}$, and define the dynamics $\widetilde{F}\colon D \to D$ by the diffeomorphisms

$$\widetilde{F}|_{D_k} \coloneqq g_k\colon D_k \longrightarrow D \qquad \text{for}\ \ k = 2, \ldots, q-2\,,$$

as well as

$$\widetilde{F}\big|_{D_1^{(n)}} \coloneqq g_1^n\colon D_1^{(n)} \longrightarrow D \setminus D_1\,, \quad \text{and}$$
$$\widetilde{F}\big|_{D_{q-1}^{(n)}} \coloneqq g_{q-1}^n\colon D_{q-1}^{(n)} \longrightarrow D \setminus D_{q-1}\,,$$

for all $n \in \mathbb{N}$. The transfer operator family associated with this dynamics is then formally given by

$$\mathcal{L}_{\widetilde{F},s} := \sum_{n=1}^{\infty} \mathbb{1}_{D\setminus D_1}\tau_s(g_1^n) + \sum_{k=2}^{q-2}\tau_s(g_k) + \sum_{n=1}^{\infty}\mathbb{1}_{D\setminus D_{q-1}}\tau_s(g_{q-1}^n)\,, \tag{4}$$

where $\mathbb{1}_M$ denotes the characteristic function of any subset M of $\mathbb{C}$. Now, a complexification of the underlying sets exists, which gives rise to a suitable Banach space of functions on which operator sums induced by (4) converge to nuclear operators $\widetilde{\mathcal{L}}_s$ of order zero for $\operatorname{Re} s$ sufficiently large, and the map $s \mapsto \widetilde{\mathcal{L}}_s$ admits a meromorphic continuation to all of $\mathbb{C}$, with values in nuclear operators of order zero. We refer the reader to [44, Section 4.2] for the details (see also Property 5 in Section 3.1 and the definitions in Section 3.3). The identity (2) is then obtained via a thermodynamic formalism approach [44, Theorem 4.15]. Therefore, the orbit space of every cofinite Hecke triangle group is admissible for Theorem A, at least for χ the trivial one-dimensional representation.

In order to extend the scope of admissible orbisurfaces we aim to generalize the approach by Möller and Pohl. Pohl's cusp expansion algorithm already applies to (almost) all non-cocompact geometrically finite Fuchsian groups with cusps, regardless of torsion and covolume. Thus, the desired approach for these groups breaks down to a generalization of the acceleration procedure and the thermodynamic formalism. Also, one would like to include non-trivial finite-dimensional representations of fitting regularity. Fortunately, for the latter two objectives a framework has already been provided by Fedosova and Pohl in [22]. Their *strict transfer operator approach* provides a list of properties that together guarantee the existence of a suitable Banach space, a family of (twisted) fast transfer operators, as well as feasibility of a thermodynamic formalism approach to the χ-twisted Selberg zeta function, for all finite-dimensional linear representations χ having non-expanding cusp monodromy. Hence, in other words, they proved the following statement.

(STOA) *Every hyperbolic orbisurface that admits a strict transfer operator approach is admissible for Theorem A, for every good representation.*

For twists without non-expanding cusp monodromy they further showed that the product in the definition of $Z_{\mathbb{X},\chi}(s)$ diverges for every choice of s. Thus, the good representations are optimal for their purpose in Theorem A.

The demands for a strict transfer operator approach are listed in Section 3.1. These demands make it possible to verify, on a group-by-group basis, that a given cross section gives rise to a fast transfer operator family suitable for a representation as in (3). If this is the case, then this family is given explicitly in terms of the transformation sets and intervals one is required to provide (i. e., in terms of the *structure tuple*, see Section 3.3). But they do not allow to directly obtain these cross sections and hence, in particular, they do not imply existence of strict transfer operator approaches.

To find such cross sections and thereby deduce such an existence statement is the main objective of this thesis. To that end, for a geometrically finite Fuchsian group Γ we introduce the notion of *sets of branches* (see Section 4.1). These are finite families of subsets of the unit tangent bundle of $\mathbb{H}$, the union of which represents a cross section for the geodesic flow on the orbit space of Γ. They are defined by purely geometric means, which makes it easy to verify the fulfillment of all requirements for given candidates. Mirroring the approach in the case of Hecke triangle groups, sets of branches give rise to a family of slow transfer operators (Section 4.7), and, depending on Γ, require an acceleration procedure for the construction of an accompanying family of fast transfer operators (Chapter 5). The notion of sets of branches therefore encapsulates the insight that the structure of the cross section conducting the discretization of the geodesic flow already completely determines the eligibility of all objects that stem from it. The central result about sets of branches reads as follows.

Theorem B. *Every Fuchsian group that admits the construction of a set of branches also admits a strict transfer operator approach.*

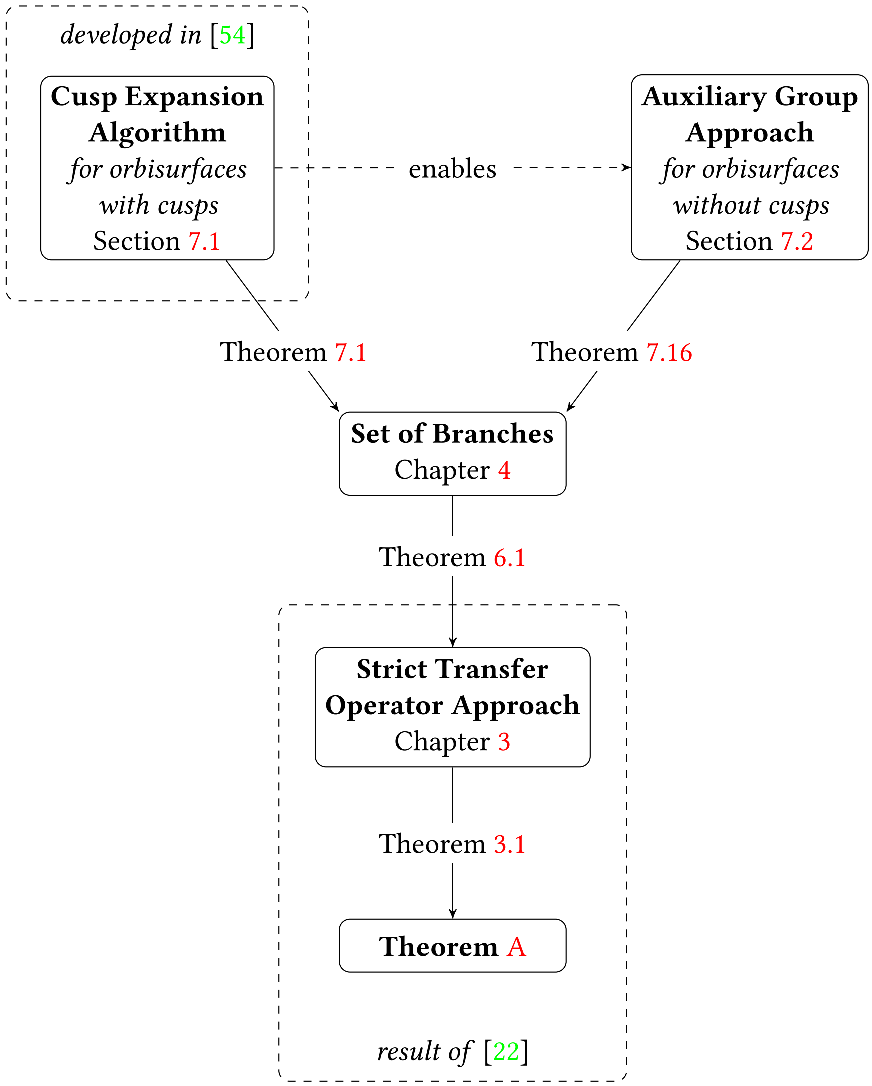
The combination of Theorem B with (STOA) implies Theorem A, wherefore it remains to investigate for which Fuchsian groups sets of branches can be obtained. For this an obvious starting point is Pohl's cusp expansion algorithm, the cross sections emerging from which are indeed seen to come from a set of branches (Section 7.1). But we are not content with Fuchsian groups with cusps, for it turns out that, while non-compactness of the orbisurface is indeed crucial for our approach, the presence of cusps is not. By an auxiliary group argument we manage to construct sets of branches also for non-compact hyperbolic orbisurfaces whose hyperbolic ends are all funnels (Section 7.2). Since we do so by essentially applying the cusp expansion algorithm "out of context" (i. e., for orbisurfaces without cusps), we inherit a technical limitation faced by the Fuchsian group Γ, simply called Condition (A) (see Section 2.1).

Finally, if Γ does not contain hyperbolic elements, then the product in (1) (and also the analogue in the definition of $Z_{\mathbb{X},\chi}$) is void, and hence $Z_{\mathbb{X},\chi} \equiv 1$. In this degenerate case it will also be impossible to construct sets of branches by virtue of their definition. In conclusion, because we provide explicit sets of branches in the settings described above, Theorem B then proves the following result.

Theorem C. *Let Γ be a geometrically finite non-cocompact Fuchsian group that contains hyperbolic elements and fulfills Condition* (A)*. Then $\mathbb{X} = \Gamma\backslash\mathbb{H}$ is an admissible orbisurface for Theorem* A.

Below we provide a diagram that visualizes the structure of proof for Theorem A, including explicit references to the results in this thesis.

We close this section with a few remarks on the accompanying family of *slow* transfer operators. This family, which we obtain not only in the case of cofinite Hecke triangle groups (see the family $\{\mathcal{L}_{F,s}\}_s$ above), but also for every set of



branches approach (see Section 4.7), is far from a superfluous by-catch. In [44] Möller and Pohl have shown that a certain subset of the eigenspace to the eigenvalue 1 of the transfer operator $\mathcal{L}_{F,s}$ from above is in bijection with the space of *Maass cusp forms* to the eigenvalue $s(1-s)$. These are a special class of automorphic forms: highly regular eigenfunctions of the Laplacian on the orbisurface $\mathbb{X}$ invariant under the fundamental group. In [56] Pohl provided this identification between Maass cusp forms and eigenfunctions of slow transfer operators for almost all nonuniform cofinite Fuchsian groups. Recently, Bruggeman and Pohl [15] managed to obtain a first such result for a class of Fuchsian groups of infinite covolume, namely the non-cofinite Hecke triangle groups, necessitating an extension to a broader class of automorphic forms in the process.

On a related note, Adam and Pohl [1] have shown that the 1-eigenspaces of fast and slow transfer operators for Hecke triangle groups are closely related as well. More precisely, the 1-eigenspace of the (unitarily twisted) fast transfer operator with parameter s is isomorphic to a certain subset of the 1-eigenspace of the (unitarily twisted) slow transfer operator with parameter s. Hence, these transfer operator techniques ultimately reveal a relation between automorphic forms and resonances of the Laplacian.

To date, we do not know whether such relations between the eigenspaces of transfer operators hold in general, and this subject is not studied in this thesis. But we expect similar results to be obtainable for various classes of hyperbolic orbisurfaces, and we do provide, in this thesis, families of closely related slow and fast transfer operators prone to spectral investigations.

Structure of the Thesis

The first three chapters of this thesis are of preliminary nature. In Chapter 1 we provide the necessary background in hyperbolic geometry, including rigorous definitions for cross sections for the geodesic flow and transfer operators associated with them, as well as for (twisted) Selberg zeta functions and their relation to the resonances of the Laplacian.

Chapter 2 recalls the cusp expansion algorithm from [54] and collects certain properties of the arising cross section. In comparison to [54] the exposition has been simplified in various regards to better fit the needs of our analysis.

In Chapter 3 we recall the concept of strict transfer operator approaches as well as the main result of [22], which constitutes a significant part of the proof of Theorem A. Chapter 3 also includes a brief survey on the definition of nuclear operators and their traces and Fredholm determinants.

Chapter 4 is dedicated to the definition and extensive study of our central object, the *set of branches*.

In Chapter 5 we show how to transform these sets of branches in order to obtain the structure necessary for a strict transfer operator approach. This is comprised of three distinct algorithms, called *branch reduction*, *identity elimination*, and *cuspidal acceleration*.

Chapter 6 consists of our first main result, the explicit version of Theorem B, as well as its proof. The proof successively verifies each of the properties demanded by the strict transfer operator approach, which is reflected in the formal structure of Chapter 6.

Finally, in Chapter 7, we construct explicit sets of branches for non-cocompact Fuchsian groups with and without cusps. This yields a constructive proof of Theorem C.

Throughout this thesis a multitude of examples and figures is provided in order to illustrate the various statements and constructions. Besides the bibliography, also an index of figures, of terminology, as well as of notations are appended.

This thesis is based on the following publications:

[63] A. Pohl and P. Wabnitz, *Selberg zeta functions, cuspidal accelerations, and existence of strict transfer operator approaches*, arXiv:2209.05927, 2022.

[79] P. Wabnitz, *Strict transfer operator approaches for non-compact hyperbolic orbisurfaces*, arXiv:2209.06601, 2022.

Chapter 1

Elements of Hyperbolic Geometry

In this chapter we present the background material on hyperbolic orbisurfaces and their geodesic flows necessary for our investigations. Comprehensive treatises, including proofs for all statements that we leave unproven, can be found in the many excellent textbooks on hyperbolic geometry. We refer in particular to [4, 6, 68, 76, 33, 35, 31].

We will use throughout standard notations such as $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ for the set of positive numbers, non-negative numbers, and all integers, respectively. We use $\mathbb{R}$ and $\mathbb{C}$ for the set of real and complex numbers, respectively, both equipped with the Euclidean topology. The induced norm in both cases is denoted by $|\cdot|$. We write $\mathrm{i} \coloneqq \sqrt{-1}$ for the imaginary unit. For any set M we denote by

$$\#M \in [0, +\infty]$$

the number of elements in M. We denote closed, semi-closed/half-open and open intervals in $\mathbb{R}$ by $[a, b]$, $(a, b]$, $[a, b)$ and (a, b) for any $a, b \in \mathbb{R}$, respectively, or, if applicable also for $a, b \in \mathbb{R} \cup \{\pm\infty\}$. We use the abbreviations $\mathbb{R}_{>0} \coloneqq (0, +\infty)$ and $\mathbb{R}_{\geq 0} \coloneqq [0, +\infty)$. For any $z \in \mathbb{C}$ we denote its real part by $\operatorname{Re} z$ and its imaginary part by $\operatorname{Im} z$. We emphasize that we consider Re and Im as projections:

$$\operatorname{Re}\colon \begin{cases} \mathbb{C} & \longrightarrow & \mathbb{R} \\ x + \mathrm{i}y & \longmapsto & x \end{cases} \qquad \text{and} \qquad \operatorname{Im}\colon \begin{cases} \mathbb{C} & \longrightarrow & \mathbb{R} \\ x + \mathrm{i}y & \longmapsto & y \end{cases} .$$

Hence, for instance, $\operatorname{Re}(M) = \{\operatorname{Re} z \mid z \in M\}$ for any subset M of $\mathbb{C}$, and

$$\operatorname{Re}^{-1} x = \{z \in \mathbb{C} \mid \operatorname{Re} z = x\} \quad \text{and} \quad \operatorname{Re}^{-1}(N) = \{z \in \mathbb{C} \mid \operatorname{Re} z \in N\} ,$$

for any $x \in \mathbb{R}$ and any subset N of $\mathbb{R}$. In particular, we allow restrictions:

For $M \subseteq \mathbb{C}$ and $N \subseteq \mathbb{R}$ we write

$$\mathrm{Re}|_M^{-1}(N) = \{z \in M \mid \mathrm{Re}\, z \in N\} .$$

We denote the sign function by $\mathrm{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$, where

$$\mathrm{sgn}(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

For M a subset of any group G we denote by $\langle M \rangle = \langle x | x \in M \rangle$ the subgroup of G generated by the elements of M. We further denote by M^* the subset of all non-neutral elements of M. Hence, for instance, for G a group of transformations with a neutral element id we have

$$M^* := M \setminus \{\mathrm{id}\} . \tag{1.1}$$

1.1 The Hyperbolic Plane

As model for the hyperbolic plane throughout this thesis we use the *upper half-plane*

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\} ,$$

endowed with the hyperbolic metric given by the line element

$$\mathrm{d}s_z^2 := (\mathrm{Im}\, z)^{-2}\, \mathrm{d}z\, \mathrm{d}\overline{z}$$

at any $z \in \mathbb{H}$. The hyperbolic distance function for this metric is given by[1]

$$\mathrm{dist}_{\mathbb{H}}(z, w) := \mathrm{arcosh}\left(1 + \frac{|z - w|^2}{2\, \mathrm{Im}\, z\, \mathrm{Im}\, w}\right) \tag{1.2}$$

for $z, w \in \mathbb{H}$ (see, e. g., [10, Proposition 2.4]), and we write

$$\mathrm{dist}_{\mathbb{H}}(z, M) := \inf_{w \in M} \mathrm{dist}_{\mathbb{H}}(z, w)$$

and

$$\mathrm{dist}_{\mathbb{H}}(M, N) := \inf_{w \in M,\, v \in N} \mathrm{dist}_{\mathbb{H}}(w, v)$$

[1] Recall that $\mathrm{arcosh}(x)$ denotes the inverse of $\cosh(x)$ in $[0, +\infty)$ and thus is well-defined on $[1, +\infty)$. We have $\mathrm{arcosh}(x) = \log\left(x + \sqrt{x^2 - 1}\right)$.

for subsets M, N of $\mathbb{H}$. The *geodesic boundary* $\partial_g \mathbb{H}$ of $\mathbb{H}$ can and shall be identified, in the obvious way, with the Alexandroff extension (one-point compactification)

$$\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$$

of the real line $\mathbb{R}$. Likewise, we understand the *geodesic closure*

$$\overline{\mathbb{H}}^g := \mathbb{H} \cup \partial_g \mathbb{H}$$

of $\mathbb{H}$ as a subset of the Alexandroff compactification

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \tag{1.3}$$

of $\mathbb{C}$, also known as the Riemann sphere. In the topology of $\widehat{\mathbb{C}}$, the geodesic boundary $\partial_g \mathbb{H}$ of $\mathbb{H}$ is indeed the topological boundary of $\mathbb{H}$, and the geodesic closure $\overline{\mathbb{H}}^g$ is the topological closure of $\mathbb{H}$. The topology of $\overline{\mathbb{H}}^g$ can also be characterized intrinsically, most conveniently by taking advantage of the Riemannian isometries of $\mathbb{H}$. We will recall this characterization in the next subsection, but will not make use of it here.

For a subset K of $\mathbb{H}$, the closure of K in the topology of $\mathbb{H}$ may differ from its closure in the topology of $\overline{\mathbb{H}}^g$. We will write $\overline{K}$ for its closure in $\mathbb{H}$, and $\overline{K}^g$ for its closure in $\overline{\mathbb{H}}^g$. Further, we will write ∂K for the boundary of K in $\mathbb{H}$, and $\partial_g K$ for its boundary in $\overline{\mathbb{H}}^g$. The *geodesic boundary of* K, that is the part of $\partial_g K$ which is contained in $\partial_g \mathbb{H}$, will be denoted by $g K$. For instance, for

$$K := \{z \in \mathbb{H} \mid 0 < \operatorname{Re} z < 1\}$$

we have

$$\begin{aligned} \overline{K} &= \{z \in \mathbb{H} \mid 0 \leq \operatorname{Re} z \leq 1\} \ , \\ \overline{K}^g &= \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1,\ \operatorname{Im} z \geq 0\} \cup \{\infty\} \\ &= \overline{K} \cup [0,1] \cup \{\infty\} \, , \end{aligned}$$

and

$$g K = [0,1] \cup \{\infty\} \, .$$

The sets of inner points of K in the two topologies coincide and are therefore denoted with the same symbol, K°. For $M \subseteq \widehat{\mathbb{R}}$ we also write M° for the inner points of M and $\overline{M}$ for the closure of M in the $\widehat{\mathbb{R}}$-topology.

We will require the following extension of the notion of intervals in $\mathbb{R}$ to *intervals in* $\widehat{\mathbb{R}}$. For any $a, b \in \mathbb{R}$, $a \neq b$, we let

$$(a,b)_c := \begin{cases} (a,b) & \text{if } a < b \\ (a,+\infty) \cup \{\infty\} \cup (-\infty, b) & \text{if } a > b \end{cases}$$

be the *open interval* in $\widehat{\mathbb{R}}$ from a to b. For $a = \infty \in \widehat{\mathbb{R}}$ and $b \in \mathbb{R}$ we set

$$(a,b)_c = (\infty, b)_c = (-\infty, b)\,,$$

and analogously we define $(b,a)_c$. We define semi-open and closed intervals in $\widehat{\mathbb{R}}$ in the obvious, analogous way. In particular, we write $(a,+\infty] \coloneqq (a,+\infty)\cup\{\infty\}$ as well as $[-\infty,b) \coloneqq \{\infty\}\cup(-\infty,b)$. The subscript c refers to the cyclic order of $\widehat{\mathbb{R}}$ that is used implicitly in this definition. We remark that singletons in $\widehat{\mathbb{R}}$ and the empty set cannot be defined consistently within this notation. For subsets M of $\widehat{\mathbb{R}}$ we write M° for the interior and $\overline{M}$ for the closure of M in $\widehat{\mathbb{R}}$.

In order to distinguish between the point ∞ in $\widehat{\mathbb{R}}$ and the two infinite endpoints of $\mathbb{R}$ with its standard order, we will write $\pm\infty$ whenever we refer to the latter ones and use the extended standard order of $\mathbb{R}\cup\{\pm\infty\}$ (i. e., $-\infty < r < +\infty$ for all $r \in \mathbb{R}$). The unsigned symbol ∞ will always refer to the point in $\widehat{\mathbb{R}}$. As usual, we consider $\mathbb{R}$ to be embedded into $\widehat{\mathbb{R}}$. In particular, we have

$$\widehat{\mathbb{R}} = (-\infty,+\infty)\cup\{\infty\}\,.$$

1.2 Classification of the Elements in $\mathrm{PSL}_2(\mathbb{R})$

For a field $\mathbb{K}$ and $n \in \mathbb{N}$ denote by $\mathbb{K}^{n\times n}$ the set of $n\times n$-matrices with entries in $\mathbb{K}$. We write

$$\mathrm{SL}_n(\mathbb{K}) \coloneqq \left\{ q \in \mathbb{K}^{n\times n} \;\middle|\; \det q = 1 \right\}$$

for the *special linear group of degree n over the field* $\mathbb{K}$. We define the *projective special linear group* by setting

$$\mathrm{PSL}_n(\mathbb{K}) \coloneqq \mathrm{SL}_n(\mathbb{K}) \big/ \mathrm{Z}(\mathrm{SL}_n(\mathbb{K}))\ , \tag{1.4}$$

where $\mathrm{Z}(\mathrm{SL}_n(\mathbb{K}))$ denotes the center of $\mathrm{SL}_n(\mathbb{K})$. In this thesis we will be concerned with the case $n = 2$ and $\mathbb{K} = \mathbb{R}$. In this setting the center takes the form

$$\mathrm{Z}(\mathrm{SL}_2(\mathbb{R})) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Hence, $\mathrm{PSL}_2(\mathbb{R})$ consists of equivalence classes of real 2×2-matrices with determinant 1 each of which has exactly two representatives in $\mathrm{SL}_2(\mathbb{R})$ that differ only in sign. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we write $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for its equivalence class in $\mathrm{PSL}_2(\mathbb{R})$. The underlying equivalence relation respects matrix multiplication, which induces a multiplication in $\mathrm{PSL}_2(\mathbb{R})$. Furthermore, for $\mathrm{tr}(\widetilde{g})$ denoting the trace of $\widetilde{g} \in \mathrm{SL}_2(\mathbb{R})$, the *unsigned trace*

$$|\mathrm{tr}(g)| \coloneqq |\mathrm{tr}(\widetilde{g})| \tag{1.5}$$

is well-defined for every $g = [\widetilde{g}] \in \mathrm{PSL}_2(\mathbb{R})$, as it is independent of the choice of representative.

Let G be any subgroup of $\mathbb{C}^{2\times 2}$. Two elements $g, h \in G$ are called *mutually conjugate* or *similar in* G if there exists a regular $q \in G$ such that

$$g = q \cdot h \cdot q^{-1}\,.$$

This defines an equivalence relation on G which we denote by $\sim_G$, where we drop the subscript G whenever we are confident that the underlying group is clear from the context. We write $g \not\sim_G h$ for g, h being non-conjugate and handle the subscript analogously. This concept directly descends to the quotient $\mathrm{PSL}_2(\mathbb{R})$: We say that two elements $g, h \in \mathrm{PSL}_2(\mathbb{R})$ are mutually conjugate, if there exists $q \in \mathrm{PSL}_2(\mathbb{R})$ such that $q \cdot g \cdot q^{-1} = h$, or, equivalently, if there exists a choice of representatives $\widetilde{g}$ of g and $\widetilde{h}$ of h such that $\widetilde{g} \sim_{\mathrm{SL}_2(\mathbb{R})} \widetilde{h}$.

Let $\widetilde{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\widetilde{q} = \left(\begin{smallmatrix} x & y \\ v & w \end{smallmatrix}\right)$ be elements in $\mathrm{SL}_2(\mathbb{R})$. Then

$$\begin{aligned} \widetilde{q} \cdot \widetilde{g} \cdot \widetilde{q}^{-1} &= \begin{pmatrix} x & y \\ v & w \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} w & -y \\ -v & x \end{pmatrix} \\ &= \begin{pmatrix} x(aw - bv) + y(cw - dv) & * \\ * & x(bv + dw) - y(av + cw) \end{pmatrix}. \end{aligned}$$

Hence,

$$\mathrm{tr}(\widetilde{q} \cdot \widetilde{g} \cdot \widetilde{q}^{-1}) = (a + d)(wx - vy) = \mathrm{tr}(\widetilde{g})\,,$$

which means that conjugation in $\mathrm{SL}_2(\mathbb{R})$ preserves traces. On the one hand, this implies that conjugation in $\mathrm{PSL}_2(\mathbb{R})$ preserves unsigned traces. On the other hand it follows that, in general, the matrices $\widetilde{g}$ and $-\widetilde{g}$ are not mutually conjugate in $\mathrm{SL}_2(\mathbb{R})$. This necessitates caution when handling conjugacy in $\mathrm{PSL}_2(\mathbb{R})$.

Furthermore, one has to be careful when applying results which are stated for $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{PSL}_2(\mathbb{C})$. Conjugation in these groups chooses from a larger pool of matrices and thereby enjoys stronger properties. Consider, for instance, [6, Theorem 4.3.1], which states that two elements $g, h \in \mathrm{PSL}_2(\mathbb{C})$ are mutually conjugate if and only if $\mathrm{tr}(g)^2 = \mathrm{tr}(h)^2$. This statement is false in $\mathrm{PSL}_2(\mathbb{R})$, as $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \not\sim \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$.

Lemma 1.1. *Let* $g \in \mathrm{PSL}_2(\mathbb{R})$, $g \neq \mathrm{id}$.

(i) *We have* $|\mathrm{tr}(g)| > 2$ *if and only if there exists* $\ell \in \mathbb{R}_{>0}$, *such that* g *is conjugate within* $\mathrm{PSL}_2(\mathbb{R})$ *to*

$$\mathrm{h}_\ell := \begin{bmatrix} e^{\frac{\ell}{2}} & 0 \\ 0 & e^{-\frac{\ell}{2}} \end{bmatrix}. \tag{1.6}$$

The number ℓ *is then uniquely determined.*

(ii) *We have* $|\mathrm{tr}(g)| = 2$ *if and only if there exists* $\kappa \in \mathbb{R} \setminus \{0\}$ *such that* g *is conjugate within* $\mathrm{PSL}_2(\mathbb{R})$ *to*

$$\mathrm{t}_\kappa := \begin{bmatrix} 1 & \kappa \\ 0 & 1 \end{bmatrix}. \tag{1.7}$$

(iii) *We have* $|\mathrm{tr}(g)| < 2$ *if and only if there exists* $\theta \in (0, 2\pi)$ *such that* g *is conjugate within* $\mathrm{PSL}_2(\mathbb{R})$ *to*

$$\mathrm{s}_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{1.8}$$

If this is the case, then there exists exactly one $\theta \in (0, \pi)$ *with that property.*

Proof. In each of the statements (i), (ii), and (iii) the converse implication follows immediately from the fact that conjugation in $\mathrm{PSL}_2(\mathbb{R})$ preserves unsigned traces. Hence, it remains to prove the conjugacy identities given the respective trace relation.

If $|\mathrm{tr}(g)| > 2$ we find a representative $\widetilde{g} \in \mathrm{SL}_2(\mathbb{R})$ of g whose eigenvalues fulfill the relations

$$0 < \lambda_2 < 1 < \lambda_1 .$$

Hence, $\widetilde{g}$ is diagonalizable over $\mathbb{R}$, meaning there exists a regular matrix $a \in \mathbb{R}^{2\times 2}$ such that

$$\widetilde{g} = a \cdot \mathrm{diag}(\lambda_1, \lambda_2) \cdot a^{-1} .$$

Since a is given by eigenvectors of $\widetilde{g}$ to the eigenvalues $\lambda_{1/2}$, it can be normalized in order to obtain $\det a = 1$, i. e., $a \in \mathrm{SL}_2(\mathbb{R})$. Thus, the claim follows with

$$\ell := 2 \log \lambda_1 . \tag{1.9}$$

The only alternative would be to conjugate $\widetilde{g}$ to $\mathrm{diag}(\lambda_2, \lambda_1)$ instead. But this leads to $\ell < 0$, proving that (1.9) yields the only choice of ℓ in the required interval. Hence, (i) follows.

Now assume $|\mathrm{tr}(g)| = 2$ and let $\widetilde{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ be the representative of g fulfilling $\mathrm{tr}(\widetilde{g}) = a + d = 2$. If $c = 0$, then, by virtue of the determinant condition, we obtain $a = d = 1$. Since g was assumed not to be the identity, we further have $b \neq 0$. Then,

$$\begin{pmatrix} \sqrt{|b|} & 0 \\ 0 & \frac{1}{\sqrt{|b|}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathrm{sgn}(b) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{|b|}} & 0 \\ 0 & \sqrt{|b|} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

If $c \neq 0$, let

$$q := \begin{pmatrix} 0 & \frac{1}{2c} \\ -2c & a - d \end{pmatrix}.$$

Then we have

$$q \cdot \widetilde{g} \cdot q^{-1} = \begin{pmatrix} 0 & \frac{1}{2c} \\ -2c & a-d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a-d & -\frac{1}{2c} \\ 2c & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4c} \\ 0 & 1 \end{pmatrix}.$$

Hence, in either case (ii) follows.

Finally, assume $|\mathrm{tr}(g)| < 2$ and let again $\widetilde{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ be a representative of g. Then $c \neq 0$ and we can choose $\widetilde{g}$ so that $c > 0$. We set $D := \sqrt{4 - \mathrm{tr}(\widetilde{g})^2}$ and define

$$q := \begin{pmatrix} \sqrt{\frac{2c}{D}} & \frac{d-a}{\sqrt{2cD}} \\ 0 & \sqrt{\frac{D}{2c}} \end{pmatrix}.$$

With that we obtain

$$\begin{aligned} q \cdot \widetilde{g} \cdot q^{-1} &= \begin{pmatrix} \sqrt{\frac{2c}{D}} & \frac{d-a}{\sqrt{2cD}} \\ 0 & \sqrt{\frac{D}{2c}} \end{pmatrix} \cdot \begin{pmatrix} a\sqrt{\frac{D}{2c}} & \frac{a(a-d)+2bc}{\sqrt{2cD}} \\ \sqrt{\frac{cD}{2}} & \mathrm{tr}(\widetilde{g})\sqrt{\frac{c}{2D}} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \mathrm{tr}(\widetilde{g}) & -D \\ D & \mathrm{tr}(\widetilde{g}) \end{pmatrix}. \end{aligned}$$

Since $\cos([0, 2\pi]) = [-1, 1]$ and $|\mathrm{tr}(g)| < 2$ we find $\theta \in (0, 2\pi)$ such that

$$\frac{\mathrm{tr}(\widetilde{g})}{2} = \cos\theta \,.$$

Furthermore,

$$\frac{D}{2} = \sqrt{1 - \frac{\mathrm{tr}(\widetilde{g})^2}{4}} = \sqrt{1 - \cos^2\theta} = |\sin\theta| \,.$$

This shows the existence statement in (iii). Uniqueness of $\theta \in (0, \pi)$ follows by observing that the above two equations have exactly two solutions $\theta_1, \theta_2 \in (0, 2\pi)$ and we have $|\theta_1 - \theta_2| = \pi$. □

Lemma 1.1 provides a complete classification of the elements of $\mathrm{PSL}_2(\mathbb{R})$. Let $g \in \mathrm{PSL}_2(\mathbb{R})$, $g \neq \mathrm{id}$. We call g

- *hyperbolic* if $|\mathrm{tr}(g)| > 2$,
- *parabolic* if $|\mathrm{tr}(g)| = 2$, and
- *elliptic* if $|\mathrm{tr}(g)| < 2$.

The identity constitutes its own class. Note that the number κ from (ii) not unique. In particular, κ can always be chosen to be either 1 or -1, depending on g.

For $g \in \mathrm{PSL}_2(\mathbb{R})$ hyperbolic the number ℓ from part (i) of Lemma 1.1 can be calculated as

$$\ell = 2\log\left(|\mathrm{tr}(g)| + \sqrt{|\mathrm{tr}(g)|^2 - 4}\right) - 2\log 2\,.$$

It is called the *displacement length* of g and we denote it by $\ell(g)$. In Section 1.7 we will recall its relation to periodic geodesics.

We end this section with a characterization of the involutions in $\mathrm{PSL}_2(\mathbb{R})$. As usual, a transformation g is called an *involution*, if $g^{-1} = g$. The following result is well-known.

Lemma 1.2. *The non-identity involutions in* $\mathrm{PSL}_2(\mathbb{R})$ *are exactly the elliptic elements* g *with* $|\mathrm{tr}(g)| = 0$.

Proof. Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Assume first that $a + d = 0$. Then

$$g^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} a^2 + bc & 0 \\ 0 & d^2 + bc \end{bmatrix}.$$

Hence,

$$\left|\mathrm{tr}(g^2)\right| = \left|a^2 + 2bc + d^2\right| = \left|(a+d)^2 - 2\right| = 2\,.$$

Together with the determinant condition it follows that $g^2 = \mathrm{id}$. Now assume that g is an involution. Then $b\,|\mathrm{tr}(g)| = c\,|\mathrm{tr}(g)| = 0$ by the calculation above. Setting $b = c = 0$ leads to $a^2 = d^2 = 1$ and thus $a, d \in \{\pm 1\}$. The determinant condition then assures $a = d$ and thus $g = \mathrm{id}$. The only alternative is $|\mathrm{tr}(g)| = 0$, which yields the assertion. □

1.3 Riemannian Isometries on $\mathbb{H}$

It is well known (see, e. g., [33, Theorems 1.1.2 and 1.3.1] or [10, Proposition 2.2]) that $\mathrm{Isom}^+(\mathbb{H})$, the *group of orientation preserving Riemannian isometries* of $\mathbb{H}$, is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, considered as acting on $\mathbb{H}$ from the left by linear fractional transformations (Möbius transformations). For that reason we usually refer to the elements of $\mathrm{PSL}_2(\mathbb{R})$ as *transformations*. With respect to this identification and the above notation, the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} . z := \frac{az + b}{cz + d}$$

for any $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ and $z \in \mathbb{H}$. Note that the determinant condition assures that $cz + d \neq 0$ for any $z \in \mathbb{H}$. The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive and faithful and every $g \in \mathrm{PSL}_2(\mathbb{R})$ is conformal and maps circles onto circles[2]. Here by circles we mean generalized circles in the Riemann sphere $\widehat{\mathbb{C}}$.

[2]Note that the hyperbolic centers of circles do not match their Euclidean ones. Therefore, even though circles are preserved by Möbius transformations, in general their Euclidean centers are not.

Each transformation $g \in \mathrm{PSL}_2(\mathbb{R})$ extends smoothly to an action on $\overline{\mathbb{H}}^g$. For every transformation $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ and $z \in \overline{\mathbb{H}}^g$ we have

$$g.z := \begin{cases} \frac{a}{c} & \text{if } z = \infty \\ \frac{az+b}{cz+d} & \text{otherwise} \end{cases}, \tag{1.10}$$

where $g.z := \infty$ whenever the denominator on the right hand side of (1.10) vanishes. As for $\mathbb{H}$, the action of $\mathrm{PSL}_2(\mathbb{R})$ restricted to $\partial_g\mathbb{H}$ is transitive and faithful. In particular, $g.\partial_g\mathbb{H} = \partial_g\mathbb{H}$ for every $g \in \mathrm{PSL}_2(\mathbb{R})$. Throughout we will identify the elements in $\mathrm{PSL}_2(\mathbb{R})$ with their action on $\overline{\mathbb{H}}^g$.

For the action of $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ on $\overline{\mathbb{H}}^g$ we have

$$g'(z) = \frac{1}{(cz+d)^2}\,.$$

This is not a linear fractional transformation anymore, which is reflected in the different choice of notation. Since $(c,d) \neq (0,0)$, g' is meromorphic (on $\widehat{\mathbb{C}}$) with a single pole in $z_0 = -\frac{d}{c}$ of order 2 and residue 0.

The classes of transformations in $\mathrm{PSL}_2(\mathbb{R})$ identified in Section 1.2 can further be characterized via the number and location of their fixed points. Each element in $\mathrm{PSL}_2(\mathbb{R})$ has at least one fixed point in $\overline{\mathbb{H}}^g$. The identity element

$$\mathrm{id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

is the unique linear fractional transformation that fixes (at least) two points in $\mathbb{H}$. It is also the unique element with three fixed points in $\overline{\mathbb{H}}^g$. Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$, $g \neq \mathrm{id}$. If $c = 0$, then, by (1.10), we see that g fixes ∞. If $d - a \neq 0$, then g has a further fixed point in $\frac{b}{d-a}$. If this is the case, then we obtain $|a+d| > 2$, meaning g is hyperbolic. If $d - a = 0$, then ∞ is the sole fixed point of g and g is parabolic. Assume now that $c \neq 0$. Then $-\frac{d}{c} \neq \infty$. Therefore, it cannot be a fixed point of g. For $z \neq -\frac{d}{c}$ the fixed point equation $g.z = z$ transforms to

$$z^2 + \frac{d-a}{c}z - \frac{b}{c} = 0\,.$$

This equation has exactly two solutions in $\mathbb{R}$, if and only if $\mathrm{tr}(g)^2 - 4 > 0$, exactly one solution in $\mathbb{R}$, if and only if $\mathrm{tr}(g)^2 - 4 = 0$, and exactly two solutions in $\mathbb{C} \setminus \mathbb{R}$, if and only if $\mathrm{tr}(g)^2 - 4 < 0$. In the latter case, the two solutions are mutually conjugate complex numbers, meaning exactly one of them is contained in $\mathbb{H}$. Hence, we have proven the following result.

Lemma 1.3. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$. If g fixes two distinct points in $\mathbb{H}$ or three distinct points in $\overline{\mathbb{H}}^g$, then $g = \mathrm{id}$. Furthermore, we have that*

(i) *g is hyperbolic if and only if it fixes exactly two distinct points in $\partial_g\mathbb{H}$,*

(ii) *g is parabolic if and only if it fixes exactly one point in $\partial_g \mathbb{H}$, and*

(iii) *g is elliptic if and only if it fixes exactly one point in $\mathbb{H}$.*

The fixed points of hyperbolic and parabolic transformations can further be characterized as limit points for $\mathrm{PSL}_2(\mathbb{R})$-orbits, that are sequences of the form $(g_n.z)_{n\in\mathbb{N}}$ for $g_n \in \mathrm{PSL}_2(\mathbb{R})$ and $z \in \mathbb{H}$. With a little caution, this characterization extends to $\overline{\mathbb{H}}^g$.

Lemma 1.4. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be hyperbolic.*

(i) *For any $z \in \mathbb{H}$, the limit of $(g^n.z)_{n\in\mathbb{N}}$ in $\overline{\mathbb{H}}^g$ exists and is independent of the choice of z. We denote this limit point by $\mathrm{f}_+(g)$.*

(ii) *For any $z \in \mathbb{H}$, the limit of $(g^{-n}.z)_{n\in\mathbb{N}}$ in $\overline{\mathbb{H}}^g$ exists and is independent of the choice of z. We denote this limit point by $\mathrm{f}_-(g)$.*

(iii) *For all $z \in \overline{\mathbb{H}}^g$ we have*

$$\lim_{n\to+\infty} g^n.z = \begin{cases} \mathrm{f}_-(g) & \text{if } z = \mathrm{f}_-(g) \\ \mathrm{f}_+(g) & \text{otherwise} \end{cases}$$

and

$$\lim_{n\to-\infty} g^n.z = \begin{cases} \mathrm{f}_+(g) & \text{if } z = \mathrm{f}_+(g) \\ \mathrm{f}_-(g) & \text{otherwise} \end{cases}.$$

Lemma 1.5. *Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ be parabolic. For any choice of $z \in \overline{\mathbb{H}}^g$ the limits of the sequences $(g^n.z)_{n\in\mathbb{N}}$ and $(g^{-n}.z)_{n\in\mathbb{N}}$ exist and are independent of the choice of z. We have*

$$\mathrm{f}(g) := \lim_{n\to+\infty} g^n.z = \lim_{n\to-\infty} g^n.z\,.$$

The statements of Lemma 1.4 resp. Lemma 1.5 are easily verified for transformations of the form h_ℓ resp. t_κ from (1.6) resp. (1.7) and remain valid under conjugation in $\mathrm{PSL}_2(\mathbb{R})$. Lemma 1.1(i) resp. (ii) then yield validity in the general case.

Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ be hyperbolic. From continuity of the transformations g and g^{-1} we see that $\mathrm{f}_+(g)$ and $\mathrm{f}_-(g)$ from Lemma 1.4 are exactly the fixed points of g and g^{-1} alike. The limit point $\mathrm{f}_+(g)$ is called the *attracting fixed point* or *attractor of g*. The limit point $\mathrm{f}_-(g)$ we call the *repelling fixed point* or the *repeller of g*. By solving the fixed point equation and comparing parts (i) and (ii) of Lemma 1.4 one derives

$$\mathrm{f}_+(g^{-1}) = \mathrm{f}_-(g) = \frac{a-d}{2c} - \frac{1}{2\,|c|}\sqrt{|\mathrm{tr}(g)|^2 - 4}\,, \tag{1.11}$$

and

$$\mathrm{f}_-(g^{-1}) = \mathrm{f}_+(g) = \frac{a-d}{2c} + \frac{1}{2\,|c|}\sqrt{|\mathrm{tr}(g)|^2 - 4}\,, \tag{1.12}$$

independently of the choice of representative of g. Analogously, if g is parabolic, then $\mathrm{f}(g)$ from Lemma 1.5 is the fixed point of g. We have

$$\mathrm{f}(g) = \begin{cases} \frac{a-d}{2c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}\,, \tag{1.13}$$

which again is obviously independent of the choice of representative.

Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ be elliptic. Then $c \neq 0$ and again from the fixed point equation one easily derives

$$\mathrm{f}(g) = \frac{a-d}{2c} + \frac{\mathrm{i}}{2\,|c|}\sqrt{4 - |\,\mathrm{tr}(g)|^2}\,, \tag{1.14}$$

where $\mathrm{f}(g)$ likewise denotes the (unique) fixed point of g. The fixed point of an elliptic transformation is also called an *elliptic point.*

Lemma 1.6 ([33, Theorem 2.3.2]). *Two non-identity elements of* $\mathrm{PSL}_2(\mathbb{R})$ *commute if and only if they have the same fixed point set.*

1.4 Topology of $\overline{\mathbb{H}}^g$

For subgroups Γ of $\mathrm{PSL}_2(\mathbb{R})$ and subsets M of $\overline{\mathbb{H}}^g$ the set

$$\Gamma.M \coloneqq \{g.x \mid g \in \Gamma,\ x \in M\}$$

is called the Γ-*orbit* of M. For M a singleton, say $M = \{z\}$, we abbreviate to

$$\Gamma.z \coloneqq \Gamma.\{z\} = \{g.z \mid g \in \Gamma\}\ .$$

Taking advantage of the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\overline{\mathbb{H}}^g$, we can conveniently provide an intrinsic characterization of the topology of $\overline{\mathbb{H}}^g$. On the subset $\mathbb{H}$ of $\overline{\mathbb{H}}^g$ the topology is given by the Euclidean topology of $\mathbb{C}$. A neighborhood basis at ∞ is given by the family

$$\mathcal{U}_\infty \coloneqq \{U_\varepsilon \mid \varepsilon > 0\}$$

consisting of the open sets

$$U_\varepsilon \coloneqq \left\{z \in \mathbb{H} \,\middle|\, \mathrm{Im}(z) > \varepsilon^{-1}\right\} \cup \{\infty\}\,, \quad \varepsilon > 0\,.$$

Finally, since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\partial_g\mathbb{H}$, the images $\mathrm{PSL}_2(\mathbb{R}).\mathcal{U}_\infty$ of this neighborhood basis at ∞ yield neighborhood bases at any point of $\partial_g \mathbb{H}$, each one consisting of open sets. Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be such that $g.\infty \neq \infty$ and let $\varepsilon > 0$. Then $g.U_\varepsilon$ is an open disk in $\mathbb{H}$ fulfilling

$$\mathrm{dist}_{\mathrm{E}}(\mathbb{R}, g.U_\varepsilon) = \mathrm{dist}_{\mathrm{E}}(g.\infty, g.U_\varepsilon) = 0\,,$$

where $\mathrm{dist}_{\mathrm{E}}$ denotes the Euclidean distance function in $\mathbb{C}$. Such disks are called *horoballs*.

For $\varepsilon > 0$ and $z \in \mathbb{H}$ we write

$$\mathrm{B}_\varepsilon(z) := \{w \in \mathbb{H} \mid \mathrm{dist}_{\mathbb{H}}(z, w) < \varepsilon\}\,. \tag{1.15}$$

for the open (hyperbolic) ball of radius ε around z. In contrast, for $z \in \partial_g \mathbb{H}$ the ball of radius ε around z in the metric of $\widehat{\mathbb{R}}$ is denoted by

$$\mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(z) := \begin{cases} (z-\varepsilon, z+\varepsilon) & \text{if } z \in \mathbb{R} \\ \widehat{\mathbb{R}} \setminus (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) & \text{if } z = \infty \end{cases}, \tag{1.16}$$

or by $\mathrm{B}_{\mathbb{R},\varepsilon}(z)$ if $z \in \mathbb{R}$.

1.5 Geodesics on the Hyperbolic Plane

A smooth curve $\gamma\colon I \subseteq \mathbb{R} \to \mathbb{M}$, $s \mapsto (x_1(s), x_2(s))$, on a two-dimensional manifold $\mathbb{M}$ and parameterized by arc length is a *geodesic* if and only if it satisfies the geodesic equations

$$\frac{\mathrm{d}^2 x_k}{\mathrm{d}s^2} + \sum_{i,j=1}^{2} \boldsymbol{\Gamma}_{ij}^k \frac{\mathrm{d}x_i}{\mathrm{d}s}\frac{\mathrm{d}x_j}{\mathrm{d}s} = 0\,, \qquad k = 1, 2\,,$$

where $\boldsymbol{\Gamma}_{ij}^k$ denote the Christoffel symbols (see, e. g., [34, Section 4B]). In the upper half-plane model $\mathbb{H}$ of the hyperbolic plane we find

$$\boldsymbol{\Gamma}_{12}^1 = \boldsymbol{\Gamma}_{21}^1 = \boldsymbol{\Gamma}_{22}^2 = -\boldsymbol{\Gamma}_{11}^2 = -\frac{1}{x_2}\,, \quad \text{and} \quad \boldsymbol{\Gamma}_{11}^1 = \boldsymbol{\Gamma}_{12}^2 = \boldsymbol{\Gamma}_{21}^2 = \boldsymbol{\Gamma}_{22}^1 = 0\,,$$

which leads to the differential equations

$$x_1'' x_2 - 2x_1' x_2' = x_2'' x_2 + \left(x_1'\right)^2 - \left(x_2'\right)^2 = 0\,, \tag{1.17}$$

where $(\,.\,)'$ denotes the derivative with respect to s. Let

$$\gamma\colon \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{H} \\ s & \longmapsto & \big(x(s), y(s)\big) \end{cases}$$

be a solution of (1.17). If $x' \equiv 0$, then $\gamma(\mathbb{R})$ is a vertical line in $\mathbb{H}$. If $x' \not\equiv 0$, then there exist constants $a, b \in \mathbb{R}$ such that

$$x^2 - ax + y^2 = b\,.$$

Hence, $\gamma(\mathbb{R})$ is a circle with center at the line $y = 0$. This means a mapping $\gamma \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{H})$ is a geodesic on $\mathbb{H}$ if and only if γ is injective and $\gamma(\mathbb{R})$ is a (generalized) semicircle perpendicular to $\partial_g \mathbb{H}$ (we refer also to [10, Proposition 2.3]). We assume all geodesics on $\mathbb{H}$ to be parameterized by arc length (*unit speed geodesics*). Despite that, we differ from tradition by denoting the parameter by t ("time") instead of the arc length s.

Convention

For $\gamma\colon I \subseteq \mathbb{R} \to \mathbb{M}$, $t \mapsto (x(t), y(t))$, a smooth curve on a two-dimensional manifold or orbifold $\mathbb{M}$ we write

$$\dot\gamma(t) := \left(\frac{\mathrm{d}x}{\mathrm{d}t}(t), \frac{\mathrm{d}y}{\mathrm{d}t}(t)\right) \in \mathbb{R}^2$$

for the derivative of γ with respect to t. We further set

$$\gamma'(t) := \big(\gamma(t), \dot\gamma(t)\big) \in \mathbb{M} \times \mathbb{R}^2\,.$$

We distinguish between geodesics, that are smooth curves $\gamma\colon \mathbb{R} \to \mathbb{H}$, and geodesic arcs, segments, and rays, that are subsets of $\gamma(\mathbb{R}) \subseteq \mathbb{H}$ for some geodesic γ. A *geodesic segment* is any connected subset of $\gamma(\mathbb{R})$ for any geodesic γ. A geodesic segment is called *complete* or a *geodesic arc* if it equals $\gamma(\mathbb{R})$. Accordingly, for γ a geodesic on $\mathbb{H}$ the subset $\gamma(\mathbb{R})$ is called the *arc* of γ. A *geodesic ray* is a geodesic segment of the form $\gamma((-\infty, r))$ or $\gamma((r, +\infty))$ with $r \in \mathbb{R}$. For γ a geodesic on $\mathbb{H}$ we denote by

$$\gamma(\pm\infty) := \lim_{r \to \pm\infty} \gamma(r)$$

the *endpoints* of γ in $\partial_g \mathbb{H}$. For any $r_1, r_2 \in \mathbb{R} \cup \{\pm\infty\}$, $r_1 < r_2$, we call

$$[z_1, z_2]_{\mathbb{H}} := \gamma([r_1, r_2])$$

the *closed geodesic segment* or *closed hyperbolic interval* from $z_1 = \gamma(r_1)$ to $z_2 = \gamma(r_2)$. Analogously, we define the *open* or *semi-open geodesic segments/hyperbolic intervals* $(z_1, z_2)_{\mathbb{H}}$, $[z_1, z_2)_{\mathbb{H}}$, and $(z_1, z_2]_{\mathbb{H}}$. We note that for any two distinct points $z_1, z_2 \in \overline{\mathbb{H}}^g$ the definition of these hyperbolic intervals does not depend on the choice of the geodesic γ as long as $\gamma(r_1) = z_1$, $\gamma(r_2) = z_2$ for some $r_1 < r_2$. The subscript $\mathbb{H}$ for hyperbolic intervals will be maintained throughout. To the contrary, intervals in $\widehat{\mathbb{R}}$ will be denoted by $[r_1, r_2]$ or by $[r_1, r_2]_{\mathbb{R}}$ (and analogous) whenever we deem it appropriate to emphasize the distinction between them and their hyperbolic counterparts or tuples. A geodesic segment β is called *vertical*, if $\mathrm{Re}(\beta)$ is a singleton in $\mathbb{R}$, and *non-vertical* otherwise. Hence, a geodesic segment is vertical if and only if it is contained in $x + \mathrm{i}\mathbb{R}_{\geq 0}$ for some $x \in \mathbb{R}$.

A subset M of $\mathbb{H}$ is called *(hyperbolically) convex* if for all $z, w \in M$ we also have $[z, w]_{\mathbb{H}} \subseteq M$. We denote by $\mathrm{conv}(M) := \bigcup_{z,w\in M}[z, w]_{\mathbb{H}}$ the *convex hull* of M in $\mathbb{H}$. Occasionally we will require the convex hull in the Euclidean sense as well. We denote it by $\mathrm{conv}_{\mathrm{E}}(M)$ in order to avoid confusion.

We denote the *unit tangent bundle* of $\mathbb{H}$ by $\mathrm{S}\mathbb{H}$, i. e.,

$$\mathrm{S}\mathbb{H} = \bigcup_{z\in\mathbb{H}} \{z\} \times \mathrm{T}_{1,z}\mathbb{H}\,, \tag{1.18}$$

where $\mathrm{T}_{1,z}\mathbb{H}$ denotes the unit sphere in the tangent space at $z \in \mathbb{H}$. We further denote by

$$\mathrm{bp}\colon\ \mathrm{S}\mathbb{H} \longrightarrow \mathbb{H} \tag{1.19}$$

the projection onto base points, that is the first component in the tuple $\nu \in \mathrm{S}\mathbb{H}$. This map is obviously well-defined by virtue of the structure of $\mathrm{S}\mathbb{H}$. We will often apply it to subsets of $\mathrm{S}\mathbb{H}$ in order to effectively characterize them by means of the set of base points of their members. The component of ν in $\mathrm{T}_{1,\mathrm{bp}(\nu)}\mathbb{H}$ we denote by $\vec{\nu}$, thus in total for $\nu \in \mathrm{S}\mathbb{H}$ we have

$$\nu = \big(\mathrm{bp}(\nu), \vec{\nu}\big)\,.$$

Each unit tangent vector $\nu \in \mathrm{S}\mathbb{H}$ uniquely determines a geodesic γ_ν on $\mathbb{H}$ via the rule

$$\gamma_\nu'(0) = \nu\,. \tag{1.20}$$

I. e., γ_ν is the unique geodesic passing through $\mathrm{bp}(\nu)$ with derivative equal to $\vec{\nu}$ at time 0. For instance, the geodesic determined by the unit tangent vector $\partial_y|_{\mathrm{i}}$ is

$$\gamma_s := \gamma_{\partial_y|_{\mathrm{i}}}\colon \left\{\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{H} \\ t & \longmapsto & \mathrm{i}e^t \end{array}\right., \tag{1.21}$$

the *standard geodesic* on $\mathbb{H}$. Since the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is by Riemannian isometries, it induces an action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{S}\mathbb{H}$, which is simply transitive. Therefore, $\mathrm{PSL}_2(\mathbb{R})$ also acts simply transitive on the set of all geodesics on $\mathbb{H}$, and hence any geodesic on $\mathbb{H}$ is a (unique) $\mathrm{PSL}_2(\mathbb{R})$-translate of the standard geodesic γ_s from (1.21).

Situated on $\mathrm{S}\mathbb{H}$ is the *(unit speed) geodesic flow* on $\mathbb{H}$, which is the dynamical system

$$\Phi\colon \left\{\begin{array}{ccc} \mathbb{R}\times\mathrm{S}\mathbb{H} & \longrightarrow & \mathrm{S}\mathbb{H} \\ (t,\nu) & \longmapsto & \gamma_\nu'(t) \end{array}\right., \tag{1.22}$$

where γ_ν is the geodesic on $\mathbb{H}$ uniquely induced by ν (see (1.20)).

We end this section by defining an equivalence relation on the set of geodesics on $\mathbb{H}$. Two geodesics γ_1 and γ_2 on $\mathbb{H}$ are considered equivalent if they differ only by a parameter change, i. e., if there exists $t_0 \in \mathbb{R}$ such that

$$\gamma_1(t) = \gamma_2(t + t_0)$$

for all $t \in \mathbb{R}$. We denote the equivalence class of a geodesic γ associated to that relation by $[\gamma]$ and set

$$\mathscr{G}(\mathbb{H}) \coloneqq \{[\gamma] \mid \gamma \text{ a geodesic on } \mathbb{H}\} \ . \tag{1.23}$$

Since every representative of $[\gamma] \in \mathscr{G}(\mathbb{H})$ traces out the same geodesic arc in $\mathbb{H}$, we may denote this arc by $[\gamma](\mathbb{R})$. Two geodesics on γ_1, γ_2 on $\mathbb{H}$ are equivalent w. r. t. this equivalence relation if and only if they have the same arc and orientation, or, equivalently, if

$$\gamma_1(\pm\infty) = \gamma_2(\pm\infty) \, .$$

In other words, an equivalence class $[\gamma] \in \mathscr{G}(\mathbb{H})$ is uniquely determined by the points

$$[\gamma](\pm\infty) \coloneqq \gamma(\pm\infty) \, , \tag{1.24}$$

for any choice of representative. The action of Γ on the set of geodesics descends to an action on $\mathscr{G}(\mathbb{H})$ by

$$g.[\gamma] \coloneqq [g.\gamma]$$

for all $g \in \Gamma$ and all geodesics γ on $\mathbb{H}$. Hence, for all $\nu \in \mathrm{S}\mathbb{H}$ and all $g \in \Gamma$ we have

$$g.[\gamma_\nu] = [\gamma_{g.\nu}] \, .$$

For M a subset of $\mathrm{S}\mathbb{H}$, γ a (unit speed) geodesic on $\mathbb{H}$, and $t \in \mathbb{R}$ we say that γ *intersects M at time t*, if

$$\gamma'(t) \in M \, . \tag{1.25}$$

Accordingly, for $[\gamma] \in \mathscr{G}(\mathbb{H})$ we say that $[\gamma]$ *intersects M*, if some (and hence any) representative of $[\gamma]$ intersects M at some time $t \in \mathbb{R}$.

1.6 Fuchsian Groups and Developable Hyperbolic Orbisurfaces

Endow $\mathrm{PSL}_2(\mathbb{R})$ with the quotient topology (see (1.4)). A subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ which is discrete with respect to that topology is called a *Fuchsian group*. For any subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$, being Fuchsian is equivalent to any of the following (equivalent) properties (see, e. g., [33, Section 2.2]):

(a) Γ acts *properly discontinuously* on $\mathbb{H}$, that is, any compact subset of $\mathbb{H}$ contains only finitely many points of each Γ-orbit.

(b) Each Γ-orbit is a discrete subset of $\mathbb{H}$.

(c) The identity element $\mathrm{id} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is isolated in Γ.

Let Γ be a Fuchsian group. We denote by

$$\mathbb{X} := \Gamma\diagdown\mathbb{H}$$

the *orbit space* of the action of Γ on $\mathbb{H}$ and by

$$\pi = \pi_\Gamma \colon \mathbb{H} \longrightarrow \mathbb{X} \tag{1.26}$$

the canonical quotient map, where we drop the subscript Γ whenever the choice of Fuchsian group is clear from the context. Since Γ acts properly discontinuously, the space $\mathbb{X}$ naturally carries the structure of a (2-dimensional) good hyperbolic Riemannian orbifold, also called a *hyperbolic orbisurface*. It inherits a hyperbolic Riemannian metric via the projection map π. For $\mathbb{X}$ a hyperbolic orbisurface, every group Γ with $\mathbb{X} = \Gamma\diagdown\mathbb{H}$ is called a *fundamental group* for $\mathbb{X}$. A hyperbolic orbisurface with a fundamental group is called *developable*. If Γ has *torsion*, that is if it contains elliptic elements, then $\mathbb{X}$ has conical singularities, and hence it is not a manifold but a genuine orbifold. If Γ does not contain elliptic elements, then we call it *torsion-free*.

Convention
In this thesis we restrict our considerations to hyperbolic orbisurfaces that are developable. We will therefore speak only of "hyperbolic orbisurfaces" and always assume, sometimes implicitly, that a fundamental group exists.

From the characterizations above it follows immediately that for any Fuchsian group Γ and any point $z \in \mathbb{H}$, the order of the stabilizer subgroup

$$\mathrm{Stab}_\Gamma(z) := \{g \in \Gamma \mid g.z = z\} \tag{1.27}$$

of Γ is finite. This is no longer true for $z \in \partial_g\mathbb{H}$. For such a point z the stabilizer subgroup $\mathrm{Stab}_\Gamma(z)$ is either trivial or *cyclic*, i. e., isomorphic to $\mathbb{Z}$, as the following result implies.

Lemma 1.7 ([33, Theorem 2.3.5]). *Let Γ be a Fuchsian group all whose non-identity elements have the same fixed-point set. Then Γ is cyclic.*

The upper half-plane model demands special attention to the point at infinity. For instance, it will be necessary to exclude group elements that stabilize ∞

from certain definitions. For this reason it will prove convenient to introduce the shorthand notation

$$\Gamma_\infty \coloneqq \mathrm{Stab}_\Gamma(\infty) \tag{1.28}$$

for the stabilizer subgroup of ∞ in Γ. Further we will often conjugate the group Γ so that ∞ is either contained in a funnel representative or is itself a representative of a cusp of $\mathbb{X}$ (see Section 1.8 below), which will make it more convenient to verbally describe certain structures. On the other hand, this leads to notions which are not invariant under conjugation. For this reason we consider every Fuchsian group to be implicitly given by a set of generators and their mutual relations. In particular, we refrain from identifying mutually conjugate Fuchsian groups with each other, even though they produce the same orbisurface.

The discreteness of a subgroup of isometries has profound consequences for the interrelation of the fixed points of its hyperbolic and parabolic elements. The following result is well-known; we provide a short proof for the convenience of the reader.

Lemma 1.8. *Let Γ be a Fuchsian group. Then the set of fixed points of its hyperbolic elements is disjoint from the set of fixed points of its parabolic elements.*

Proof. Assume that Γ contains a hyperbolic element h as well as parabolic element p, for otherwise there is nothing to show. Assume $\mathrm{f}(p) = \mathrm{f}_+(h)$, which, because of (1.11) and (1.12), entails no loss of generality. By Lemma 1.1(i) there exists $a \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$a \cdot h \cdot a^{-1} = \mathrm{h}_{\ell(h)}$$

and since discreteness is preserved by conjugation, we may consider the subgroup $a\Gamma a^{-1}$ instead of Γ. Let $\widetilde{p} \coloneqq a \cdot p \cdot a^{-1}$. Then

$$\mathrm{f}(\widetilde{p}) = a\mathbin{.}\mathrm{f}(p) = a\mathbin{.}\mathrm{f}_+(h) = \mathrm{f}_+(\mathrm{h}_{\ell(h)}) = \infty\,,$$

hence, $\widetilde{p} = \left[\begin{smallmatrix} 1 & \kappa \\ 0 & 1 \end{smallmatrix}\right]$ for some $\kappa \in \mathbb{R} \setminus \{0\}$. Then, for $n \in \mathbb{N}$,

$$\mathrm{h}_{\ell(h)}^{-n} \cdot \widetilde{p} \cdot \mathrm{h}_{\ell(h)}^{n} = \begin{bmatrix} 1 & \kappa e^{-n\ell(h)} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{as} \quad n \to +\infty\,,$$

since $\ell(h) > 0$. Hence, the identity is not isolated in $a\Gamma a^{-1}$ which is equivalent to the group being non-discrete (see (c) above). □

Let $g \in \mathrm{PSL}_2(\mathbb{R})$. If there exists $\sigma \in \mathbb{N}$ such that $g^\sigma = \mathrm{id}$, then we say that g is *of finite order*. If g is of finite order and σ is the smallest positive integer such that $g^\sigma = \mathrm{id}$, then we call $\sigma(g) \coloneqq \sigma$ the *order* of g. If no such σ exists, we say that g is *of infinite order*. Obviously, the identity is the unique element of order 1. The elements in $\mathrm{PSL}_2(\mathbb{R})$ of order 2, that are the involutions, are exactly the elliptic elements with vanishing trace (see Lemma 1.2). Note that, if g is of (finite) order σ, then

$$g^{k\sigma+\ell}.z = (g^{\sigma})^k g^{\ell}.z = g^{\ell}.z$$

for all $k \in \mathbb{Z}$, $\ell \in \mathbb{N}$, and $z \in \overline{\mathbb{H}}^g$, hence, every sequence of the form $(g^n.z)_{n\in\mathbb{Z}}$ is periodic with minimal period σ. Therefore, Lemma 1.4 resp. Lemma 1.5 imply that every hyperbolic resp. parabolic element in $\mathrm{PSL}_2(\mathbb{R})$ is of infinite order and for every $z \in \mathbb{H}$ the subgroup $\mathrm{Stab}_\Gamma(z)$ is either trivial or all its non-identity elements are elliptic.

Lemma 1.9 ([33, Theorem 2.2.3])**.** *Let Γ be a Fuchsian group. Then every elliptic element in Γ is of finite order.*

Remark 1.10. Let $g \in \Gamma$ be elliptic of order $\sigma = \sigma(g)$. By Lemma 1.1(iii) there exists[3] $\theta = \theta(g) \in (0, \pi/2) \cup (3\pi/2, 2\pi)$ such that g is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to s_θ. Since the order is preserved under conjugation, we obtain

$$\mathrm{id} = \mathrm{s}_\theta^\sigma = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^\sigma = \begin{bmatrix} \cos(\sigma\theta) & -\sin(\sigma\theta) \\ \sin(\sigma\theta) & \cos(\sigma\theta) \end{bmatrix}.$$

Hence, $\theta = \frac{k}{\sigma}\pi$, for some $k \in \left(\left(0, \frac{\sigma}{2}\right) \cup \left(\frac{3\sigma}{2}, 2\sigma\right)\right) \cap \mathbb{Z}$. From the proof of Lemma 1.1(iii) we can therefore read off a formula for the order of g:

$$\sigma(g) = k\pi \left(\arccos\left(\tfrac{|\mathrm{tr}(g)|}{2}\right)\right)^{-1}, \tag{1.29}$$

where k is the smallest positive integer for which the right hand side becomes a positive integer. Or in other words, $\sigma(g)$ equals the denominator of the fully reduced fraction k/σ representing $\pi^{-1}\arccos(|\mathrm{tr}(g)|/2)$. Thus, the integer k is uniquely determined and we denote it by $k(g)$ for later use. Note that, since we have $|\mathrm{tr}(g)| < 2$ and $\arccos([0,1)) = (0, \pi/2]$, we also have $k(g) \le \sigma(g)/2$ with equality only if $\sigma(g) = 2$.

1.7 Geodesics on Hyperbolic Orbisurfaces

Let Γ be a Fuchsian group and let $\mathbb{X} = \Gamma\backslash\mathbb{H}$ be the associated hyperbolic orbisurface. The (unit speed) *geodesics* on $\mathbb{X}$ are the images of the geodesics on $\mathbb{H}$ under the canonical quotient map π from (1.26). Thus, if γ is a geodesic on $\mathbb{H}$, then

$$\widehat{\gamma} := \pi \circ \gamma \colon \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{X} \\ t & \longmapsto & \pi(\gamma(t)) \end{array} \right., \tag{1.30}$$

is the induced geodesic on $\mathbb{X}$. In this case, we say that γ is a *representative* or a *lift* of $\widehat{\gamma}$ to $\mathbb{H}$. We emphasize that all geodesics on $\mathbb{X}$ arise in this way.

[3]The constraint on the set from which θ is chosen corresponds to selecting the representative of g with positive trace.

As for $\mathbb{H}$, geodesics on $\mathbb{X}$ are determined by any of their tangent vectors. To simplify the further exposition, we recall from Section 1.5 that $\mathrm{PSL}_2(\mathbb{R})$, and hence Γ, act on $\mathrm{S}\mathbb{H}$ and we identify the *unit tangent bundle* $\mathrm{S}\mathbb{X}$ of $\mathbb{X}$ with the Γ-orbit space of $\mathrm{S}\mathbb{H}$:

$$\mathrm{S}\mathbb{X} = \Gamma\diagdown \mathrm{S}\mathbb{H}\,. \tag{1.31}$$

We let

$$\pi\colon\ \mathrm{S}\mathbb{H} \longrightarrow \mathrm{S}\mathbb{X} \tag{1.32}$$

denote the canonical quotient map, which is indeed the tangent map of the quotient map from (1.26). The context will always clarify whether π refers to the map in (1.26) or (1.32). If an object on $\mathbb{X}$ (or related to $\mathbb{X}$) is defined as the π-image of a corresponding object of $\mathbb{H}$, then we usually denote the object on $\mathbb{X}$ by the name of the object on $\mathbb{H}$ decorated with $\widehat{\ }$. One example for that is the notation in (1.30) for geodesics on $\mathbb{X}$. Analogously, we denote an element in $\mathrm{S}\mathbb{X}$ by $\widehat{\nu}$ if it is represented by $\nu \in \mathrm{S}\mathbb{H}$, thus

$$\widehat{\nu} = \pi(\nu)\,.$$

Each unit tangent vector $\widehat{\nu} \in \mathrm{S}\mathbb{X}$ uniquely determines a geodesic $\widehat{\gamma}_{\widehat{\nu}}$ on $\mathbb{X}$ via

$$\widehat{\gamma}_{\widehat{\nu}} := \pi(\gamma_\nu)\,,\quad \pi(\nu) = \widehat{\nu}\,,$$

which is independent of the choice of the representative $\nu \in \mathrm{S}\mathbb{H}$ and thus well-defined. For that reason we will omit from the notation the $\widehat{\ }$ in the index. Also the *geodesic flow* on $\mathbb{X}$, denoted $\widehat{\Phi}$, is the π-image of the geodesic flow Φ on $\mathbb{H}$ (see (1.22)). Thus,

$$\widehat{\Phi} := \pi \circ \Phi \circ \left(\mathrm{id} \times \pi_0^{-1}\right) : \begin{cases} \mathbb{R} \times \mathrm{S}\mathbb{X} & \longrightarrow & \mathrm{S}\mathbb{X} \\ (t, \widehat{\nu}) & \longmapsto & \widehat{\gamma}'_\nu(t) \end{cases}, \tag{1.33}$$

where π_0^{-1} is an arbitrary section of π. (It is straightforward to check that $\widehat{\Phi}$ does not depend on the choice of the section π_0^{-1}.)

Whereas the arcs of geodesics on $\mathbb{H}$ are always generalized semicircles, the arcs of geodesics on $\mathbb{X}$ enjoy a greater variety of forms. Of particular interest for us are the periodic geodesics, which we will discuss now. We say that a geodesic $\widehat{\gamma}$ on $\mathbb{X}$ is *periodic* if there exists $\delta > 0$ such that

$$\widehat{\gamma}(t+\delta) = \widehat{\gamma}(t)$$

for all $t \in \mathbb{R}$. Any such δ is called a *period* for $\widehat{\gamma}$.

Analogous to the situation on $\mathbb{H}$, we call any two geodesics $\widehat{\gamma}_1, \widehat{\gamma}_2$ on $\mathbb{X}$ equivalent if they differ only by a parameter change. One immediately observes that the two geodesics $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are equivalent if and only if $\Gamma.[\gamma_1] = \Gamma.[\gamma_2]$ for any choice of representatives γ_1 of $\widehat{\gamma}_1$ and γ_2 on $\widehat{\gamma}_2$. Further, periodicity and periods of geodesics on $\mathbb{X}$ are stable under descend to equivalence classes. For the equivalence classes we have

$$[\widehat{\gamma}] = [\pi(\gamma)] = \pi([\gamma])\,.$$

We denote by $\mathscr{G}(\mathbb{X})$ the set of all equivalence classes of geodesics on $\mathbb{X}$, and by $\mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ the subset of the equivalence classes of periodic geodesics. Further, we denote by

$$\mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H}) \coloneqq \{[\gamma] \in \mathscr{G}(\mathbb{H}) \mid \pi_\Gamma([\gamma]) \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})\} \tag{1.34}$$

the subset of equivalence classes of lifts of periodic geodesics from $\mathbb{X}$ into $\mathbb{H}$. From the properties of Möbius transformations we infer that the sets $\mathscr{G}(\mathbb{H})$ and $\mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ are invariant under Γ-actions.

Periodic geodesics (or, more precisely, equivalence classes thereof) are closely related to hyperbolic elements in Γ in a way that we recall now. Let $h \in \mathrm{PSL}_2(\mathbb{R})$ be hyperbolic. By the discussion at the end of Section 1.5 there exists exactly one $[\gamma] \in \mathscr{G}(\mathbb{H})$ such that

$$[\gamma](+\infty) = \mathrm{f}_+(h) \qquad \text{and} \qquad [\gamma](-\infty) = \mathrm{f}_-(h)\,, \tag{1.35}$$

where the points $[\gamma](\pm\infty)$ are as in (1.24). We call

$$\alpha(h) \coloneqq [\gamma] \tag{1.36}$$

the *axis* of h. From (1.11) and (1.12) it follows that $\alpha(h^{-1})$ consists exactly of the representatives of $\alpha(h)$ with their orientations reversed. Hence, in particular

$$\alpha(h)(\mathbb{R}) = \alpha(h^{-1})(\mathbb{R})\,. \tag{1.37}$$

With the following lemma we recall a well-known first observation on the relation between periodic geodesics on $\mathbb{X}$ and hyperbolic elements in Γ as well as between axes and displacement lengths of different hyperbolic elements. A proof can be deduced, e. g., from [2, Observations 3.28, 3.29].

Lemma 1.11. *Let $h \in \Gamma$ be hyperbolic with displacement length $\ell(h)$ and axis $\alpha(h)$. Then the following statements hold true.*

(i) *The geodesics in the equivalence class $\pi(\alpha(h))$ are periodic with period $\ell(h)$.*

(ii) *For all $g \in \Gamma$ the element ghg^{-1} is hyperbolic with displacement length $\ell(h)$ and axis $g.\alpha(h)$.*

(iii) *For all $n \in \mathbb{N}$ we have $\alpha(h^n) = \alpha(h)$ and $\ell(h^n) = n\ell(h)$.*

Remark 1.12. Recall the standard hyperbolic element h_ℓ, $\ell > 0$ from Lemma 1.1(i). We have

$$\alpha(\mathrm{h}_\ell) = [\gamma_s]\,,$$

where γ_s denotes the standard geodesic from (1.21). Let $z \in [\gamma_s](\mathbb{R})$, i. e., $z = \mathrm{i}y$ for some $y > 0$. Then $\mathrm{h}_\ell . z = \mathrm{i}e^\ell y$ and thus

$$\begin{aligned}\mathrm{dist}_{\mathbb{H}}(z, \mathrm{h}_\ell . z) &= \operatorname{arcosh}\left(1 + \frac{\left|\mathrm{i}y - \mathrm{i}e^\ell y\right|^2}{2y^2 e^\ell}\right) \\ &= 2\log\left(\frac{\sqrt{(e^\ell y - y)^2} + \sqrt{(e^\ell y + y)^2}}{2\sqrt{e^\ell y^2}}\right) \\ &= 2\log e^{\frac{\ell}{2}} = \ell\,.\end{aligned}$$

This justifies the notion “displacement length”: Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be hyperbolic. Then, by Lemma 1.1(i), $g = q \cdot \mathrm{h}_{\ell(g)} \cdot q^{-1}$ for some $q \in \mathrm{PSL}_2(\mathbb{R})$. By Lemma 1.11(ii) we have $\alpha(g) = q.[\gamma_s]$. Since Möbius transformations are isometries it follows that

$$\mathrm{dist}_{\mathbb{H}}(z, g.z) = \ell(g) \tag{1.38}$$

for all $z \in \alpha(g)(\mathbb{R})$.

We denote by $[\Gamma]_\mathrm{h}$ the set of all Γ-conjugacy classes of hyperbolic elements in Γ. Lemma 1.11 yields that all representatives of $[g] \in [\Gamma]_\mathrm{h}$ give rise to the same equivalence class of geodesics in $\mathscr{G}_{\mathrm{Per}}(\mathbb{X})$. Thus, this relation gives rise to the map

$$\varrho\colon \left\{\begin{array}{ccc} [\Gamma]_\mathrm{h} & \longrightarrow & \mathscr{G}_{\mathrm{Per}}(\mathbb{X}) \\ [h] & \longmapsto & \pi(\alpha(h)) \end{array}\right. . \tag{1.39}$$

Lemma 1.11 further shows that for each hyperbolic $h \in \Gamma$, the displacement length is constant in $[h]$ and is one of the possible periods of the geodesics representing $\varrho([h])$. However, since the displacement length scales with powers of h but the image of ϱ remains unchanged, ϱ is not a bijection, but an infinite covering.

For each hyperbolic element $h \in \Gamma$ there exists a unique pair $(h_0, n) \in \Gamma \times \mathbb{N}$ such that h_0 is hyperbolic and n is maximal with the property that $h = h_0^n$. The displacement length of h_0 as well as the value of n are invariants of the equivalence class $[h]$. We set

$$\mathrm{ct}([h]) := n \tag{1.40}$$

and

$$\ell_0([h]) := \ell(h_0)\,.$$

Further, we call the element h_0 the *primitive* of h.

Proposition 1.13 (Theorem 3.30 in [2]). *The map*

$$\varrho \times \mathrm{ct}\colon \left\{\begin{array}{ccc} [\Gamma]_\mathrm{h} & \longrightarrow & \mathscr{G}_{\mathrm{Per}}(\mathbb{X}) \times \mathbb{N} \\ [h] & \longmapsto & \big(\pi(\alpha(h)), \mathrm{ct}([h])\big) \end{array}\right.$$

is a bijection.

If $\widehat{\gamma}$ is a periodic geodesic on $\mathbb{X}$ and δ_0 is its minimal period, then for any $n \in \mathbb{N}$, also $n\delta_0$ is a period of $\widehat{\gamma}$. In view of Proposition 1.13, we may understand $([\widehat{\gamma}], n) \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X}) \times \mathbb{N}$ as the equivalence class $[\widehat{\gamma}]$ of periodic geodesics endowed with a fixed choice of period, namely $n\delta_0$. We may further define the *length* of $(\widehat{\gamma}, n) \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X}) \times \mathbb{N}$ as

$$\ell(\widehat{\gamma}, n) := \ell([h])$$

for any $h \in \Gamma$ such that

$$(\varrho \times \mathrm{ct})([h]) = (\widehat{\gamma}, n)\,.$$

If $n = 1$ then we call $([\widehat{\gamma}], n)$ *prime* or *primitive*. Likewise, we call h and $[h]$ *primitive* if $\mathrm{ct}([h]) = 1$.

Convention
Up until now we have considered $\mathscr{G}(\mathbb{X})$ and $\mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ as sets. For convenience, in what follows, we will often consider them as multisets and we will refer to their elements as geodesics. Thus, by slight abuse of notion, we identify geodesics with their equivalence classes, and we will often indicate the choice of period only implicitly. In particular, in order to ease notation, we will denote the elements of $\mathscr{G}(\mathbb{X})$ and $\mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ by γ instead of $[\gamma]$.

Corollary 1.14. *The conjugacy classes of primitive hyperbolic elements in Γ are in bijection with the prime periodic geodesics on $\mathbb{X}$.*

1.8 Geometry at Infinity

Let Γ be a Fuchsian group and let $\mathbb{X} = \Gamma \diagdown \mathbb{H}$ be the associated (hyperbolic developable) orbisurface. The geometry of $\mathbb{X}$ allows us to find a (large) compact subset K of $\mathbb{X}$ such that, for all compact subsets $\widetilde{K}$ with $K \subseteq \widetilde{K}$, the spaces $\mathbb{X} \setminus K$ and $\mathbb{X} \setminus \widetilde{K}$ have the same number of connected components. For definiteness we may take for K the *compact core* of $\mathbb{X}$. As we will not use any further properties of the compact core other than this separation property, we refer to [10] for a definition. The connected components of $\mathbb{X} \setminus K$ are the *(hyperbolic) ends* of $\mathbb{X}$. (We ignore here the slight inexactness in that this notion of ends depends on the choice of K if we do not pick the compact core, as we will need only the general concept.) The geometric finiteness of Γ yields that $\mathbb{X}$ has only a finite number of ends.

The hyperbolic orbisurface $\mathbb{X}$ has at least one periodic geodesic if Γ contains a hyperbolic element (see Proposition 1.13). Therefore, $\mathbb{X}$ admits only two types of ends:

(a) Ends of finite hyperbolic area are called *cusps*. Via the canonical quotient map π from (1.26), each cusp of $\mathbb{X}$ can be identified with the Γ-orbit of the fixed point c of some parabolic transformation in Γ. The stabilizer group $\mathrm{Stab}_\Gamma(c)$ of c is a cyclic subgroup of Γ. In particular, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ and a unique $\lambda > 0$ such that $\mathrm{Stab}_\Gamma(c)$ is generated by

$$g \cdot \mathrm{t}_\lambda \cdot g^{-1} ,$$

with t_λ as in (1.7). We call c a *cusp representative* or a *cuspidal point* and denote the corresponding cusp by $\widehat{c}$ (see also Section 1.7). Further, we call λ the *cusp width* of $\widehat{c}$.

(b) Ends of infinite area are called *funnels*. Funnels can be identified with certain subsets of the geodesic boundary of a fundamental domain for Γ (see Section 1.10). Further below, after the introduction of the limit set of Γ, we will give a second characterization.

Each funnel is characterized by a *funnel bounding geodesic*. That is the equivalence class of a periodic geodesic γ on $\mathbb{X}$ that is furthest into the funnel, in the sense that every geodesic that intersects γ cannot be periodic. The funnel bounding geodesic of a given funnel is unique up to orientation.

The hyperbolic orbisurface $\mathbb{X}$ is compact if and only if it has neither cusps nor funnels. In this case, Γ is called *cocompact* or *uniform*. If $\mathbb{X}$ is not compact, then Γ is called *non-cocompact* or *non-uniform*. If $\mathbb{X}$ has no cusps, but probably funnels, and is a proper surface (i. e., Γ has no elliptic elements), then Γ is called *convex cocompact*. (The naming originates from the fact that the convex core of $\mathbb{X}$ is compact in this case. We refer to [10] for the definition of the convex core.) Furthermore, the area of $\mathbb{X}$ is called the *covolume* of Γ. If $\mathbb{X}$ has finite area—which is the case if and only if it has no funnel ends—then Γ is said to be *cofinite*.

A hyperbolic orbisurface $\mathbb{X}$ is called *geometrically finite*, if it has no more than finitely many hyperbolic ends and conical singularities and is of finite genus.

Crucial for our investigations will be the fact that the periodic geodesics on $\mathbb{X}$ lie dense in the set of all geodesics on $\mathbb{X}$ in a certain sense which we describe in the following. Since Γ is discrete, Γ-orbits do not accumulate in $\mathbb{H}$. But they may do in $\overline{\mathbb{H}}^g$. We denote by $\Lambda(\Gamma)$ the set of all limit points (accumulation points) of Γ-orbits in $\overline{\mathbb{H}}^g$, the *limit set* of Γ. The set $\Lambda(\Gamma)$ equals the set of all limit points of the single orbit $\Gamma.z$ for any $z \in \mathbb{H}$ with trivial stabilizer subgroup in Γ. Since Γ acts properly discontinuously on $\mathbb{H}$ and transitively on $\overline{\mathbb{H}}^g$, this implies that $\Lambda(\Gamma)$ is a Γ-invariant subset of $\partial_g \mathbb{H}$. Because of Lemma 1.4 and Lemma 1.5 the limit set contains in particular all hyperbolic and parabolic fixed points. The complement of the limit set

$$\Omega(\Gamma) \coloneqq \partial_g \mathbb{H} \setminus \Lambda(\Gamma)$$

is called the *ordinary set* of Γ.

The Fuchsian group Γ is called *elementary* if $\Lambda(\Gamma)$ is finite, and *non-elementary* otherwise. The elementary Fuchsian groups are completely understood: If Γ is elementary, then it is either cyclic, or there exists $\lambda > 1$ such that Γ is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to the group generated by the two transformations

$$\mathrm{s}_{\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathrm{h}_\lambda = \begin{bmatrix} e^{\frac{\lambda}{2}} & 0 \\ 0 & e^{-\frac{\lambda}{2}} \end{bmatrix}$$

(see [33, Theorem 2.4.3]). The cyclic elementary groups generated by a hyperbolic/parabolic element are also called *hyperbolic/parabolic cylinders*. If Γ is non-elementary, then it necessarily contains hyperbolic elements [33, Theorem 2.4.4] and the limit set $\Lambda(\Gamma)$ is either all of $\partial_g\mathbb{H}$ or a perfect, nowhere dense subset of $\partial_g\mathbb{H}$ [33, Theorem 3.4.6].

Consider the set

$$E(\mathbb{X}) := \left\{ \left(\gamma(+\infty), \gamma(-\infty)\right) \;\middle|\; \gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H}) \right\} , \tag{1.41}$$

where $\mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ is as in (1.34). By Proposition 1.13, this set can also be characterized as

$$E(\mathbb{X}) = \left\{ \left(\mathrm{f}_+(h), \mathrm{f}_-(h)\right) \;\middle|\; [h] \in [\Gamma]_{\mathrm{h}} \right\} .$$

In particular, $E(\mathbb{X})$ is a subset of $\Lambda(\Gamma) \times \Lambda(\Gamma)$. The following result now establishes the density of periodic geodesics.

Proposition 1.15 ([20], [33, Theorem 3.4.4])**.** *For any geometrically finite Fuchsian group Γ with hyperbolic elements the set $E(\mathbb{X})$ is dense in $\Lambda(\Gamma) \times \Lambda(\Gamma)$.*

The limit set $\Lambda(\Gamma)$ allows for another interpretation of funnels as follows. The set

$$\widehat{\mathbb{R}} \setminus \Lambda(\Gamma)$$

decomposes into countably many connected (open) components. Each such component we call a *funnel interval*. Further, we call each interval that is contained in a funnel interval consisting of points that are pairwise non-equivalent under the Γ-action and that is maximal with these properties a *funnel representative*. One easily sees that each funnel interval is the union of several funnel representatives. The Γ-orbits of funnel intervals coincide with the Γ-orbits of funnel representatives (as sets or as equivalence classes), and each such Γ-orbit corresponds to a unique funnel of $\mathbb{X}$. We may identify each funnel of $\mathbb{X}$ with such a Γ-orbit (understood as an equivalence class) or with a funnel representative.

Finally, we introduce a few more definitions that are not classical but will be used extensively throughout. As before, for any parabolic element $p \in \Gamma$ we denote its fixed point by $\mathrm{f}(p)$. We set

$$\widehat{\mathbb{R}}_{\mathrm{st}} := \Lambda(\Gamma) \setminus \{\mathrm{f}(p) \mid p \in \Gamma \text{ parabolic}\} . \tag{1.42}$$

Lemma 1.16. *The set* $\widehat{\mathbb{R}}_{\mathrm{st}}$ *is* Γ*-invariant and* $E(\mathbb{X}) \subseteq \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}$.

Proof. Let $g, p \in \Gamma$. Since conjugation preserves traces, if p is parabolic, so is $g{\cdot}p{\cdot}g^{-1}$, and we have

$$(g \cdot p \cdot g^{-1}).(g.\,\mathrm{f}(p)) = (g \cdot p).\,\mathrm{f}(p) = g.\,\mathrm{f}(p)\,.$$

Hence, the set $\{\mathrm{f}(p) \mid p \in \Gamma \text{ parabolic}\}$ is Γ-invariant. Since the limit set is Γ-invariant as well, this yields the first claim. The second claim is now immediate from Lemma 1.8. $\square$

For every subset $M \subseteq \widehat{\mathbb{R}}$ we further set

$$M_{\mathrm{st}} \coloneqq M \cap \widehat{\mathbb{R}}_{\mathrm{st}}\,,$$

and for elements of some family $\{M_j\}_j$ of subsets of $\widehat{\mathbb{R}}$ we abbreviate

$$M_{j,\mathrm{st}} \coloneqq (M_j)_{\mathrm{st}}\,.$$

Lemma 1.17. *A Fuchsian group* Γ *is cocompact if and only if* $\widehat{\mathbb{R}} = \widehat{\mathbb{R}}_{\mathrm{st}}$.

Proof. Assume first that Γ is cocompact, meaning that $\mathbb{X}$ has no hyperbolic ends. By the above this is equivalent to

$$\Omega(\Gamma) = \varnothing \qquad \text{and} \qquad \{\mathrm{f}(p) \mid p \in \Gamma \text{ parabolic}\} = \varnothing\,.$$

These two statements are equivalent to

$$\Lambda(\Gamma) = \widehat{\mathbb{R}} \qquad \text{and} \qquad \widehat{\mathbb{R}}_{\mathrm{st}} = \Lambda(\Gamma)\,,$$

respectively. For the converse implication it now suffices to observe that

$$\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \Lambda(\Gamma) \subseteq \widehat{\mathbb{R}}\,.$$

$\square$

1.9 Isometric Spheres

Let Γ be a non-cocompact Fuchsian group and denote by Γ_∞ the stabilizer subgroup of ∞ in Γ (see also (1.28)). As before we denote by $\mathbb{X}$ the orbit space of Γ. Recall the limit set $\Lambda(\Gamma)$ as well as the ordinary set $\Omega(\Gamma)$ of Γ from Section 1.8. In order to avoid dealing with a change of charts, we assume that ∞ is "contained in a hyperbolic end". By that we mean that ∞ is either a representative of a cusp of $\mathbb{X}$ or an inner point of some funnel interval, i. e., $\Omega(\Gamma)$ contains an interval of the form $(r, +\infty) \cup \{\infty\} \cup (-\infty, -r)$ for some $r \in \mathbb{R}_{>0}$. In the latter case we say that $\Omega(\Gamma)$ *contains a neighborhood of* ∞. In either case we have $\infty \in \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$ and

(S) Every element of Γ_∞ is of the form $\mathrm{t}_\lambda = \left[\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right]$ with some $\lambda \in \mathbb{R}$.

To be more precise, whenever $\Omega(\Gamma)$ contains a neighborhood of ∞ as above, then ∞ is not a fixed point for any non-identity element in Γ. Hence, $\Gamma_\infty = \{\mathrm{id}\}$. If ∞ is a representative of a cusp, then Γ contains a parabolic transformation that fixes ∞. Every transformation in $\mathrm{PSL}_2(\mathbb{R})$ with those two properties is necessarily of the form t_κ for some $\kappa \in \mathbb{R}\setminus\{0\}$ (see Lemma 1.5). In particular, Γ_∞ is cyclic and generated by t_λ for $\lambda \in \mathbb{R}$ uniquely determined up to sign, only depending on Γ. Hence, the elements of Γ_∞ are all translations $z \mapsto z + n\lambda$, $n \in \mathbb{Z}$, and thus not only isometries w. r. t. the hyperbolic metric, but also w. r. t. to the Euclidean metric in $\mathbb{C}$. The elements of Γ_∞ are the only transformations in Γ with that property. One may ask on which subsets of $\mathbb{H}$ a transformation $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma \setminus \Gamma_\infty$ behaves as a Euclidean isometry. For $z, w \in \mathbb{H}$ one calculates

$$\begin{aligned} |g.z - g.w| &= \frac{|acwz + adz + bcw + bd - (acwz + adw + bcw + bd)|}{|cz+d|\,|cw+d|} \\ &= \frac{|z-w|}{|cz+d|\,|cw+d|}\,. \end{aligned}$$

Hence, $|g.z - g.w| = |z - w|$ if and only if $|g'(z)| = |g'(w)|^{-1}$. Comparison of three distinct points in $\mathbb{H}$ shows that the complete locus of points where g acts as a Euclidean isometry is given by

$$\mathrm{I}(g) := \left\{ z \in \mathbb{H} \;\middle|\; \left|g'(z)\right| = 1 \right\} = \{ z \in \mathbb{H} \mid |cz+d| = 1 \}\,. \tag{1.43}$$

This is a semicircle of radius $1/|c|$ around the center $-d/c$ and thus is called the *isometric sphere* of g. It is immediately clear that each isometric sphere is a geodesic arc in $\mathbb{H}$. We denote by

$$\mathrm{ISO}(\Gamma) := \{ \mathrm{I}(g) \mid g \in \Gamma \setminus \Gamma_\infty \} \tag{1.44}$$

the set of all isometric spheres of admissible elements in Γ. For $\mathrm{I} \in \mathrm{ISO}(\Gamma)$ and $g \in \Gamma \setminus \Gamma_\infty$ with $\mathrm{I} = \mathrm{I}(g)$ the element g is called a *generator* of I. In general, generators of isometric spheres are not uniquely determined within $\Gamma \setminus \Gamma_\infty$. In Lemma 1.20 below we elaborate on this point in more detail.

Every geodesic arc $(z_1, z_2)_{\mathbb{H}}$, $z_1, z_2 \in \partial_g \mathbb{H}$, divides the upper half-plane into two open half-spaces H_1, H_2 so that we have the disjoint decomposition

$$\mathbb{H} = H_1 \cup (z_1, z_2)_{\mathbb{H}} \cup H_2$$

in which each of the sets involved is convex. In the case of an isometric sphere those half-spaces can be characterized via the derivative of g. For $\mathrm{I} \in \mathrm{ISO}(\Gamma)$ and $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ a generator of I we call

$$\operatorname{int} \mathrm{I} = \operatorname{int} \mathrm{I}(g) := \left\{ z \in \mathbb{H} \;\middle|\; \left|g'(z)\right| > 1 \right\} = \{ z \in \mathbb{H} \mid |cz+d| < 1 \} \tag{1.45}$$

the *interior* and

$$\operatorname{ext} \mathrm{I} = \operatorname{ext} \mathrm{I}(g) := \left\{ z \in \mathbb{H} \;\middle|\; \left| g'(z) \right| < 1 \right\} = \{ z \in \mathbb{H} \mid |cz + d| > 1 \} \tag{1.46}$$

the *exterior* of $\mathrm{I}(g)$.

Lemma 1.18 ([55, Lemma 3.11 and Proposition 3.12])**.** *Let $M \subseteq \mathrm{ISO}(\Gamma)$. Then we have*

$$\overline{\bigcap_{\mathrm{I} \in M} \operatorname{ext} \mathrm{I}} = \bigcap_{\mathrm{I} \in M} \overline{\operatorname{ext} \mathrm{I}} = \mathbb{H} \setminus \bigcup_{\mathrm{I} \in M} \operatorname{int} \mathrm{I} \,.$$

In the remainder of this section we collect several results about isometric spheres that will be needed on several occasions throughout the upcoming chapters. Most of these results are well-known or straightforward consequences of well-known properties of Fuchsian groups. Nevertheless, the majority of the proofs is provided. We start with a set of identifications for the half-spaces introduced above.

Lemma 1.19. *Let $g \in \Gamma \setminus \Gamma_\infty$.*

(i) *The center of $\mathrm{I}(g^{-1})$ is given by $g.\infty$. Likewise, the center of $\mathrm{I}(g)$ is given by $g^{-1}.\infty$.*

(ii) *For every $z \in \mathrm{I}(g)$ we have $\operatorname{Im} g.z = \operatorname{Im} z$.*

(iii) *We have the identities*

$$\begin{aligned} g.\mathrm{I}(g) &= \mathrm{I}(g^{-1})\,, \\ g.\operatorname{int} \mathrm{I}(g) &= \operatorname{ext} \mathrm{I}(g^{-1})\,, \quad \textit{and} \\ g.\operatorname{ext} \mathrm{I}(g) &= \operatorname{int} \mathrm{I}(g^{-1})\,. \end{aligned}$$

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a representative of g such that $c > 0$. Then $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ is a representative of g^{-1}. Hence, $g^{-1}.\infty = -d/c$ and the center of $\mathrm{I}(g^{-1})$ is given by $a/c = g.\infty$. Because of (1.43) this yields (i). Let $z \in \mathrm{I}(g)$. There exists a unique $\theta \in (0, \pi)$ such that

$$z = -\frac{d}{c} + \frac{e^{\mathrm{i}\theta}}{c}\,.$$

Then

$$g.z = \left(-\frac{ad}{c} + \frac{ae^{\mathrm{i}\theta}}{c} + b \right) e^{-\mathrm{i}\theta} = \frac{a}{c} - \frac{ad - bc}{c} e^{-\mathrm{i}\theta} = \frac{a}{c} + \frac{e^{\mathrm{i}(\pi - \theta)}}{c}\,.$$

This yields (ii) as well as the first equation in (iii). The other two equations in (iii) now follow from (i) and continuity of g. □

Let $\mathrm{I} \in \mathrm{ISO}(\Gamma)$. Because of Lemma 1.19(i) the interior of I can be characterized as the half-space H in $\mathbb{H}$ induced by I such that the center of I is contained in $\mathrm{g}H$. Since the action of $\mathrm{PSL}_2(\mathbb{R})$ extends smoothly to $\overline{\mathbb{H}}^{g}$, we obtain from Lemma 1.19(iii) that

$$g.\mathrm{g}\operatorname{int}\mathrm{I}(g) = \mathrm{g}\operatorname{ext}\mathrm{I}(g^{-1}) \qquad \text{and} \qquad g.\mathrm{g}\operatorname{ext}\mathrm{I}(g) = \mathrm{g}\operatorname{int}\mathrm{I}(g^{-1})\,, \tag{1.47}$$

for every $g \in \Gamma \setminus \Gamma_\infty$. This implies

$$\infty \notin \mathrm{g}\operatorname{int}\mathrm{I} \tag{1.48}$$

for any $\mathrm{I} \in \mathrm{ISO}(\Gamma)$.

The following result is an amalgamation of the Lemmas 6.1.2, 6.1.3, and 6.1.28 in [54]. The proves for all statements are straightforward calculations and can be found ibid.

Lemma 1.20. *Let $g, h \in \Gamma \setminus \Gamma_\infty$ and let $\lambda > 0$ be such that $\mathrm{t}_\lambda \in \Gamma_\infty$.*

(i) *We have $\mathrm{I}(g) = \mathrm{I}(h)$ if and only if $gh^{-1} \in \Gamma_\infty$.*

(ii) *For all $n \in \mathbb{Z}$ we have $\mathrm{I}(g\mathrm{t}_\lambda^n) = \mathrm{t}_\lambda^{-n}.\mathrm{I}(g)$.*

(iii) *Suppose $\mathrm{I}(g) \cap \operatorname{int}\mathrm{I}(h) \neq \varnothing$. Then $hg^{-1} \in \Gamma \setminus \Gamma_\infty$ and*

$$g.\big(\mathrm{I}(g) \cap \operatorname{int}\mathrm{I}(h)\big) = \mathrm{I}(g^{-1}) \cap \operatorname{int}\mathrm{I}(hg^{-1})\,.$$

We define the maps

$$r\colon \begin{cases} \Gamma \setminus \Gamma_\infty & \longrightarrow \quad \mathbb{R}_{>0} \\ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] & \longmapsto \quad 1/|c| \end{cases} \tag{1.49}$$

and

$$c\colon \begin{cases} \Gamma \setminus \Gamma_\infty & \longrightarrow \quad \mathbb{R} \\ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] & \longmapsto \quad -d/c \end{cases} . \tag{1.50}$$

Both maps are obviously well-defined and they give the radius and the center of $\mathrm{I}(g)$, respectively. Since isometric spheres are Euclidean semicircles centered at $\mathbb{R}$, we have

$$\max\{\operatorname{Im} z \mid z \in \mathrm{I}(g)\} = r(g) \tag{1.51}$$

and the point $z \in \mathrm{I}(g)$ attaining this maximum is uniquely given by

$$s(g) := c(g) + \mathrm{i}r(g)\,. \tag{1.52}$$

We call $s(g)$ the *summit* of $\mathrm{I}(g)$. From Lemma 1.19(ii) it follows immediately that

$$s(g^{-1}) = g.s(g)\,. \tag{1.53}$$

Lemma 1.20(i) implies that generators of isometric spheres are uniquely determined up to left multiplication with elements in Γ_∞, or in other words, there is a bijection between $\Gamma_\infty\diagdown(\Gamma\setminus\Gamma_\infty)$ and $\mathrm{ISO}(\Gamma)$. Because of (S) and

$$\begin{bmatrix}1 & \lambda\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a+\lambda c & b+\lambda d\\ c & d\end{bmatrix}$$

for all $a, b, c, d, \lambda \in \mathbb{R}$, the maps r, c, s are constant on cosets $\Gamma_\infty g$, $g \in \Gamma\setminus\Gamma_\infty$. Hence, each of the maps r, c, s induces a map on $\mathrm{ISO}(\Gamma)$:

$$r(\mathrm{I}) \coloneqq r(g)\,, \quad c(\mathrm{I}) \coloneqq c(g)\,, \quad \text{and} \quad s(\mathrm{I}) \coloneqq s(g)\,, \tag{1.54}$$

where g is any generator of I.

With the concept of isometric spheres at hand we can give another characterization of the different types of transformations in $\mathrm{PSL}_2(\mathbb{R})$. To ease the notation, for $g \in \Gamma\setminus\Gamma_\infty$ we define the open interval

$$\mathscr{W}(g) \coloneqq (\underline{g}\,\mathrm{int}\,\mathrm{I}(g))^\circ\ . \tag{1.55}$$

Then from (1.47) it follows immediately that

$$g.\mathscr{W}(g) = \big(\widehat{\mathbb{R}}\setminus\mathscr{W}(g^{-1})\big)^\circ \quad \text{and} \quad g.\big(\widehat{\mathbb{R}}\setminus\mathscr{W}(g)\big) = \overline{\mathscr{W}(g^{-1})}\,. \tag{1.56}$$

Lemma 1.21. *Let $g \in \Gamma\setminus\Gamma_\infty$.*

(i) *The transformation g is elliptic if and only if $\mathrm{I}(g)\cap\mathrm{I}(g^{-1}) \neq \varnothing$, parabolic if and only if $\mathrm{I}(g)\cap\mathrm{I}(g^{-1}) = \varnothing$ and $\underline{g}\mathrm{I}(g)\cap\underline{g}\mathrm{I}(g^{-1}) \neq \varnothing$, and hyperbolic if and only if $\overline{\mathrm{I}(g)}^{g}\cap\overline{\mathrm{I}(g^{-1})}^{g} = \varnothing$.*

(ii) *If g is elliptic, then $\mathrm{I}(g) = \mathrm{I}(g^{-1})$ if and only if g is an involution. In this case, $\mathrm{f}(g) = s(g)$. If g is elliptic with $|\mathrm{tr}(g)| \in (0,2)$, then $\mathrm{I}(g)$ and $\mathrm{I}(g^{-1})$ intersect each other exactly in $\mathrm{f}(g)$.*

(iii) *If g is parabolic, then $\mathrm{f}(g)$ is the unique point in $\underline{g}\mathrm{I}(g)\cap\underline{g}\mathrm{I}(g^{-1})$.*

(iv) *If g is hyperbolic, then $\alpha(g)(\mathbb{R})$ is the unique geodesic arc perpendicular to both $\mathrm{I}(g)$ and $\mathrm{I}(g^{-1})$. In particular, $\mathrm{f}_-(g) \in \mathscr{W}(g)$ and $\mathrm{f}_+(g) \in \mathscr{W}(g^{-1})$.*

Proof. Let $a, b, c, d \in \mathbb{R}$ be such that $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right] = g$. By assumption, $c \neq 0$, and we have

$$\big|c(g) - c(g^{-1})\big| = \left|\frac{a+d}{c}\right| = |\mathrm{tr}(g)|\cdot r(g)\,.$$

Together with Lemma 1.1 this yields (i).

Assume that g is elliptic. Since g fixes some point in $\mathbb{H}$, so does g^n for any integer $n \in \mathbb{Z}$, meaning g^n is either elliptic or the identity. If $\mathrm{I}(g) = \mathrm{I}(g^{-1})$, then Lemma 1.20(i) implies $g^2 \in \Gamma_\infty$. Since every non-identity element of Γ_∞

is parabolic, this implies $g = g^{-1}$. In this case $\mathrm{f}(g) = s(g)$ follows immediately from (1.53). Now assume that $|\mathrm{tr}(g)| \in (0,2)$, i. e., g is elliptic but no involution. From (i) and convexity we obtain that $\#\left(\mathrm{I}(g) \cap \mathrm{I}(g^{-1})\right) = 1$. Using (1.14) we calculate for $x \in \{-\frac{d}{c}, \frac{a}{c}\}$,

$$
\begin{aligned}
|\mathrm{f}(g) - x| &= \frac{1}{|c|}\left|\frac{\mathrm{tr}(g)}{2} \pm \frac{\mathrm{i}}{2}\sqrt{4 - |\,\mathrm{tr}(g)|^2}\right| \\
&= \frac{1}{4\,|c|}\left(|\mathrm{tr}(g)|^2 + 4 - |\mathrm{tr}(g)|^2\right) = r(g)\,.
\end{aligned}
$$

This implies (ii).

Similarly, if g is parabolic, the fixed point of g is

$$z = \frac{a-d}{2c} = \frac{c(g) + c(g^{-1})}{2}\,.$$

Combining this with (i) and $r(g) = r(g^{-1})$ yields (iii).

In order to verify (iv) we first note that for any two geodesics $\gamma_1, \gamma_2 \in \mathscr{G}(\mathbb{H})$ with $\mathrm{dist}_{\mathbb{H}}(\gamma_1(\mathbb{R}), \gamma_2(\mathbb{R})) > 0$ there exists exactly one geodesic arc in $\mathbb{H}$ perpendicular to both $\gamma_1(\mathbb{R})$ and $\gamma_2(\mathbb{R})$ (see, e. g., [7, Section 1.2]). Let g be hyperbolic. We transform g into its standard form $\mathrm{h}_{\ell(g)}$ given by (1.6) via conjugation by some $q \in \mathrm{PSL}_2(\mathbb{R})$. Then, by Lemma 1.11(ii),

$$q.\alpha(g)(\mathbb{R}) = \alpha(\mathrm{h}_{\ell(g)})(\mathbb{R}) = \gamma_s(\mathbb{R}) = (0,\infty)_{\mathbb{H}}\,,$$

with γ_s the standard geodesic given by (1.21) (see also Remark 1.12). In particular, any suitable transformation q necessarily maps $\mathrm{f}_-(g)$ to 0 and $\mathrm{f}_+(g)$ to ∞. From this we derive a possible choice

$$q = \begin{bmatrix} -\frac{c}{D} & \frac{a-d}{2D} - \frac{1}{2} \\ 1 & \frac{d-a}{2c} - \frac{D}{2c} \end{bmatrix}\,,$$

with $D := \sqrt{\mathrm{tr}(g)^2 - 4}\,$. By Lemma 1.1 and since g is hyperbolic, we have $D > 0$. Let $\varphi := \mathrm{tr}(g) + 2$ and $\psi := \mathrm{tr}(g) - 2$. Then $\varphi\psi = D^2$ and we calculate

$$
\begin{aligned}
&q.\,(c(g) + r(g)) + q.\,(c(g) - r(g)) \\
&\quad = \frac{-\frac{c}{D}\left(\frac{-d+1}{c}\right) + \frac{a-d}{2D} - \frac{1}{2}}{\frac{-d+1}{c} + \frac{d-a}{2c} - \frac{D}{2c}} + \frac{-\frac{c}{D}\left(\frac{-d-1}{c}\right) + \frac{a-d}{2D} - \frac{1}{2}}{\frac{-d-1}{c} + \frac{d-a}{2c} - \frac{D}{2c}} \\
&\quad = \frac{c}{D}\left(\frac{\psi - D}{-\psi - D} + \frac{\varphi - D}{-\varphi - D}\right) \\
&\quad = \frac{c\left(-\varphi\psi - \psi D + \varphi D + D^2 - \varphi\psi + \psi D - \varphi D + D^2\right)}{D\,(-\psi - D)\,(-\varphi - D)} = 0
\end{aligned}
$$

as well as

$$\begin{aligned}
&q\boldsymbol{.}\left(c(g^{-1})+r(g)\right)+q\boldsymbol{.}\left(c(g^{-1})-r(g)\right)\\
&\quad=\frac{-\frac{c}{D}\left(\frac{a+1}{c}\right)+\frac{a-d}{2D}-\frac{1}{2}}{\frac{a+1}{c}+\frac{d-a}{2c}-\frac{D}{2c}}+\frac{-\frac{c}{D}\left(\frac{a-1}{c}\right)+\frac{a-d}{2D}-\frac{1}{2}}{\frac{a-1}{c}+\frac{d-a}{2c}-\frac{D}{2c}}\\
&\quad=\frac{c}{D}\left(\frac{-\varphi-D}{\varphi-D}+\frac{-\psi-D}{\psi-D}\right)\\
&\quad=\frac{c\left(-\varphi\psi-\psi D+\varphi D+D^2-\varphi\psi+\psi D-\varphi D+D^2\right)}{D\left(\varphi-D\right)\left(\psi-D\right)}=0\,.
\end{aligned}$$

This means q maps the two points in $\underline{g}\mathrm{I}(g)$ to a pair of points symmetric to the origin, and the same holds true for the two points in $\underline{g}\mathrm{I}(g^{-1})$. Since Möbius transformations map circles onto circles and preserve $\widehat{\mathbb{R}}$, the above calculations show that $q\boldsymbol{.}\mathrm{I}(g)$ and $q\boldsymbol{.}\mathrm{I}(g^{-1})$ are semicircles centered at the origin, respectively, and thus both are orthogonal to $\alpha(\mathrm{h}_{\ell(g)})(\mathbb{R})$. Since Möbius transformations are also conformal, this orthogonality is preserved under action of q^{-1}. This implies the first claim of (iv), from which the second claim follows by convexity. □

Lemma 1.22. *Let $g \in \Gamma\setminus\Gamma_\infty$ be elliptic of order $\sigma = \sigma(g) \geq 3$ and let $k = k(g)$ be the unique positive integer from Remark 1.10. Then $\mathrm{I}(g)$ and $\mathrm{I}(g^{-1})$ intersect each other at an angle of $\frac{2k\pi}{\sigma}$, measured above the spheres.*

Proof. Since Möbius transformations are conformal, by Lemma 1.1(iii) and Remark 1.10 it suffices to consider the case

$$g = \mathrm{s}_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}, \qquad \theta = \frac{k}{\sigma}\pi\,.$$

Recall from Remark 1.10 that $k < \frac{\sigma}{2}$ and $(k,\sigma) = 1$. I. e., $\sin\theta > 0$. By Lemma 1.2 the order of g being larger than 2 is equivalent to $|\mathrm{tr}(g)| \in (0,2)$. Hence, by Lemma 1.21(ii) the spheres $\mathrm{I}(g)$ and $\mathrm{I}(g^{-1})$ intersect each other in a single point, which is fixed by g, and we denote the intersection angle by β. By (1.14) we have

$$\mathrm{f}(\mathrm{s}_\theta) = \frac{\mathrm{i}}{2\,|\sin\theta|}\sqrt{4-4|\cos\theta|^2} = \frac{\mathrm{i}}{|\sin\theta|}\sqrt{1-\cos^2\theta} = \mathrm{i}\,. \tag{1.57}$$

Consider $\mathrm{I}(g)$ and $\mathrm{I}(g^{-1})$ as contained in two circles in $\mathbb{C}$, say S_1 and S_2. More precisely,

$$S_{1/2} := \overline{\left(\mathrm{I}(g^{\pm1}) \cup \{-z \mid z \in \mathrm{I}(g^{\pm1})\}\right)}\,,$$

where the closure is taken with respect to the Euclidean topology in $\mathbb{C}$. Those two circles are of equal radius with centers on the real line which lie symmetric to the origin. Hence, the angle between the real line and the line connecting $\mathrm{f}(g)$

with $c(g)$ equals the angle between the real line and the line connecting $\mathrm{f}(g)$ with $c(g^{-1})$. Denote this angle by α. Obviously, $\alpha \in (0, \frac{\pi}{2})$. We have

$$\sin\alpha = \frac{\operatorname{Im}\mathrm{f}(g)}{r(g)} \overset{(1.57)}{=} \sin\theta\,,$$

which, in the required interval, has the sole solution $\alpha = \theta$. Finally, by basic planimetrics we obtain

$$\beta = \pi - 2\left(\frac{\pi}{2} - \alpha\right) = 2\theta = \frac{2k\pi}{\sigma}\,. \qquad \square$$

For the following statements we need to impose additional structure on the subgroup, namely that it is finitely generated. By [68, Theorem 6.6.3] the group Γ being finitely generated is sufficient for it to be Fuchsian. In the next section we will give a geometric characterization of finitely generated Fuchsian groups.

A family $\{M_j\}_{j\in J}$ of subsets of $\mathbb{H}$ to an arbitrary index set J is called *locally finite* if for each $z \in \mathbb{H}$ there exists a neighborhood $\mathcal{U}$ of z in $\mathbb{H}$ such that

$$\#\{j \in J \mid M_j \cap \mathcal{U}\} < +\infty\,.$$

Proposition 1.23. *The sets* $\operatorname{ISO}(\Gamma)$ *and* $\{\operatorname{int} \mathrm{I} \mid \mathrm{I} \in \operatorname{ISO}(\Gamma)\}$ *are locally finite, respectively.*

We refer the reader to [54, Proposition 6.1.5 and Lemma 6.1.11] for a proof of Proposition 1.23.

Proposition 1.24. *The set* $\{r(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$ *attains its maximum.*

Proof. First assume that ∞ represents a cusp of $\mathbb{X}$. Lemma 3.7 in [8] implies that for every $R > 0$ there exist only finitely many $g \in \Gamma \setminus \Gamma_\infty$ for which $r(g) \geq R$. This yields the assertion in this case.

Now assume that $\Omega(\Gamma)$ contains a neighborhood of ∞. By [35, IV.1D.5] the set $\{r(g) \mid g \in \Gamma \setminus \Gamma_\infty\}$ is bounded from above, say by $R_1 > 0$. Furthermore, by [35, IV.1D.3], all centers of isometric spheres lie within a bounded distance from the origin. Combining those two statements yields the existence of $R_2 \in \mathbb{R}_{>0}$ such that

$$\bigcup \operatorname{ISO}(\Gamma) \subseteq \operatorname{Re}|_{\mathbb{H}}^{-1}\big((-R_2, R_2)\big)\,.$$

Consider the set $M_r := (-R_2, R_2) + \mathrm{i}(r, R_1 + \varepsilon)$ for $r \in (0, R_1)$ and some arbitrary $\varepsilon > 0$. Proposition 1.23 implies that for every $z \in \overline{M_r}$ there exists an open neighborhood $\mathcal{U}_z$ which intersects only finitely many members of the family $\mathcal{J} := \big\{\operatorname{int}\mathrm{I} \,\big|\, \mathrm{I} \in \operatorname{ISO}(\Gamma)\big\}$. Since $\overline{M_r}$ is compact, we find finitely many points $z_1, \ldots, z_n \in \overline{M_r}$ such that

$$\overline{M_r} \subseteq \bigcup_{k=1}^{n} \mathcal{U}_{z_k}\,.$$

Since each of the sets $\mathcal{U}_{z_k}$ intersects only finitely many members of $\mathcal{I}$, so does $\overline{M_r}$. Hence, only finitely many isometric spheres exceed a height of r, for any $r > 0$. By (1.51), this finishes the proof. $\square$

We conclude this section with two more observations about isometric spheres, which, to our knowledge, are not yet to be found in the literature.

Proposition 1.25. *Let* $\mathrm{I}, \mathrm{J} \in \mathrm{ISO}(\Gamma)$ *be concentric, i. e.,* $c(\mathrm{I}) = c(\mathrm{J})$. *Then* $\mathrm{I} = \mathrm{J}$.

Proof. Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma \setminus \Gamma_\infty$ be a generator of I. Without loss of generality we assume $c > 0$. By Lemma 1.19(i) the center of I is then given by $-d/c$ and its radius by $1/c$. By assumption, a generator of J must be of the form $h := \left[\begin{smallmatrix} x & y \\ rc & rd \end{smallmatrix}\right]$ with $x, y \in \mathbb{R}$ and $r \in \mathbb{R} \setminus \{0\}$. The determinant condition on h yields

$$dx - cy = \frac{1}{r}\,,$$

with which we calculate

$$gh^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} rd & -y \\ -rc & x \end{bmatrix} = \begin{bmatrix} adr - bcr & bx - ay \\ 0 & dx - cy \end{bmatrix} = \begin{bmatrix} r & bx - ay \\ 0 & \frac{1}{r} \end{bmatrix}.$$

Hence, $gh^{-1} \in \Gamma_\infty$, and from (S) we obtain $r = 1$. This yields the assertion. $\square$

The final result of this section studies the relationship between the sets $\mathcal{W}(g)$ from (1.55) and the limit set $\Lambda(\Gamma)$. It will later be applied to locate limit points in certain regions of $\partial_g \mathbb{H}$.

Proposition 1.26. *Assume that* Γ *is either non-elementary or a hyperbolic cylinder. Then for every* $g \in \Gamma \setminus \Gamma_\infty$ *we have*

$$\mathcal{W}(g) \cap \Lambda(\Gamma) \neq \varnothing\,. \tag{1.58}$$

Additionally, if g *is elliptic of order* $\sigma \geq 3$ *and* $c(g) \notin \mathcal{W}(g^{-1})$, *then*

$$\left(\mathcal{W}(g) \setminus \mathcal{W}(g^{-1})\right)^\circ \cap \Lambda(\Gamma) \neq \varnothing\,.$$

The demand that $c(g) \notin \mathcal{W}(g^{-1})$ for $g \in \Gamma \setminus \Gamma_\infty$ elliptic is equivalent to $k(g) \leq \sigma(g)/3$, with $k(g)$ as in Remark 1.10 (see also Lemma 1.22). For the proof of Proposition 1.26 we require the following auxiliary result.

Lemma 1.27. *Let* $g \in \Gamma \setminus \Gamma_\infty$ *be hyperbolic or parabolic. Then* $\mathrm{I}(g^{n+1}) \subseteq \operatorname{int} \mathrm{I}(g^n)$ *and* $\mathrm{I}(g^{-n-1}) \subseteq \operatorname{int} \mathrm{I}(g^{-n})$ *for all* $n \in \mathbb{N}$. *The sequence of radii,* $(r(g^n))_{n \in \mathbb{N}}$, *tends to zero.*

Proof. Let $m, n \in \mathbb{Z}$ be such that $\mathrm{I}(g^n) = \mathrm{I}(g^m)$. By Lemma 1.20(i) it follows that $g^{m-n} \in \Gamma_\infty$, which, since g is of infinite order, contradicts the choice of g

unless $m = n$. Hence, no two of the isometric spheres $\mathrm{I}(g^n)$, $n \in \mathbb{Z}\setminus\{0\}$, coincide. Therefore, [33, Theorem 3.3.7] implies that the sequence $(r(g^n))_{n\in\mathbb{N}}$ tends to zero.

If g is hyperbolic, then from Lemma 1.11(iii) we obtain $\alpha(g^n) = \alpha(g)$. Hence, Lemma 1.21(iv) implies that, for every $n \in \mathbb{N}$, the geodesic arc $\alpha(g)(\mathbb{R})$ intersects both $\mathrm{I}(g^n)$ and $\mathrm{I}(g^{-n})$. Denote by ξ_n the intersection point of $\alpha(g)(\mathbb{R})$ with $\mathrm{I}(g^n)$, for $n \in \mathbb{Z} \setminus \{0\}$. Since g^n stabilizes $\alpha(g)(\mathbb{R})$, we deduce from Lemma 1.19(iii) that $g^n.\xi_n = \xi_{-n}$, for all $n \in \mathbb{Z} \setminus \{0\}$. Hence, in particular, $\operatorname{Im}\xi_n = \operatorname{Im}\xi_{-n}$ (Lemma 1.19(ii)) and by (1.38) and Lemma 1.11(iii),

$$\operatorname{dist}_{\mathbb{H}}(\xi_n, \xi_{-n}) = \ell(g^n) = n\ell(g) = n \operatorname{dist}_{\mathbb{H}}(\xi_1, \xi_{-1})$$

for all $n \in \mathbb{N}$. Hence, the geodesic segment $[\xi_1, \xi_{-1}]_{\mathbb{H}}$ lies symmetric in each segment $[\xi_n, \xi_{-n}]_{\mathbb{H}}, n \in \mathbb{N}$. This together with $(\mathrm{f}_+(g), \mathrm{f}_-(g)) = (\mathrm{f}_+(g^n), \mathrm{f}_-(g^n))$ for all $n \in \mathbb{N}$ and again Lemma 1.21(iv) shows the assertion in the case that g is hyperbolic.

Now assume that g is parabolic and let $\widetilde{g} \in \mathrm{SL}_2(\mathbb{R})$ be the representative of g fulfilling $\operatorname{tr}(\widetilde{g}) = 2$. By virtue of the determinant condition $\widetilde{g}$ admits the representation

$$\widetilde{g} = \begin{pmatrix} a & -\frac{1}{c}(a-1)^2 \\ c & 2-a \end{pmatrix} .$$

Note that $c \neq 0$ by assumption. One shows by induction that

$$\widetilde{g}^n = \begin{pmatrix} (a-1)n+1 & -\frac{n}{c}(a-1)^2 \\ cn & -an+n+1 \end{pmatrix} .$$

Indeed,

$$\begin{aligned}
&\begin{pmatrix} (a-1)n+1 & -\frac{n}{c}(a-1)^2 \\ cn & -an+n+1 \end{pmatrix} \cdot \begin{pmatrix} a & -\frac{1}{c}(a-1)^2 \\ c & 2-a \end{pmatrix} \\
&= \begin{pmatrix} a(a-1)n + a - (a-1)^2 n & -\frac{(a-1)^2}{c}\big((a-1)n+1+(2-a)n\big) \\ cn+c & -(a-1)^2 n + (2-a)(n-an) + 2 - a \end{pmatrix} \\
&= \begin{pmatrix} (a-1)(n+1)+1 & -\frac{n+1}{c}(a-1)^2 \\ c(n+1) & -a(n+1)+(n+1)+1 \end{pmatrix} .
\end{aligned}$$

Therefore, the radius of $\mathrm{I}(g^n)$ for $n \in \mathbb{Z} \setminus \{0\}$ is given by

$$r(g^n) = \frac{1}{n\,|c|} = \frac{r(g)}{n} ,$$

which means the sequence $(r(g^n))_{n\in\mathbb{N}}$ is strictly decreasing. Combining this with Lemma 1.21(iii) and $\mathrm{f}(g^n) = \mathrm{f}(g)$ for all $n \in \mathbb{N}$, implies that

$$\mathrm{I}(g^{n+1}) \subseteq \operatorname{int}\mathrm{I}(g^n) \cup \operatorname{int}\mathrm{I}(g^{-n}) ,$$

for all $n \in \mathbb{Z} \setminus \{0\}$. Assume for contradiction that $\mathrm{I}(g^n) \subseteq \operatorname{int} \mathrm{I}(g^{-1})$ for some $n \in \mathbb{N}$. From Lemma 1.20(iii) we then obtain

$$\mathrm{I}(g^{-n}) \subseteq \operatorname{int} \mathrm{I}(g^{-(n+1)}) ,$$

in contradiction to the relation of the radii. Hence, the assertion follows in the parabolic case as well. □

Proof of Proposition 1.26. For $g \in \Gamma \setminus \Gamma_\infty$ hyperbolic (1.58) is a direct consequence of Lemma 1.21(iv).

Let g be parabolic. Since Γ is not a parabolic cylinder, there exist further limit points which are no Γ-translates of $\mathrm{f}(g)$, and we choose x to be such. Then

$$|\mathrm{f}(g) - x| =: \varepsilon > 0 .$$

By Lemma 1.27 there exists $N \in \mathbb{N}$ such that $r(g^n) < \frac{\varepsilon}{2}$ for all $n \geq N$. From Lemma 1.21(iii) and $\mathrm{f}(g^n) = \mathrm{f}(g)$ for all $n \in \mathbb{N}$ we deduce $x \notin \overline{\mathscr{W}(g^{-n})}$. Therefore, (1.56) and again Lemma 1.27 imply that

$$g^{-n}.x \in \mathscr{W}(g^n) \subseteq \mathscr{W}(g) .$$

Since $\Lambda(\Gamma)$ is Γ-invariant, this implies (1.58) for g parabolic.

Now assume that g is elliptic and denote by $\sigma = \sigma(g)$ the order of g, which is finite by Lemma 1.9. Further, Γ is not a hyperbolic cylinder and thus non-elementary by assumption, meaning $\Lambda(\Gamma)$ is of infinite cardinality. If $\sigma = 2$, i. e., g is an involution, then $\mathrm{I}(g) = \mathrm{I}(g^{-1})$. Hence, $\overline{\mathscr{W}(g)} \cup \big(\widehat{\mathbb{R}} \setminus \mathscr{W}(g^{-1})\big) = \widehat{\mathbb{R}}$ and application of (1.56) if necessary yields (1.58). Thus, assume $\sigma > 2$. Then, by Lemma 1.21(ii), $\mathrm{I}(g)$ and $\mathrm{I}(g^{-1})$ intersect each other in exactly one point, $\mathrm{f}(g)$. By convexity,

$$\partial_g \mathrm{I}(g) \setminus \mathscr{W}(g^{-1}) = \{\xi(g)\} \qquad \text{and} \qquad \partial_g \mathrm{I}(g) \cap \mathscr{W}(g^{-1}) = \{\xi'(g)\} ,$$

for two points $\xi(g), \xi'(g) \in \mathbb{R}$ only depending on g. By renaming g to g^{-1} if necessary we may assume the ordering

$$\xi(g) < \xi'(g^{-1}) < \xi'(g) < \xi(g^{-1}) . \tag{1.59}$$

From Lemma 1.19(iii) and (1.56) we obtain

$$g.\xi(g) = \xi(g^{-1}) \qquad \text{and} \qquad g.\xi'(g) = \xi'(g^{-1}) . \tag{1.60}$$

The orbit $(g^n.\xi(g))_{n\in\mathbb{N}}$ is periodic with minimal period σ (see the discussion right before Lemma 1.9). Hence, $\bigcup_{i=1}^{\sigma-1}\{g^i.\xi(g)\}$ dissects $\widehat{\mathbb{R}}$ into σ intervals, whose structure we study now. Define

$$\mathscr{V} := \widehat{\mathbb{R}} \setminus \left(\overline{\mathscr{W}(g)} \cup \overline{\mathscr{W}(g^{-1})}\right) = \left(\xi(g^{-1}), +\infty\right) \cup \{\infty\} \cup \left(-\infty, \xi(g)\right).$$

Then, by (1.56), $g.\mathscr{V} \subseteq \mathscr{W}(g^{-1})$, and thus $g.\mathscr{V} \cap \mathscr{V} = \varnothing$. But (1.60) implies that

$$g.\mathscr{V} = \left(g.\xi(g^{-1}), \xi(g^{-1})\right),$$

which in turn shows that $g.\mathscr{V} \cup \{\xi(g^{-1})\} \cup \mathscr{V}$ is an interval in $\mathbb{R}$. The structure of $g.\mathscr{V}$ immediately implies that this argument applies iteratively, meaning $g^2.\mathscr{V} \cup \{g.\xi(g^{-1})\} \cup g.\mathscr{V}$ is again an interval in $\mathbb{R}$ and so on. Continuing in this way yields the decomposition

$$\widehat{\mathbb{R}} = \bigcup_{i=1}^{\sigma-1} g^i.\mathscr{V} \cup \bigcup_{i=1}^{\sigma-1} \{g^i.\xi(g)\}.$$

Note that this decomposition is not necessarily disjoint, but rather covers $\widehat{\mathbb{R}}$ exactly $k(g)$-times, with $k(g)$ as in Remark 1.10. Since Γ is non-elementary and thus contains infinitely many non-conjugate hyperbolic elements (see [33, Theorem 2.4.4]), we have

$$\Lambda(\Gamma) \setminus \Gamma.\{\xi(g), \xi'(g)\} \neq \varnothing. \tag{1.61}$$

Hence, in particular, for

$$x \in \Lambda(\Gamma) \setminus \bigcup_{i=1}^{\sigma-1} g^i.\{\xi(g)\}$$

we have $x \in g^i.\mathscr{V}$ for at least one $i \in \{1, \dots, \sigma\}$ according to the above decomposition. By construction there exists $j \in \mathbb{N}$, $j < \sigma$, such that $g^j.x \in \mathscr{V}$, and thus, $g^{j+1}.x \in \mathscr{W}(g^{-1})$. By symmetry, this yields (1.58) in the elliptic case.

Finally, assume that g is elliptic of order $\sigma \geq 3$ and that $c(g) \notin \mathscr{W}(g)$. Because of (1.59) this means that

$$\Xi := \overline{\mathscr{W}(g) \cap \mathscr{W}(g^{-1})} = \left[\xi'(g^{-1}), \xi'(g)\right] \subseteq \left[c(g), c(g^{-1})\right].$$

Consider further the intervals

$$\Theta := \left(\xi(g), c(g)\right] \qquad \text{and} \qquad \Psi := \left(c(g), \xi'(g^{-1})\right).$$

Then $\Theta \cup \Psi = \left(\mathscr{W}(g) \setminus \mathscr{W}(g^{-1})\right)^\circ$ and the union $\Theta \cup \Psi \cup \Xi$ is disjoint and constitutes an interval in $\mathbb{R}$, where Ψ might be the empty set. By the above it follows that $\Lambda(\Gamma) \cap \mathscr{W}(g) \neq \varnothing$. Hence, it remains to show that $\Lambda(\Gamma) \cap \mathscr{W}(g) \nsubseteq \Xi$.

Because of (1.60) and Lemma 1.19(i) we have

$$\widehat{\mathbb{R}} = \big(\Xi \cup \Psi\big) \cup g.\big(\Xi \cup \Psi\big) \cup g^2.\big(\Xi \cup \Psi\big) \cup \big(\xi'(g), c(g^{-1})\big], \tag{1.62}$$

where the union on the right hand side is disjoint. Furthermore,

$$\Theta = \big(g.(\Xi \cup \Psi)\big) \cap \mathscr{W}(g)$$

and hence, by (1.56),

$$g.\Theta = \big(\xi(g^{-1}), +\infty\big] = \big(g^2.(\Xi \cup \Psi)\big) \setminus \overline{\mathscr{W}(g^{-1})}\,.$$

From the disjointness of the union in (1.62) we therefore obtain

$$\big(g^2.(\Xi \cup \Psi)\big) \setminus g.\Theta \subseteq \overline{\mathscr{W}(g^{-1})} \setminus \mathscr{W}(g)\,.$$

Because of (1.61) this implies $\Lambda(\Gamma) \cap \big(\mathscr{W}(g^{-1}) \setminus \mathscr{W}(g)\big)^\circ \neq \varnothing$, and switching the roles of g and g^{-1} yields the assertion. □

1.10 Fundamental Domains

Let Γ be a Fuchsian group. A subset $\mathcal{F}$ of $\mathbb{H}$ is called a *fundamental region* for (the action of) Γ in $\mathbb{H}$ if $\mathcal{F}$ is an open set such that

(F1) any two Γ-translates of $\mathcal{F}$ are disjoint, i. e., for all $g \in \Gamma \setminus \{\mathrm{id}\}$,

$$g.\mathcal{F} \cap \mathcal{F} = \varnothing\,,$$

(F2) the Γ-translates of $\overline{\mathcal{F}}$ cover all of $\mathbb{H}$:

$$\mathbb{H} = \bigcup_{g\in\Gamma} g.\overline{\mathcal{F}}\,.$$

Property (F2) is called the *tessellation property*. Every set $M \subseteq \mathbb{H}$ for which the union $\bigcup \Gamma.\overline{M}$ covers $\mathbb{H}$ is said to *tessellate* $\mathbb{H}$ *under* Γ. Property (F1) implies that the union in (F2) is *essentially disjoint* for every fundamental region $\mathcal{F}$. A family of subsets M_j of some finite dimensional vector space V, or the union thereof, with j in some index set J, is called *essentially disjoint*, if, for all $j, k \in J$, $j \neq k$, the intersection $M_j \cap M_k$ is of dimension at most $\dim V - 1$.

The following observation is immediate.

Lemma 1.28. *Let $n \in \mathbb{N}$ and let $M_1, \ldots, M_n$ be mutually disjoint, open subsets of $\mathbb{H}$ such that $\bigcup_{k=1}^n M_k$ is a fundamental region for some Fuchsian group Γ. Further let $g_1, \ldots, g_n \in \Gamma$. Then $\bigcup_{k=1}^n g_k.M_k$ is again a fundamental region for Γ.*

A connected fundamental region is called a *fundamental domain*. Each Fuchsian group admits a fundamental domain (see [33, Theorem 3.2.2]). Standard choices are Dirichlet or Ford fundamental domains, of which we will use the latter and which are examples of fundamental domains in the shape of (interiors of) exact, convex polygons (in the sense of [68, §6.3]). We assemble the necessary notions.

Definition 1.29. Let $\mathcal{F}$ be a convex, open, nonempty subset of $\mathbb{H}$ and let Γ be a Fuchsian group.

(a) A *side of* $\mathcal{F}$ is a maximally convex subset of $\partial\mathcal{F}$ of positive length.

(b) $\mathcal{F}$ is called a *convex polygon* in $\mathbb{H}$ if the set of sides of $\mathcal{F}$ is locally finite in $\mathbb{H}$. (See the exposition right before Proposition 1.23 in Section 1.9.)

(c) $\mathcal{F}$ is called *geometrically finite* or a *geometrically finite polygon* in $\mathbb{H}$ if the set of its sides is finite.

(d) Let $\mathcal{F}$ be a convex polygon and assume that $\mathcal{F}$ is a fundamental domain for Γ. Then $\mathcal{F}$ is called a *convex fundamental polygon* for Γ.

(e) Assume that $\mathcal{F}$ is a convex fundamental polygon for Γ. Then $\mathcal{F}$ is called *exact*, if for each side β of $\mathcal{F}$ there exists $g \in \Gamma$ such that $\beta = \overline{\mathcal{F}} \cap g.\overline{\mathcal{F}}$.

For $\mathcal{F}$ a convex polygon in $\mathbb{H}$ we denote by $S_{\mathcal{F}}$ the set of its sides.

Definition 1.30. Let $\mathcal{F}$ be a convex polygon in $\mathbb{H}$. A subset G of $\mathrm{PSL}_2(\mathbb{R})$ is called a *side-pairing for* $\mathcal{F}$, if there exists a surjective map $\rho\colon S_{\mathcal{F}} \to G$ such that for all $\beta \in S_{\mathcal{F}}$

(I) we have

$$\rho(\beta).\beta \in S_{\mathcal{F}} \qquad \text{and} \qquad \rho\big(\rho(\beta).\beta\big) = \rho(\beta)^{-1}\,,$$

(II) there exists a neighborhood $\mathcal{U}$ of β in $\overline{\mathbb{H}}^g$ such that

$$\mathcal{F} \cap \rho(\beta).\big(\mathcal{U} \cap \mathcal{F}\big) = \varnothing\,.$$

If G is a side-pairing for $\mathcal{F}$ then each element of G is called a *side-pairing transformation of* $\mathcal{F}$.

Let $\mathcal{F}$, $S_{\mathcal{F}}$, G, and ρ be as in Definition 1.30. Property (I) induces an involution on $S_{\mathcal{F}}$: every side β of $\mathcal{F}$ is paired with exactly one side $\beta' = \rho(\beta).\beta$. A side β being paired to itself is not prohibited; however (II) prevents G from containing the identity. Hence, no side of $\mathcal{F}$ is fixed by a side-pairing transformation ($S_{\mathcal{F}}$ is locally finite by assumption). The surjectivity of ρ assures that G is minimal for its purpose. In particular, if $\mathcal{F}$ is geometrically finite, then

$$\#G \leq \#S_{\mathcal{F}} < +\infty\,.$$

Lemma 1.31 ([68, Theorem 6.7.5]). *A convex fundamental polygon $\mathcal{F}$ for Γ is exact if and only if there exists a side-pairing of $\mathcal{F}$ in Γ.*

The Fuchsian group Γ is called *geometrically finite* if it admits a geometrically finite fundamental domain. This fundamental domain is then automatically in the form of a convex fundamental polygon. In particular, every $\beta \in S_{\mathcal{F}}$ is a geodesic segment, and is closed if and only if both its endpoints are contained in $\mathbb{H}$.

Lemma 1.32 ([68, Theorem 12.4.5]). *Let Γ be a geometrically finite Fuchsian group. Then every exact convex fundamental polygon for Γ is geometrically finite.*

Corollary 1.33. *A Fuchsian group is geometrically finite if and only if its orbit space is geometrically finite.*

For geometrically finite Fuchsian groups a full set of generators and their relations can be re-obtained from the fundamental polygon and its side-pairing in Γ. This is the quintessence of Poincaré's fundamental polygon theorem, which we tend to formulate in the following. In order to do so, the concept of vertex cycles is required, which we recall now.

Let $\mathcal{F}$ be a geometrically finite polygon in $\mathbb{H}$ with side-pairing G. A *(finite) vertex* of $\mathcal{F}$ is a point $v \in \mathbb{H}$ that is the common endpoint of two distinct sides of $\mathcal{F}$. Equivalently, a vertex is every point $v \in \mathbb{H}$ for which there exist $g, h \in \Gamma$, $g \neq h$, such that

$$\{v\} = \overline{\mathcal{F}} \cap g.\overline{\mathcal{F}} \cap h.\overline{\mathcal{F}}$$

(see also [6, Definition 9.3.2]). Assume that $S_{\mathcal{F}}$ contains two elements of the form $[z_1, x)_{\mathbb{H}}, [z_2, x)_{\mathbb{H}}$ with some $x \in \partial_g \mathbb{H}$. Then x is called an *infinite vertex of* $\mathcal{F}$. We denote by $V_{\mathcal{F}}$ the set of finite and by $V_{\mathcal{F}}^g$ the set of infinite vertices of $\mathcal{F}$. For $v \in V_{\mathcal{F}} \cup V_{\mathcal{F}}^g$ we define its *vertex cycle* as

$$C(v) := \overline{\mathcal{F}}^g \cap G.v\,.$$

Since $\mathrm{PSL}_2(\mathbb{R}).\mathbb{H} = \mathbb{H}$ and $\mathrm{PSL}_2(\mathbb{R}).\partial_g \mathbb{H} = \partial_g \mathbb{H}$, the set $C(v)$ consists solely of finite vertices of $\mathcal{F}$ if v is finite, and solely of infinite vertices if v is infinite. Since $\mathcal{F}$ is geometrically finite and thus $\#G < +\infty$, we have $\#C(v) < +\infty$ for every vertex $v \in V_{\mathcal{F}} \cup V_{\mathcal{F}}^g$. For each finite vertex $v \in V_{\mathcal{F}}$ we denote by $\theta(v)$ the angle $\mathcal{F}$ subtends at v. We define the *angle sum* of $C(v)$ for $v \in V_{\mathcal{F}}$ to be

$$\theta(C(v)) := \sum_{w \in C(v)} \theta(w)\,. \tag{1.63}$$

Lemma 1.34 ([6, Theorem 9.3.5]). *For every geometrically finite Fuchsian group Γ, every geometrically finite fundamental domain $\mathcal{F}$ of Γ, and every $v \in V_{\mathcal{F}}$ there exists $\omega \in \mathbb{N}$ such that*

$$\theta(C(v)) = \frac{2\pi}{\omega}\,.$$

Let $v \in V_{\mathcal{F}} \cup V_{\mathcal{F}}^{g}$. Further let β_1, β_2 be the two distinct sides of $\mathcal{F}$ such that v is the common endpoint of β_1 and β_2. Recall the map $\rho\colon S_{\mathcal{F}} \to G$ from Definition 1.30. By construction, $\rho(\beta_1).v \in C(v)$, and we set $g_1 := \rho(\beta_1)$. Let β_3, β_4 be the two sides whose common endpoint is $g_1.v$. Then one of them, say β_4, is equal to $g_1.\beta_1$ By the second condition in Definition 1.30(I) this implies $\rho(\beta_4) = \rho(\beta_1)^{-1}$. We set $g_2 := \rho(\beta_3)g_1$. Again, $g_2.v \in C(v)$. Proceeding in this manner successively generates every element of $C(v)$. We stop once we obtain $g_n.v = v$ for $n \in \mathbb{N}$. We call

$$c_v := g_n$$

cycle transformation of v. Another cycle transformation c_v' is obtained by starting with $g_1 = \rho(\beta_2)$ instead. Applying the second condition in Definition 1.30(I) for every $w \in C(v)$ yields $c_v' = c_v^{-1}$. Because of Lemma 1.34 for $v \in V_{\mathcal{F}}$ the associated cycle transformations are of finite order, hence either elliptic or the identity (Lemma 1.9; see also [38]). In particular, one obtains $c_v^{\sigma} = \mathrm{id}$ with $\sigma \in \mathbb{N}$. These relations for all $v \in V_{\mathcal{F}}$, where σ is chosen minimal respectively, are called the *cycle relations for $\mathcal{F}$*.

Remark 1.35. The above treatment of angle sums and cycle transformations cuts short in various regards, most profoundly in terms of justifications. This curtailment was deemed appropriate, for in the analysis that follows those objects are not applied beyond their mere concepts, which are required for the formulation of the Poincaré theorem (Proposition 1.36 below) and its application (Section 7.2). We refer the reader to [6, §9.3] for an in-depth discussion of all these objects.

Furthermore, some of our notions here differ from what is usually encountered in the literature. For instance, a convex polygon $\mathcal{F}$ is usually defined as a *closed* subset of $\mathbb{H}$ whose interior might be a fundamental domain for some Fuchsian group. We omitted that distinction here for we do not require it. Moreover, infinite vertices in the literature refer to a larger class of ideal points than they do here. Usually a distinction is made between two-sided (or proper) and one-sided (or improper) infinite vertices, of which only the former are infinite vertices in our sense. One-sided infinite vertices do not demand any special treatment in light of Poincaré's theorem and thus are omitted here.

Proposition 1.36 (Poincaré's theorem on fundamental polygons, [38])**.** *Let $\mathcal{F}$ be a convex polygon in $\mathbb{H}$ for which there exists a side-pairing G fulfilling the following conditions:*

(I) *For every $v \in V_{\mathcal{F}}$ there exists $\omega \in \mathbb{N}$ such that $\omega\theta(C(v)) = 2\pi$.*

(II) *For every $v \in V_{\mathcal{F}}^{g}$ its cycle transformations are parabolic.*

Then $\langle G\rangle$ is Fuchsian, $\mathcal{F}$ is a convex fundamental polygon for $\langle G\rangle$, and the cycle relations for $\mathcal{F}$ form a full set of relations for $\langle G\rangle$.

From Proposition 1.36 we infer that every geometrically finite Fuchsian group is finitely generated. The converse is also true (see [6, Theorem 10.1.2]), meaning a Fuchsian group is finitely generated if and only if it possesses a geometrically finite fundamental polygon.

In what follows we will exclusively consider geometrically finite Fuchsian groups (of finite or infinite covolume) that contain hyperbolic elements (and possibly elliptic and parabolic ones as well). For these groups geometrically finite fundamental polygons can be constructed rather easily, in form of Dirichlet or Ford fundamental domains. Here we will work with the latter. The remainder of this section is dedicated to a study of these domains. To that end we adopt the concept of the common exterior and relevant isometric spheres from [54]. We recall these objects in the following.

Let Γ be Fuchsian. As before, denote by Γ_∞ the stabilizer of ∞ in Γ. Then Γ_∞ is again Fuchsian. As in Section 1.9, we assume that ∞ is contained in a hyperbolic end, that is, $\infty \notin \widehat{\mathbb{R}}_{\mathrm{st}}$ and, consequentially, condition (S) holds true. We additionally assume that, if $\mathbb{X}$ has cusps, then ∞ does indeed represent a cusp of $\mathbb{X}$. In this case let $\lambda > 0$ be its cusp width, that is,

$$\langle \mathrm{t}_\lambda \rangle = \Gamma_\infty\,,$$

with t_λ as in (1.7).

Lemma 1.37. *Assume that $\mathbb{X}$ has cusps and that ∞ represents a cusp of $\mathbb{X}$. For every $r \in \mathbb{R}$ the set*

$$\mathcal{F}_\infty(r) := \mathrm{Re}|_{\mathbb{H}}^{-1}\big((r, r+\lambda)\big) \tag{1.64}$$

is a fundamental domain for Γ_∞.

Proof. For $z \in \mathbb{H}$ and $n \in \mathbb{Z}$ we have

$$\mathrm{Re}\, \mathrm{t}_\lambda^n . z = \mathrm{Re}\, z + n\lambda = \mathrm{t}_\lambda^n .(\mathrm{Re}\, z)\,.$$

With that one immediately verifies (F1) and (F2). □

Remark 1.38. If Γ has no cusps and thus a neighborhood of ∞ is contained in $\Omega(\Gamma)$, then the stabilizer subgroup Γ_∞ is trivial. In this case

$$\mathcal{F}_\infty := \mathbb{H} \tag{1.65}$$

trivially fulfills (F1) and (F2), and thus is a fundamental domain for $\Gamma_\infty = \{\mathrm{id}\}$.

Recall the set $\mathrm{ISO}(\Gamma)$ from (1.44). The set

$$\mathcal{K} = \mathcal{K}_\Gamma := \bigcap_{\mathrm{I} \in \mathrm{ISO}(\Gamma)} \mathrm{ext}\, \mathrm{I} \tag{1.66}$$

is called the *common exterior of* $\mathrm{ISO}(\Gamma)$. We drop the index Γ whenever the associated Fuchsian group is clear from the context. One shows (see [54, Proposition 6.1.12]) that

$$\partial\mathcal{K} \subseteq \bigcup_{g\in\Gamma\setminus\Gamma_\infty} \mathrm{I}(g)\,. \tag{1.67}$$

Hence, the boundary of $\mathcal{K}$ in $\mathbb{H}$ is piece-wise given by geodesic segments. Since the set $\operatorname{ext}\mathrm{I}$ is convex for every $\mathrm{I} \in \mathrm{ISO}(\Gamma)$, so is $\mathcal{K}$. In particular, because of (1.48), the geodesic ray $(z,\infty)_{\mathbb{H}}$ is contained in $\mathcal{K}$ for every $z \in \mathcal{K}$, or in other words, $\infty \in g\mathcal{K}$. Further, as a consequence of the following observation, $\mathcal{K}$ is open.

Lemma 1.39. *Let $\{M_j\}_{j\in J}$ be a family of open sets in a path-connected metric space (X,d) such that the family of their boundaries $\{\partial M_j\}_{j\in J}$ is locally finite. Then $\bigcap_{j\in J} M_j$ is open.*

Proof. Since X is path-connected, it is connected. Hence, a set M_k being closed (as well as open by assumption) for some $k \in J$ implies $M_k \in \{\varnothing, X\}$, Hence, for all $j \in J$ with $M_j \notin \{\varnothing, X\}$ we therefore have $\partial M_j \neq \varnothing$. Without loss of generality we may assume $M_j \neq \varnothing$ for all $j \in J$, for otherwise the assertion is obvious. For the same reason we may assume that $\bigcap_{j\in J} M_j \neq \varnothing$.

Let $x \in \bigcap_{j\in J} M_j$. By assumption there exists an open neighborhood $\mathcal{U}$ of x in X such that

$$\#J_{\mathcal{U}} < +\infty\,, \qquad \text{where } \; J_{\mathcal{U}} \coloneqq \{j \in J \mid \mathcal{U} \cap \partial M_j \neq \varnothing\}\;.$$

Without loss of generality we may assume that $\mathcal{U}$ is path-connected, for otherwise we may pass to the path-connected component of $\mathcal{U}$ containing x. (Note that, since X is path-connected by assumption, every connected component of every open subset of X is again open.) First consider the case that $J_{\mathcal{U}} = \varnothing$. Assume that there exists $y \in \mathcal{U}$ such that $y \notin \bigcap_{j\in J} M_j$. This means there exists $j' \in J$ such that $y \notin M_{j'}$. Clearly, $y \notin \partial M_{j'}$, for then $\mathcal{U} \cap \partial M_{j'} \neq \varnothing$, contradicting $J_{\mathcal{U}} = \varnothing$. Hence, y is contained in the open set $X \setminus \overline{M_{j'}}$. By assumption, there exists a path from x to y in $\mathcal{U}$, i.e., a continuous function $f\colon [0,1] \to \mathcal{U}$ fulfilling $f(0) = x$ and $f(1) = y$. Since $x \in M_{j'}$, $M_{j'}$ and $X\setminus\overline{M_{j'}}$ are both open, and f is continuous, there exists $r \in (0,1)$ such that

$$\mathrm{B}_{d,\varepsilon}(f(r)) \cap M_{j'} \neq \varnothing \qquad \text{and} \qquad \mathrm{B}_{d,\varepsilon}(f(r)) \cap X \setminus \overline{M_{j'}} \neq \varnothing\,,$$

for every choice of $\varepsilon > 0$, where $\mathrm{B}_{d,\varepsilon}(z)$ denotes the open ε-ball centered at z with respect to the metric d. Hence, $f(r) \in \partial M_{j'}$. But since $f([0,1]) \subseteq \mathcal{U}$, this contradicts $J_{\mathcal{U}} = \varnothing$. In turn, $\mathcal{U} \subseteq \bigcap_{j\in J} M_j$. Since x has been chosen arbitrarily, this yields the assertion in this case.

Now assume that $J_{\mathcal{U}} \neq \varnothing$ and denote its elements by $j_1,\ldots,j_n$ (recall from before that $\#J_{\mathcal{U}} < +\infty$ by assumption). Then, in particular, $x \in \bigcap_{i=1}^n M_{j_i}$. Since each of the sets M_j is open, we find $d(x, \partial M_{j_i}) > 0$ for every $i = 1, \ldots, n$. We now set

$$\varepsilon := \frac{1}{2} \min \left\{ d(x, M_{j_i}) \;\middle|\; i \in \{1, \ldots, n\} \right\} .$$

Then $\mathrm{B}_{d,\varepsilon}(x) \subseteq \bigcap_{i=1}^{n} M_{j_i}$ and as in the proof in the case $J_{\mathcal{U}} = \varnothing$ we conclude that $\mathrm{B}_{d,\varepsilon}(x)$ cannot contain any points exterior to M_j, for any $j \in J$. This yields the assertion in this case and thereby finishes the proof. $\square$

By convexity, every isometric sphere induces at most one maximally convex component of $\partial\mathcal{K}$. An isometric sphere $\mathrm{I} \in \mathrm{ISO}(\Gamma)$ that does so, that is, if $\mathrm{I} \cap \partial\mathcal{K}$ consists of more than one point, is called *relevant*. In this case we also say that I *contributes non-trivially* to $\partial\mathcal{K}$. We denote by $\mathrm{REL}(\Gamma)$ the subset of $\mathrm{ISO}(\Gamma)$ of all relevant isometric spheres. Because of Proposition 1.23 we have $\mathrm{REL}(\Gamma) \neq \varnothing$ whenever $\mathrm{ISO}(\Gamma) \neq \varnothing$. For each $\mathrm{I} \in \mathrm{REL}(\Gamma)$ the set

$$\beta_{\mathrm{I}} := \mathrm{I} \cap \partial\mathcal{K} \tag{1.68}$$

is a geodesic segment in $\mathbb{H}$, which we call the *relevant part* of I.

The proofs of the following two results are the same as for [54, Proposition 6.1.26] resp. [54, Proposition 6.1.29].

Proposition 1.40. *Suppose* $\mathrm{ISO}(\Gamma) \neq \varnothing$. *Then*

$$\partial\mathcal{K} = \bigcup_{\mathrm{I} \in \mathrm{REL}(\Gamma)} \beta_{\mathrm{I}}$$

and for each choice of $\mathrm{I}, \mathrm{J} \in \mathrm{REL}(\Gamma)$, $\mathrm{I} \neq \mathrm{J}$, *the intersection* $\beta_{\mathrm{I}} \cap \beta_{\mathrm{J}}$ *is either empty or a singleton in* $\mathbb{H}$.

Recall the notion of generators of isometric spheres from (1.43). We denote by Γ_{REL} the subset of $\Gamma \setminus \Gamma_\infty$ of all generators of relevant isometric spheres.

Proposition 1.41. *With* $g \in \Gamma_{\mathrm{REL}}$ *we also have* $g^{-1} \in \Gamma_{\mathrm{REL}}$. *Furthermore,*

$$\beta_{\mathrm{I}(g^{-1})} = g \boldsymbol{.} \beta_{\mathrm{I}(g)} \,.$$

The common exterior $\mathcal{K} = \mathcal{K}_\Gamma$ naturally contains fundamental domains for Γ. Consider the decomposition of $\partial\mathcal{K}$ from Proposition 1.40 and denote by $S_{\mathcal{K}}$ the full set of relevant parts of relevant isometric spheres, i. e.,

$$S_{\mathcal{K}} = \{ \beta_{\mathrm{I}} \mid \mathrm{I} \in \mathrm{REL}(\Gamma) \} \,. \tag{1.69}$$

We also call the elements of $S_{\mathcal{K}}$ *sides* of $\mathcal{K}$, which is justified by the boundary structure of $\mathcal{K}$ and Proposition 1.40. Denote further by $W_{\mathcal{K}}$ the set of endpoints of the elements of $S_{\mathcal{K}}$ in $\overline{\mathbb{H}}^g$.

Proposition 1.42. *Let*

$$r \in \mathrm{Re}|_{\overline{\mathbb{H}}^g}(W_{\mathcal{K}}) \cup \left(g\mathcal{K} \setminus \{\infty\}\right) \in \mathbb{R}$$

and let $\mathcal{F}_\infty = \mathcal{F}_\infty(r)$ be as in (1.64), if $\mathbb{X}$ has cusps, or as in (1.65), if $\mathbb{X}$ has no cusps. Then

$$\mathcal{F} := \mathcal{F}_\infty \cap \mathcal{K} \tag{1.70}$$

is an exact convex fundamental polygon for Γ. If Γ is geometrically finite, then $\mathcal{F}$ is geometrically finite.

Proof. In the case that $\mathbb{X}$ has cusps, all claims follow from [54, Proposition 6.1.36] and [54, Theorem 6.1.38]. Thus, assume that $\mathbb{X}$ has no cusps and that Γ_∞ is trivial. A combination of Propositions 1.23 and 1.24 with [55, Corollary 3.20] shows that $\mathcal{F} = \mathcal{K}$ is a fundamental region for Γ. Since $\mathcal{K}$ is convex and open, Proposition 1.23 further shows that it is a convex fundamental polygon for Γ. Further, Lemma 1.20(i) implies that

$$\begin{cases} \Gamma & \longrightarrow & \mathrm{ISO}(\Gamma) \\ g & \longmapsto & \mathrm{I}(g) \end{cases} \quad \text{and thus} \quad \begin{cases} \Gamma_{\mathrm{REL}} & \longrightarrow & \mathrm{REL}(\Gamma) \\ g & \longmapsto & \mathrm{I}(g) \end{cases} \tag{1.71}$$

are both bijections. Hence, the Propositions 1.40 and 1.41 yield a unique side-pairing for $\mathcal{K}$ in Γ, which means that $\mathcal{K}$ is also exact. The last claim now follows from Lemma 1.32. $\square$

The domain $\mathcal{F}$ from Proposition 1.42 is called a *Ford fundamental domain* or a *fundamental domain of the Ford type* for Γ. This type will constitute the fundamental domains of choice in this thesis. We will often choose a Ford fundamental domain $\mathcal{F} = \mathcal{F}(r)$ and will indicate the choice of r only implicitly.

From Proposition 1.42 we read off the structure of the boundary of a Ford fundamental domain in $\mathbb{H}$ and in $\partial_g\mathbb{H}$. With $\mathcal{F}$ as in (1.70) and $\mathcal{F}_\infty$ as in (1.64) resp. (1.65) we find (see [54, Theorem 6.1.15]) that $\partial\mathcal{F}$ decomposes disjointly as

$$\partial\mathcal{F} = \left(\partial\mathcal{F}_\infty \cap \mathcal{K}\right) \cup \left(\overline{\mathcal{F}_\infty} \cap \partial\mathcal{K}\right). \tag{1.72}$$

Concerning vertices and the boundary of $\mathcal{F}$ in $\partial_g\mathbb{H}$ we have the following result.

Proposition 1.43. *Let $\mathcal{F}$ be a Ford fundamental domain for a geometrically finite Fuchsian group Γ. Then the following statements hold true.*

(i) *Every $v \in V_{\mathcal{F}}$ is a fixed point of some elliptic element in Γ. For every elliptic fixed point ξ of Γ there exists $g \in \Gamma$ such that $g.\xi \in V_{\mathcal{F}}$.*

(ii) *Every $v \in V_{\mathcal{F}}^g$ is a fixed point of some parabolic element in Γ. For every parabolic fixed point x of Γ there exists $h \in \Gamma$ such that $h.x \in V_{\mathcal{F}}^g$. None of the elements of $V_{\mathcal{F}}^g$ is fixed by a hyperbolic element in Γ.*

(iii) *Assume that $\mathcal{gF} \cap \partial_g \mathbb{H}$ contains an interval $I \subseteq \widehat{\mathbb{R}}$ of positive length. Then $I \subseteq \Omega(\Gamma)$. In particular, $\mathbb{X}$ has funnels and I intersects a funnel representative. The boundary points of I in $\widehat{\mathbb{R}}$ are no parabolic fixed points and for every funnel of $\mathbb{X}$ the set $\mathcal{gF} \cap \partial_g \mathbb{H}$ contains a representative of it.*

Proof. In the proof of [33, Theorem 3.3.5] it is shown that every Ford fundamental domain is a Dirichlet fundamental domain. Hence, all statements about Dirichlet fundamental domains can be applied here as well. Statement (i) therefore follows from the discussion after [33, Theorem 3.5.1], while the first two statements of (ii) are shown in [33, Theorem 4.2.5]. The final statement of (ii) is thus a consequence of Lemma 1.8. Statement (iii) then follows from the combination of statement (ii) with [6, Theorems 10.2.3 and 10.2.5] and [35, IV.7E and IV.7G]. $\square$

Proposition 1.42 further has profound implications for the structure of $\mathcal{K}$ for geometrically finite Fuchsian groups Γ: If $\mathbb{X}$ is void of cusps, then $\mathcal{K}$ itself is a fundamental domain for Γ, and thus geometrically finite, i. e., finite-sided. Hence,

$$\#\mathrm{REL}(\Gamma) < +\infty$$

and thus, by the second map in (1.71) being bijective, $\#\Gamma_{\mathrm{REL}} < +\infty$. In particular, there exist $a, b \in \mathbb{R}$, $a < b$, such that

$$\partial\mathcal{K} = \bigcup_{g \in \Gamma_{\mathrm{REL}}} \beta_{\mathrm{I}(g)} \subseteq \mathrm{Re}|_{\mathbb{H}}^{-1}\big([a, b]\big)\,. \tag{1.73}$$

If Γ has cusps and ∞ represents a cusp of Γ with cusp width λ, then, by (F2), $\mathcal{K}$ is invariant under transformations in Γ_∞, meaning

$$\mathrm{t}_\lambda^n.\mathcal{K} = \mathcal{K}\,, \tag{1.74}$$

for all $n \in \mathbb{Z}$. Because of that and Proposition 1.42, the statements of Proposition 1.43 also apply, *mutatis mutandis*, to $\mathcal{K}$ instead of $\mathcal{F}$.

Ford fundamental domains provide models for the orbisurface $\mathbb{X}$ for which, in a sense, scaling distortions alongside the virtual boundary are minimized. The following result refines this statement.

Lemma 1.44. *Let $\mathcal{F}$ be a Ford fundamental domain for Γ and let $z \in \partial_g \mathcal{F}$. Then for all $w \in \Gamma.z \cap \partial_g \mathcal{F}$ we have*

$$\mathrm{Im}\, w = \mathrm{Im}\, z\,.$$

Proof. Since $\Gamma.\mathbb{H} \subseteq \mathbb{H}$ and $\Gamma.\partial_g \mathbb{H} \subseteq \partial_g \mathbb{H}$, it suffices to consider $z \in \partial\mathcal{F}$. Assume first that $z \notin \partial\mathcal{K}$. By (1.72) this is only possible if $\mathbb{X}$ has cusps and $z \in \partial\mathcal{F}_\infty \cap \mathcal{K}$. From (1.66), Lemma 1.18, and Lemma 1.19(iii) we obtain

$$h.z \in \bigcup_{g \in \Gamma \setminus \Gamma_\infty} \mathrm{int}\,\mathrm{I}(g) \subseteq \mathbb{H} \setminus \mathcal{K}\,,$$

for all $h \in \Gamma \setminus \Gamma_\infty$. Since $\operatorname{Im} t_\lambda^n . z = \operatorname{Im} z$ for all $n \in \mathbb{Z}$, the claim follows in this case. Hence, assume $z \in \partial\mathcal{K}$ and let $h \in \Gamma$. There are three possible scenarios for the interrelation of z and h: either $h \in \Gamma_\infty$, $h \in \Gamma_{\mathrm{REL}}$ and $z \in \mathrm{I}(h)$, or $h \in \Gamma\setminus\Gamma_\infty$ and $z \notin \mathrm{I}(h)$. In the first case we may argue as before, while in the second case the claim follows from Lemma 1.19(ii). This leaves the third case. Observe that (1.66) and Lemma 1.18 imply that

$$\partial\mathcal{K} \subseteq \overline{\bigcap_{g\in\Gamma\setminus\Gamma_\infty} \operatorname{ext} \mathrm{I}(g)}\,.$$

Hence, in particular $z \in \operatorname{ext} \mathrm{I}(h)$. Again by Lemma 1.19(iii) and Lemma 1.18 we therefore obtain

$$h.z \in \operatorname{int} \mathrm{I}(h^{-1}) \subseteq \bigcup_{g\in\Gamma\setminus\Gamma_\infty} \operatorname{int} \mathrm{I}(g) = \mathbb{H} \setminus \Bigg(\overline{\bigcap_{g\in\Gamma\setminus\Gamma_\infty} \operatorname{ext} \mathrm{I}(g)}\Bigg) = \mathbb{H} \setminus \overline{\mathcal{K}}\,.$$

Hence, $h.z \notin \partial\mathcal{F}$ by (1.72) and the assertion follows. □

Suppose that Γ is geometrically finite and such that $\mathbb{X}$ has no cusps. Suppose further that a neighborhood of ∞ is contained in $\Omega(\Gamma)$. Let a^* be the maximum of all a and b^* be the minimum of all b for which (1.73) holds, meaning for any choice of $\varepsilon_1, \varepsilon_2 \geq 0$, not both equal to zero, we have

$$\partial\mathcal{K} \nsubseteq \operatorname{Re}|_{\mathbb{H}}^{-1}\big([a^* + \varepsilon_1, b^* - \varepsilon_1]\big)\,.$$

Since Γ_{REL} and $\mathrm{REL}(\Gamma)$ are in bijection, there then exist exactly one $g_1 \in \Gamma_{\mathrm{REL}}$ and exactly one $g_2 \in \Gamma_{\mathrm{REL}}$ such that

$$a^* \in \underline{g}\,\mathrm{I}(g_1) \qquad \text{and} \qquad b^* \in \underline{g}\,\mathrm{I}(g_2)\,. \tag{1.75}$$

In the case that $g_1 = g_2^{-1}$ we can infer further information about the boundary structure of $\mathcal{K}$. To that end recall the notion of the summit of an isometric sphere from (1.52) and (1.54). For $f\colon \mathbb{R} \supseteq I \to \mathbb{R}$ continuous we call $x_0 \in I$ a *strict local maximum*, if there exists $\varepsilon > 0$ such that

$$f(x_0) > f(x) \qquad \text{for all} \quad x \in \big((x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}\big) \cap I\,.$$

Every local maximum of f that is not strict is called *non-strict*.

Lemma 1.45. *Let $g \in \Gamma$. Further let $\mathrm{I}_1, \mathrm{I}_2 \in \mathrm{REL}(\Gamma)$ be such that $a^* \in \underline{g}\,\mathrm{I}_1$ and $b^* \in \underline{g}\,\mathrm{I}_2$ and suppose $\mathrm{I}_1 = \mathrm{I}(g)$ and $\mathrm{I}_2 = \mathrm{I}(g^{-1})$. Then either $\Gamma = \langle g\rangle$, or there exists $\mathrm{I}_3 \in \mathrm{REL}(\Gamma) \setminus \{\mathrm{I}_1, \mathrm{I}_2\}$ such that $s(\mathrm{I}_3) \in \partial\mathcal{K}$. In the latter case we have*

$$\operatorname{Re} s(\mathrm{I}_1) < \operatorname{Re} s(\mathrm{I}_3) < \operatorname{Re} s(\mathrm{I}_2)\,. \tag{1.76}$$

Proof. If Γ is cyclic, then $\mathrm{REL}(\Gamma) = \{\mathrm{I}(h), \mathrm{I}(h^{-1})\}$, where h is such that $\langle h\rangle = \Gamma$.

The converse is also true. By assumption $a^* \in \mathpzc{g}\mathrm{I}(g)$ and $b^* \in \mathpzc{g}\mathrm{I}(g^{-1})$, the definition of a^* and b^*, and the bijection between $\mathrm{REL}(\Gamma)$ and Γ_{REL} it follows that $g = h$.

Thus, assume that Γ is non-cyclic. Let δ be a curve in $\mathbb{C}$ that traces out the boundary of $\mathcal{K}$ in $\overline{\mathbb{H}}^{\mathpzc{g}}$ between a^* and b^*, i. e., let $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, and let

$$\delta\colon I := [t_0, t_1] \longrightarrow \mathbb{C}$$

be a continuous map such that $\delta(t_0) = a^*$, $\delta(t_1) = b^*$, and $\delta(I) \subseteq \partial_{\mathpzc{g}}\mathcal{K}$. In particular, the boundary structure of $\mathcal{K}$ allows δ to be chosen as an injective map, for instance by imposing δ to be piece-wise parameterized by arc length (with respect to the Euclidean metric in $\mathbb{C}$). Then $\delta(I)$ is piece-wise given by either intervals in $\mathbb{R}$ or geodesic segments in $\mathbb{H}$. The function

$$f\colon \begin{cases} I & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \operatorname{Im}\delta(x) \end{cases}$$

is continuous and for every strict local maximum of f the point $\delta(x_0)$ coincides with the summit of some relevant isometric sphere. Furthermore, all non-strict local maxima of f are likewise zeros of it.

We start by considering the case $\mathrm{I}_1 \subseteq \partial\mathcal{K}$. Proposition 1.41 then implies that $\mathrm{I}_2 \subseteq \partial\mathcal{K}$. Since Γ is non-cyclic, $\mathrm{REL}(\Gamma)$ consists of further spheres besides I_1 and I_2. By construction, we find $x_1, x_2 \in (t_0, t_1)$ such that

$$\mathpzc{g}\,\mathrm{I}_1 \setminus \{a^*\} = \{\delta(x_1)\} \qquad \text{and} \qquad \mathpzc{g}\,\mathrm{I}_2 \setminus \{b^*\} = \{\delta(x_2)\}\,.$$

Define Δ to be the (Euclidean) line segment in $\mathbb{C}$ connecting $\delta(x_1)$ and $\delta(x_2)$, i. e.,

$$\Delta := \{(1-\tau)\delta(x_1) + \tau\delta(x_2) \mid \tau \in (0,1)\}\,. \tag{1.77}$$

Then $\overline{\Delta} = [\delta(x_1), \delta(x_2)] \subseteq \mathbb{R}$ and, by assumption,

$$\bigcup \mathrm{REL}(\Gamma) \cap \operatorname{Re}|_{\mathbb{H}}^{-1}(\Delta) \neq \varnothing\,.$$

Thus, there exists $x_3 \in (x_1, x_2)$ such that $f(x_3) > 0$. Since f is continuous, it assumes its maximum in the compact interval $[x_1, x_2]$, say x_4, and we have that $f(x_4) \geq f(x_3) > 0$. Hence, x_4 is strict, and by the above this implies the existence of $\mathrm{I}_3 \in \mathrm{REL}(\Gamma)$ such that

$$\delta(x_4) = s(\mathrm{I}_3) \in \partial\mathcal{K} \cap \operatorname{Re}|_{\mathbb{H}}^{-1}(\Delta)\,.$$

This yields the claim in the case $\mathrm{I}_1 \subseteq \partial\mathcal{K}$.

Assume now that $\mathrm{I}_1 \nsubseteq \partial\mathcal{K}$. In this case we find $x_1, x_2 \in (t_0, t_1)$ such that $\delta(x_1) \in \mathrm{I}_1$ and $\delta(x_2) \in \mathrm{I}_2$, but

$$\delta\big((x_1, x_1 + \varepsilon)\big) \cap \mathrm{I}_1 = \delta\big((x_2, x_2 + \varepsilon)\big) \cap \mathrm{I}_2 = \varnothing\,,$$

for any choice of $\varepsilon > 0$. Since Γ is non-cyclic and thus $\mathrm{REL}(\Gamma) \setminus \{\mathrm{I}_1, \mathrm{I}_2\} \neq \varnothing$, we have $\delta(x_1) \neq \delta(x_2)$ and hence $x_1 < x_2$. But because of Proposition 1.41 and Lemma 1.19(ii), we have

$$f(x_1) = \operatorname{Im}\delta(x_1) = \operatorname{Im}\delta(x_2) = f(x_2)\,.$$

Thus, if we again define Δ as in (1.77), this time Δ is a horizontal line segment in the upper half-plane. We obtain another continuous path in $\mathbb{C}$ by connecting the segments $\delta\big((t_0, x_1)\big)$, Δ, and $\delta\big((x_2, t_1)\big)$. The angles this path assumes at the points $\delta(x_1)$ and $\delta(x_2)$ (measured above the curve) are equal by virtue of Lemma 1.19(ii), and we denote this angle by ϑ. Necessarily,

$$\frac{\pi}{2} < \vartheta < \pi\,. \tag{1.78}$$

By assumption, there exists $\mathrm{I}_3 \in \mathrm{REL}(\Gamma) \setminus \{\mathrm{I}_1, \mathrm{I}_2\}$ such that

$$\delta\big([x_1, x_1 + \varepsilon)\big) \in \mathrm{I}_3\,,$$

for $\varepsilon > 0$ sufficiently small. The points $v_1 := \delta(x_1)$ and $v_2 := \delta(x_2)$ are finite vertices of $\mathcal{K}$ and $v_2 \in C(v_1)$ by assumption. Clearly

$$\theta(v_\iota) < \pi\,, \tag{1.79}$$

for $\iota \in \{1, 2\}$. If $\theta(v_1) < \vartheta$, then $\delta\big((x_1, x_1 + \varepsilon)\big)$ lies above Δ. This allows us to proceed as above in order to find a strict local maximum of f in the interval $[x_1, x_2]$ and thereby a summit of some relevant isometric sphere contained in $\partial\mathcal{K} \cap \operatorname{Re}|_{\mathbb{H}}^{-1}(\operatorname{Re}(\Delta))$. If $\theta(v_1) = \theta$, then Δ is contained in a line tangent to I_3. Since Δ is horizontal, this means $\delta(x_1)$ is the summit of I_3 and thus fulfills the assertion. The same arguments apply if $\theta(v_2)$ either falls below or equals ϑ. This leaves only the case of both these angles exceeding ϑ, which, because of (1.78) and (1.79), implies

$$\pi < \theta(v_1) + \theta(v_2) < 2\pi\,. \tag{1.80}$$

Since Γ is geometrically finite, Lemma 1.34 implies that $\theta(C(v_1)) = \frac{2\pi}{\omega}$ for some $\omega \in \mathbb{N}$. Because of (1.80) this equation can only hold for $\omega = 1$ and $C(v_1)$ consisting of further vertices besides v_1 and v_2. Thus, let $v_3 \in C(v_1) \setminus \{v_1, v_2\}$. Then, by (1.80),

$$\theta(v_3) \leq 2\pi - \theta(v_1) - \theta(v_2) < \pi\,. \tag{1.81}$$

We further find $x_3 \in (x_1, x_2)$ such that $\delta(x_3) = v_3$. Lemma 1.19(ii) now implies $f(x_3) = f(x_1)$, or in other words, $v_3 \in \Delta$. But now, because of (1.81), we find some small $\varepsilon' > 0$ such at least one of the segments $\delta\big((x_3 - \varepsilon', x_3)\big)$, $\delta\big((x_3, x_3 + \varepsilon')\big)$ lies above Δ. Hence, the same argument as before allows us to deduce the existence of a strict local maximum of f, either in $[x_1, x_3]$, or in $[x_3, x_2]$. This yields the assertion in the case $\mathrm{I}_1 \not\subseteq \partial\mathcal{K}$.

In order to verify (1.76), we argue indirectly and assume that it is not the case. Because of symmetry it suffices to consider the case $\operatorname{Re} s(I_3) \leq \operatorname{Re} s(\mathrm{I}_1)$. Since $s(\mathrm{I}_3) \in \partial\mathcal{K}$, in particular $s(\mathrm{I}_3) \notin \operatorname{int} \mathrm{I}_1$. But since isometric spheres are geodesic arcs, by convexity it follows that $\cancel{g}\ \mathrm{I}_3 \cap (-\infty, a^*) \neq \varnothing$. This contradicts the choice of I_1 and thereby finishes the proof. □

We close this section with an example that introduces a family of Fuchsian groups to which we will return several times in examples throughout this thesis. It constitutes a—in some sense—minimal example of Fuchsian groups for which the associated orbisurface exhibits all three types of pertinent features: it has one funnel, one cusp, and one conical singularity of arbitrary order $\sigma \geq 2$. The order of the conical singularity implies that only those members of the family with parameter $\sigma = 2$ are Hecke triangle group (of infinite covolume).

Example 1.46. Let $\sigma \in \mathbb{N} \setminus \{1\}$ and consider

$$g_\sigma := \begin{bmatrix} \cos\big(\frac{\pi}{\sigma}\big) & \cos\big(\frac{\pi}{\sigma}\big) - 1 \\ \cos\big(\frac{\pi}{\sigma}\big) + 1 & \cos\big(\frac{\pi}{\sigma}\big) \end{bmatrix}.$$

Then

$$\det(g_\sigma) = \cos^2\big(\tfrac{\pi}{\sigma}\big) - \big(\cos^2\big(\tfrac{\pi}{\sigma}\big) - 1\big) = 1\,.$$

Hence, for all σ we have $g_\sigma \in \mathrm{PSL}_2(\mathbb{R})$. Further,

$$|\operatorname{tr}(g_\sigma)| = 2\cos\big(\tfrac{\pi}{\sigma}\big) \in [0, 2)$$

for all $\sigma \in \mathbb{N} \setminus \{1\}$. Thus, g_σ is elliptic and is an involution if and only if $\sigma = 2$. For $\sigma \geq 3$ we further calculate

$$\frac{1}{\pi}\arccos\left(\tfrac{|\operatorname{tr}(g_\sigma)|}{2}\right) = \frac{1}{\pi}\frac{\pi}{\sigma} = \frac{1}{\sigma}\,.$$

Note that $\sigma \geq 3$ implies $\pi/\sigma \in [0, \pi]$, where the cosine is bijective. Hence, by (1.29), the order of g_σ equals σ and $k(g) = 1$, where $k(g)$ is as in Remark 1.10. Hence, by Lemma 1.22, the isometric spheres $\mathrm{I}(g_\sigma)$ and $\mathrm{I}(g_\sigma^{-1})$ intersect each other at an angle of $2\pi/\sigma$. The fixed point of g_σ we obtain by applying formula (1.14):

$$\mathrm{f}(g_\sigma) = \frac{\mathrm{i}}{2\left(\cos\big(\frac{\pi}{\sigma}\big) + 1\right)}\sqrt{4 - 4\cos^2\big(\tfrac{\pi}{\sigma}\big)} = \mathrm{i}\cdot\tan\big(\tfrac{\pi}{2\sigma}\big)\,.$$

Furthermore,

$$c(g_\sigma) - r(g_\sigma) = -\frac{\cos\left(\frac{\pi}{\sigma}\right)}{\cos\left(\frac{\pi}{\sigma}\right)+1} - \frac{1}{\cos\left(\frac{\pi}{\sigma}\right)+1} = -1\,,$$

and analogously it follows that

$$c(g_\sigma^{-1}) + r(g_\sigma) = 1\,.$$

Hence, $\mathrm{I}(g_\sigma)\cup\mathrm{I}(g_\sigma^{-1}) \subseteq \mathrm{Re}|_{\mathbb{H}}^{-1}\big([-1,1]\big)$ for any σ. Further consider the parabolic transformation $\mathrm{t}_\lambda = \left[\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right]$ with $\lambda > 2$ and define the family of groups

$$\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\,\lambda>2}\,, \qquad \text{where} \qquad \Gamma_{\sigma,\lambda} \coloneqq \langle g_\sigma, \mathrm{t}_\lambda \,|\, g_\sigma^\sigma = \mathrm{id}\rangle\ . \tag{1.82}$$

The associated orbisurfaces we denote by $\mathbb{X}_{\sigma,\lambda}$. Each of these orbisurfaces has a cusp, represented by ∞. We assign to each group $\Gamma_{\sigma,\lambda}$ the Ford fundamental domain $\mathcal{F}_{\sigma,\lambda}$ given by

$$\mathcal{F}_{\sigma,\lambda} \coloneqq \mathcal{F}_\infty\left(-\tfrac{\lambda}{2}\right) \cap \operatorname{ext}\mathrm{I}(g_\sigma) \cap \operatorname{ext}\mathrm{I}(g_\sigma^{-1})\,, \tag{1.83}$$

where $\mathcal{F}_\infty(r)$, $r\in\mathbb{R}$, is as in (1.64). From the above it follows that $\mathcal{F}_{\sigma,\lambda}$ is a Ford fundamental domain and thus, in particular, a geometrically finite fundamental polygon for $\Gamma_{\sigma,\lambda}$. By Proposition 1.43 we read off from $\mathcal{F}_{\sigma,\lambda}$ that $\mathbb{X}$ has exactly one cusp (represented by ∞), exactly one conical singularity (represented by $\mathrm{f}(g_\sigma)$), and exactly one funnel. Two examples of fundamental domains are depicted in Figures 2 and 3. Note that, for $\sigma = 2$, the group $\Gamma_{\sigma,\lambda}$ is a (non-cofinite) Hecke triangle group. For all other choices of σ it is not.

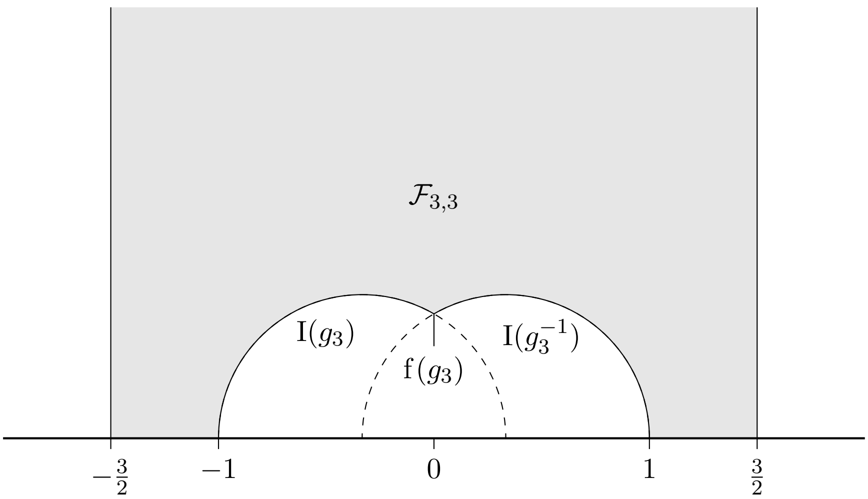


Figure 2: The Ford fundamental domain $\mathcal{F}_{3,3}$ for $\Gamma_{3,3}$.

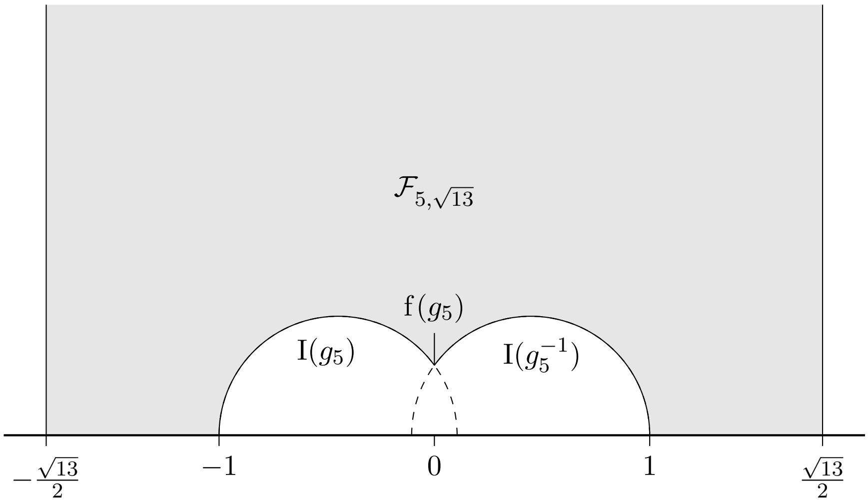


Figure 3: The Ford fundamental domain $\mathcal{F}_{5,\sqrt{13}}$ for $\Gamma_{5,\sqrt{13}}$.

1.11 Cross Sections and Transfer Operators

Let Γ be a geometrically finite, non-cocompact Fuchsian group with hyperbolic elements and denote by $\mathbb{X}$ the associated hyperbolic orbisurface. Recall the sets of geodesics $\mathscr{G}(\mathbb{X})$ and $\mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ from Section 1.7 as well as the unit tangent bundle $\mathrm{S}\mathbb{X}$ of $\mathbb{X}$ from (1.31). Let $M \subseteq \mathrm{S}\mathbb{X}$, $\widehat{\gamma} \in \mathscr{G}(\mathbb{X})$, and $t \in \mathbb{R}$. We say that $\widehat{\gamma}$ *intersects M at time t* if

$$\widehat{\gamma}'(t) \in M\,. \tag{1.84}$$

We say that $\widehat{\gamma}$ *intersects M transversally at time t* if there exists $\varepsilon > 0$ such that

$$\left\{\widehat{\gamma}'(\tau) \;\middle|\; \tau \in (t-\varepsilon, t+\varepsilon)\right\} \cap M = \{\widehat{\gamma}'(t)\}\,. \tag{1.85}$$

Further recall the (unit speed) geodesic flow on $\mathbb{X}$,

$$\widehat{\Phi}\colon \left\{ \begin{array}{ccc} \mathbb{R} \times \mathrm{S}\mathbb{X} & \longrightarrow & \mathrm{S}\mathbb{X} \\ (t, \widehat{\nu}) & \longmapsto & \widehat{\gamma}_\nu'(t) \end{array} \right. ,$$

from (1.33). This is a time-continuous flow on $\mathbb{X}$ with phase space equal to $\mathrm{S}\mathbb{X}$. For many applications (of which the constructions in this thesis are one example) a time-discrete counterpart of $\widehat{\Phi}$ is required. Such can be obtained via introduction of a *Poincaré cross section*, a certain submanifold of $\mathrm{S}\mathbb{X}$ together with a discrete dynamics induced by $\widehat{\Phi}$. This approach is called a *discretization* of the geodesic flow $\widehat{\Phi}$.

Usually, for a Poincaré cross section C one demands that every geodesic on $\mathbb{X}$ intersects C transversally infinitely often in past and future (i. e., infinitely often with $t > 0$ and infinitely often with $t < 0$). Because of $\mathbb{X}$ bearing hyperbolic ends, this framework is not fully suitable in our case. For this reason we instead borrow, and use throughout, the concept of cross sections from [54], which presumes a measure μ on $\mathscr{G}(\mathbb{X})$ in order to single out those geodesics whose behavior is essential for the applications in mind. For the general statement of the definition of a cross section, we neither require the measure μ to be finite or even a probability measure, nor do we ask for any specific properties of the implicitly fixed σ-algebra on $\mathscr{G}(\mathbb{X})$. We refer to the discussion below Definition 1.47 and to Section 4.6 for the class of measures relevant for our applications.

Definition 1.47. A subset $\widehat{\mathrm{C}}$ of $\mathrm{S}\mathbb{X}$ is called a *cross section for* $\widehat{\Phi}$ *with respect to* μ if

(CS1) μ-almost every geodesic $\widehat{\gamma}$ on $\mathbb{X}$ intersects $\widehat{\mathrm{C}}$ infinitely often in past and future, i. e., there exists a two-sided sequence $(t_n)_{n\in\mathbb{Z}}$ with

$$\lim_{n\to\pm\infty} t_n = \pm\infty$$

such that for each $n \in \mathbb{Z}$ the geodesic $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}$ at time t_n, and

(CS2) each intersection of any geodesic $\widehat{\gamma}$ on $\mathbb{X}$ and $\widehat{\mathrm{C}}$ is transversal.

A cross section $\widehat{\mathrm{C}}$ for $\widehat{\Phi}$ is called *strong* if it additionally satisfies that

(CS3) every geodesic on $\mathbb{X}$ that intersects $\widehat{\mathrm{C}}$ at all, intersects $\widehat{\mathrm{C}}$ infinitely often both in past and future.

We emphasize that this notion of a cross section deviates from the classical notion of Poincaré cross sections in that it does not require that every geodesic intersects the set $\widehat{\mathrm{C}}$. For the applications that motivate this thesis we may restrict to certain measures whose support contains $\mathscr{G}_{\mathrm{Per}}(\mathbb{X})$, and we may relax (CS1) to

(CS1′) Every periodic geodesic $\widehat{\gamma}$ on $\mathbb{X}$ intersects $\widehat{\mathrm{C}}$.

In [58, 44, 57, 56, 15] it has been seen that cross sections of this kind capture just the right amount of geometry and simultaneously allow for sufficient freedom to construct discretizations of the geodesic flow for which the associated transfer operators mediate between the geodesic flow and the Laplace eigenfunctions of $\mathbb{X}$.

Substituting (CS1) by (CS1′) would allow us to omit the choice of a measure from the definition of cross sections. However, to achieve greater flexibility in view of potential further applications, we will work with a larger class of measures. Starting with Section 4.6, we will consider all those measures that do not assign positive mass to the geodesics that "vanish" into a hyperbolic end of $\mathbb{X}$. We refer to Proposition 4.36 for a precise statement. In what follows we will often suppress the choice of the measure μ from the notation.

Suppose that $\widehat{\mathrm{C}}$ is a strong cross section for $\widehat{\Phi}$. An immediate consequence of (CS3) is that for any $\widehat{\nu} \in \widehat{\mathrm{C}}$, the *first return time of* $\widehat{\nu}$ *with respect to* $\widehat{\mathrm{C}}$,

$$t^{+}_{\widehat{\mathrm{C}}}(\nu) := \min\left\{ t > 0 \,\middle|\, \widehat{\gamma}'_{\nu}(t) \in \widehat{\mathrm{C}} \right\}\,, \tag{1.86}$$

exists. Hence, the *first return map*

$$\widehat{\mathscr{R}} \colon \left\{ \begin{array}{ccc} \widehat{\mathrm{C}} & \longrightarrow & \widehat{\mathrm{C}} \\ \widehat{\nu} & \longmapsto & \widehat{\gamma}'_{\nu}\big(t^{+}_{\widehat{\mathrm{C}}}(\widehat{\nu})\big) \end{array} \right. \tag{1.87}$$

is well-defined. The dynamical system

$$\left\{ \begin{array}{ccc} \mathbb{Z} \times \widehat{\mathrm{C}} & \longrightarrow & \widehat{\mathrm{C}} \\ (n, \widehat{\nu}) & \longmapsto & \widehat{\mathscr{R}}^{n}(\widehat{\nu}) \end{array} \right. ,$$

for short $(\widehat{\mathrm{C}}, \widehat{\mathscr{R}})$, constitutes the discretization of the geodesic flow $\widehat{\Phi}$ on $\mathbb{X}$ mentioned above. We will apply the notions of first return time and first return map also to cross sections $\widehat{\mathrm{C}}$ that are not necessarily strong. In this case, the first return time and the first return map might be defined only on a subset of $\widehat{\mathrm{C}}$, resulting in partial maps.

Let $\widehat{\mathrm{C}}$ now be a cross section that may not be strong. Recall the quotient map $\pi \colon \mathrm{S}\mathbb{H} \to \mathrm{S}\mathbb{X}$ from (1.32). We call a subset C of $\mathrm{S}\mathbb{H}$ a *set of representatives* for $\widehat{\mathrm{C}}$ if C and $\widehat{\mathrm{C}}$ are bijective via π, i. e., $\pi(\mathrm{C}) = \widehat{\mathrm{C}}$ and the restricted map

$$\pi|_{\mathrm{C}} \colon \mathrm{C} \longrightarrow \widehat{\mathrm{C}}$$

is a bijection. For any set of representatives C, the first return map $\widehat{\mathscr{R}}$ induces a *first return map* $\mathscr{R}$ *on* C via

$$\mathscr{R} := \pi|_{\mathrm{C}}^{-1} \circ \widehat{\mathscr{R}} \circ \pi\,. \tag{1.88}$$

In other words, the diagram

$$\begin{tikzcd} \mathrm{C} \arrow[r, "\mathscr{R}"] \arrow[d, "\pi"'] & \mathrm{C} \arrow[d, "\pi"] \\ \widehat{\mathrm{C}} \arrow[r, "\widehat{\mathscr{R}}"] & \widehat{\mathrm{C}} \end{tikzcd}$$

commutes. If $\widehat{\mathrm{C}}$ is not strong and hence $\widehat{\mathscr{R}}$ is only partially defined, then $\mathscr{R}$ is also only partially defined. Sometimes it is possible to find a partition of C into (finitely or infinitely) many subsets, say

$$\mathrm{C} = \bigcup_{a \in A} \mathrm{C}_a\,, \tag{1.89}$$

such that for each $a \in A$, the map

$$\psi\colon \begin{cases} \mathrm{C}_a & \longrightarrow & \widehat{\mathbb{R}} \\ \nu & \longmapsto & \gamma_\nu(+\infty) \end{cases}$$

is injective. In this case, we set

$$D_a \coloneqq \{(\gamma_\nu(+\infty), a) \mid \nu \in \mathrm{C}_a\}$$

and

$$D \coloneqq \bigcup_{a \in A} D_a\,. \tag{1.90}$$

We emphasize that the union in (1.90) is disjoint. Then $\mathscr{R}$ induces a (well-defined, unique) map

$$F\colon D \longrightarrow D$$

that makes the diagram

$$\begin{tikzcd} \mathrm{C} \arrow[r, "\mathscr{R}"] \arrow[d, "\psi"'] & \mathrm{C} \arrow[d, "\psi"] \\ D \arrow[r, "F"] & D \end{tikzcd} \tag{1.91}$$

commutative. In the first component, the map F is piece-wise given by the action of certain elements of Γ on $\mathbb{H}$. In the second component, F is a certain symbol transformation. We call (D, F) the *discrete dynamical system* induced by C.

A standard tool for the study of time-discrete dynamical systems like (D, F) from statistical mechanics is the *transfer operator*: Let V be a finite-dimensional complex vector space and denote by $\mathrm{GL}(V)$ the group of automorphisms of V. Let $\varrho\colon D \to \mathbb{C}$ and $\omega\colon D \to \mathrm{GL}(V)$ be some functions. The *transfer operator of* (D, F) *with potential* ϱ *and weight* ω is the operator defined by

$$\mathcal{L} f(x) \coloneqq \sum_{y \in F^{-1}(x)} \omega(y) e^{\varrho(y)} f(y) \tag{1.92}$$

on some Banach space of functions $f\colon D \to V$. In this thesis we will focus on families of weighted transfer operators $\{\mathcal{L}_s\}_{s\in\mathbb{C}}$ characterized by the potentials

$$\varrho(y) \coloneqq -s \log \left|F'(y)\right|$$

and weights ω given by representations of elements of Γ in $\mathrm{GL}(V)$. We refer to Section 4.7 for an exact definition.

Transfer operators are widely applied in order to find invariant measures. However, as described in the introduction, we are mostly interested in the representation of the Selberg zeta function associated to $\mathbb{X}$ by means of Fredholm determinants for the operators in $\{\mathcal{L}_s\}_{s\in\mathbb{C}}$. Hence, we require this operator family to fulfill a certain set of properties (see Section 3.1) that guarantee the existence of these determinants. The above suggests that the endeavor to fulfill these properties translates to the search for a suitable cross section or a set of representatives thereof. This constitutes the main objective of this thesis.

1.12 Selberg Zeta Function and Resonances

In this section we briefly recall the (twisted) Selberg zeta function for a given hyperbolic orbisurface with fundamental group $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, and its relation to the resonances and resonance states of the Laplacian. So let $\mathbb{X}$ and Γ be as such and denote, as before, by $[\Gamma]_{\mathrm{h}}$ the set of conjugacy classes of hyperbolic elements in Γ and by $\ell([h])$ the displacement length of $[h] \in [\Gamma]_{\mathrm{h}}$. Recall the map $\mathrm{ct}\colon [\Gamma]_{\mathrm{h}} \to \mathbb{N}$ from (1.40) and let

$$[\Gamma]_{\mathrm{P}} \coloneqq \{[h] \in [\Gamma]_{\mathrm{h}} \mid \mathrm{ct}([h]) = 1\} \,,$$

that is the subset of primitive elements in $[\Gamma]_{\mathrm{h}}$. Denote by δ the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ.

Proposition 1.48 ([10, Section 2.5.2]). *The infinite product*

$$Z_{\mathbb{X}}(s) \coloneqq \prod_{[h]\in[\Gamma]_{\mathrm{P}}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell([h])}\right) \tag{1.93}$$

converges absolutely for $\operatorname{Re} s > \delta$.

From Corollary 1.14 we know that $[\Gamma]_{\mathrm{h}}$ is in bijection with the set of prime geodesics on $\mathbb{X}$. This justifies the notation $Z_{\mathbb{X}}(s)$, as the value of the product in (1.93) does not depend on the choice of the fundamental group Γ.

Proposition 1.49 ([27]). *Let Γ be geometrically finite. Then the product $Z_{\mathbb{X}}(s)$ from* (1.93) *admits an analytic continuation to a meromorphic function on $\mathbb{C}$.*

The meromorphic continuation of $Z_{\mathbb{X}}(s)$ to $\mathbb{C}$, which we denote by the same symbol, is called the *Selberg zeta function on* $\mathbb{X}$. It has been introduced by Atle Selberg [74] as an analogue to zeta- and L-functions in analytic number theory, in particular the famous Riemann zeta function ζ. In fact, since $[\Gamma]_{\mathrm{h}}$ is in bijection with the prime periodic geodesics on $\mathbb{X}$, $Z_{\mathbb{X}}(s)$ is defined purely in terms of the *prime geodesic length spectrum*, that is the multiset of the lengths of prime geodesics, which hence can be viewed as playing the role of the prime numbers in the Euler product for ζ.

The outstanding significance of the Selberg zeta function stems from its set of zeros, as it is known to contain the resonances of the Laplacian ([74, 52, 11], see also Theorem 1.50 below). For a more detailed exposition, we denote by $\Delta_{\mathbb{H}}$

the (positive) Laplacian on $\mathbb{H}$, or more precisely, the *Laplace–Beltrami operator*, defined as (minus) the divergence of the gradient. With respect to the hyperbolic metric (see Section 1.1) and in the coordinates $z = x + \mathrm{i}y$ it takes the form

$$\Delta_{\mathbb{H}} = -y^2\left(\partial_x^2 + \partial_y^2\right),$$

defined on $C^\infty(\mathbb{H})$ and thus, by extension, on $L^2(\mathbb{H})$ (the space of equivalence classes of square-Lebesgue-integrable functions). The Laplacian is intrinsic to the Riemannian metric, i. e., invariant under the action of isometries. This means that, for every $g \in \mathrm{PSL}_2(\mathbb{R})$ and τ_g the operator on $\mathrm{Fct}(\mathbb{H})$ defined by

$$\tau_g f(z) = f(g^{-1}.z),$$

whenever this makes sense, we have

$$\tau_g \Delta_{\mathbb{H}} = \Delta_{\mathbb{H}} \tau_g\,. \tag{1.94}$$

This property, in a sense, characterizes the Laplacian: The differential operators which commute with all operators τ_g, $g \in \mathrm{PSL}_2(\mathbb{R})$, form a polynomial algebra in $\Delta_{\mathbb{H}}$ [7, 30]. The identity (1.94) further induces differential operators on (developable) hyperbolic orbisurfaces: For every C^∞-function f on $\mathbb{X} = \Gamma\backslash\mathbb{H}$, if viewed as a Γ-periodic C^∞-function on $\mathbb{H}$ (i. e., $f(g.z) = f(z)$ for all $g \in \Gamma$ and $z \in \mathbb{H}$), the function $\Delta_{\mathbb{H}} f$ is again Γ-periodic on $\mathbb{H}$ by virtue of (1.94), and thus can be viewed as an element of $C^\infty(\mathbb{X})$. Hence, we obtain a differential operator on $C^\infty(\mathbb{X})$, which we denote by $\Delta_{\mathbb{X}}$. Again, by extension, $\Delta_{\mathbb{X}}$ can be defined for all L^2-functions [7], where it becomes an unbounded positive self-adjoint operator.

Now denote by $H^2(\mathbb{X})$ the Sobolev space for $p = 2$ on $\mathbb{X}$ and consider the *resolvent of* $\Delta_{\mathbb{X}}$, that is

$$R_{\mathbb{X}}(s) \coloneqq \left(\Delta_{\mathbb{X}} - s(1-s)\right)^{-1} \colon L^2(\mathbb{X}) \longrightarrow H^2(\mathbb{X}),$$

defined for all $s \in \mathbb{C}$, $\operatorname{Re} s > 1/2$, for which $s(1-s)$ is not an L^2-eigenvalue of $\Delta_{\mathbb{X}}$. For $\operatorname{Re} s \gg 1$ the range of the restricted operators $R_{\mathbb{X}}(s)|_{C_c^\infty(\mathbb{X})}$ is contained in $C^\infty(\mathbb{X})$. These restricted operators extend to a meromorphic family of operators

$$\mathrm{R}_{\mathbb{X}}(s) \colon L^2_c(\mathbb{X}) \longrightarrow H^2_{\mathrm{loc}}(\mathbb{X})$$

for all $s \in \mathbb{C}$, where $L^2_c(\mathbb{X})$ denotes the space of the compactly supported elements of $L^2(\mathbb{X})$, and $H^2_{\mathrm{loc}}(\mathbb{X})$ is the space of functions that are locally in $H^2(\mathbb{X})$ [43, 29]. The poles of the map $s \mapsto \mathrm{R}_{\mathbb{X}}(s)$ are called the *resonances of* $\mathbb{X}$ and the generalized eigenfunctions of $\Delta_{\mathbb{X}}$ at a resonance s are called the *resonant states with spectral parameter* s. We denote by $\mathcal{R}_{\mathbb{X}}$ the multiset of the resonances of $\mathbb{X}$, that is, the set of resonances with multiplicities. The following result establishes the relation between the Selberg zeta function $Z_{\mathbb{X}}$ and the resonances of $\mathbb{X}$ in the case that $\mathbb{X}$ is a proper geometrically finite surface (i. e., free of conical singularities) for which its fundamental group Γ is non-elementary and non-cofinite [11], or convex cocompact [52], or cocompact [74]. Note that, in the cocompact case all resonances are L^2-eigenvalues of $\Delta_{\mathbb{X}}$.

Theorem 1.50 ([11, Theorem 1.1], [52, Theorem 1.9], [74])**.** *Let $\mathbb{X}$ and Γ be either of the above and denote by $\chi_{\mathrm{E}}^{\mathrm{top}}(\mathbb{X})$ the topological Euler characteristic of $\mathbb{X}$. For every $s \in \mathcal{R}_{\mathbb{X}}$ we have $Z_{\mathbb{X}}(s) = 0$, and the multiplicity of s as a resonance of $\mathbb{X}$ matches its order as a zero of $Z_{\mathbb{X}}$ for all up to finitely many s. Furthermore, $Z_{\mathbb{X}}$ vanishes on every $k \in -\mathbb{N}_0$ to the order $-\chi_{\mathrm{E}}^{\mathrm{top}}(\mathbb{X}) \cdot (2k+1)$. Besides those, $Z_{\mathbb{X}}$ has no further zeros.*

The finitely many resonances for which equality of multiplicities fails are well understood. They stem from the collision of certain zeros and poles in the factorization of $Z_{\mathbb{X}}$, in the cases where such a factorization is available (see also Remark 1.53 below).

In the study of resonances and related applications often a *twisted variant* of the Selberg zeta function appears. For its definition let Γ be geometrically finite, V be a finite-dimensional Hermitian vector space, $\mathrm{GL}(V)$ be the group of automorphisms on V, and

$$\chi\colon \Gamma \longrightarrow \mathrm{GL}(V)$$

be a linear representation (i. e., for all $g, h \in \Gamma$ we have $\chi(g) \circ \chi(h) = \chi(gh)$). We say that χ has *non-expanding cusp monodromy*, if for each parabolic element $p \in \Gamma$ the endomorphism $\chi(p)$ has only eigenvalues with absolute value 1. (We refer to [21] and [22] for an extended discussion of this property.) Every unitary representation has non-expanding cusp monodromy.

The infinite Euler product

$$Z_{\mathbb{X},\chi}(s) \coloneqq \prod_{[h]\in[\Gamma]_{\mathrm{P}}} \prod_{k=0}^{\infty} \det\left(\mathrm{Id}_V - \chi([h])e^{-(s+k)\ell([h])}\right) \tag{1.95}$$

converges for $\operatorname{Re} s > \delta$ if and only if χ has non-expanding cusp monodromy [22, Proposition 6.1]. In this case, $Z_{\mathbb{X},\chi}$ is known to continue analytically to a meromorphic function on $\mathbb{C}$ for various combinations of Γ and χ [74, 78, 27, 22]. This continuation of the product in (1.95), which we again denote by $Z_{\mathbb{X},\chi}(s)$, is called the *χ-twisted Selberg zeta function*. One immediately sees that (1.93) emerges from (1.95) for χ the trivial one-dimensional representation.

As for the zeta function, also twisted versions of the Laplacian and its resolvent can be considered. Let χ be a unitary representation of Γ on V. Then χ induces a Hermitian vector orbibundle E_χ with typical fiber V, that is,

$$E_\chi \coloneqq \Gamma \backslash \left(\mathbb{H} \times V\right),$$

where the action of Γ extends from $\mathbb{H}$ to $\mathbb{H} \times V$ by virtue of χ: For all $g \in \Gamma$

and $(z, v) \in \mathbb{H} \times V$ we set

$$g_{\bullet}(z, v) := \big(g_{\bullet}z, \chi(g)v\big)\,.$$

To each function $f \in C^\infty(\mathbb{H}; V)$ one then associates functions $f_j \colon \mathbb{H} \to V$, $j = 1, \ldots, \dim V$, such that, for $z \in \mathbb{H}$,

$$f(z) = \sum_{j=1}^{\dim V} f_j(z) v_j\,,$$

for a given basis $(v_j)_{j=1}^{\dim V}$ of V. The operator Δ given by

$$\Delta f(z) := \sum_{j=1}^{\dim V} \big(\Delta_{\mathbb{H}} f_j(z)\big) v_j$$

is then independent of the basis and induces a self-adjoint operator

$$\Delta_{\mathbb{X},\chi} \colon C_c^\infty(\mathbb{X}; E_\chi) \longrightarrow L^2(\mathbb{X}; E_\chi)\,,$$

which we call the Laplacian on E_χ. It again extends to an unbounded positive self-adjoint operator on $L^2(\mathbb{X}; E_\chi)$, and its resolvent

$$R_{\mathbb{X},\chi}(s) = \big(\Delta_{\mathbb{X},\chi} - s(1-s)\big)^{-1} \colon L^2(\mathbb{X}; E_\chi) \longrightarrow L^2(\mathbb{X}; E_\chi)$$

is well-defined for $\operatorname{Re} s > 1/2$ and $s(1-s)$ not in the spectrum of $\Delta_{\mathbb{X},\chi}$, and is a bounded operator. We refer to [18] for the details.

Proposition 1.51 ([18, Theorem A]). *The resolvent $R_{\mathbb{X},\chi}(s)$ admits a meromorphic continuation to $s \in \mathbb{C}$ with poles of finite multiplicity as an operator*

$$\mathrm{R}_{\mathbb{X},\chi}(s) \colon L_c^2(\mathbb{X}; E_\chi) \longrightarrow L^2_{\mathrm{loc}}(\mathbb{X}; E_\chi)\,,$$

where $L_c^2(\mathbb{X}; E_\chi)$ is the subspace of $L^2(\mathbb{X}; E_\chi)$ of compactly supported functions and $L^2_{\mathrm{loc}}(\mathbb{X}; E_\chi)$ is the space of functions that are locally in $L^2(\mathbb{X}; E_\chi)$.

Again, we denote the multiset of resonances, i. e., the poles of the function $s \mapsto \mathrm{R}_{\mathbb{X},\chi}(s)$, by $\mathcal{R}_{\mathbb{X},\chi}$. The following very recent result of Doll and Pohl constitutes a version of Theorem 1.50 taking non-trivial finite-dimensional unitary representations into account. But even for χ the trivial one-dimensional representation, it provides a proper generalization of Theorem 1.50, for we may now drop the assumption that Γ is torsion-free.

Theorem 1.52 ([19]). *For all $s \in \mathcal{R}_{\mathbb{X},\chi}$ we have $Z_{\mathbb{X},\chi}(s) = 0$, and the multiplicity of s as a resonance matches its order as a zero of $Z_{\mathbb{X},\chi}$, except for finitely many s. Furthermore, $Z_{\mathbb{X},\chi}$ vanishes on $-\mathbb{N}_0$. Besides those, $Z_{\mathbb{X},\chi}$ has no further zeros.*

Remark 1.53. Both Theorems 1.50 and 1.52 are proven by a factorization of the zeta function. For instance, in the latter case we have

$$Z_{\mathbb{X},\chi}(s) = e^{p(s)} \cdot G_\chi(s) \cdot G_\infty(s)^{-\dim(V)\,\chi_{\mathrm{E}}^{\mathrm{top}}(\mathbb{X})} \cdot \Gamma(s-\tfrac{1}{2})^{n_{\mathrm{P}}} \cdot \mathcal{P}_{\mathbb{X},\chi}(s)\,, \tag{1.96}$$

where

- p is a polynomial of degree ≤ 2,
- $G_\chi(s)$ is an entire function with zeros in $-\mathbb{N}_0$ whose order depend on χ and the elliptic elements in Γ (for Γ without torsion and χ the trivial one-dimensional representation one has $G_\chi \equiv 1$),
- $G_\infty(s)$ is a meromorphic function without zeros,
- $\Gamma(s)$ denotes the (meromorphic continuation of the) gamma function,
- n_{P} is an integer depending on the parabolic elements in Γ, and
- $\mathcal{P}_{\mathbb{X},\chi}(s)$ is the Weierstraß product over the resonances (with multiplicities).

Each of the objects, $G_\chi(s)$, $G_\infty(s)$, n_{P}, is explicitly known. Hence, (1.96) not only yields information about the zeros of $Z_{\mathbb{X},\chi}(s)$ and their orders, but also about its poles, including residues. We omit this here for we do not require it, and refer the reader to [11] and [19] for the details. The finitely many zeros s for which equality of multiplicities fails stem from the poles of the factor $G_\infty(s)$ colliding with certain zeros of $\mathcal{P}_{\mathbb{X},\chi}(s)$.

Chapter 2

The Cusp Expansion Algorithm

Let Γ be a geometrically finite, non-cocompact Fuchsian group containing hyperbolic elements and let $\mathbb{X}$ be the associated hyperbolic orbisurface. This chapter is dedicated to a brief review of the cusp expansion algorithm developed in [54]. This algorithm offers a geometrical approach for the construction of cross sections for the geodesic flow together with suitable representatives of them in $\mathrm{S}\mathbb{H}$ (see Section 1.11). It does so by identifying a finite set of vertical[1] geodesic arcs, each endowed with a sense of direction. The unit tangent vectors based on these arcs and pointing into the respective direction then constitute a cross section for the geodesic flow. The cross sections arising in this way in turn give rise to discrete dynamics on subsets of $\widehat{\mathbb{R}}$ and associated families of transfer operators.

This chapter does not trace out the actual constructions undertaken in [54], but rather collects properties of the cross section and its representatives obtained by a cusp expansion procedure. Some of these are statements which were included in [54] already, for others a reformulation or extraction is required in order to fit our needs. However, all arguments are based on investigations from [54], for the most part from Sections 6.2, 6.4, 6.6, and 6.7 ibid.[2] Evidently, we refer the reader to [54] for the actual constructions of the objects discussed. Our proofs will include more precise references.

The cusp expansion algorithm constituted the starting point of our studies. The notion of sets of branches introduced in Chapter 4 below emerged as a collection of conditions one needs to impose upon the cross section representatives. It identifies the key aspects necessary for the approach outlined in the introduction and in Section 1.11. Thereby, it enables us to prove that cross sections emerging from a cusp expansion procedure bear the structure required for a strict transfer operator approach (see Chapter 3).

As the name suggests, the cusp expansion algorithm presumes the orbisur-

[1] Recall that a geodesic segment β is called *vertical* if $\mathrm{Re}(\beta)$ is a singleton in $\mathbb{R}$, and *non-vertical* otherwise.

[2] The necessary background information from [54, Section 6.1] has already been included in Section 1.9.

face $\mathbb{X}$ to have cusps. Hence, for the duration of this chapter we assume that Γ contains parabolic elements. We further assume that ∞ represents a cusp of $\mathbb{X}$, which can always be achieved by conjugation of Γ with a suitable element in $\mathrm{PSL}_2(\mathbb{R})$. However, later on (see Section 7.2) we will demonstrate how to apply the algorithm for non-cocompact Fuchsian groups without parabolic elements, thereby establishing strict transfer operator approaches for a large class of isometry subgroups.

2.1 Construction of the Cross Section

Let Γ and $\mathbb{X}$ be as before. Since ∞ represents a cusp of $\mathbb{X}$, by (S), there exists a unique $\lambda > 0$ such that

$$\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty) = \langle \mathrm{t}_\lambda \rangle \ ,$$

with t_λ as in (1.7). The starting point of the cusp expansion algorithm is the set $\mathcal{K} = \mathcal{K}_\Gamma$ from (1.66),

$$\mathcal{K} = \bigcap_{\mathrm{I} \in \mathrm{ISO}(\Gamma)} \operatorname{ext} \mathrm{I} = \bigcap_{\mathrm{I} \in \mathrm{REL}(\Gamma)} \operatorname{ext} \mathrm{I} \,,$$

with $\mathrm{ISO}(\Gamma)$ and $\mathrm{REL}(\Gamma)$ as in Section 1.10. Recall further the relevant part β_I of a relevant isometric sphere $\mathrm{I} \in \mathrm{REL}(\Gamma)$ from (1.68) as well as the summit $s(\mathrm{I})$ of an isometric sphere I from (1.52). We need to impose the following restriction:

(A) For every $\mathrm{I} \in \mathrm{REL}(\Gamma)$ there exists $\varepsilon > 0$ such that

$$\mathrm{B}_\varepsilon(s(\mathrm{I})) \cap \mathrm{I} \subseteq \beta_\mathrm{I} \,.$$

Remark 2.1. In [54, Section 6.3] an example is given for a group that does not satisfy (A). Hence, it is a proper restriction. As of now, the cusp expansion algorithm requires this assumption. However, it is conjectured that it is not necessary for the construction of cross sections. For the group from [54, Section 6.3] for instance it has been shown in [53] how to circumvent this issue by a cut-and-project deformation of the considered fundamental domain. Furthermore, beyond application of the cusp expansion algorithm, we will not make further use of condition (A) here. In fact, we go to some lengths in order to avoid additional use of (A). Lemma 1.45 is required solely for that purpose. This means that, once the cusp expansion algorithm (or an equivalent approach) has been shown to work regardless of condition (A), it may be safely removed from here as well and all subsequent constructions apply without changes.

Recall $S_\mathcal{K}$, the set of sides of $\mathcal{K}$, from (1.69), as well as the set $W_\mathcal{K}$ of endpoints of sides of $\mathcal{K}$. We construct a new set

$$\widetilde{W}_{\mathcal{K}} := \big(W_{\mathcal{K}} \setminus \{\mathrm{f}(g) \mid g \in \Gamma \text{ elliptic}\}\big) \cup \{c(\mathrm{I}) \mid \mathrm{I} \in \mathrm{REL}(\Gamma)\}\ , \tag{2.1}$$

which, because of Proposition 1.43(i), is a discrete subset of $\mathbb{R}$. By fixing $x_0 \in \widetilde{W}_{\mathcal{K}}$ one may write $\widetilde{W}_{\mathcal{K}} = \{x_j \mid j \in \mathbb{Z}\}$, where x_j, $j \in \mathbb{Z} \setminus \{0\}$, is inductively defined by

$$x_{j+1} := \min\left\{x \in \widetilde{W}_{\mathcal{K}} \;\middle|\; x > x_j\right\} \qquad \text{for} \quad j \geq 0 \qquad \text{and}$$
$$x_{j-1} := \max\left\{x \in \widetilde{W}_{\mathcal{K}} \;\middle|\; x < x_j\right\} \qquad \text{for} \quad j \leq 0\,.$$

Since ∞ represents a cusp of $\mathbb{X}$, (1.74) implies that

$$\bigcup_{j\in\mathbb{Z}} \mathrm{Re}|_{\mathbb{H}}^{-1}\big((x_j, x_{j+1}]\big) = \mathbb{H}\,,$$

where the union on the left is disjoint. We use this to slice up the common exterior: We call

$$\mathcal{A} := \left\{\mathcal{K} \cap \mathrm{Re}|_{\mathbb{H}}^{-1}\big([x_j, x_{j+1}]\big) \;\middle|\; j \in \mathbb{Z}\right\} \tag{2.2}$$

the *set of precells of* Γ and each element of $\mathcal{A}$ a *precell of* Γ. Obviously, the set $\mathcal{A}$ is independent of the particular choice of x_0. Proposition 1.42 and Lemma 1.28 imply that there exists a subset $\mathbb{A}$ of $\mathcal{A}$ such that $(\bigcup \mathbb{A})^\circ$ is a fundamental region for Γ (see also [54, Theorem 6.2.20]). Every such set $\mathbb{A}$ is called a *basal family of precells of* Γ. Since Γ is geometrically finite, each basal family of precells is of the same finite cardinality.

Lemma and Definition 2.2 ([54, Propositions 6.4.11 – 6.4.13])**.** *Let $\mathbb{A}$ be a basal family of precells of Γ and let $A_0 \in \mathbb{A}$. Then there exist $A_1, \ldots, A_n \in \mathbb{A}$ and $g_1, \ldots, g_n \in \Gamma$, with $n \in \mathbb{N}$ only depending on A_0, such that*

$$B(A_0) := A_0 \cup \bigcup_{j=1}^{n} g_j.A_j \tag{2.3}$$

is the closure of a convex polygon in $\mathbb{H}$ every side of which is a geodesic arc, the union on the right hand side of (2.3) is essentially disjoint, and

$$B(A_0) \subseteq \mathrm{Re}|_{\mathbb{H}}^{-1}\big(\mathrm{Re}(A_0)\big)\,.$$

The set $B(A_0)$ is called the cell induced by A_0 *and we denote*

$$\mathbb{B} = \mathbb{B}(\mathbb{A}) := \{B(A) \mid A \in \mathbb{A}\}\ . \tag{2.4}$$

Let $B = B(A) \in \mathbb{B}$ and let $x, y \in \mathbb{R}$ be such that

$$A = \mathcal{K} \cap \mathrm{Re}|_{\mathbb{H}}^{-1}\big([x, y]\big)$$

(cf. (2.2)). Then either

$$B = \mathrm{Re}|_{\mathbb{H}}^{-1}\big(\mathrm{Re}(A)\big)\,,$$

B is the hyperbolic triangle with the vertices x, y, ∞, or the vertices of B are given by

$$g.\infty, g^2.\infty, \ldots, g^\sigma.\infty$$

for some elliptic $g \in \Gamma$ of order σ. In the latter case, $\{x, y\} = \{g.\infty, g^{-1}.\infty\}$.

Remark 2.3. The definition of cells in that way is what requires the assumption of condition (A). By construction, every precell $A \in \mathbb{A}$ that is not a strip in $\mathbb{H}$ (i. e., of the form $\mathrm{Re}|_{\mathbb{H}}^{-1}(I)$ for some interval I in $\mathbb{R}$) has at least one side that can be written as $[s(\mathrm{I}), \infty)_{\mathbb{H}}$, for some $\mathrm{I} \in \mathrm{REL}(\Gamma)$, where $s(\mathrm{I})$ denotes the summit of I as in (1.54). Another side β of A is then contained in β_{I}, with endpoint $s(\mathrm{I})$. Hence, in order for $\mathcal{K}$ to be decomposable into an essentially disjoint union of subsets of this kind, for every relevant isometric sphere its summit must be contained in its relevant part without being an endpoint of it, which is exactly what (A) demands. Furthermore, because of Proposition 1.41, there then exists a generator g of I and exactly one precell $A' \in \mathbb{A}$ with sides $[s(\mathrm{I}(g^{-1})), \infty)_{\mathbb{H}}$ and $g^{-1}.\beta$. By Lemma 1.19(i) and (1.53) we find

$$\begin{aligned}
[s(\mathrm{I}(g)), \infty)_{\mathbb{H}} \cup g.[s(\mathrm{I}(g^{-1})), \infty)_{\mathbb{H}} &= [s(\mathrm{I}(g)), \infty)_{\mathbb{H}} \cup (c(\mathrm{I}(g)), s(\mathrm{I}(g))]_{\mathbb{H}} \\
&= (c(\mathrm{I}(g)), \infty)_{\mathbb{H}}\,,
\end{aligned}$$

and since $g.(g^{-1}.\beta) = \beta$, the union $A \cup g.A'$ is thus connected with at least one side given by a geodesic arc. This way the claimed properties of cells are assured (for a complete discussion we refer the reader to the proofs of [54, Propositions 6.4.11 – 6.4.13]).

Let $\mathbb{A}$ be a basal family of precells and denote by $\mathbb{B}$ the set of cells it induces. Then the map

$$\begin{cases} \mathbb{A} & \longrightarrow & \mathbb{B}(\mathbb{A}) \\ A & \longmapsto & B(A) \end{cases}$$

is a bijection ([54, Corollary 6.4.14]). Hence, in particular

$$\#\mathbb{B}(\mathbb{A}) = \#\mathbb{A} < +\infty\,. \tag{2.5}$$

Since $\mathbb{A}$ tessellates $\mathbb{H}$ under Γ, the set of Γ-translates of $\mathbb{B}$ covers $\mathbb{H}$. However, the union of these translates does not need to be essentially disjoint anymore. Denote by $S_{\mathbb{B}}$ the set of sides of cells in $\mathbb{B}$. By Lemma 2.2, for every $\beta \in S_{\mathbb{B}}$ there exists $\gamma \in \mathscr{G}(\mathbb{H})$ such that

$$\beta = \gamma(\mathbb{R})\,.$$

Lemma 2.4 ([54, Proposition 6.4.15]). *Let $B_1, B_2 \in \mathbb{B}$ and $g \in \Gamma$. If $B_1 \neq g.B_2$ and $B_1 \cap g.B_2 \neq \varnothing$, then B_1 and $g.B_2$ coincide in exactly one side of B_1. In particular, if the intersection of B_1 and $g.B_2$ contains an inner point of B_1, then $B_1 = g.B_2$.*

The sides of cells will give rise to the cross section we are seeking. Hence, further study of the structure of cells and their boundary is appropriate. To that end we define

$$\mathbb{Q} := \underline{g}\mathcal{K} \cup \{c(\mathrm{I}) \mid \mathrm{I} \in \mathrm{REL}(\Gamma)\} \subseteq \mathbb{R} \setminus \mathbb{R}_{\mathrm{st}} . \tag{2.6}$$

Lemma 2.5. *We have $\Gamma.\mathbb{Q} = \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$.*

Proof. Since $\widehat{\mathbb{R}}_{\mathrm{st}}$ is Γ-invariant, so is $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$. Hence, $\Gamma.\mathbb{Q} \subseteq \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$. The converse inclusion follows from the Propositions 1.42 and 1.43. □

Let $S_{\mathbb{B}}^{\mathrm{v}}$ be the subset of $S_{\mathbb{B}}$ of vertical arcs. The following observation is immediate from Lemma 2.2, Lemma 2.5, and the constructions above.

Corollary 2.6. *The set $S_{\mathbb{B}}$ is finite. We further have*

$$\mathrm{Re}\Big(\bigcup S_{\mathbb{B}}^{\mathrm{v}}\Big) \subseteq \mathbb{Q} \qquad \text{and} \qquad \underline{g}\Big(\Gamma.\bigcup \mathbb{B}\Big) \subseteq \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}} .$$

The orbits of the members of $S_{\mathbb{B}}^{\mathrm{v}}$ generate all sides of cells in the following sense.

Lemma 2.7 ([54, Corollary 6.4.18]). *Let $\beta \in S_{\mathbb{B}}$ be a side of $B \in \mathbb{B}$. Then there exists $(\beta', B', g) \in S_{\mathbb{B}}^{\mathrm{v}} \times \mathbb{B} \times \Gamma$ such that β' is a side of B' and*

$$\beta = g.\beta' \qquad \text{and} \qquad B \cap g.B' = \beta .$$

Proposition 2.8 ([54, Propositions 6.5.2 and 6.5.3]). *The Γ-orbit of $\bigcup S_{\mathbb{B}}$ equals the Γ-orbit of $\bigcup S_{\mathbb{B}}^{\mathrm{v}}$ and is a totally geodesic submanifold of $\mathbb{H}$ of codimension 1 and independent of the choice of the basal family $\mathbb{A}$.*

Recall the map $\mathrm{bp}\colon S\mathbb{H} \to \mathbb{H}$ from (1.19). Let $M \subseteq \mathbb{H}$ be open and let $\nu \in S\mathbb{H}$ be such that $\mathrm{bp}(\nu) \in \partial M$. Recall further the unique geodesic γ_ν on $\mathbb{H}$ determined by ν as in (1.20). We say that ν *points into* M, if for $\varepsilon > 0$ sufficiently small we have

$$\gamma_\nu\big((0,\varepsilon)\big) \subseteq M .$$

We are now ready to derive a cross section via a set of representatives in $S\mathbb{H}$. Let $\mathbb{A}$ be a basal family of precells of Γ and let $\mathbb{B} = \mathbb{B}(\mathbb{A})$ be as in (2.4). For each $B \in \mathbb{B}$ denote its two vertical sides by $\beta_B, \beta_B' \in S_{\mathbb{B}}^{\mathrm{v}}$ and let

$$\mathrm{BM} := \bigcup_{B \in \mathbb{B}} \big\{(B, \beta_B), (B, \beta_B')\big\} . \tag{2.7}$$

For any choice of tuple $b = (B, \beta) \in \mathrm{BM}$ we define

$$\mathrm{C_P}(b) := \{\nu \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(\nu) \in \beta \text{ and } \nu \text{ points into } B^\circ\} .$$

Let $\mathcal{C}(\mathrm{BM}) := \{\mathrm{C_P}(b) \mid b \in \mathrm{BM}\}$. By Lemma 2.7 there exists a minimal subset

$$\mathcal{C}_\mathrm{P} \subseteq \mathcal{C}(\mathrm{BM}) \tag{2.8}$$

such that the Γ-orbit of

$$\mathrm{C_P} := \bigcup \mathcal{C}_\mathrm{P} \tag{2.9}$$

contains all of $\bigcup \mathcal{C}(\mathrm{BM})$. Here by "minimal" we mean that any proper subset of $\mathcal{C}_\mathrm{P}$ does not have this property. Or in other words, $\mathcal{C}_\mathrm{P}$ is a representative of $\pi\big(\mathcal{C}(\mathrm{BM})\big)$ in $\mathbb{H}$, where π denotes the canonical quotient map from (1.32). Because of (2.5) the set BM is of finite cardinality, and subsequently so are the sets $\mathcal{C}(\mathrm{BM})$ and $\mathcal{C}_\mathrm{P}$. Hence, we may enumerate

$$\mathcal{C}_\mathrm{P} = \{\mathrm{C}_{\mathrm{P},1}, \ldots, \mathrm{C}_{\mathrm{P},N}\} , \tag{2.10}$$

with some $N \in \mathbb{N}$ which does not depend on the choice of $\mathcal{C}_\mathrm{P}$.

Remark 2.9. In the notation of [54] we have

$$\mathcal{C}_\mathrm{P} = \left\{ \mathrm{CS}'(\widetilde{\mathcal{B}}) \;\middle|\; \widetilde{\mathcal{B}} \in \widetilde{\mathbb{B}}_{\mathbb{S},\mathbb{T}} \right\} , \tag{2.11}$$

where the subscript $\mathbb{S}$ fixes a sequence of choices to be made during the construction of these sets (which translate to the choice of the basal family $\mathbb{A}$ and the representative $\mathcal{C}_\mathrm{P}$), and the subscript $\mathbb{T}$ indicates that arbitrary translations of the sets $\mathrm{C}_{\mathrm{P},j}$ by elements of Γ_∞ are permitted and a collection of such translations is chosen and applied. The statements that follow are meant to be understood *"for all possible choices of* $\mathbb{S}$ *and* $\mathbb{T}$*"*.

We now write

$$\widehat{\mathrm{C}}_\mathrm{P} := \pi\big(\mathrm{C_P}\big) \subseteq \mathrm{S}\mathbb{X} .$$

This constitutes our cross section. However, it is not a cross section in the traditional sense (a Poincaré cross section), for the first return map is not well-defined for all tangent vectors $\widehat{\nu} \in \widehat{\mathrm{C}}_\mathrm{P}$. More precisely, the issue is with vectors $\widehat{\nu}$ for which

$$\gamma_\nu(\pm\infty) \in g\big(\Gamma.\bigcup S_{\mathbb{B}}^{\mathrm{v}}\big) \subseteq \Gamma.\mathbb{Q} = \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_\mathrm{st} ,$$

for some (and hence any) representative ν of $\widehat{\nu}$ in $\mathrm{S}\mathbb{H}$. Therefore, by a slight abuse of notation we define

$$\mathrm{S}\mathbb{H}_\mathrm{st} := \left\{ \nu \in \mathrm{S}\mathbb{H} \;\middle|\; \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in \widehat{\mathbb{R}}_\mathrm{st} \times \widehat{\mathbb{R}}_\mathrm{st} \right\} . \tag{2.12}$$

By definition of $\widehat{\mathbb{R}}_\mathrm{st}$, we have $\{\nu \in \mathrm{S}\mathbb{H} \mid (\gamma_\nu(+\infty), \gamma_\nu(-\infty)) \in E(\mathbb{X})\} \subseteq \mathrm{S}\mathbb{H}_\mathrm{st}$, with $E(\mathbb{X})$ as in (1.41). Or in other words, for every geodesic $\gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ we have $\gamma'(t) \in \mathrm{S}\mathbb{H}_{\mathrm{st}}$ for any time $t \in \mathbb{R}$, with $\mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ the set of all representatives on $\mathbb{H}$ of periodic geodesics on $\mathbb{X}$ (cf. (1.34)). Finally, set

$$\mathrm{C}_{\mathrm{P,st}} := \mathrm{C}_{\mathrm{P}} \cap \mathrm{S}\mathbb{H}_{\mathrm{st}} \qquad \text{and} \qquad \widehat{\mathrm{C}}_{\mathrm{P,st}} := \pi\big(\mathrm{C}_{\mathrm{P,st}}\big)\,,$$

and recall the notion of a (strong) cross section for $\widehat{\Phi}$ from Definition 1.47.

Proposition 2.10 ([54, Theorem 6.7.17 and Corollary 6.7.18])**.** *Let μ be a measure on $\mathscr{G}(\mathbb{X})$ such that*

$$\mu\big(\{\widehat{\gamma}_\nu \mid \nu \in \mathrm{S}\mathbb{H} \setminus \mathrm{S}\mathbb{H}_{\mathrm{st}}\}\big) = 0\,.$$

Then $\widehat{\mathrm{C}}_{\mathrm{P}}$ is a cross section for $\widehat{\Phi}$ with respect to μ. Moreover, $\widehat{\mathrm{C}}_{\mathrm{P,st}}$ is a strong cross section for $\widehat{\Phi}$ with respect to μ.

Example 2.11. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ from Example 1.46. We infer the set $\widetilde{W}_{\mathcal{K}}$ to be given by

$$\widetilde{W}_{\mathcal{K}} = \bigcup_{n\in\mathbb{Z}} \mathrm{t}_\lambda^n\boldsymbol{.}\{-1, c(g_\sigma), -c(g_\sigma), 1\}\,,$$

where

$$c(g_\sigma) = -\frac{\cos\big(\frac{\pi}{\sigma}\big)}{\cos\big(\frac{\pi}{\sigma}\big)+1}\,.$$

This yields the four precells

$$A' := \mathrm{Re}|_{\mathbb{H}}^{-1}\big([\mathrm{t}_\lambda^{-1}\boldsymbol{.}1, -1]\big)\,, \qquad A_2 := \mathcal{K} \cap \mathrm{Re}|_{\mathbb{H}}^{-1}\big([-1, c(g_\sigma)]\big)\,,$$
$$A_3 := \mathcal{K} \cap \mathrm{Re}|_{\mathbb{H}}^{-1}\big([c(g_\sigma), -c(g_\sigma)]\big)\,, \quad \text{and} \quad A_4 := \mathcal{K} \cap \mathrm{Re}|_{\mathbb{H}}^{-1}\big([-c(g_\sigma), 1]\big)\,.$$

We dissect the precell A' into the essentially disjoint union $\mathrm{Re}|_{\mathbb{H}}^{-1}([\mathrm{t}_\lambda^{-1}\boldsymbol{.}1, -\lambda/2]) \cup \mathrm{Re}|_{\mathbb{H}}^{-1}([-\lambda/2, -1])$ and translate the left half of it by t_λ in order to obtain the two further precells

$$A_1 := \mathrm{Re}|_{\mathbb{H}}^{-1}\big([-\tfrac{\lambda}{2}, -1]\big) \qquad \text{and} \qquad A_5 := \mathrm{Re}|_{\mathbb{H}}^{-1}\big([1, -\tfrac{\lambda}{2}]\big)\,.$$

This has the advantage that, by comparing to (1.83), we immediately see that the set $\{A_1, \ldots, A_5\}$ constitutes a basal family of precells. Furthermore, we retain a certain symmetry in the sketches below. We excluded the fact that we are allowed to do this from the discussion above in order to cut short on exposition. The arising cells are

$$B_1 := A_1\,, \qquad B_2 := A_2 \cup g_\sigma^{-1}\boldsymbol{.}A_4\,, \qquad B_3 := \bigcup_{k=1}^{\sigma} g_\sigma^k\boldsymbol{.}A_3\,,$$
$$B_4 := A_4 \cup g_\sigma\boldsymbol{.}A_2\,, \qquad \text{and} \qquad B_5 := A_5\,.$$

From these we can now define the representative. To that end let

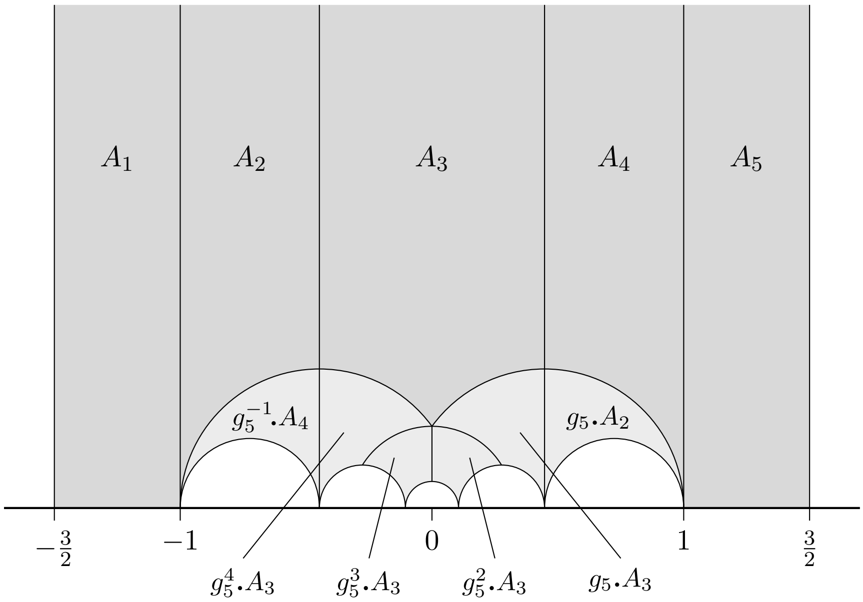


Figure 4: The basal family of precells for $\Gamma_{5,3}$ and the translates forming the associated set of cells.

$$\beta_1 := \big(-\tfrac{\lambda}{2}, \infty\big)_{\mathbb{H}}\,, \qquad \beta_2 := \beta_8 := \big(-1, \infty\big)_{\mathbb{H}}\,, \qquad \beta_3 := \big(c(g_\sigma), \infty\big)_{\mathbb{H}}\,,$$
$$\beta_4 := \big(-c(g_\sigma), \infty\big)_{\mathbb{H}}\,, \qquad \beta_5 := \beta_7 := \big(1, \infty\big)_{\mathbb{H}}\,, \qquad \beta_6 := \big(\tfrac{\lambda}{2}, \infty\big)_{\mathbb{H}}\,,$$

and

$$I_1 := \big(-\tfrac{\lambda}{2}, +\infty\big)\,, \qquad I_2 := \big(-1, +\infty\big)\,, \qquad I_3 := \big(c(g_\sigma), +\infty\big)\,,$$
$$I_4 := \big(-c(g_\sigma), +\infty\big)\,, \qquad I_5 := \big(1, +\infty\big)\,, \qquad I_6 := \big(-\infty, \tfrac{\lambda}{2}\big)\,,$$
$$I_7 := \big(-\infty, 1\big)\,, \qquad I_8 := \big(-\infty, -1\big)\,.$$

With those we define for every $j \in \{1, \ldots, 8\}$,

$$\mathrm{C}_{\mathrm{P},j} := \{\nu \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(\nu) \in \beta_j\,,\ \gamma_\nu(+\infty) \in I_j\}\ .$$

Then $\mathrm{C}_\mathrm{P} := \bigcup_{j=1}^{8} \mathrm{C}_{\mathrm{P},j}$ is a representative in $\mathrm{S}\mathbb{H}$ for a cross section for the geodesic flow $\widehat{\Phi}$ on $\Gamma_{\sigma,\lambda}\diagdown\mathbb{H}$ (see also Figure 5).

2.2 Properties of the Representative

In this section we collect a few further properties of the cross section representative yielded by the cusp expansion algorithm which we will require for arguments

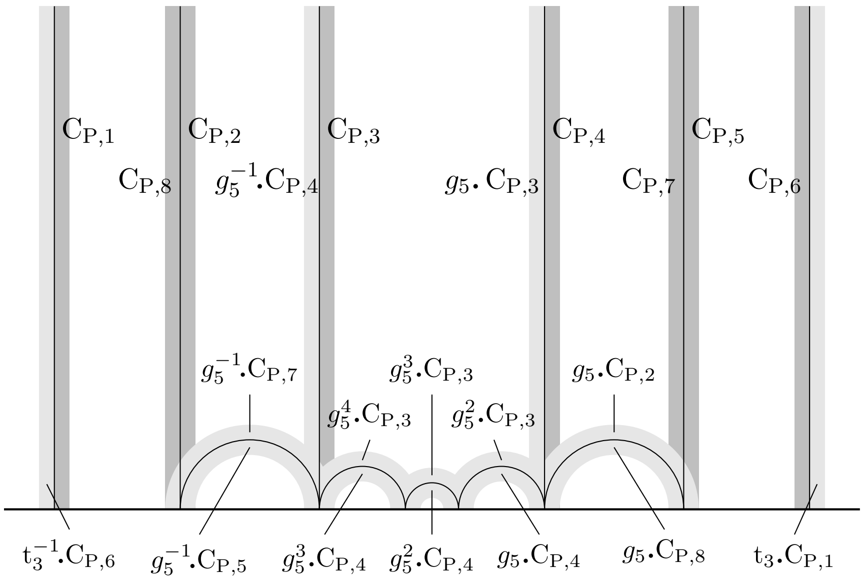


Figure 5: The representative of the cross section yielded by the cusp expansion algorithm for $\Gamma_{5,3}$. The gray stripes indicate that the respective set consists of unit tangent vectors based on the adjacent geodesic and pointing into the indicated half-space. The components of the cross section representative are colored in dark gray, their translates in light gray.

later on. We thus let Γ be a Fuchsian groups with cusps as before, assume that ∞ represents a cusp of Γ with cusp width $\lambda > 0$, define

$$A := \{1, \ldots, N\}\,,$$

and let $\mathcal{C}_{\mathrm{P}}$, C_{P} and $\mathrm{C}_{\mathrm{P},j}$, $j \in A$, be as in (2.8), (2.9), and (2.10), respectively. Then, by construction, for every $j \in A$ there exists $x_j \in \mathbb{Q}$ such that

$$\mathrm{bp}(\mathrm{C}_{\mathrm{P},j}) = (x_j, \infty)_{\mathbb{H}}\,, \tag{2.13}$$

and all vectors of $\mathrm{C}_{\mathrm{P},j}$ point into the same open half-space relative to $(x_j, \infty)_{\mathbb{H}}$, which we denote by $\mathrm{H}_+^{\mathrm{P}}(j)$. We further set $\mathrm{H}_-^{\mathrm{P}}(j) := \mathbb{H}\setminus\overline{\mathrm{H}_+^{\mathrm{P}}(j)}$ and denote by $I_{\mathrm{P},j}$ and $J_{\mathrm{P},j}$ the largest open interval contained in $\partial_g \mathrm{H}_+^{\mathrm{P}}(j)$ and $\partial_g \mathrm{H}_-^{\mathrm{P}}(j)$, respectively. Furthermore, for $j \in A$ we set

$$\mathrm{C}_{\mathrm{P},j,\mathrm{st}} := \mathrm{C}_{\mathrm{P},j} \cap \mathrm{S}\mathbb{H}_{\mathrm{st}}\,,$$

with $\mathrm{S}\mathbb{H}_{\mathrm{st}}$ as in (2.12).

Lemma 2.12. *For every $j \in A$ the map*

$$\phi_j \colon \begin{cases} \mathrm{C}_{\mathrm{P},j} & \longrightarrow & I_{\mathrm{P},j} \times J_{\mathrm{P},j} \\ \nu & \longmapsto & \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \end{cases} \tag{2.14}$$

is a bijection. Furthermore, $\phi_j(\mathrm{C}_{\mathrm{P},j,\mathrm{st}}) = I_{\mathrm{P},j,\mathrm{st}} \times J_{\mathrm{P},j,\mathrm{st}}$.

Proof. That ϕ_j is surjective is clear from the definition of the sets involved. Injectivity is immediate from the uniqueness of γ_ν for $\nu \in \mathrm{S}\mathbb{H}$ and geodesics in $\mathbb{H}$ being uniquely given by their endpoints. The last statement is again clear from the definition of the sets under consideration. □

Lemma 2.13. *Let $j \in A$. Then there exists a unique pair $(k, g) \in A \times \Gamma$ such that*

$$I_{\mathrm{P},j} = g.J_{\mathrm{P},k} \qquad \text{and} \qquad J_{\mathrm{P},j} = g.I_{\mathrm{P},k}\,.$$

Proof. Let $b_j = (B_j, \beta_j) \in \mathrm{BM}$ be such that $\mathrm{C}_{\mathrm{P},j} = \mathrm{C}_{\mathrm{P}}(b_j)$. By Lemma 2.7 there exists $(B, h_1) \in \mathbb{B} \times \Gamma$ such that

$$B_j \cap h_1.B = \beta_j$$

and $h_1^{-1}.\beta_j$ is a vertical side of B. Then, by definition, there exists exactly one pair $(k, h_2) \in A \times \Gamma$ such that

$$\mathrm{C}_{\mathrm{P},k} = h_2^{-1}.\,\mathrm{C}_{\mathrm{P}}((B, h_1^{-1}.\beta_j))\,.$$

Define $g \coloneqq h_1 h_2$. Then

$$g.\,\mathrm{C}_{\mathrm{P},k} = \left\{\nu \in \mathrm{S}\mathbb{H} \;\middle|\; \mathrm{bp}(\nu) \in \beta_j\,,\ \big(\gamma_\nu(-\infty), \gamma_\nu(+\infty)\big) \in \phi_j(\mathrm{C}_{\mathrm{P},j})\right\}\,,$$

implying the asserted identities by Lemma 2.12. Since $\mathcal{C}_{\mathrm{P}}$ was chosen minimal, the pair (k, g) is unique with that property. □

The following result is immediate from Proposition 2.10. See also [54, Proposition 6.7.12].

Lemma 2.14. *Let $\nu \in \mathrm{C}_{\mathrm{P},\mathrm{st}}$. Then both the values*

$$t_{\mathrm{P}}^{+}(\nu) \coloneqq \min\left\{t > 0 \;\middle|\; \gamma_\nu'(t) \in \Gamma.\mathrm{C}_{\mathrm{P}}\right\} \tag{2.15}$$

and

$$t_{\mathrm{P}}^{-}(\nu) \coloneqq \max\left\{t < 0 \;\middle|\; \gamma_\nu'(t) \in \Gamma.\mathrm{C}_{\mathrm{P}}\right\} \tag{2.16}$$

are finite, and $\{\gamma_\nu'(t_{\mathrm{P}}^{+}(\nu)), \gamma_\nu'(t_{\mathrm{P}}^{-}(\nu))\} \subseteq \Gamma.\mathrm{C}_{\mathrm{P},\mathrm{st}}$.

We will require the following observations, which relate the geodesics on which the sets $\mathrm{C}_{\mathrm{P},j}$ are based to the relevant isometric spheres. Hence, recall the sets $\mathrm{REL}(\Gamma)$ and Γ_{REL} from Section 1.10 as well as the relevant part β_{I} of $\mathrm{I} \in \mathrm{REL}(\Gamma)$ from (1.68). Further recall the point x_j from (2.13), for $j \in A$.

Lemma 2.15. *Let $j \in A$. Then for every $g \in \Gamma$ with $g.x_j = \infty$ we have $g \in \Gamma_{\mathrm{REL}}$.*

Proof. Since $x_j \in \mathbb{Q} \subseteq \mathbb{R}$ for every $j \in A$, we have $g \in \Gamma \setminus \Gamma_\infty$. Hence, the isometric sphere $\mathrm{I}(g)$ is well-defined with center $g^{-1}.\infty = x_j$. Since $g\mathcal{K}$ does not contain centers of isometric spheres, (2.6) implies that the point x_j is also the center of some relevant isometric sphere. Proposition 1.25 now yields the assertion. ☐

Lemma 2.16. *For every $\mathrm{I} \in \mathrm{REL}(\Gamma)$ for which its summit $s(\mathrm{I})$ is contained in β_{I} but is not an endpoint of it, there exists a pair $(j, g) \in A \times \Gamma$ such that*

$$g.\,\mathrm{bp}(\mathrm{C}_{\mathrm{P},j}) = \big(\operatorname{Re} s(\mathrm{I}), \infty\big)_{\mathbb{H}}\,.$$

We further have $g \in \Gamma_\infty \cup \Gamma_\infty \Gamma_{\mathrm{REL}}$.

Proof. From the assumption it follows that $\operatorname{Re} s(\mathrm{I}) = c(\mathrm{I}) \in \widetilde{W_{\mathcal{K}}}$. Hence, there exist precells A_1, A_2 of the form $\mathcal{K} \cap \operatorname{Re}|_{\mathbb{H}}^{-1}\big([x, \operatorname{Re} s(\mathrm{I})]\big)$ and $\mathcal{K} \cap \operatorname{Re}|_{\mathbb{H}}^{-1}([\operatorname{Re} s(\mathrm{I}), y])$, respectively, with some $x, y \in \mathbb{R}$. From this and (1.74) we derive

$$\big(\operatorname{Re} s(\mathrm{I}), \infty\big)_{\mathbb{H}} \in \Gamma_\infty.S_{\mathbb{B}}^{\mathrm{v}}\,.$$

Hence, there exists $\beta \in S_{\mathbb{B}}^{\mathrm{v}}$ such that $\mathrm{t}_\lambda^n.\beta = (\operatorname{Re} s(\mathrm{I}), \infty)_{\mathbb{H}}$ for some $n \in \mathbb{Z}$, and thus the first statement follows from Lemma 2.7. Let $g \in \Gamma$ be such that $g.\,\mathrm{bp}(\mathrm{C}_j) = (\operatorname{Re} s(\mathrm{I}), \infty)_{\mathbb{H}}$ for some $j \in A$. By the above we may write

$$g = \mathrm{t}_\lambda^n h\,,$$

with $h \in \Gamma$ such that $h.\,\mathrm{bp}(\mathrm{C}_j) = \beta$. It suffices to show that $h \in \Gamma_\infty \cup \Gamma_{\mathrm{REL}}$. If $\beta = \mathrm{bp}(\mathrm{C}_j)$, then obviously $h = \mathrm{id}$. So assume that this is not the case. Since $h.\,\mathrm{bp}(\mathrm{C}_j) = \beta$ and both geodesics are vertical, either $h.x_j = \operatorname{Re}(\beta)$, or $h.x_j = \infty$. In the former case we further have $h.\infty = \infty$ and hence, $h \in \Gamma_\infty$. In the latter case $h \in \Gamma_{\mathrm{REL}}$ follows from Lemma 2.15. ☐

Lemma 2.17. *For every $r \in \mathbb{R}$ there exist $i_1, \ldots, i_N \in \mathbb{Z}$ such that*

$$\bigcup_{j=1}^{n} \mathrm{t}_\lambda^{i_j}.\,\mathrm{bp}(\mathrm{C}_j) \subseteq \operatorname{Re}|_{\mathbb{H}}^{-1}\big([r, r+\lambda]\big)\,.$$

Proof. By Lemma 1.37 the set $\operatorname{Re}|_{\mathbb{H}}^{-1}\big([r, r+\lambda]\big)$ is the closure of a fundamental domain for the stabilizer subgroup Γ_∞ of ∞ in Γ. Therefore, the assertion follows from (F2) (the tessellation property of fundamental domains). ☐

Chapter 3

Strict Transfer Operator Approaches and Fast Transfer Operators

This chapter serves to recall, in Section 3.1, the concept of strict transfer operator approaches from [22]. We further recall, in Section 3.3, the main result of [22], for it is of utmost importance to our proof of Theorem A, as described in the introduction. As already stated in the introduction, the aim of this thesis is to construct strict transfer operator approaches for a large class of Fuchsian groups. Hence, the list of properties constituting such an approach preempts the structure of our argumentation. Throughout this chapter let Γ be a geometrically finite Fuchsian group and denote by $\mathbb{X} = \Gamma\backslash\mathbb{H}$ its orbit space.

3.1 Strict Transfer Operator Approaches

We say that Γ admits a *strict transfer operator approach* if there exists a *structure tuple*

$$\mathcal{S} := \left(\widehat{A}, \{\widehat{I}_a\}_{a\in\widehat{A}}, \{P_{a,b}\}_{a,b\in\widehat{A}}, \{C_{a,b}\}_{a,b\in\widehat{A}}, \{\{g_p\}_{p\in P_{a,b}}\}_{a,b\in\widehat{A}}\right)$$

consisting of

- a finite set $\widehat{A}$,
- a family $\{\widehat{I}_a\}_{a\in\widehat{A}}$ of (not necessarily disjoint) intervals in $\widehat{\mathbb{R}}$,
- a family $\{P_{a,b}\}_{a,b\in\widehat{A}}$ of finite (possibly empty) sets of parabolic elements in Γ,
- a family $\{C_{a,b}\}_{a,b\in\widehat{A}}$ of finite (possibly empty) subsets of Γ, and

- a family $\{\{g_p\}_{p\in P_{a,b}}\}_{a,b\in\widehat{A}}$ of elements of Γ (which may be the identity),

which satisfies the following five properties.

Property 1. *For all $a, b \in \widehat{A}$*

(I) *we have $p^{-n}g_p^{-1}.\widehat{I}_{a,\mathrm{st}} \subseteq \widehat{I}_{b,\mathrm{st}}$ for all $p \in P_{a,b}$ and $n \in \mathbb{N}$, and $p^n \notin P_{a,b}$ for $n \geq 2$,*

(II) *we have $g^{-1}.\widehat{I}_{a,\mathrm{st}} \subseteq \widehat{I}_{b,\mathrm{st}}$ for all $g \in C_{a,b}$,*

(III) *the sets in the family*

$$\left\{g^{-1}.\widehat{I}_{j,\mathrm{st}} \;\middle|\; j \in \widehat{A}, g \in C_{j,b}\right\} \cup \left\{p^{-n}g_p^{-1}.\widehat{I}_{j,\mathrm{st}} \;\middle|\; j \in \widehat{A},\, p \in P_{j,b},\, n \in \mathbb{N}\right\}$$

are pairwise disjoint and

$$\widehat{I}_{b,\mathrm{st}} = \bigcup_{j\in\widehat{A}}\left(\bigcup_{g\in C_{j,b}} g^{-1}.\widehat{I}_{j,\mathrm{st}} \cup \bigcup_{p\in P_{j,b}} \bigcup_{n=1}^{\infty} p^{-n}g_p^{-1}.\widehat{I}_{j,\mathrm{st}}\right).$$

Property 1 induces a discrete dynamical system (D, F), where

$$D := \bigcup_{a\in\widehat{A}} \widehat{I}_{a,\mathrm{st}} \times \{a\},$$

and F splits into the submaps (bijections, that are local parts of the map F)

$$\begin{cases} g^{-1}.\widehat{I}_{a,\mathrm{st}} \times \{b\} & \longrightarrow & \widehat{I}_{a,\mathrm{st}} \times \{a\} \\ (x,b) & \longmapsto & (g.x, a) \end{cases}$$

and

$$\begin{cases} p^{-n}g_p^{-1}.\widehat{I}_{a,\mathrm{st}} \times \{b\} & \longrightarrow & \widehat{I}_{a,\mathrm{st}} \times \{a\} \\ (x,b) & \longmapsto & (g_p p^n.x, a) \end{cases}$$

for all $a, b \in \widehat{A}$, $g \in C_{a,b}$, $p \in P_{a,b}$ and $n \in \mathbb{N}$, which completely determine F.

Property 2. *For $n \in \mathbb{N}$ denote by Per_n the subset of Γ of all g for which there exists $a \in \widehat{A}$ such that*

$$\begin{cases} g^{-1}.\widehat{I}_{a,\mathrm{st}} \times \{a\} & \longrightarrow & \widehat{I}_{a,\mathrm{st}} \times \{a\} \\ (x,a) & \longmapsto & (g.x, a) \end{cases}$$

is a submap of F^n. Then the union

$$\mathrm{Per} := \bigcup_{n=1}^{\infty} \mathrm{Per}_n$$

is disjoint.

As before (see Section 1.7), we denote by $[\Gamma]_\mathrm{h}$ the set of all Γ-conjugacy classes of hyperbolic elements in Γ.

Property 3. *Let* Per *be as in Property 2. Then*

(I) *all elements of* Per *are hyperbolic,*

(II) *for each* $h \in \mathrm{Per}$ *also its primitive* h_0 *is contained in* Per*,*

(III) *for each* $[g] \in [\Gamma]_\mathrm{h}$ *there exists a unique element* $n \in \mathbb{N}$ *such that* Per_n *contains an element that represents* $[g]$.

Suppose that $[g] \in [\Gamma]_\mathrm{h}$ is represented by $h \in \mathrm{Per}_n$, $n \in \mathbb{N}$. Because of Property 2 we shall define the *word length of* h as

$$\omega(h) := n\,. \tag{3.1}$$

We denote by $m = m(h) \in \mathbb{N}$ the unique number such that $h = h_0^m$ for a primitive hyperbolic element $h_0 \in \Gamma$, and we set

$$p(h) := \tfrac{\omega(h)}{m(h)}\,. \tag{3.2}$$

Further we set $\omega(g) := \omega(h)$ as well as $m(g) := m(h)$ and $p(g) := p(h)$. By Property 3 these values are well-defined.

Property 4. *For each element* $[g] \in [\Gamma]_\mathrm{h}$ *there are exactly* $p(g)$ *distinct elements* $h \in \mathrm{Per}_{\omega(g)}$ *such that* $h \in [g]$.

Property 5. *There exists a family* $\{\mathcal{E}_a\}_{a\in\widehat{A}}$ *of open, bounded, connected and simply connected sets in* $\widehat{\mathbb{C}}$ *such that*

(I) *for all* $a \in \widehat{A}$ *we have*

$$\overline{\widehat{I}_{a,\mathrm{st}}} \subseteq \mathcal{E}_a\,,$$

(II) *there exists* $q \in \mathrm{PSL}_2(\mathbb{R})$ *such that for all* $a \in \widehat{A}$ *we have* $q.\overline{\mathcal{E}_a} \subseteq \mathbb{C}$*, and for all* $b \in \widehat{A}$ *and all* $g \in C_{a,b}$ *we have*

$$gq^{-1}.\infty \notin \overline{\mathcal{E}_a}\,,$$

(III) *for all* $a, b \in \widehat{A}$ *and all* $g \in C_{a,b}$ *we have*

$$g^{-1}.\overline{\mathcal{E}_a} \subseteq \mathcal{E}_b\,,$$

(IV) *for all* $a, b \in \widehat{A}$ *and all* $p \in P_{a,b}$ *there exists a compact subset* $K_{a,b,p}$ *of* $\widehat{\mathbb{C}}$ *such that for all* $n \in \mathbb{N}$ *we have*

$$p^{-n}g_p^{-1}.\overline{\mathcal{E}_a} \subseteq K_{a,b,p} \subseteq \mathcal{E}_b\,,$$

(V) *for all* $a, b \in \widehat{A}$ *and all* $p \in P_{a,b}$ *the set* $g_p^{-1}.\overline{\mathcal{E}_a}$ *does not contain the fixed point of* p.

3.2 Nuclear Operators

In this section we briefly recall an object crucial for the understanding of Theorem 3.1 presented in the next section, namely nuclear operators on Banach spaces. We refer the reader to [26, 25, 45, 72] for more extensive treatises of this subject.

Nuclear operators have been introduced by Alexander Grothendieck in his dissertation thesis (see [26]) as a generalization of trace-class operators to Banach spaces. Let B be an arbitrary Banach space equipped with some norm $\|\,.\,\|$ and denote by B^* its dual, that is the space of bounded linear functionals on B, equipped with the usual dual norm

$$\|f\|_* := \sup\left\{|f(x)| \mid x \in B\,,\ \|x\| \leq 1\right\}\,,$$

for $f \in B^*$. The tensor product $B^* \otimes B$ has a completion under the norm

$$\|F\|_p = \inf \sum_i \|f_i\|_* \|e_i\|\,,$$

where the infimum is taken over all finite representations

$$F = \sum_i f_i \otimes e_i \in B^* \otimes B\,.$$

This completion is called the *projective topological tensor product* of B and its elements are called *Fredholm kernels* on B. Every Fredholm kernel admits a representation

$$F = \sum_{i=1}^{\infty} \lambda_i f_i \otimes e_i\,,$$

with $e_i \in B$, $f_i \in B^*$, $\|e_i\| = \|f_i\|_* = 1$, and an absolutely summable sequence $(\lambda_i)_{i\in\mathbb{N}}$ of complex numbers. Assigned to each Fredholm kernel is a *trace*

$$\operatorname{tr}(F) := \sum_{i=1}^{\infty} \lambda_i f_i(e_i)$$

and an *order*

$$\operatorname{ord}(F) := \inf\left\{0 < q \leq 1 \mid \textstyle\sum_i |\lambda_i|^q < +\infty\right\}\,,$$

which are both independent of the representation. Furthermore, associated to each Fredholm kernel $F = \sum_i \lambda_i f_i \otimes e_i$ is a compact linear operator

$$\mathcal{L}_F \colon B \longrightarrow B\,, \qquad \mathcal{L}_F\varphi = \sum_{i=1}^{\infty} \lambda_i f_i(\varphi) e_i\,.$$

Linear operators arising in this way are called *nuclear operators*. These operators may inherit order and trace from the Fredholm kernel. But in general, for a given nuclear operator $\mathcal{L}$ there might be more than one Fredholm kernel F such that $\mathcal{L} = \mathcal{L}_F$, and thus the trace need not be unique. However, in [26] Grothendieck showed that if $\mathcal{L} = \mathcal{L}_F$ with $\operatorname{ord}(F) \leq 2/3$, then the trace is unique. This includes *nuclear operators of order zero*, that is

$$\sum_{i=1}^{\infty} |\lambda_i|^q < +\infty$$

for all $q > 0$. Given a unique trace, which we denote by $\operatorname{Tr}\mathcal{L}$, we have

$$\operatorname{Tr}\mathcal{L} = \sum_i \rho_i\,,$$

where ρ_i are the eigenvalues of $\mathcal{L}$ counted with multiplicities. Then the *Fredholm determinant of* $\mathcal{L}$ can be defined as

$$\det(1 - z\mathcal{L}) \coloneqq \prod_i (1 - \rho_i z) = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr}\mathcal{L}^n \right)\,, \tag{3.3}$$

with $z \in \mathbb{C}$. This is an entire function in z. Furthermore, for a family of nuclear operators $\{\mathcal{L}_s\}_s$ for which the parameterization $s \mapsto \mathcal{L}_s$ is holomorphic (meromorphic) on some domain, the Fredholm determinants $\det(1 - z\mathcal{L}_s)$ are holomorphic (meromorphic) in s from the same domain, for every $z \in \mathbb{C}$.

3.3 Fast Transfer Operators, Representation and Meromorphic Continuation of the Selberg Zeta Function

In this section we explain the use of strict transfer operator approaches for Selberg zeta functions and, for this purpose, introduce the notion of fast transfer operators. To that end we suppose that the Fuchsian group Γ admits a strict transfer operator approach with structure tuple

$$\mathcal{S} \coloneqq \left(\widehat{A}, \{\widehat{I}_a\}_{a\in\widehat{A}}, \{P_{a,b}\}_{a,b\in\widehat{A}}, \{C_{a,b}\}_{a,b\in\widehat{A}}, \{\{g_p\}_{p\in P_{a,b}}\}_{a,b\in\widehat{A}}\right),$$

as defined in Section 3.1. We let V be a finite-dimensional vector space and let

$$\chi\colon \Gamma \longrightarrow \mathrm{GL}(V)$$

be a representation of Γ on V. Recall from Section 1.12 that χ is said to have non-expanding cusp monodromy, if for each parabolic element $p \in \Gamma$ all eigenvalues of the endomorphism $\chi(p)$ are of modulus 1. We assume that χ has this property.

Let $\widehat{\mathbb{C}}$ be as in (1.3). For $U \subseteq \widehat{\mathbb{C}}$ denote by $\mathrm{Fct}(U;V)$ the space of functions $f\colon U \to V$ and by $C(U;V)$ its subspace of continuous functions. For any choice of $s \in \mathbb{C}$, $U \subseteq \widehat{\mathbb{C}}$, $f \in \mathrm{Fct}(U;V)$, $g \in \Gamma$, and $z \in U$ we set

$$\alpha_s(g^{-1})f(z) := (g'(z))^s \chi(g) f(g.z)\,, \tag{3.4}$$

whenever this is well-defined. (We note that α_s is typically not a representation of Γ on $\mathrm{Fct}(U;V)$, but it satisfies some restricted homomorphism properties, which motivated the notation. We refer to the discussion in [15, Section 6.3] for details.) For any open set $U \subseteq \mathbb{C}$ we set

$$\mathcal{B}(U;V) := \left\{ f \in C(\overline{U};V) \;\middle|\; f|_U \text{ holomorphic} \right\}\,.$$

Then $\mathcal{B}(U;V)$, endowed with the supremum norm, is a Banach space. We write

$$\mathcal{B}(\mathcal{E}_{\widehat{A}};V) := \bigoplus_{a\in\widehat{A}} \mathcal{B}(\mathcal{E}_a;V)$$

for the product space, where $\mathcal{E}_{\widehat{A}} = \{\mathcal{E}_a\}_{a\in\widehat{A}}$ is a family of open sets as provided by Property 5. We identify the elements $f \in \mathcal{B}(\mathcal{E}_{\widehat{A}};V)$ with the function vectors $f = (f_a)_{a\in\widehat{A}}$, where

$$f_a\colon \widehat{I}_{a,\mathrm{st}} \longrightarrow V$$

for $a \in \widehat{A}$. Then we define the *(fast) transfer operator* $\mathcal{L}_{s,\chi}$ *with parameter* s *associated to* $\mathcal{S}$ *and* χ by

$$\mathcal{L}_{s,\chi} := \left(\sum_{g\in C_{a,b}} \alpha_s(g) + \sum_{p\in P_{a,b}} \sum_{n\in\mathbb{N}} \alpha_s(g_p p^n) \right)_{a,b\in\widehat{A}}.$$

We call $\{\mathcal{L}_{s,\chi}\}_s$ the *fast transfer operator family* for Γ associated to $\mathcal{S}$.

We are now ready to formulate the main result of [22]. It shows that (fast) transfer operator families arising from strict transfer operator approaches provide Fredholm determinant representations of Selberg zeta functions.

Theorem 3.1 ([22, Theorem 4.2]). *Let Γ be a geometrically finite Fuchsian group which admits a strict transfer operator approach, and let $\chi\colon \Gamma \to \mathrm{GL}(V)$ be a finite-dimensional representation of Γ on the finite-dimensional vector space V having non-expanding cusp monodromy. Let $\mathcal{S}$ be a structure tuple for Γ with associated* *fast transfer operator family* $\{\mathcal{L}_{s,\chi}\}_s$. *Then we have:*

(i) *There exists* $\delta > 0$*, only depending on* Γ *and* (V, χ)*, such that for* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > \delta$ *the operator* $\mathcal{L}_{s,\chi}$ *on* $\mathcal{B}(\mathcal{E}_{\widehat{A}}; V)$ *is bounded and nuclear of order zero, independently of the choice of the family* $\mathcal{E}_{\widehat{A}}$*.*

(ii) *The map* $s \mapsto \mathcal{L}_{s,\chi}$ *extends meromorphically to all of* $\mathbb{C}$ *with values in nuclear operators of order zero on* $\mathcal{B}(\mathcal{E}_{\widehat{A}}; V)$*. All poles are simple. There exists* $d \in \mathbb{N}$ *such that each pole is contained in* $\frac{1}{2}(d - \mathbb{N}_0)$*.*

(iii) *For* $\operatorname{Re} s \gg 1$*, we have*

$$Z_{\mathbb{X},\chi}(s) = \det(1 - \mathcal{L}_{s,\chi}) \,.$$

(iv) *The Selberg zeta function* $Z_{\mathbb{X},\chi}$ *extends to a meromorphic function on* $\mathbb{C}$ *with poles contained in* $\frac{1}{2}(d - \mathbb{N}_0)$ *and the identity in* (iii) *extends to all of* $\mathbb{C}$*.*

We reduced the statement of Theorem 3.1 to match our needs here. Theorem 4.2 in [22] contains further information about the rank of the operators $\mathcal{L}_{s,\chi}$ as well as the order of the poles of $Z_{\mathbb{X},\chi}$. In particular, explicit values for δ and d are given. However, all of these additional informations are not needed for our purposes here.

Chapter 4

Sets of Branches

Throughout this chapter let Γ be a geometrically finite, non-cocompact Fuchsian group containing hyperbolic elements, and denote by $\mathbb{X}$ the associated hyperbolic orbisurface with geodesic flow $\widehat{\Phi}$. In this chapter we present the starting point of our constructions, the so-called *sets of branches*, and prove various crucial properties of these seminal objects.

These sets of branches will be seen to give rise to a cross section for $\widehat{\Phi}$ as presented in Section 1.11 (see Definition 1.47 in particular). We will define any cross section by choosing a set of representatives for it, i. e., a subset of the unit tangent bundle $\mathrm{S}\mathbb{H}$ that is bijective to the cross section. More precisely, we may and will consider the set of representatives as the primary object and the cross section as a consequential object that inherits all its properties from the set of representatives. The starting point of our constructions are well-structured sets of representatives—the aforementioned sets of branches—which we introduce in Section 4.1 and whose first essential properties we discuss in Sections 4.2–4.7.

In a nutshell, the notion of sets of branches constitutes an equilibrium between our wish to keep the framework as general as possible and the requirements of a descent algorithm of cuspidal acceleration and a nicely structured passage from slow to fast transfer operators. In Chapter 7 we show that cross sections yielded by the cusp expansion algorithm do indeed come from sets of branches. More precisely, the set $\mathcal{C}_{\mathrm{P}}$ as given in (2.8) and (2.10) *is* a set of branches. Therefore, all sets of representatives for cross sections in [58, 15, 54] decompose, in a straightforward way, into sets of branches. Moreover, also many more choices of sets of branches with much different properties are possible. A first indication of this is provided in Example 4.21. On the other hand, the notion of a set of branches is sufficiently rigid to give rise to well-structured families of slow transfer operators, as we show in Section 4.7. Further, it allows for an acceleration/induction algorithm that will enable us to set up a strict transfer operator approach as defined in Section 3.1. Given Theorem B (see Theorem 6.1 below), this reduces the proof of Theorem A to the (purely geometric) task of constructing sets of branches.

4.1 Definition and First Observations

In order to elaborate further on the intended setup of cross sections, let C be a set of representatives for a cross section $\widehat{\mathrm{C}}$ in the sense of Definition 1.47. Then C completely determines $\widehat{\mathrm{C}}$. For that reason, we may turn around the order of definitions. That means, for defining a cross section, we may start by picking a subset C of $\mathrm{S}\mathbb{H}$ such that the quotient map $\pi|_{\mathrm{C}} \colon \mathrm{C} \to \pi(\mathrm{C})$ is bijective and the image set $\pi|_{\mathrm{C}}(\mathrm{C})$ is a cross section. Then all properties of $\widehat{\mathrm{C}} = \pi(\mathrm{C})$ are controlled by the properties of C, and specific requirements on a set of representatives can sometimes be guaranteed by a suitable choice of C.

The concept of sets of branches, which we will introduce in this section, implements this idea. A set of branches is a family of subsets of $\mathrm{S}\mathbb{H}$ that serves as a set of representatives with a decomposition as in (1.89), namely the elements of this family, and which induces a nicely structured discrete dynamical system as in (1.91). This concept takes advantage of points in $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$, whose existence is equivalent to the non-cocompactness of Γ by Lemma 1.17. This also explains why we restrict our considerations to non-cocompact Fuchsian groups.

Recall the projection onto base points, $\mathrm{bp} \colon \mathrm{S}\mathbb{H} \to \mathbb{H}$, from (1.19), as well as the definition of intersections between subsets of $\mathrm{S}\mathbb{H}$ (or $\mathrm{S}\mathbb{X}$) and (equivalence classes of) geodesics on $\mathbb{H}$ (or $\mathbb{X}$) from (1.25) (or (1.84)). Finally, denote by Γ^* the subset of all non-identity elements of Γ as in (1.1).

Definition 4.1. Let $N \in \mathbb{N}$ and let $\mathrm{C}_1, \ldots, \mathrm{C}_N$ be subsets of $\mathrm{S}\mathbb{H}$. Set $A := \{1, \ldots, N\}$,

$$\mathcal{C} := \{\mathrm{C}_j \mid j \in A\} \qquad \text{and} \qquad \mathrm{C} := \bigcup \mathcal{C}\,.$$

We call $\mathcal{C}$ a *set of branches for the geodesic flow on* $\mathbb{X}$ if it satisfies the following properties:

(B1) For each $j \in A$ there exists $\nu \in \mathrm{C}_j$ such that $\widehat{\gamma}_\nu$ is a periodic geodesic on $\mathbb{X}$.

(B2) For each $j \in A$, the set $\overline{\mathrm{bp}(\mathrm{C}_j)}$ is a complete geodesic segment in $\mathbb{H}$ and its endpoints are in $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$. In particular, for each $j \in A$, the set $\mathbb{H} \setminus \overline{\mathrm{bp}(\mathrm{C}_j)}$ decomposes uniquely into two (geodesically) convex open half-spaces.

(B3) For each $j \in A$, all elements of C_j point into the same open half-space relative to $\overline{\mathrm{bp}(\mathrm{C}_j)}$. We denote this half-space by $\mathrm{H}_+(j)$ and set

$$\mathrm{H}_-(j) := \mathbb{H} \setminus \left(\overline{\mathrm{bp}(\mathrm{C}_j)} \cup \mathrm{H}_+(j)\right).$$

Further, we denote by I_j the largest open subset of $\widehat{\mathbb{R}}$ that is contained in $\partial_g \mathrm{H}_+(j)$, and by J_j the largest open subset of $\widehat{\mathbb{R}}$ contained in $\partial_g \mathrm{H}_-(j)$.

(B4) The Γ-orbit of $\{I_j \mid j \in A\}$ covers the set $\widehat{\mathbb{R}}_{\mathrm{st}}$, i. e.,

$$\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \bigcup_{j\in A} \bigcup_{g\in\Gamma} g.I_j \,.$$

(B5) For each $j \in A$ and each pair $(x,y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$ there exists a (unique) vector $\nu \in \mathrm{C}_j$ such that

$$(x,y) = \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big)\,.$$

(B6) If $\overline{\mathrm{bp}(\mathrm{C}_j)} \cap g.\overline{\mathrm{bp}(\mathrm{C}_k)} \neq \varnothing$ for some $j,k \in A$ and $g \in \Gamma$, then either $j = k$ and $g = \mathrm{id}$, or $\mathrm{H}_\pm(j) = g.\mathrm{H}_\mp(k)$.

(B7) For each pair $(a,b) \in A \times A$ there exists a (possibly empty) subset $\mathcal{G}(a,b)$ of Γ such that

(a) for all $j \in A$ we have

$$\bigcup_{k\in A} \bigcup_{g\in\mathcal{G}(j,k)} g.I_k \subseteq I_j$$

and

$$\bigcup_{k\in A} \bigcup_{g\in\mathcal{G}(j,k)} g.I_{k,\mathrm{st}} = I_{j,\mathrm{st}}\,,$$

and these unions are disjoint,

(b) for each pair $(j,k) \in A \times A$, each $g \in \mathcal{G}(j,k)$ and each pair of points $(z,w) \in \mathrm{bp}(\mathrm{C}_j) \times g.\,\mathrm{bp}(\mathrm{C}_k)$, the geodesic segment $(z,w)_{\mathbb{H}}$ is nonempty, is contained in $\mathrm{H}_+(j)$ and does not intersect $\Gamma.\mathrm{C}$,

(c) for all $j \in A$ we have

$$J_{j,\mathrm{st}} \subseteq \bigcup_{k\in A} \bigcup_{h\in\mathcal{G}(k,j)} h^{-1}.J_{k,\mathrm{st}}\,.$$

We call the sets C_j, $j \in A$, the *branches* of $\mathcal{C}$, and C the *branch union*. Further, we call the sets $\mathcal{G}(j,k)$, $j,k \in A$, from (B7) the *(forward) transition sets* of $\mathcal{C}$, with $\mathcal{G}(j,k)$ being the *(forward) transition set from* C_j *to* C_k.

A set of branches is called *admissible* if it satisfies the following property:

(B8) There exist a point $q \in \widehat{\mathbb{R}}$ and an open neighborhood $\mathcal{U}$ of q in $\widehat{\mathbb{R}}$ such that

$$\mathcal{U} \cap \bigcup_{j\in A} I_{j,\mathrm{st}} = \varnothing \qquad \text{and} \qquad q \notin I_j\,,$$

for every $j \in A$.

A set of branches $\mathcal{C}$ is called *non-collapsing* if it satisfies

(B9) For all $n \in \mathbb{N}$, every choice of $j_1, \ldots, j_{n+1} \in A$ such that $\mathcal{G}(j_i, j_{i+1}) \neq \varnothing$ for all $i \in \{1, \ldots, n\}$, and every choice of elements $g_i \in \mathcal{G}(j_i, j_{i+1})$ for $i \in \{1, \ldots, n\}$, we have
$$g_1 \cdots g_n \neq \mathrm{id}\ .$$

If $\mathcal{C}$ does not satisfy (B9), then it is called *collapsing*.

Remark 4.2. We comment on some properties of a set of branches that will be used throughout and that are immediate by its definition. We resume the notation from Definition 4.1.

(a) A close relationship between the set of branches $\mathcal{C}$ and periodic geodesics on $\mathbb{X}$ is guaranteed by (B1) and, in fact, (B4). The property (B1) assures that every branch contributes in a meaningful way to the complete collection of branches by detecting at least one periodic geodesic on $\mathbb{X}$ or, more precisely, a lift to $\mathbb{H}$ of a periodic geodesic on $\mathbb{X}$. In particular, it implies that each branch is a nonempty set. Therefore, for orbifolds without periodic geodesics (e. g., a parabolic cylinder, see [10]) a set of branches in the sense of Definition 4.1 does not exist. On the contrary, property (B4) has the consequence that every periodic geodesic on $\mathbb{X}$ is detected by $\mathcal{C}$. See Proposition 4.8 below.

(b) The emphasis on periodic geodesics is due to our applications. For a strict transfer operator approach and hence a representation of the Selberg zeta function as a Fredholm determinant of a transfer operator family, we need to provide a certain symbolic presentation of each periodic geodesic on $\mathbb{X}$ by means of iterated intersections with a cross section. For more details we refer to the brief discussion in Section 1.11 as well as to [44, 61, 22].

(c) Properties (B2)–(B5) determine the structure of branches. Each branch partitions the hyperbolic plane into two geodesically convex half-spaces and a complete geodesic segment. The requirement that, for each $j \in A$, the endpoints of $\overline{\mathrm{bp}(\mathrm{C}_j)}$ are in $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$ implies that geodesics γ on $\mathbb{H}$ with $\gamma(\mathbb{R}) = \overline{\mathrm{bp}(\mathrm{C}_j)}$ do not represent any periodic geodesic on $\mathbb{X}$. This condition further implies that $I_{j,\mathrm{st}} \cap J_{j,\mathrm{st}} = \varnothing$.

(d) For each $j \in A$, the requirements of (B3)–(B5) yield that the union
$$I_j \cup J_j \cup_{g} \mathrm{bp}(\mathrm{C}_j)$$
is disjoint and equals $\widehat{\mathbb{R}}$. The set $\overline{\mathrm{bp}(\mathrm{C}_j)}$ is the complete geodesic segment in $\mathbb{H}$ that connects the two endpoints of I_j or, equivalently, of J_j. The

boundary of the half-spaces $\mathrm{H}_+(j)$ and $\mathrm{H}_-(j)$ in $\mathbb{H} \cup \partial_g \mathbb{H}$ is

$$\partial_g \mathrm{H}_+(j) = \overline{\mathrm{bp}(\mathrm{C}_j)} \cup \overline{I_j}^g \quad \text{and} \quad \partial_g \mathrm{H}_-(j) = \overline{\mathrm{bp}(\mathrm{C}_j)} \cup \overline{J_j}^g \,,$$

respectively. It will be useful to fix the following notation for the endpoints of $\overline{\mathrm{bp}(\mathrm{C}_j)}$: let $\mathrm{X}_j, \mathrm{Y}_j$ be the elements in $\widehat{\mathbb{R}}$ such that

$$\{\mathrm{X}_j, \mathrm{Y}_j\} = \partial_g \overline{\mathrm{bp}(\mathrm{C}_j)}$$

and that, when traveling along the geodesic segment $\overline{\mathrm{bp}(\mathrm{C}_j)}$ from X_j to Y_j, the half-space $\mathrm{H}_+(j)$ lies to the right of the path of travel. See Figure 6.

(e) Property (B5) further has the following consequence for all $j \in A$: Let $(x, y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$ and let γ be a geodesic on $\mathbb{H}$ from x to y. The unique vector $\nu \in \mathrm{C}_j$ with $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) = (x, y)$ is then

$$\nu = \gamma'(t)\,,$$

where $t \in \mathbb{R}$ is the unique time such that $\gamma(t) = \mathrm{bp}(\nu) \in \mathrm{bp}(\mathrm{C}_j)$. We emphasize that (B5) does not prevent the branches from containing vectors ν such that $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) \notin \mathbb{R}_{\mathrm{st}} \times \mathbb{R}_{\mathrm{st}}$.

(f) Properties (B6) and (B7) describe the mutual interplay of the branches. Property (B6) implies that a set of branches $\{\mathrm{C}_1, \ldots, \mathrm{C}_n\}$ is pairwise disjoint, which will be crucial for the well-definedness of the intersection sequences in Section 4.3 below. A stronger statement is shown in Proposition 4.9(i). Property (B7) uses the close relation between each branch C_j, $j \in A$, and its associated sets I_j and J_j in $\widehat{\mathbb{R}}$ in order to provide the tools necessary to track the behavior of geodesics which intersect C_j in future and past time directions. As we will see in Sections 4.3 and 4.7, the rather precise tracking makes it possible to deduce an explicit discrete model of the geodesic flow, or in other words, of the arising symbolic dynamics or intersection sequences.

(g) In the situation of (B6) we always have $\overline{\mathrm{bp}(\mathrm{C}_j)} = g.\overline{\mathrm{bp}(\mathrm{C}_k)}$. However, it does not necessarily follow that $\mathrm{bp}(\mathrm{C}_j) = g.\,\mathrm{bp}(\mathrm{C}_k)$. This subtle difference makes it necessary to be rather careful with choices and argumentation at some places.

(h) We emphasize that the uniqueness or non-uniqueness of the forward transition sets is not part of the requirements in (B7). For the moment and in particular in isolated consideration of (B7) it may well be that different choices for the families of forward transition sets $(\mathcal{G}(j,k))_{j,k\in A}$ can be made. However, in Proposition 4.15 we will see that the interplay of all properties of a set of branches enforces uniqueness of these sets.

We further emphasize that the transition sets $\mathcal{G}(j,k)$, $j,k \in A$, are allowed to be infinite. In Example 4.21 we show that there are sets of branches with finite as well as with infinite transition sets. In Section 4.4 we provide a characterization of sets of branches with infinite transition sets.

(i) Property (B8) allows us to suppose that for all $j \in A$, the intervals I_j are contained in $\mathbb{R}$. For this we possibly need to conjugate Γ by some element $g \in \mathrm{PSL}_2(\mathbb{R})$, translate $\mathcal{C}$ by g and consider a set of branches for $g\Gamma g^{-1}$. In other words, (B8) allows us to suppose without loss of generality that the discrete dynamical system induced by $\mathcal{C}$ is completely defined within $\mathbb{R}$ and any handling of a second manifold chart to investigate neighborhoods of ∞ can be avoided. This often simplifies the discussion, in particular in Section 6.1.

It is immediately clear that a sufficient (but not necessary) condition for (B8) is that $\widehat{\mathbb{R}} \setminus \bigcup_{k=1}^{N} I_k$ contains an open interval. Let $j \in A$. Property (B1) is equivalent to the existence of an equivalence class of geodesics $[\gamma] \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ such that

$$\big(\gamma(+\infty), \gamma(-\infty)\big) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}} \subseteq I_j \times J_j\,,$$

for every representative γ of $[\gamma]$. The class $[\gamma]$ is then the axis of some hyperbolic transformation $h \in \Gamma$, which, because of Lemma 1.11, contracts the interval I_j towards $\mathrm{f}_+(h) = \gamma(+\infty)$. For every $j \in A$ a hyperbolic transformation $h_j \in \Gamma$ can be found in this way, and the contracting behavior assures that we find $i_1, \ldots, i_N \in \mathbb{N}$ such that

$$\widehat{\mathbb{R}} \setminus \bigcup_{k=1}^{N} h_k^{i_k} \boldsymbol{.} I_k$$

contains an open interval. Lemma 4.10 below implies that

$$\{h_k^{i_1} \boldsymbol{.}\, \mathrm{C}_1, \ldots, h_N^{i_N} \boldsymbol{.}\, \mathrm{C}_N\}$$

is again a set of branches for the geodesic flow on $\mathbb{X}$. For orbisurfaces with cusps it is often possible and appropriate to work with parabolic instead of hyperbolic transformations. We refer to Proposition 4.35 below for a rigorous treatment of this aspect.

(j) Indispensable for our approach is a *unique* coding of periodic geodesics in terms of the chosen generators of the Fuchsian group. This requires in particular that the identity transformation will not be encountered during the tracking of geodesics. This property is formulated as (B9). Even though (B9) will eventually be fundamental, this property need not be guaranteed immediately during the construction of a set of branches. Indeed, as we will see, every set that fulfills (B1)–(B7) can be transformed into one that fulfills (B9) (at the tolerable cost of weakening others, most profoundly (B5)). This is done by means of a reduction procedure, which we call *identity elimination*. It is discussed in Section 5.2 below.

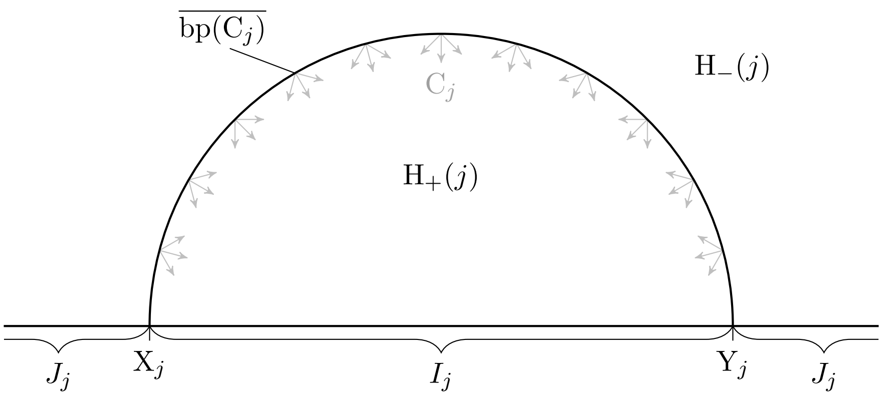


Figure 6: The relationship between the sets $\overline{\mathrm{bp}(\mathrm{C}_j)}$, $\mathrm{H}_+(j)$, $\mathrm{H}_-(j)$, I_j, J_j and the points X_j and Y_j for a branch C_j, $j \in A$.

Examples for sets of branches can be found in [58, 61, 54, 63, 79]. Indeed, all cross sections constructed there arise from sets of branches. We end this section with two examples of sets of branches, one of them for Schottky surfaces.

Example 4.3. Let Γ_S be a Schottky group, that is, a geometrically finite, non-cofinite Fuchsian group consisting solely of hyperbolic elements and the identity. By [16], we may associate to Γ_S a choice of *Schottky data*, that is a tuple

$$\left(r, \{\mathcal{D}_j, \mathcal{D}_{-j}\}_{j=1}^r, \{s_j, s_{-j}\}_{j=1}^r\right),$$

where $r \in \mathbb{N}$, $\{s_1, \ldots, s_r\} \subseteq \mathrm{PSL}_2(\mathbb{R})$ is a set of generators of Γ_S, $s_{-j} := s_j^{-1}$ for $j \in \{1, \ldots, r\}$, and $\mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{D}_{-1}, \ldots, \mathcal{D}_{-r}$ are mutually disjoint open Euclidean disks in $\mathbb{C}$ centered on $\mathbb{R}$ such that for each $j = 1, \ldots, r$, the element s_j maps the exterior of $\mathcal{D}_j$ to the interior of $\mathcal{D}_{-j}$, and such that

$$\mathbb{H} \setminus \bigcup_{j=1}^{r} \left(\overline{\mathcal{D}_j} \cup \overline{\mathcal{D}_{-j}}\right)$$

is a fundamental domain for Γ_S. For $j \in \{\pm 1, \ldots, \pm r\}$ we let C_j be the set of unit tangent vectors $\nu \in \mathrm{S}\mathbb{H}$ that are based on the boundary $\partial\mathcal{D}_j$ of $\mathcal{D}_j$ and that point into $\mathcal{D}_j$ (thus, $\gamma_\nu(+\infty) \in \mathrm{Re}\,(\mathcal{D}_j)$). Then

$$\{\mathrm{C}_1, \ldots, \mathrm{C}_r, \mathrm{C}_{-1}, \ldots, \mathrm{C}_{-r}\}$$

is a set of branches for the geodesic flow on the hyperbolic surface $\Gamma_{\mathrm{S}}\backslash\mathbb{H}$.

Example 4.4. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ from Example 1.46 and the set

$$\mathcal{C}_{\mathrm{P}} := \{\mathrm{C}_{\mathrm{P},1}, \ldots, \mathrm{C}_{\mathrm{P},8}\}$$

from Example 2.11. One shows that $\mathcal{C}_{\mathrm{P}}$ is a set of branches for the geodesic flow on $\mathbb{X}_{\sigma,\lambda}$, the orbit space of $\Gamma_{\sigma,\lambda}$, for every choice of σ and λ. We omit the proof here, for later on in Chapter 7 we will show that, in fact, each set $\mathcal{C}_{\mathrm{P}}$ constructed by the cusp expansion algorithm as described in Section 2.1 is a set of branches.

4.2 Elementary Properties of Sets of Branches

Throughout this section let

$$\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$$

be a set of branches for the geodesic flow on $\mathbb{X}$, set $A := \{1, \ldots, N\}$, and let $\mathrm{C} = \bigcup \mathcal{C}$ denote the branch union of $\mathcal{C}$. In the course of this section and the following two sections we will show that

$$\widehat{\mathrm{C}} := \pi(\mathrm{C})$$

is a cross section with respect to any measure in a certain class and also in the sense of (CS1$'$) and (CS2). See Proposition 4.36. We will further find a subset of $\widehat{\mathrm{C}}$ that is a strong cross section (Corollary 4.37).

For any $j \in A$, we resume the notation for the sets I_j, J_j, $\mathrm{H}_+(j)$ and $\mathrm{H}_-(j)$ from (B3). We fix a family of forward transition sets $\{\mathcal{G}(j,k)\}_{j,k\in A}$ as given by (B7). Further, for $j, k \in A$ we set

$$\mathcal{V}(k,j) := \mathcal{G}(k,j)^{-1} = \left\{g^{-1} \;\middle|\; g \in \mathcal{G}(k,j)\right\}, \tag{4.1}$$

which we call the *backward transition set from* C_k *to* C_j.

Proposition 4.5. *The backward transition sets satisfy the properties dual to* (B7). *That is,* (B7) *holds also for* $\{\mathcal{V}(k,j)\}_{j,k\in A}$ *in place of* $\{\mathcal{G}(j,k)\}_{j,k\in A}$ *and the roles of* $\{I_j\}_{j\in A}$ *and* $\{J_j\}_{j\in A}$ *as well as* $\{\mathrm{H}_+(j)\}_{j\in A}$ *and* $\{\mathrm{H}_-(j)\}_{j\in A}$ *interchanged. More precisely:*

(i) *For all* $j \in A$ *we have*

$$\bigcup_{k\in A}\bigcup_{g\in\mathcal{V}(k,j)} g.J_k \subseteq J_j$$

and

$$\bigcup_{k\in A}\bigcup_{g\in\mathcal{V}(k,j)} g.J_{k,\mathrm{st}} = J_{j,\mathrm{st}}\,,$$

and these unions are disjoint,

(ii) *for each pair* $(j,k) \in A\times A$*, each* $g\in\mathcal{V}(k,j)$ *and each pair of points* $(v,z)\in g.\,\mathrm{bp}(\mathrm{C}_k)\times\mathrm{bp}(\mathrm{C}_j)$*, the geodesic segment* $(v,z)_{\mathbb{H}}$ *is nonempty, contained in* $\mathrm{H}_-(j)$*, and does not intersect* $\Gamma.\mathrm{C}$*.*

Proof. We first establish (i). Let $j\in A$. For any $k\in A$ and $g\in\mathcal{V}(k,j) = \mathcal{G}(k,j)^{-1}$ we have

$$g^{-1}.I_j\subseteq I_k$$

by (B7a). Therefore $g^{-1}.J_j\supseteq J_k$ and hence

$$g.J_k\subseteq J_j\,.$$

It follows that

$$\bigcup_{k\in A}\bigcup_{g\in\mathcal{V}(k,j)} g.J_k\subseteq J_j \tag{4.2}$$

and further

$$\bigcup_{k\in A}\bigcup_{g\in\mathcal{V}(k,j)} g.J_{k,\mathrm{st}}\subseteq J_{j,\mathrm{st}}\,.$$

Combining the latter with (B7c) shows the claimed equality of sets. It remains to show that the unions in (4.2) are disjoint. To that end let $k_1,k_2\in A$ and $g_1\in\mathcal{V}(k_1,j)$, $g_2\in\mathcal{V}(k_2,j)$ such that

$$g_1.J_{k_1}\cap g_2.J_{k_2}\neq\varnothing.$$

If we assume that

$$g_1.J_{k_1}\neq g_2.J_{k_2}\,,$$

then

$$g_2.I_{k_2}\cap g_1.J_{k_1}\neq\varnothing\qquad\text{and}\qquad g_2.I_{k_2}\cap g_1.I_{k_1}\neq\varnothing\,,$$

from which we obtain

$$\begin{aligned} g_2.\mathrm{H}_+(k_2)\cap g_1.\mathrm{H}_-(k_1)&\neq\varnothing\,,\\ g_2.\mathrm{H}_+(k_2)\cap g_1.\mathrm{H}_+(k_1)&\neq\varnothing\,,\end{aligned}$$

and

$$g_2.\overline{\mathrm{bp}(\mathrm{C}_{k_2})}\cap g_1.\overline{\mathrm{bp}(\mathrm{C}_{k_1})}\neq\varnothing\,.$$

Property (B6) implies that this constellation is impossible. In turn,

$$g_1.J_{k_1} = g_2.J_{k_2}\,.$$

It follows that $g_1.I_{k_1} = g_2.I_{k_2}$ and further

$$g_1.\overline{\mathrm{bp}(\mathrm{C}_{k_1})} = g_2.\overline{\mathrm{bp}(\mathrm{C}_{k_2})}\,,$$

as well as

$$g_1.\mathrm{H}_+(k_1) = g_2.\mathrm{H}_+(k_2)\,.$$

Thus, $k_1 = k_2$ and $g_1 = g_2$ by (B6), which shows that the unions in (4.2) are disjoint.

We now show (ii). To that end let $j,k \in A$, $g \in \mathcal{V}(k,j)$ and $(v,z) \in g.\,\mathrm{bp}(\mathrm{C}_k) \times \mathrm{bp}(\mathrm{C}_j)$. Then, since

$$g^{-1}.(v,z) = (g^{-1}.v, g^{-1}.z) \in \mathrm{bp}(\mathrm{C}_k) \times g^{-1}.\,\mathrm{bp}(\mathrm{C}_j)$$

and $g^{-1} \in \mathcal{G}(k,j)$, (B7b) shows that $g^{-1}.(v,z)_{\mathbb{H}}$ (and hence $(v,z)_{\mathbb{H}}$) is nonempty and does not intersect $\Gamma.\mathrm{C}$. By (B7a), $g^{-1}.I_j \subseteq I_k$ and therefore $\mathrm{H}_+(j) \subseteq g.\mathrm{H}_+(k)$. Combination with (B7b) yields

$$(v,z)_{\mathbb{H}} \subseteq g.\mathrm{H}_+(k) \setminus \mathrm{H}_+(j) \subseteq \mathrm{H}_-(j)\,.$$

This completes the proof. □

We recall from Section 1.9 that a family $\mathfrak{B}$ of subsets of $\mathbb{H}$ is called *locally finite in* $\mathbb{H}$ if for each $z \in \mathbb{H}$ there exists an open neighborhood $\mathcal{U}$ of z in $\mathbb{H}$ such that at most finitely many members of $\mathfrak{B}$ intersect $\mathcal{U}$.

Proposition 4.6. *The family*

$$\mathfrak{B} := \left\{ g.\overline{\mathrm{bp}(\mathrm{C}_j)} \;\middle|\; g \in \Gamma,\ j \in A \right\}$$

of Γ-translates of the closures of the base sets of the branches in $\mathcal{C}$ is locally finite in $\mathbb{H}$.

Proof. Let $\sigma \subseteq \mathbb{H}$ be a geodesic arc with both its endpoints in $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$. In what follows we show that the family of Γ-translates of σ is locally finite in $\mathbb{H}$. Since the family of the closures of the base sets of the branches $\mathcal{C}$ consists of finitely many of such geodesic arcs, the statement of the proposition follows then immediately.

Let $\mathcal{F}$ be a Ford fundemental domain for Γ in $\mathbb{H}$ as defined in Section 1.10 (see (1.70)). By Proposition 1.43(ii), each cusp $\widehat{c}$ of $\mathbb{X}$ is represented by a point, say c, in $g\mathcal{F}$. By construction, the part of $\mathcal{F}$ near c is of the form

$$g.\left\{ z \in \mathbb{H} \mid |\operatorname{Re} z| < w,\ \operatorname{Im} z > h \right\}\,,$$

for some $g = g(c) \in \mathrm{PSL}_2(\mathbb{R})$ fulfilling $g^{-1}.c = \infty$, $w = w(c) > 0$ and $h = h(c) > 0$. The neighboring translates of $\mathcal{F}$ at c are given by $p.\mathcal{F}$ and $p^{-1}.\mathcal{F}$, where $p \in \Gamma$ is a generator of $\mathrm{Stab}_\Gamma(c)$. By Proposition 1.43(iii), each funnel of $\mathbb{X}$ is represented by a funnel representative, say I, in $\mathcal{F}$, and the part of $\mathcal{F}$ near I is bounded by two geodesic segments, each one having one of the two boundary points of I as an endpoint. The neighboring translates of $\mathcal{F}$ at this part are of the form $b.\mathcal{F}$ and $b^{-1}.\mathcal{F}$, where b is a primitive hyperbolic element of Γ whose axis represents a funnel bounding geodesic (see [24] for cofinite Fuchsian groups and [6, Section 10] for general, geometrically finite Fuchsian groups).

Since each endpoint of σ is either cuspidal or contained in a funnel representative, the shape of $\mathcal{F}$ implies that we find $g_1, g_2 \in \Gamma$ such that $g_1.\mathcal{F} \cup g_2.\mathcal{F}$ covers "most" of σ. In other words,

$$\beta \coloneqq \sigma \setminus (g_1.\mathcal{F} \cup g_2.\mathcal{F})$$

is a geodesic segment of finite hyperbolic length. Again, the shape of $\mathcal{F}$ implies that β can be covered by finitely many Γ-translates of $\overline{\mathcal{F}}$. In total, the geodesic arc σ intersects only finitely many Γ-translates of $\overline{\mathcal{F}}$. Equivalently, $\overline{\mathcal{F}}$ contains only finitely many Γ-translates of σ. Since σ, and each of its Γ-translates, is a geodesic segment, it immediately follows that the family $\Gamma.\sigma$ is locally finite in $\overline{\mathcal{F}}$, and hence in all of $\mathbb{H}$ by the tessellation property. This completes the proof. □

We now aim to prove that the set of branches $\mathcal{C}$ accounts for all periodic geodesics on $\mathbb{X}$, in the sense that every $\widehat{\gamma} \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ has a representative $\gamma \in \mathscr{G}(\mathbb{H})$ that intersects C. For this we take advantage of the following equivalent formulation of (B4). Recall the subset $\mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ of geodesics on $\mathbb{H}$ which, through Γ, are lifts of periodic geodesics on $\mathbb{X}$ (see (1.34)).

Lemma 4.7. *Property (B4) is equivalent to the following statement:*

(B4*) *For all $\gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ we have $\gamma(+\infty) \in \Gamma.\bigcup_{j\in A} I_j$.*

Proof. Recall from (1.41) the set

$$E(\mathbb{X}) = \{(\gamma(+\infty), \gamma(-\infty)) \mid \gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})\}\ .$$

The obvious inclusion relation $E(\mathbb{X}) \subseteq \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}$ immediately shows that (B4) implies (B4*). Since the set $E(\mathbb{X})$ is dense in $\Lambda(\Gamma) \times \Lambda(\Gamma)$ by Proposition 1.15 and $\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \Lambda(\Gamma)$, also $\widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}$ is dense in $\Lambda(\Gamma) \times \Lambda(\Gamma)$. Thus, the openness of the sets I_j, $j \in A$, yields that (B4*) implies (B4). □

Proposition 4.8. *Under the assumption of (B5) and (B7), property (B4) is equivalent to the following statement:*

($\mathrm{B_{Per}}$) *For all $\widehat{\gamma} \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ there exists $\gamma \in \mathscr{G}(\mathbb{H})$ such that $\pi(\gamma) = \widehat{\gamma}$ and γ intersects C.*

Proof. Let $\gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ and set $\widehat{\gamma} := \pi(\gamma) \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$. By combining Proposition 1.13 and Lemma 1.11(ii) one sees that the set of all representatives of $\widehat{\gamma}$ is given by $\Gamma.\gamma$. Hence, if one representative of $\widehat{\gamma}$ intersects C, then all its representatives intersect Γ.C. Hence, given ($\mathrm{B}_{\mathrm{Per}}$), there exists $(j,g) \in A \times \Gamma$ such that γ intersects $g.\,\mathrm{C}_j$, i. e.,

$$\big(\gamma(+\infty), \gamma(-\infty)\big) \in g.I_{j,\mathrm{st}} \times g.J_{j,\mathrm{st}} \subseteq g.I_j \times g.J_j \,.$$

Thus, ($\mathrm{B}_{\mathrm{Per}}$) implies (B4*). For the converse implication we suppose that (B4*) is satisfies and let $\widehat{\gamma} \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$. As discussed above, it suffices to find a representative of $\widehat{\gamma}$ that intersects Γ.C. Let $\gamma \in \mathscr{G}(\mathbb{H})$ be any representative of $\widehat{\gamma}$. Then $\gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$. Further, (B4*) yields the existence of a pair $(k_1, g_1) \in A \times \Gamma$ such that $\gamma(+\infty) \in g_1.I_{k_1}$. By (B5), for a geodesic $\eta \in \mathscr{G}(\mathbb{H})$ to intersect $g.\,\mathrm{C}_k$, $(k,g) \in A \times \Gamma$, it suffices to have

$$\big(\eta(+\infty), \eta(-\infty)\big) \in g.I_{k,\mathrm{st}} \times g.J_{k,\mathrm{st}} \,.$$

Since $E(\mathbb{X}) \subseteq \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}$ we immediately have $(\gamma(+\infty), \gamma(-\infty)) \in \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}$. Therefore, if $\gamma(-\infty) \in g_1.J_{k_1}$, the statement of ($\mathrm{B}_{\mathrm{Per}}$) follows. In order to seek a contradiction, we assume that this is not the case. Since $\mathrm{X}_j, \mathrm{Y}_j \in \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$, it follows that $\gamma(-\infty) \in g_1.I_{k_1}$. By (B7a) we find $(k_2, g_2) \in (A \times \Gamma) \setminus \{(k_1, g_1)\}$ such that

$$g_1^{-1} g_2 \in \mathcal{G}(k_1, k_2) \qquad \text{and} \qquad \gamma(+\infty) \in g_2.I_{k_2} \,.$$

Now the same argumentation as before applies: If $\gamma(-\infty) \in g_2.J_{k_2}$, then ($\mathrm{B}_{\mathrm{Per}}$) follows. If this is not the case, then necessarily $\gamma(-\infty) \in g_2.I_{k_2}$ and we apply (B7a) to find $(k_3, g_3) \in (A \times \Gamma) \setminus \{(k_1, g_1), (k_2, g_2)\}$ such that

$$g_1^{-1} g_2^{-1} g_3 \in \mathcal{G}(k_2, k_3) \qquad \text{and} \qquad \gamma(+\infty) \in g_3.I_{k_3} \,.$$

We now show that iteration of this procedure must terminate after finitely many steps by finding $\gamma(-\infty) \in g_i.J_{k_i}$ for some $i \in \mathbb{N}$. Assume for contradiction that this is not the case. Thus, the above procedure yields a sequence $((k_n, g_n))_{n \in \mathbb{N}}$ in $A \times \Gamma$ such that

$$\{\gamma(+\infty), \gamma(-\infty)\} \subseteq \bigcap_{n \in \mathbb{N}} g_n.I_{k_n} \,.$$

Then Proposition 1.13 provides a hyperbolic element $h \in \Gamma$ such that

$$\gamma(+\infty) = \mathrm{f}_+(h) \neq \mathrm{f}_-(h) = \gamma(-\infty) \,,$$

meaning the geodesic arc $\gamma(\mathbb{R})$ is non-degenerate. Further, from the construction it is clear that

$$g_{i+1}.I_{k_{i+1}} \subseteq g_i.I_{k_i}$$

for all $i \in \mathbb{N}$. Hence, the sequence $((g_n.\mathrm{X}_{k_n}, g_n.\mathrm{Y}_{k_n}))_{n\in\mathbb{N}}$ converges to some pair $(x,y) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ with $x, y \in \overline{(g_1.I_{k_1})} \setminus \mathrm{Re}\,(\gamma(\mathbb{R}))$. But this entails the convergence of the sequence $(g_n.\overline{\mathrm{bp}(\mathrm{C}_{k_n})})_{n\in\mathbb{N}}$ to the geodesic arc $(x,y)_{\mathbb{H}}$. Any neighborhood of any point $z \in (x,y)_{\mathbb{H}}$ therefore intersects infinitely many members of the family $\{g_n.\overline{\mathrm{bp}(\mathrm{C}_{k_n})}\}_{n\in\mathbb{N}}$, which contradicts the local finiteness of its superset $\mathfrak{B}$ ensured by Proposition 4.6. In turn, the above procedure must terminate after finitely many steps, thereby showing that γ intersects $\Gamma.\mathrm{C}$. Combining this with Lemma 4.7 finishes the proof. $\square$

The following proposition is the first immediate step towards proving that $\widehat{\mathrm{C}}$ is a cross section with C as set of representatives. We show that the intersections of geodesics on $\mathbb{X}$ with $\widehat{\mathrm{C}}$ are bijective to the intersections of geodesics on $\mathbb{H}$ with C. This observation will be crucial for establishing discreteness of intersections. We show further that the map

$$\pi|_{\mathrm{C}} \colon \begin{cases} \mathrm{C} & \longrightarrow & \widehat{\mathrm{C}} \\ \nu & \longmapsto & \pi(\nu) \end{cases} \tag{4.3}$$

is a bijection. Hence, as soon as $\widehat{\mathrm{C}}$ is known to be a cross section, C constitutes a representative for it. To simplify the exposition, we will already call C a set of representatives, thereby refering to (4.3).

Proposition 4.9. *The set C, the family of the Γ-translates of its elements, and its image $\widehat{\mathrm{C}}$ under π satisfy the following properties:*

(i) *The members of the family $\{g.\nu \mid g \in \Gamma,\ j \in A,\ \nu \in \mathrm{C}_j\}$ are pairwise distinct. In particular, C is a set of representatives for $\widehat{\mathrm{C}}$.*

(ii) *Let $\widehat{\gamma}$ be a geodesic on $\mathbb{X}$ that intersects $\widehat{\mathrm{C}}$ at time t. Then there exists a unique geodesic γ on $\mathbb{H}$ such that $\pi(\gamma) = \widehat{\gamma}$ and γ intersects C at time t.*

Proof. In order to prove (i), let $j,k \in A$, $\nu \in \mathrm{C}_j$, $\eta \in \mathrm{C}_k$ and $g \in \Gamma$ such that $\nu = g.\eta$. Thus $\mathrm{bp}(\mathrm{C}_j) \cap g.\,\mathrm{bp}(\mathrm{C}_k) \neq \varnothing$. Then (B6) implies $\mathrm{bp}(\mathrm{C}_j) = g.\,\mathrm{bp}(\mathrm{C}_k)$. Since $\nu = g.\eta$, we have $\mathrm{H}_+(j) \cap g.\mathrm{H}_+(k) \neq \varnothing$ by (B3). Using again (B6), we obtain $j = k$ and $g = \mathrm{id}$. This shows (i).

In order to prove (ii) let $\widehat{\gamma}$ be a geodesic on $\mathbb{X}$ that intersects $\widehat{\mathrm{C}}$ at t. Without loss of generality, we may suppose that $t = 0$ (otherwise we apply a reparametrization of $\widehat{\gamma}$). Let $\widehat{\nu} := \widehat{\gamma}'(0)$. Since C is a set of representatives for $\widehat{\mathrm{C}}$ by (i), there exists a unique element $\nu \in \mathrm{C}$ such that $\pi(\nu) = \widehat{\nu}$. Thus, γ_ν is the unique lift of $\widehat{\gamma}$ to $\mathbb{H}$ that intersects $\widehat{\mathrm{C}}$ at $t = 0$. This completes the proof. $\square$

The final result of this section shows that there is no unique choice of a set of branches for a given set $\widehat{\mathrm{C}} = \pi(\mathrm{C})$.

Lemma 4.10. *Let $g_1, \ldots, g_N \in \Gamma$. Then $C' := \{g_1.\,\mathrm{C}_1, \ldots, g_N.\,\mathrm{C}_N\}$ is a set of branches for $\widehat{\Phi}$, and $\pi(\bigcup_{j=1}^N g_j.\,\mathrm{C}_j) = \widehat{\mathrm{C}}$.*

Proof. The second statement is clear. Thus it suffices to check validity of the properties (B1)–(B7). In order to distinguish the properties fulfilled by $\mathcal{C}$ from those we aim to prove for $\mathcal{C}'$, we denote the latter ones by (B1′)–(B7′), respectively.

Let $j \in A$ and $\nu \in \mathrm{C}_j$. Then

$$\widehat{\gamma}_{g_j.\nu} = \pi(\gamma_{g_j.\nu}) = \pi(g_j.\gamma_\nu) = \pi(\gamma_\nu) = \widehat{\gamma}_\nu\,.$$

Therefore, (B1′) follows from (B1). The properties (B2′), (B3′), (B5′), and (B6′) are immediate from (B2), (B3), (B5), and (B6), respectively, combined with the conformity of Möbius transformations. Property (B4′) follows from (B4) and the Γ-invariance of the set $\widehat{\mathbb{R}}_{\mathrm{st}}$ (see Lemma 1.16). Finally, let $I_j' := g_j.I_j$ for all $j \in A$. Then (B7a) yields

$$p_j.I_{j,\mathrm{st}} = p_j.\Big(\bigcup_{k\in A}\ \bigcup_{g\in\mathcal{G}(j,k)} g.I_{k,\mathrm{st}}\Big) = \bigcup_{k\in A}\ \bigcup_{g\in\mathcal{G}(j,k)} p_j g p_k^{-1}.I'_{k,\mathrm{st}}\,,$$

for all $j,k \in A$. From that we obtain the updated transition sets

$$\mathcal{G}'(j,k) := g_j \cdot \mathcal{G}(j,k) \cdot g_k^{-1}\,,$$

for all $j,k \in A$, with whom (B7′) is easily derived from (B7). □

4.3 Strong Sets of Representatives and Iterated Intersections

Throughout this section we continue to let $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ be a set of branches, where $A := \{1,\ldots,N\}$, and let $\mathrm{C} = \bigcup \mathcal{C}$ denote the branch union of $\mathcal{C}$. We pick again a family of forward transition sets $(\mathcal{G}(j,k))_{j,k\in A}$ and denote the family of backward transition sets by $(\mathcal{V}(k,j))_{j,k\in A}$ (cf. (4.1)). For any $j \in A$, we resume the notation for the sets I_j, J_j, $\mathrm{H}_+(j)$ and $\mathrm{H}_-(j)$ from (B3).

In this section we show that the transition sets are indeed unique, and we provide an alternative characterization of them. See Proposition 4.15. Moreover, we prepare the ground for showing that $\widehat{\mathrm{C}} := \pi(\mathrm{C})$ is intersected by almost all geodesics infinitely often in future and past, for finding a strong cross section as a subset of $\widehat{\mathrm{C}}$, and for determining the induced discrete dynamical system on subsets of $\widehat{\mathbb{R}}$.

Recall the subset $\mathrm{S}\mathbb{H}_{\mathrm{st}}$ of the unit tangent bundle $\mathrm{S}\mathbb{H}$ from (2.12). A branch C_j, $j \in A$, is called a *strong branch*, if $\mathrm{C}_j \subseteq \mathrm{S}\mathbb{H}_{\mathrm{st}}$, and we define

$$\mathrm{C}_{j,\mathrm{st}} := \mathrm{C}_j \cap \mathrm{S}\mathbb{H}_{\mathrm{st}} = \Big\{\nu \in \mathrm{C}_j \;\Big|\; \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}\Big\}\,, \tag{4.4}$$

We call
$$\mathrm{C}_{\mathrm{st}} := \bigcup_{j\in A} \mathrm{C}_{j,\mathrm{st}} \tag{4.5}$$
the *strong set of representatives.*

Our first goal is to show that for each vector $\nu \in \mathrm{C}_{\mathrm{st}}$, the geodesic γ_ν on $\mathbb{H}$ has a (well-defined) minimal intersection time $t^+ > 0$ with $\Gamma.\mathrm{C}_{\mathrm{st}}$, the *next intersection time*, as well as a maximal intersection time $t^- < 0$ with $\Gamma.\mathrm{C}_{\mathrm{st}}$, the *previous intersection time*. We start with a remark and some preparatory lemmas.

Remark 4.11. After the restriction to strong sets and branches, we observe the following relations:

(i) For each $j \in A$, property (B1) implies that $I_{j,\mathrm{st}} \neq \varnothing$ and $J_{j,\mathrm{st}} \neq \varnothing$.

(ii) For each $j \in A$ and each pair $(x, y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$, there exists a unique vector $\nu \in \mathrm{C}_j$ such that
$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) = (x, y)$$
by (B5). Clearly, $\nu \in \mathrm{C}_{j,\mathrm{st}}$.

(iii) Let $j \in A$ and $\nu \in \mathrm{C}_j$. If the geodesic γ_ν represents a periodic geodesic on $\mathbb{X}$, then $\nu \in \mathrm{C}_{j,\mathrm{st}}$ (see Proposition 1.15 and (1.42)).

From the definition of the transition sets $\mathcal{G}(j, k)$ it is obvious that the associated half-spaces fulfill $g.\mathrm{H}_+(k) \subseteq \mathrm{H}_+(j)$, for all $g \in \mathcal{G}(j, k)$. The following lemma shows that this inclusion is indeed proper, and that also the dual property holds with the backwards transition sets.

Lemma 4.12. *Let $j, k \in A$.*

(i) *For all $g \in \mathcal{G}(j, k)$ we have $g.\mathrm{H}_+(k) \subsetneq \mathrm{H}_+(j)$.*

(ii) *For all $g \in \mathcal{V}(k, j)$ we have $g.\mathrm{H}_-(k) \subsetneq \mathrm{H}_-(j)$.*

Proof. It suffices to establish (i) as the proof of (ii) is analogous. Let $g \in \mathcal{G}(j, k)$. Then (B7a) shows that $g.I_k \subseteq I_j$. Remark 4.2(d) and the convexity of the half-spaces $\mathrm{H}_+(.)$ imply that
$$g.\mathrm{H}_+(k) \subseteq \mathrm{H}_+(j)\,.$$
We now assume that $g.\mathrm{H}_+(k) = \mathrm{H}_+(j)$, in order to seek a contradiction. Then
$$g.\overline{\mathrm{bp}(\mathrm{C}_k)} \cup g.\overline{I_k}^{g} = g.\partial_g \mathrm{H}_+(k) = \partial_g \mathrm{H}_+(j) = \overline{\mathrm{bp}(\mathrm{C}_j)} \cup \overline{I_j}^{g}$$
by Remark 4.2(d) and the continuity of the action of g. Since $\mathbb{H}$ and $\partial_g \mathbb{H}$ are both stable under the action of $\mathrm{PSL}_2(\mathbb{R})$, it follows that $g.\overline{\mathrm{bp}(\mathrm{C}_k)} = \overline{\mathrm{bp}(\mathrm{C}_j)}$. Thus, for any $z \in \overline{\mathrm{bp}(\mathrm{C}_j)}$ we have $z \in g.\overline{\mathrm{bp}(\mathrm{C}_k)}$ and $(z, z)_{\mathbb{H}} = \varnothing$, in contradiction to (B7b). In turn, $g.\mathrm{H}_+(k) \subsetneq \mathrm{H}_+(j)$. □

The following two lemmas allow us to establish next and previous intersections and to narrow down their locations by coarse-locating the endpoints of geodesics. We emphasize that in those two lemmas, g is not required to be an element of $\mathcal{G}(j,k)$ or $\mathcal{V}(k,j)$, respectively.

Lemma 4.13. *Let $j,k \in A$ and let $g \in \Gamma$ be such that $g.I_k \subseteq I_j$. Then we have*

(i) $g.\mathrm{H}_+(k) \subseteq \mathrm{H}_+(j)$.

(ii) *For all $\nu \in \mathrm{C}_{j,\mathrm{st}}$ with $\gamma_\nu(+\infty) \in g.I_k$ there exists $t \geq 0$ such that*

$$\gamma_\nu'(t) \in g.\,\mathrm{C}_k \ .$$

(iii) *If $g.I_k \subsetneq I_j$, then $g.\mathrm{H}_+(k) \subsetneq \mathrm{H}_+(j)$ and, in (ii), $t > 0$.*

Proof. For the proof of (i) and the first part of (iii) we use the characterization of the half-space $\mathrm{H}_+(j)$ from Remark 4.2(d). The continuity of g implies

$$g.\overline{I_k}^{g} = \overline{g.I_k}^{g} \subseteq \overline{I_j}^{g} \ .$$

We denote the two endpoints of I_k by x and y. From $(x,y)_{\mathbb{H}} = \overline{\mathrm{bp}(\mathrm{C}_k)}$ and $g.x,\, g.y \in \overline{I_j}^{g}$ we obtain $g.\overline{\mathrm{bp}(\mathrm{C}_k)} \subseteq \overline{\mathrm{H}_+(j)}^{g}$. Thus, $g.\mathrm{H}_+(k) \subseteq \mathrm{H}_+(j)$. If $g.I_k \neq I_j$, then $g.(x,y)_{\mathbb{H}}$ passes through (the interior) of $\mathrm{H}_+(j)$. Further, at least one of the points $g.x$ and $g.y$ is in I_j. Thus, in this case, $g.\mathrm{H}_+(k) \subsetneq \mathrm{H}_+(j)$.

For the proof of (ii) and the second part of (iii) let $\nu \in \mathrm{C}_{j,\mathrm{st}}$ be such that $\gamma_\nu(+\infty) \in g.I_k$. The hypothesis $g.I_k \subseteq I_j$ implies

$$g.I_{k,\mathrm{st}} \subseteq I_{j,\mathrm{st}}\ , \quad g.J_k \supseteq J_j\ , \quad \text{and} \quad g.J_{k,\mathrm{st}} \supseteq J_{j,\mathrm{st}}\ .$$

Since $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$ as $\nu \in \mathrm{C}_{j,st}$, it follows that

$$\gamma_\nu(+\infty) \in \widehat{\mathbb{R}}_{\mathrm{st}} \cap g.I_k = g.I_{k,\mathrm{st}}\ .$$

Therefore, $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) \in g.I_{k,\mathrm{st}} \times g.J_{k,\mathrm{st}}$ or, equivalently,

$$g^{-1}.\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in I_{k,\mathrm{st}} \times J_{k,\mathrm{st}}\ .$$

By (B5) there exists a (unique) vector $\eta \in \mathrm{C}_k$ such that

$$\big(\gamma_\eta(+\infty), \gamma_\eta(-\infty)\big) = g^{-1}.\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big)\ .$$

The uniqueness of geodesics connecting two points in $\overline{\mathbb{H}}^{g}$ implies that there exists $t \in \mathbb{R}$ such that $\gamma_\nu'(t) = g.\eta$. The combination of (B3), (i), and the first part of (iii) yield that $t \geq 0$ and, in case of $g.I_k \subsetneq I_j$, $t > 0$. $\square$

The proof of the following result is analogous to that of Lemma 4.13, for which reason we omit it.

Lemma 4.14. *Let $j, k \in A$ and let $g \in \Gamma$ be such that $g.J_k \subseteq J_j$. Then we have*

(i) $g.\mathrm{H}_-(k) \subseteq \mathrm{H}_-(j)$,

(ii) *for all $\nu \in \mathrm{C}_{j,\mathrm{st}}$ with $\gamma_\nu(-\infty) \in g.J_k$ there exists $t \leq 0$ such that*
$$\gamma'_\nu(t) \in g.\,\mathrm{C}_k\,,$$

(iii) *if $g.J_k \subsetneq J_j$, then $g.\mathrm{H}_-(k) \subsetneq \mathrm{H}_-(j)$ and, in (ii), $t < 0$.*

For each $j \in A$ and $\nu \in \mathrm{C}_j$ we set
$$t^+_{\mathrm{C}}(\nu) \coloneqq \min\left\{t > 0 \;\middle|\; \gamma'_\nu(t) \in \Gamma.\mathrm{C}\right\} \tag{4.6}$$
and
$$t^-_{\mathrm{C}}(\nu) \coloneqq \max\left\{t < 0 \;\middle|\; \gamma'_\nu(t) \in \Gamma.\mathrm{C}\right\} \tag{4.7}$$
whenever the respective element exists. In this case, we call $t^+_{\mathrm{C}}(\nu)$ the *next intersection time of ν in $\mathrm{S}\mathbb{H}$ with respect to* C, and $t^-_{\mathrm{C}}(\nu)$ the *previous intersection time of ν in $\mathrm{S}\mathbb{H}$ with respect to* C.

With these preparations we can now establish the existence of next and previous intersection times. That also allows us to present a characterization of the transition sets, which in turn implies their uniqueness.

Proposition 4.15. *Let $j \in A$.*

(i) *We have*
$$I_{j,\mathrm{st}} = \{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_{j,\mathrm{st}}\}$$
and
$$J_{j,\mathrm{st}} = \{\gamma_\nu(-\infty) \mid \nu \in \mathrm{C}_{j,\mathrm{st}}\}\ .$$

(ii) *For each $\nu \in \mathrm{C}_{j,\mathrm{st}}$, the next intersection time $t^+_{\mathrm{C}}(\nu)$, as well as the previous intersection time $t^-_{\mathrm{C}}(\nu)$, exists.*

(iii) *For each $k \in A$ we have*
$$\mathcal{G}(j,k) = \left\{g \in \Gamma \;\middle|\; \exists\, \nu \in \mathrm{C}_{j,\mathrm{st}} \colon \gamma'_\nu(t^+_{\mathrm{C}}(\nu)) \in g.\,\mathrm{C}_k\right\}$$
and
$$\mathcal{V}(k,j) = \left\{g \in \Gamma \;\middle|\; \exists\, \nu \in \mathrm{C}_{j,\mathrm{st}} \colon \gamma'_\nu(t^-_{\mathrm{C}}(\nu)) \in g.\,\mathrm{C}_k\right\}\ .$$

Proof. For the proof of (i), we recall from Remark 4.11 that $J_{j,\mathrm{st}} \neq \varnothing$ and fix any $y_0 \in J_{j,\mathrm{st}}$. For each $x \in I_{j,\mathrm{st}}$, the combination of (B5) and Remark 4.11 implies the existence of $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that

$$(x, y_0) = \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big)\,.$$

Thus,

$$x \in \{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_{j,\mathrm{st}}\}\ ,$$

and hence

$$I_{j,\mathrm{st}} \subseteq \{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_{j,\mathrm{st}}\}\ .$$

Conversely, since $\{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_j\} \subseteq I_j$ by (B3),

$$\{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_{j,\mathrm{st}}\} \subseteq I_j \cap \widehat{\mathbb{R}}_{\mathrm{st}} = I_{j,\mathrm{st}}\,.$$

This shows the first part of (i). The second part follows analogously.

For the proof of (ii) we fix $\nu \in \mathrm{C}_{j,\mathrm{st}}$. Then $\gamma_\nu(+\infty) \in I_{j,\mathrm{st}}$ by (i), and (B7a) shows the existence of unique elements $k \in A$ and $g \in \mathcal{G}(j,k)$ such that

$$\gamma_\nu(+\infty) \in g.I_k\,.$$

From Lemma 4.12(i) we obtain $g.\mathrm{H}_+(k) \subsetneq \mathrm{H}_+(j)$ and hence $g.I_k \subsetneq I_j$. Therefore we find $t > 0$ such that $\gamma_\nu'(t) \in g.\,\mathrm{C}_k$, as proven in Lemma 4.13. Thus,

$$t \in \big\{s > 0 \,\big|\, \gamma_\nu'(s) \in \Gamma.\mathrm{C}\big\}\ . \tag{4.8}$$

In order to show that the minimum of this set exists and is assumed by t, we set $z \coloneqq \gamma_\nu(0)$ and $w \coloneqq \gamma_\nu(t)$ and observe that

$$(z, w)_{\mathbb{H}} \cap \Gamma.\mathrm{C} = \varnothing$$

by using (B7b) and the fact that $g \in \mathcal{G}(j,k)$. Thus, there is no "earlier" intersection between γ_ν and $\Gamma.\mathrm{C}$, and hence $t_{\mathrm{C}}^+(\nu)$ exists and equals t. The existence of $t_{\mathrm{C}}^-(\nu)$ follows analogously by taking advantage of Proposition 4.5, Lemma 4.12(ii) and Lemma 4.14.

In order to establish (iii) we fix $k \in A$ and set

$$\mathrm{G}_{j,k} \coloneqq \big\{g \in \Gamma \,\big|\, \exists\, \nu \in \mathrm{C}_{j,\mathrm{st}} \colon \gamma_\nu'(t_{\mathrm{C}}^+(\nu)) \in g.\,\mathrm{C}_k\big\}\ .$$

We first aim at showing that $\mathrm{G}_{j,k} = \mathcal{G}(j,k)$. Let $g \in \mathrm{G}_{j,k}$ and $\nu \in \mathrm{C}_{j,\mathrm{st}}$ be such that $\gamma_\nu'(t_{\mathrm{C}}^+(\nu)) \in g.\,\mathrm{C}_k$. As in the proof of (ii) we obtain the existence and uniqueness of $\ell \in A$ and $h \in \mathcal{G}(j,\ell)$ such that $\gamma_\nu'(t_{\mathrm{C}}^+(\nu)) \in h.\,\mathrm{C}_\ell$. Thus, $g.\,\mathrm{C}_k \cap h.\,\mathrm{C}_\ell \neq \varnothing$, which yields $g = h$ and $k = \ell$ by (B6). In turn, $g \in \mathcal{G}(j,k)$, and hence

$$\mathrm{G}_{j,k} \subseteq \mathcal{G}(j,k)\,.$$

For the converse inclusion relation, we pick $g \in \mathcal{G}(j,k)$. From (B7), in combination with (B6), we get

$$g.I_{k,\mathrm{st}} \subsetneq I_{j,\mathrm{st}} .$$

We pick any $(x,y) \in g.I_{k,\mathrm{st}} \times J_{j,\mathrm{st}}$. Then $(x,y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$, and hence, by (B5) there exists a unique vector $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that

$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) = (x,y) .$$

By Lemma 4.13 there exists $t > 0$ such that $\gamma_\nu'(t) \in g.\mathrm{C}_k$. As in the proof of (ii) we obtain $t = t_{\mathrm{C}}^+(\nu)$. Therefore, $g \in \mathrm{G}_{j,k}$ and hence

$$\mathcal{G}(j,k) \subseteq \mathrm{G}_{j,k} .$$

This proves the first part of (iii). For the second part we observe that

$$\exists\, \nu \in \mathrm{C}_j \colon \gamma_\nu'(t_{\mathrm{C}}^-(\nu)) \in g.\mathrm{C}_k \quad \Longleftrightarrow \quad \exists\, \eta \in \mathrm{C}_k \colon \gamma_\eta'(t_{\mathrm{C}}^+(\eta)) \in g^{-1}.\mathrm{C}_j .$$

The equality $\mathcal{V}(k,j) = \mathcal{G}(k,j)^{-1}$ now completes the proof. $\square$

Remark 4.16. Let $j,k \in A$. We note that for the characterization of the transition sets $\mathcal{G}(j,k)$ and $\mathcal{V}(k,j)$ in Proposition 4.15(iii) we used the strong branch $\mathrm{C}_{j,\mathrm{st}}$ instead of the orginal branch C_j. This is necessary due to the possibility that branches are *not full*, i.e., there might exist $k \in A$ and a geodesic γ on $\mathbb{H}$ with $\gamma(+\infty) \in I_k$ and $\gamma(-\infty) \in J_k$ that does not intersect C_k. In other words, the geodesic γ is passing through the geodesic segment $\overline{\mathrm{bp}(\mathrm{C}_k)}$ from the half-space $\mathrm{H}_-(k)$ into $\mathrm{H}_+(k)$ and hence has the potential to intersect C_k, but the necessary vector at the intersection point with $\overline{\mathrm{bp}(\mathrm{C}_k)}$ is not contained in C_k. If we now have a vector $\nu \in \mathrm{C}_j \setminus \mathrm{C}_{j,\mathrm{st}}$ such that the next intersection time $t_{\mathrm{C}}^+(\nu)$ exists but the next intersection between γ_ν and $\Gamma.\overline{\mathrm{bp}(\mathrm{C})}$ is at an earlier time (due to a “missing” vector in C as above), then

$$\gamma_\nu'(t_{\mathrm{C}}^+(\nu)) \in h.\mathrm{C}_k$$

for some $k \in A$ and $h \in \Gamma$ with h typically not in $\mathcal{G}(j,k)$. However, if all branches are full, then

$$\mathcal{G}(j,k) = \big\{ g \in \Gamma \;\big|\; \exists\, \nu \in \mathrm{C}_j \colon t_{\mathrm{C}}^+(\nu) \text{ exists and } \gamma_\nu'(t_{\mathrm{C}}^+(\nu)) \in g.\mathrm{C}_k \big\}$$

and

$$\mathcal{V}(k,j) = \big\{ g \in \Gamma \;\big|\; \exists\, \nu \in \mathrm{C}_j \colon t_{\mathrm{C}}^-(\nu) \text{ exists and } \gamma_\nu'(t_{\mathrm{C}}^-(\nu)) \in g.\mathrm{C}_k \big\} .$$

The next observation follows immediately from Proposition 4.15.

Corollary 4.17. *Let $j,k \in A$, $\nu \in \mathrm{C}_{j,\mathrm{st}}$ and $g \in \Gamma$. Then*

(i) $\gamma'_\nu(t^+_{\mathrm{C}}(\nu)) \in g\boldsymbol{.}\,\mathrm{C}_k$ *if and only if* $g \in \mathcal{G}(j,k)$ *and* $\gamma_\nu(+\infty) \in g\boldsymbol{.}I_k$,

(ii) $\gamma'_\nu(t^-_{\mathrm{C}}(\nu)) \in g\boldsymbol{.}\,\mathrm{C}_k$ *if and only if* $g \in \mathcal{V}(k,j)$ *and* $\gamma_\nu(-\infty) \in g\boldsymbol{.}J_k$.

We now associate to each element $\nu \in \mathrm{C}_{\mathrm{st}}$ three sequences, which completely characterize the geodesic γ_ν in terms of the strong branches $\mathrm{C}_{1,\mathrm{st}}, \ldots, \mathrm{C}_{N,\mathrm{st}}$. The combination of the Propositions 4.9 and 4.15, Corollary 4.17, and Remark 4.11 shows their existence, well-definedness and the claimed properties.

We define the *sequence* $(\mathrm{t}_{\mathrm{C},n}(\nu))_{n\in\mathbb{Z}}$ *of iterated intersection times of* ν *with respect to* C by

$$\mathrm{t}_{\mathrm{C},0}(\nu) \coloneqq 0 \tag{4.9}$$

and

$$\mathrm{t}_{\mathrm{C},n}(\nu) \coloneqq \begin{cases} \min\left\{t > \mathrm{t}_{\mathrm{C},n-1}(\nu) \mid \gamma'_\nu(t) \in \Gamma\boldsymbol{.}\mathrm{C}\right\} & \text{for } n \geq 1\,, \\ \max\left\{t < \mathrm{t}_{\mathrm{C},n+1}(\nu) \mid \gamma'_\nu(t) \in \Gamma\boldsymbol{.}\mathrm{C}\right\} & \text{for } n \leq -1\,. \end{cases} \tag{4.10}$$

This sequence is strictly increasing. For each $n \in \mathbb{Z}$ we have

$$\operatorname{sgn}(\mathrm{t}_{\mathrm{C},n}(\nu)) = \operatorname{sgn}(n)$$

and

$$\mathrm{t}_{\mathrm{C},n}(\nu) = t^+_{\mathrm{C}}\big(\gamma'_\nu(\mathrm{t}_{\mathrm{C},n-1}(\nu))\big) = t^-_{\mathrm{C}}\big(\gamma'_\nu(\mathrm{t}_{\mathrm{C},n+1}(\nu))\big)\,. \tag{4.11}$$

Given the sequence $(\mathrm{t}_{\mathrm{C},n}(\nu))_{n\in\mathbb{Z}}$, for each $n \in \mathbb{Z}$ the branch translate $g\boldsymbol{.}\,\mathrm{C}_k$ containing the vector $\gamma'_\nu(\mathrm{t}_{\mathrm{C},n}(\nu))$ is uniquely determined. This allows us to define the *sequence* $(\mathrm{k}_{\mathrm{C},n}(\nu))_{n\in\mathbb{Z}}$ *of iterated intersection branches of* ν *with respect to* $\mathcal{C}$ as the sequence in A given by

$$\mathrm{k}_{\mathrm{C},n}(\nu) = k \qquad :\Longleftrightarrow \qquad \exists\, g \in \Gamma\colon \gamma'_\nu(\mathrm{t}_{\mathrm{C},n}(\nu)) \in g\boldsymbol{.}\,\mathrm{C}_k \tag{4.12}$$

for all $n \in \mathbb{Z}$. The *sequence* $(\mathrm{g}_{\mathrm{C},n}(\nu))_{n\in\mathbb{Z}}$ *of iterated intersection transformations of* ν *with respect to* C is a sequence in Γ that is given by

$$\mathrm{g}_{\mathrm{C},0}(\nu) \coloneqq \mathrm{id} \tag{4.13}$$

and for $n \in \mathbb{Z}$, $n \neq 0$, by

$$\begin{aligned} &\mathrm{g}_{\mathrm{C},n}(\nu) = g \\ &:\Longleftrightarrow \quad \begin{cases} \gamma'_\nu(\mathrm{t}_{\mathrm{C},n}(\nu)) \in \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},n-1}(\nu)g\boldsymbol{.}\,\mathrm{C}_{\mathrm{k}_{\mathrm{C},n}(\nu)} & \text{for } n \geq 1\,, \\ \gamma'_\nu(\mathrm{t}_{\mathrm{C},n}(\nu)) \in \mathrm{g}_{\mathrm{C},-1}(\nu)\cdots\mathrm{g}_{\mathrm{C},n+1}(\nu)g\boldsymbol{.}\,\mathrm{C}_{\mathrm{k}_{\mathrm{C},n}(\nu)} & \text{for } n \leq -1\,. \end{cases} \end{aligned} \tag{4.14}$$

For each $n \in \mathbb{N}$ we then have

$$\mathrm{g}_{\mathrm{C},n}(\nu) \in \mathcal{G}(\mathrm{k}_{\mathrm{C},n-1}(\nu), \mathrm{k}_{\mathrm{C},n}(\nu))$$

and for each $n \in -\mathbb{N}$ we have

$$\mathrm{g}_{\mathrm{C},n}(\nu)^{-1} \in \mathcal{G}(\mathrm{k}_{\mathrm{C},n}(\nu), \mathrm{k}_{\mathrm{C},n+1}(\nu))\,.$$

We call the ordered set

$$[(\mathrm{t}_{\mathrm{C},n}(\nu))_n, (\mathrm{k}_{\mathrm{C},n}(\nu))_n, (\mathrm{g}_{\mathrm{C},n}(\nu))_n] \tag{4.15}$$

the *system of iterated sequences of* ν *with respect to* $\mathcal{C}$.

Lemma 4.18. *For each* $\nu \in \mathrm{C}_{\mathrm{st}}$ *we have*

$$\lim_{n\to\pm\infty} \mathrm{t}_{\mathrm{C},n}(\nu) = \pm\infty\,.$$

Proof. We establish the claims via a proof by contradiction. Let $\nu \in \mathrm{C}_{\mathrm{st}}$ and consider the system

$$[(\mathrm{t}_{\mathrm{C},n}(\nu))_n, (\mathrm{k}_{\mathrm{C},n}(\nu))_n, (\mathrm{g}_{\mathrm{C},n}(\nu))_n]$$

of iterated sequences of ν from (4.15). For $n \in \mathbb{N}$ set

$$t_n \coloneqq \mathrm{t}_{\mathrm{C},n}(\nu)$$

and recall that $(t_n)_{n\in\mathbb{N}}$ is strictly increasing. We assume that $(t_n)_n$ converges in $\mathbb{R}$, say

$$\lim_{n\to+\infty} t_n = \tau \quad \in \mathbb{R}\,.$$

Since the map

$$\begin{cases} \mathbb{R} & \longrightarrow & \mathbb{H} \\ t & \longmapsto & \gamma_\nu(t) \end{cases}$$

is an isometric embedding, it follows that the sequence $(\gamma_\nu(t_n))_n$ converges in $\mathbb{H}$, namely

$$\lim_{n\to+\infty} \gamma_\nu(t_n) = \gamma_\nu(\tau)\,,$$

and the elements of the sequence $(\gamma_\nu(t_n))_n$ are pairwise distinct. For $n \in \mathbb{N}$ let

$$k_n \coloneqq \mathrm{k}_{\mathrm{C},n}(\nu)$$

and

$$h_n \coloneqq \mathrm{g}_{\mathrm{C},1}(\nu)\cdots \mathrm{g}_{\mathrm{C},n}(\nu)\,.$$

Then

$$\gamma_\nu(t_n) \in h_n . \overline{\mathrm{bp}(\mathrm{C}_{k_n})}$$

for each $n \in \mathbb{N}$ (see (4.13)–(4.14)). Further, the shape of geodesics in $\mathbb{H}$ implies that the tuples (k_n, h_n), $n \in \mathbb{N}$, are pairwise distinct. Hence, each neighborhood of $\gamma_\nu(\tau)$ in $\mathbb{H}$ intersects infinitely many members of the family

$$\left\{ h_n . \overline{\mathrm{bp}(\mathrm{C}_{k_n})} \;\middle|\; n \in \mathbb{N} \right\} .$$

This contradicts Proposition 4.6. In turn,

$$\lim_{n \to +\infty} t_n = +\infty .$$

The statement for $n \to -\infty$ follows analogously. □

The following proposition shows that each intersection between $\Gamma . \mathrm{C}_{\mathrm{st}}$ and a geodesic determined by an element of C_{st} is indeed (uniquely) detected by the iterated sequences. This observation will be crucial for establishing that $\widehat{\mathrm{C}}$ is a cross section for the geodesic flow on $\mathbb{X}$.

Proposition 4.19. *Let $\nu \in \mathrm{C}_{\mathrm{st}}$, $k \in A$, $t \in \mathbb{R}$, and $g \in \Gamma$ be such that*

$$\gamma_\nu'(t) \in g . \mathrm{C}_k .$$

Then there exists a unique element $n \in \mathbb{Z}$ such that $\mathrm{sgn}(n) = \mathrm{sgn}(t)$ and

$$k = \mathrm{k}_{\mathrm{C},n}(\nu) , \quad t = \mathrm{t}_{\mathrm{C},n}(\nu) , \quad \textit{and} \quad g = \mathrm{g}_{\mathrm{C},\mathrm{sgn}(t)}(\nu) \mathrm{g}_{\mathrm{C},2\,\mathrm{sgn}(t)}(\nu) \cdots \mathrm{g}_{\mathrm{C},n}(\nu) .$$

Proof. It suffices to show the uniqueness of $n \in \mathbb{Z}$ with $t = \mathrm{t}_{\mathrm{C},n}(\nu)$. The remaining statements are then immediate from the definitions. By the strict monotony of the sequence $(\mathrm{t}_{\mathrm{C},n}(\nu))_{n \in \mathbb{Z}}$ and Lemma 4.18 we find exactly one $n \in \mathbb{Z}$ such that

$$\mathrm{t}_{\mathrm{C},n-1}(\nu) < t \leq \mathrm{t}_{\mathrm{C},n}(\nu) .$$

If $t < \mathrm{t}_{\mathrm{C},n}(\nu)$, then the hypothesis $\gamma_\nu'(t) \in \Gamma . \mathrm{C}$ implies

$$\mathrm{t}_{\mathrm{C},n}(\nu) \neq t_{\mathrm{C}}^{+}(\gamma_\nu'(\mathrm{t}_{\mathrm{C},n-1}(\nu))) .$$

This contradicts (4.11). Hence, $t = \mathrm{t}_{\mathrm{C},n}(\nu)$. □

4.4 Ramification of Branches

Let $\mathcal{C} := \{\mathrm{C}_j \mid j \in A\}$ be a set of branches for the geodesic flow on $\mathbb{X}$ and let $\mathcal{G}(.,.)$ denote the forward transition sets. Example 4.21 below shows that it is possible for transition sets to be infinite. Those situations cause issues in terms of convergence of the arising transfer operators (see the discussion before Example 4.38 in Section 4.7). Therefore, for our constructions of strict transfer operator approaches further below, we will suppose that all transition sets are finite.

The purpose of this section is twofold. We first present a simple-to-check criterium that allows us to distinguish sets of branches with infinite transition sets from those for which all transition sets are finite. Subsequently, we provide an algorithm that turns each set of branches with infinite transition sets into one with only finite transition sets by adding a limited number of specific branches. This shows that the assumption that all transition sets are finite does not limit the scope of Fuchsian groups to which our results apply.

Definition 4.20. For $j \in A$ we define its *ramification number* by

$$\mathrm{ram}(j) \coloneqq \sum_{k \in A} \#\mathcal{G}(j,k)$$

and the *ramification in* $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ by

$$\mathrm{Ram}_{\mathcal{C}} \coloneqq \sup_{j \in A} \mathrm{ram}(j)\,.$$

A set of branches $\mathcal{C}$ is called *infinitely ramified* if $\mathrm{Ram}_{\mathcal{C}} = +\infty$, and *finitely ramified* otherwise. If we need to emphasize the choice of the Fuchsian group Γ for the ramification (as in Example 4.21, for instance), then we write $\mathrm{Ram}_{\mathcal{C},\Gamma}$ for $\mathrm{Ram}_{\mathcal{C}}$.

Let $j \in A$. Starting on the branch C_j the number $\mathrm{ram}(j)$ encodes the number of distinct directions in which one can travel with regard to the next intersection branches. Or in other words,

$$\mathrm{ram}(j) = \#\left\{ \left(\mathrm{k}_{\mathrm{C},1}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\right) \in A \times \Gamma \;\middle|\; \nu \in \mathrm{C}_j \right\}\,,$$

where $\mathrm{k}_{\mathrm{C},n}(\nu)$ and $\mathrm{g}_{\mathrm{C},n}(\nu)$ are as in (4.12)–(4.14) for $n \in \mathbb{Z}$.

With the following example we illustrate that the ramification heavily depends on the combination of Fuchsian group *and* set of branches. We provide two Fuchsian groups that admit the same set of branches, but the ramification is finite only for one of them. We also show that a different choice of set of branches may yield a finite ramification.

Example 4.21. We consider the modular group

$$\Gamma_1 \coloneqq \mathrm{PSL}_2(\mathbb{Z})$$

and the projective Hecke congruence group of level 2

$$\Gamma_2 \coloneqq \mathrm{P}\Gamma_0(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod 2 \right\}\,.$$

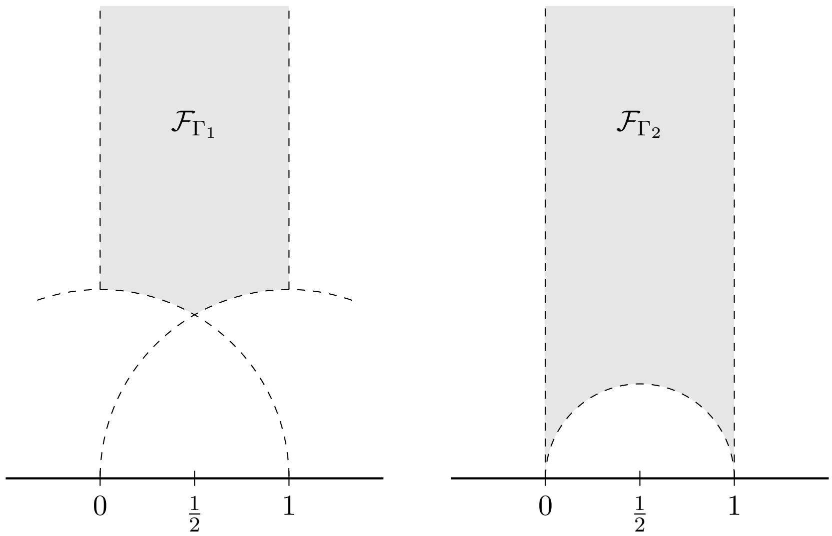


Figure 7: Examples of fundamental domains for Γ_1 and Γ_2, respectively.

These groups are well-known to be discrete and geometrically finite. Fundamental domains for them are indicated in Figure 7. We fix the elements

$$s_1 := \mathrm{s}_{\frac{3\pi}{2}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad s_2 := \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad t := \mathrm{t}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

of $\mathrm{PSL}_2(\mathbb{R})$. Then

$$\Gamma_1 = \langle s_1, t\rangle \qquad \text{and} \qquad \Gamma_2 = \langle s_2, t\rangle\,.$$

(i) We let $\gamma := (0,1)_{\mathbb{H}}$ be the geodesic segment from 0 to 1 and define $\mathrm{C}_1 \subseteq \mathrm{S}\mathbb{H}$ to be the set of all vectors based on γ and pointing into the half-space to the right of it. Thus,

$$I_1 = \{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_1\} = (0,1)$$

and

$$J_1 = \{\gamma_\nu(-\infty) \mid \nu \in \mathrm{C}_1\} = (-\infty, 0) \cup (1, +\infty)$$

(see also Figures 8 and 9). Then $\{\mathrm{C}_1\}$ is a set of branches for both groups, Γ_1 and Γ_2. However,

$$\mathrm{Ram}_{\{\mathrm{C}_1\},\Gamma_1} = 2\,, \quad \text{while} \quad \mathrm{Ram}_{\{\mathrm{C}_1\},\Gamma_2} = +\infty\,,$$

as indicated in Figures 8 and 9.

(ii) We now let $\eta := (0, 1/2)_{\mathbb{H}}$ be the geodesic segment from 0 to $1/2$, and let

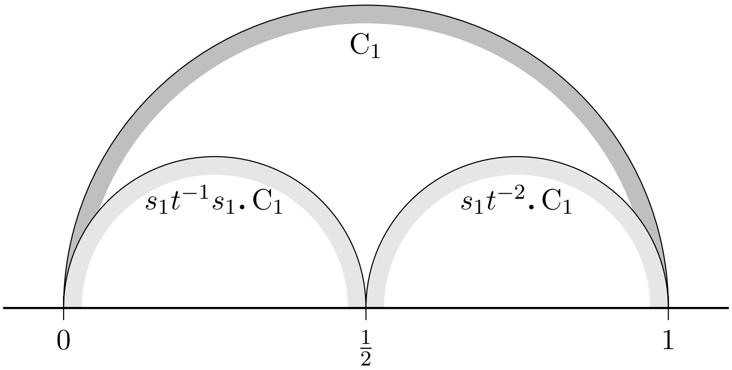


Figure 8: The set of branches $\{C_1\}$ for Γ_1 and its successors.

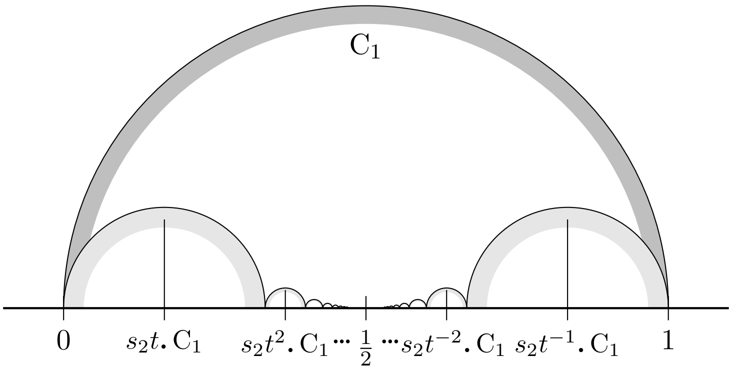


Figure 9: The set of branches $\{C_1\}$ for Γ_2 and its successors.

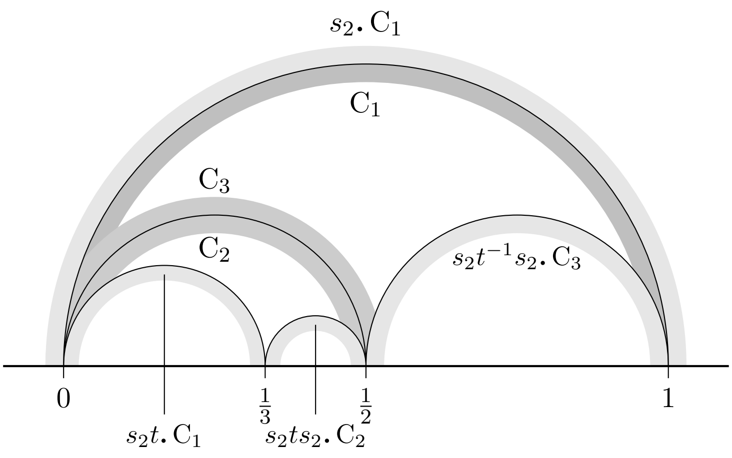


Figure 10: The set of branches $\mathcal{C} = \{C_1, C_2, C_3\}$ for Γ_2 and its successors.

C_2 and C_3 be the set of unit tangent vectors based on η that point into the half-space to the right or left of η, respectively. Then $\mathcal{C} = \{\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3\}$ is a set of branches for Γ_2, and $\mathrm{Ram}_{\mathcal{C},\Gamma_2} = 2$ (see Figure 10).

The following result shows that ramification is invariant under the action of Γ. It follows immediately from the representation of the (updated) transition sets in the proof of Lemma 4.10.

Lemma 4.22. *Let $\{p_j \mid j \in A\} \subseteq \Gamma$. Then the ramification of the set of branches $\mathcal{C}' := \{p_j\boldsymbol{.}\,\mathrm{C}_j \mid j \in A\}$ equals the one of $\mathcal{C}$, i. e.,*

$$\mathrm{Ram}_{\mathcal{C}'} = \mathrm{Ram}_{\mathcal{C}}\,.$$

In what follows, we determine the geometric structure of finitely and infinitely ramified sets of branches and find that the cusps of $\mathbb{X}$ play a central role. It is therefore convenient to first study the case that $\mathbb{X}$ has no cusps.

Lemma 4.23. *Let Γ be a geometrically finite Fuchsian group without parabolic elements. Then every set of branches for the geodesic flow on $\mathbb{X}$ is finitely ramified.*

Proof. Let $\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$ be a set of branches for the geodesic flow on $\mathbb{X}$ and adopt the standard notation from the beginning of this section. In order to seek a contradiction, we assume that $\mathrm{Ram}_{\mathcal{C}} = +\infty$. Then we find and fix $j, k \in A$ such that $\#\mathcal{G}(j,k) = +\infty$.

In what follows, we will take advantage of the Euclidean structure and the standard ordering of $\mathbb{R}$ to simplify the argumentation. To that end, we may suppose without loss of generality that the interval I_j is contained in $\mathbb{R}$ and bounded (if necessary, we conjugate Γ and the set of branches by a suitable element $g \in \mathrm{PSL}(2,\mathbb{R})$).

From (B7) we obtain that the open, nonempty intervals

$$g\boldsymbol{.}I_k\,, \qquad \text{for } g \in \mathcal{G}(j,k)\,, \tag{4.16}$$

are pairwise disjoint and all contained in I_j. In particular, for all $g \in \mathcal{G}(j,k)$ the two boundary points $g\boldsymbol{.}\mathrm{X}_k$, $g\boldsymbol{.}\mathrm{Y}_k$ of $g\boldsymbol{.}I_k$ are contained in $\overline{I_j}$. These properties, together with the boundedness of I_j, allow us to find a strictly increasing, convergent sequence in $\{g\boldsymbol{.}\mathrm{X}_k \mid g \in \mathcal{G}(j,k)\}$, say $(g_n\boldsymbol{.}\mathrm{X}_k)_{n\in\mathbb{N}}$ with

$$\lim_{n\to+\infty} g_n\boldsymbol{.}\mathrm{X}_k =: a\,.$$

We note that $a \in \overline{I_j}$. Further, the disjointness and convexity of the intervals in (4.16) and the strict monotony of the sequence $(g_n\boldsymbol{.}\mathrm{X}_k)_{n\in\mathbb{N}}$ imply that, for each index $n \in \mathbb{N}$, the point $g_n\boldsymbol{.}\mathrm{Y}_k$ is contained in the interval $(g_n\boldsymbol{.}\mathrm{X}_k, g_{n+1}\boldsymbol{.}\mathrm{X}_k)$. Therefore

$$\lim_{n\to+\infty} g_n\boldsymbol{.}\mathrm{Y}_k = a\,.$$

We fix a point on the geodesic segment $\overline{\mathrm{bp}(\mathrm{C}_k)}$, say $z_0 \in \mathbb{H}$. Then $(g_n.z_0)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{H}$ with

$$\lim_{n\to+\infty} g_n.z_0 = a\,.$$

Thus, $a \in \Lambda(\Gamma)$. Since Γ contains no parabolic elements, we have $\Lambda(\Gamma) = \widehat{\mathbb{R}}_{\mathrm{st}}$ (see (1.42)) and hence

$$a \in \widehat{\mathbb{R}}_{\mathrm{st}} \cap \overline{I_j} = I_{j,\mathrm{st}}\,.$$

By (B7) we find a (unique) pair $(\ell, h) \in A \times \mathcal{G}(j,\ell)$ such that $a \in h.I_{\ell,\mathrm{st}} \subseteq h.I_\ell$. Since $h.I_\ell$ is open and a is the limit of the sequences from above, we find $n \in \mathbb{N}$ such that $g_n.\mathrm{X}_k, g_n.\mathrm{Y}_k \in h.I_\ell$. Thus,

$$g_n.I_k \cap h.I_\ell \neq \varnothing\,.$$

This contradicts the disjointness of the unions in (B7). In turn, $\mathcal{C}$ is finitely ramified. $\square$

We now consider the case where $\mathbb{X}$ is allowed to have cusps. In Proposition 4.26 we will see that the ramification with respect to a given set of branches $\mathcal{C}$ depends on how thoroughly $\mathcal{C}$ accounts for the cusps of $\mathbb{X}$. To make this statement rigorous we require the following notion regarding the local structure of sets of branches in the vicinity of cusps.

Definition 4.24. Let $\widehat{c}$ be a cusp of $\mathbb{X}$, let $c \in \widehat{\mathbb{R}}$ be a representative of $\widehat{c}$, and let

$$\mathrm{Att}_{\mathcal{C}}(c) := \left\{ (j,h) \in A \times \Gamma \;\middle|\; c \in h.\overline{\mathrm{bp}(\mathrm{C}_j)}_{\!\!\!\!\!\!\!\!\!\!\!\!\mathcal{g}} \right\}\,.$$

We say that $\widehat{c}$ is *attached to the set of branches* $\mathcal{C}$ if the interval

$$I(\mathrm{Att}_{\mathcal{C}}(c)) := \overline{\bigcup_{(j,h)\in\mathrm{Att}_{\mathcal{C}}(c)} h.I_j}$$

contains a full neighborhood of c in $\widehat{\mathbb{R}}$.

Obviously, the definition of $\mathrm{Att}_{\mathcal{C}}$ is independent of the choice of the representative c of $\widehat{c}$ and the notion of attachedness is well-defined. If $\widehat{c}$ is attached to $\mathcal{C}$, then $\#\mathrm{Att}_{\mathcal{C}}(c) \geq 2$, because for each $j \in A$ the set $\partial_g \overline{\mathrm{bp}(\mathrm{C}_j)}$ consists of exactly the two boundary points of the interval I_j.

The following result is a technical observation that comes in handy for the remaining proofs of this section.

Lemma 4.25. *Suppose that the hyperbolic orbisurface $\mathbb{X}$ has cusps and that one of them, say $\widehat{c}$, is represented by ∞. Denote by λ the cusp width of $\widehat{c}$. Suppose further that the set of branches $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ satisfies* $\mathrm{Att}_{\mathcal{C}}(\infty) = \varnothing$. *Then,* *for each $j \in A$, the maximum*

$$\hbar_{\widehat{C}}(j) := \max\left\{ \operatorname{Im} z \;\middle|\; z \in \Gamma.\overline{\mathrm{bp}(\mathrm{C}_j)} \right\}$$

exists and is bounded from above by $\lambda/2$.

Proof. Let $j \in A$. For each $g \in \Gamma$, the hypothesis $\mathrm{Att}_{\mathcal{C}}(\infty) = \varnothing$ implies that the set $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ is a Euclidean semicircle. Hence, the maximum of the set

$$\left\{ \operatorname{Im} z \;\middle|\; z \in g.\overline{\mathrm{bp}(\mathrm{C}_j)} \right\}$$

exists and equals the radius of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$. Let $x_g, y_g \in \mathbb{R}$, $x_g < y_g$, denote the two endpoints of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$, i. e.,

$$(x_g, y_g)_{\mathbb{H}} = g.\overline{\mathrm{bp}(\mathrm{C}_j)}\,.$$

By assumption, the cyclic subgroup Γ_∞ is generated by t_λ, with Γ_∞ as in (1.28) and t_λ as in (1.7). Then the set $\left\{ \operatorname{Im} z \;\middle|\; z \in \Gamma.\overline{\mathrm{bp}(\mathrm{C}_j)} \right\}$ is bounded from above by $\lambda/2$, because otherwise we would find $h \in \Gamma$ such that $y_h - x_h > \lambda$ and then

$$x_h < x_h + \lambda = t_\lambda.x_h = x_{t_\lambda h} < y_h < t_\lambda.y_h = y_{t_\lambda h}\,.$$

Hence, the geodesic segments $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ and $t_\lambda g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ would intersects without coinciding, contradicting (B6).

Now consider the strip $\mathcal{S} := \operatorname{Re}|_{\mathbb{H}}^{-1}((-\lambda, \lambda))$. Let $\Lambda \subseteq \Gamma$ be the set of elements $g \in \Gamma$ such that

$$g.\overline{\mathrm{bp}(\mathrm{C}_j)} \subseteq \mathcal{S}\,.$$

For each $g \in \Lambda$ set

$$\hbar_g := \max\left\{ \operatorname{Im} z \;\middle|\; z \in g.\overline{\mathrm{bp}(\mathrm{C}_j)} \right\}\,.$$

In order to seek a contradiction, we suppose that there exists a sequence $(g_m)_{m\in\mathbb{N}}$ in Λ such that the sequence of maxima $(\hbar_{g_m})_{m\in\mathbb{N}}$ is strictly increasing. By the previous considerations, $(\hbar_{g_m})_{m\in\mathbb{N}}$ is bounded from above by $\lambda/2$ and hence convergent. Further, $\hbar_{g_m} > 0$ for all $m \in \mathbb{N}$. Since the union $\bigcup_{g\in\Lambda} g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ is disjoint by (B6) and contained in the strip $\mathcal{S}$ of finite width, we find a subsequence $(g_{m_k})_{k\in\mathbb{N}}$ of $(g_m)_{m\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$,

$$g_{m_k}.\overline{\mathrm{bp}(\mathrm{C}_j)} \subseteq \mathrm{conv}_{\mathrm{E}}\left(g_{m_{k+1}}.\overline{\mathrm{bp}(\mathrm{C}_j)} \right)\,,$$

where $\mathrm{conv}_{\mathrm{E}}(M)$ denotes the convex hull of the set M in $\mathbb{C}$ with respect to the Euclidean metric. Since $(\hbar_{g_{m_k}})_{k\in\mathbb{N}}$ converges, the family $\{g_{m_k}.\overline{\mathrm{bp}(\mathrm{C}_j)}\}_k$ is not locally finite in $\mathbb{H}$, which contradicts Proposition 4.6. Therefore, such a sequence $(g_m)_{m\in\mathbb{N}}$ cannot exist. In turn, there exists an element $g^* \in \Lambda$ such that

$$\hbar_{g^*} = \max_{g\in\Lambda} \hbar_g\,.$$

The same argument applies to all strips of the form

$$\mathrm{Re}|_{\mathbb{H}}^{-1}\big((n-1)\lambda, (n+1)\lambda\big) = t_\lambda^n.\mathcal{S}$$

with $n \in \mathbb{Z}$ and Λ replaced by $t_\lambda^n.\Lambda$. The t_λ-invariance of $\hbar$ yields $\hbar_g = \hbar_{t_\lambda g}$ for all $g \in \Gamma$, wherefore

$$\hbar_{g^*} = \max_{g\in t_\lambda^n.\Lambda} \hbar_g$$

for all $n \in \mathbb{Z}$. Since $\Gamma = \bigcup_{n\in\mathbb{Z}} t_\lambda^n.\Lambda$, the existence of $\hbar_{\widehat{c}}(j)$ is shown. □

Proposition 4.26. *The set of branches $\mathcal{C}$ is finitely ramified if and only if all cusps of $\mathbb{X}$ are attached to $\mathcal{C}$.*

Proof. We suppose first that $\mathcal{C}$ is finitely ramified. In order to seek a contradiction we assume that there exists a cusp of $\mathbb{X}$ that is not attached to $\mathcal{C}$. Without loss of generality we may suppose that this cusp is represented by ∞. Then $\infty \in \Lambda(\Gamma)$. Moreover, ∞ is approximated from both sides by suitable sequences in $\widehat{\mathbb{R}}_{\mathrm{st}}$, as can be seen by taking any element $w \in \widehat{\mathbb{R}}_{\mathrm{st}}$ (which is necessarily not ∞), any element t in Γ_∞ and considering the two sequences $(t^n.w)_{n\in\mathbb{N}}$ and $(t^{-n}.w)_{n\in\mathbb{N}}$, which both converge to ∞ but from different sides.

We now claim that there exists a pair $(j, g) \in A \times \Gamma$ such that

$$\infty \in g.I_j\,. \tag{4.17}$$

In order to see this, we pick a periodic geodesic $\widehat{\gamma}$ on $\mathbb{X}$. By ($\mathrm{B}_{\mathrm{Per}}$) we find a geodesic γ on $\mathbb{H}$ representing $\widehat{\gamma}$ and a pair $(j, p) \in A \times \Gamma$ such that γ intersects $p.\,\mathrm{C}_{j,\mathrm{st}}$. Then $\gamma(+\infty) \in p.I_{j,\mathrm{st}}$ and $\gamma(-\infty) \in p.J_{j,\mathrm{st}}$. Further we find a hyperbolic element $h \in \Gamma$ such that $\mathrm{f}_\pm(h) = \gamma(\pm\infty)$. In other words, the geodesic γ represents the axis $\alpha(h)$ of h. Under the iterated action of h^{-1}, the two endpoints $p.\mathrm{X}_j$ and $p.\mathrm{Y}_j$ of $p.J_j$ tend to $\mathrm{f}_-(h) = \gamma(-\infty)$. More precisely,

$$h^{-(n+1)}p.J_j \subsetneq h^{-n}p.J_j\,,$$

for all $n \in \mathbb{N}$, and

$$\bigcap_{n\in\mathbb{N}} h^{-n}p.J_j = \{\gamma(-\infty)\}$$

(see Lemma 1.4). Since $\widehat{\gamma}$ is periodic, $\gamma(-\infty)$ is a hyperbolic fixed point and hence cannot coincide with the cuspidal point ∞ by Lemma 1.8. Therefore, for some sufficiently large $N \in \mathbb{N}$, the interval $h^{-N}p.J_j = (h^{-N}p.\mathrm{Y}_j, h^{-N}p.\mathrm{X}_j)_c$ is the real interval $(h^{-N}p.\mathrm{Y}_j, h^{-N}p.\mathrm{X}_j)$. Thus, $h^{-N}p.I_j = (h^{-N}p.\mathrm{X}_j, h^{-N}p.\mathrm{Y}_j)_c$ contains ∞. This establishes the existence of a pair $(j, g) \in A\times\Gamma$ satisfying (4.17).

We now fix such a pair (j, g). Without loss of generality (using Lemma 4.22), we may suppose that $g = \mathrm{id}$. Since $\widehat{\mathbb{R}}_{\mathrm{st}}$ lies dense in small neighborhoods of ∞, we obtain that $\infty \in \overline{I_{j,\mathrm{st}}}$. Since $\mathcal{C}$ is finitely ramified, (B7a) implies that

$$\overline{I_{j,\mathrm{st}}} = \bigcup_{k \in A} \bigcup_{h \in \mathcal{G}(j,k)} h.\overline{I_{k,\mathrm{st}}}\,.$$

Thus, we find $k \in A$ and $h \in \mathcal{G}(j,k)$ such that $\infty \in h.\overline{I_k}$.

We suppose first that ∞ is an endpoint, hence a boundary point, of the interval $h.I_k$. Without loss of generality, we may suppose that $\infty = h.\mathrm{Y}_k$. Then

$$h.\overline{I_k} \cap I_j \subseteq (\mathrm{X}_j, \infty]_c\,.$$

Since ∞ is contained in the open interval I_j and is approximated within $\widehat{\mathbb{R}}_{\mathrm{st}}$ from both sides, we see that

$$(\infty, \mathrm{Y}_j)_c \cap I_{j,\mathrm{st}} \neq \varnothing$$

(the part of $I_{j,\mathrm{st}}$ at the other side of ∞) and we find $\ell \in A$ and $p \in \mathcal{G}(j,\ell)$ with $(\ell, p) \neq (k, h)$ such that $\infty \in p.\overline{I_\ell}$. Since $p.I_\ell$ and $h.I_k$ are disjoint by (B7a), we obtain that ∞ is a boundary point of $p.I_\ell$ and further that $(k,h), (\ell,p) \in \mathrm{Att}_{\mathcal{C}}(\infty)$ and $p.\overline{I_\ell} \cup h.\overline{I_k}$ is a neighborhood of ∞ in $\widehat{\mathbb{R}}$. This contradicts our hypothesis that $\pi(\infty)$ is not attached to $\mathcal{C}$. Thus, ∞ must be an inner point of the interval $h.I_k$.

In turn, since $h.I_k$ is open, $\infty \in h.I_k$ (and hence (k,h) is uniquely determined). The combination of (B7b) and (B6) implies that

$$h.I_k \subsetneq I_j\,.$$

Inductively we obtain a sequence $((k_n, g_n))_{n\in\mathbb{N}}$ in $A \times \Gamma$ such that for each $n \in \mathbb{N}$,

$$\infty \in g_n.I_{k_n}$$

and

$$g_{n+1}.I_{k_{n+1}} \subsetneq g_n.I_{k_n}\,.$$

Since the family of the geodesic segments $g_n.\overline{\mathrm{bp}(\mathrm{C}_{k_n})}$, $n \in \mathbb{N}$, is locally finite by Proposition 4.6, the intervals $g_n.I_{k_n}$, $n \in \mathbb{N}$, zero in on ∞. But this implies that the family of maxima

$$\hbar_{k_n,g_n} := \max\left\{ \mathrm{Im}\, z \;\middle|\; z \in g_n.\overline{\mathrm{bp}(\mathrm{C}_{k_n})} \right\}, \qquad n \in \mathbb{N}\,,$$

is unbounded, which contradicts Lemma 4.25. Thus, the assumption that $\pi(\infty)$ is not attached to $\mathcal{C}$ fails. This completes the proof that $\mathcal{C}$ being finitely ramified implies that all cusps of $\mathbb{X}$ are attached to $\mathcal{C}$.

In the case that $\mathbb{X}$ does not have cusps, the converse implication (i. e., if all cusps are attached to $\mathcal{C}$, then $\mathcal{C}$ is finitely ramified) has already been established in Lemma 4.23. For its proof in the general case we suppose that $\mathbb{X}$ has cusps and that every cusp of $\mathbb{X}$ is attached to $\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$. We aim to show that $\mathcal{C}$ is finitely ramified. However, in order to seek a contradiction we assume that $\mathcal{C}$ is infinitely ramified. As in the proof of Lemma 4.23 we find and fix $j, k \in A$ such that $\#\mathcal{G}(j,k) = +\infty$, we may suppose that the interval I_j is contained in $\mathbb{R}$ and bounded, and we find a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}(j,k)$ such that the endpoint sequences $(g_n.\mathrm{X}_k)_{n\in\mathbb{N}}$ and $(g_n.\mathrm{Y}_k)_{n\in\mathbb{N}}$ are contained in I_j and converge to an element

$$a \in \Lambda(\Gamma) \cap \overline{I_j}\,.$$

From the proof of Lemma 4.23 we obtain further that $a \notin \widehat{\mathbb{R}}_{\mathrm{st}}$. Thus, it remains to consider the case that a is a parabolic fixed point (cf. (1.42)). By hypothesis, the cusp $\widehat{a}$ of $\mathbb{X}$ is attached to $\mathcal{C}$. In preparation for the following considerations, we now pick $(i,h) \in \mathrm{Att}_{\mathcal{C}}(a)$ such that $I_j \subseteq h.I_i$ (if such a pair exists, otherwise we omit this step) and show that we also find $(\ell, g) \in \mathrm{Att}_{\mathcal{C}}(a)$ such that $g.I_\ell \subseteq I_j$. To that end we note that, since $a \in \overline{I_j}$ and a is an endpoint of the interval $h.I_i$, the point a is also an endpoint of I_j. Let $p \in \Gamma$ be a parabolic element that fixes a. Then the pairs (j,p) and (j,p^{-1}) belong to $\mathrm{Att}_{\mathcal{C}}(a)$ and either

$$p.I_j \subseteq I_j \qquad \text{or} \qquad p^{-1}.I_j \subseteq I_j\,,$$

which shows the existence of such a pair (ℓ, g).

From the attachment property of the cusp $\widehat{a}$ it follows that we find two distinct pairs $(\ell_1, h_1), (\ell_2, h_2) \in A \times \Gamma$ such that a is a joint endpoint of the intervals $h_1.I_{\ell_1}$ and $h_2.I_{\ell_2}$, and

$$h_1.\overline{I_{\ell_1}} \cup h_2.\overline{I_{\ell_2}}$$

is a neighborhood of a in $\mathbb{R}$. By the previous argument we may further suppose that at least one of these intervals intersects I_j but does not cover I_j. Without loss of generality, we suppose it to be $h_1.I_{\ell_1}$.

We suppose first that $h_1.I_{\ell_1} \subseteq I_j$ and fix $n \in \mathbb{N}$ such that $g_n.I_k \subsetneq h_1.I_{\ell_1}$ (the existence of n follows directly from the properties of the two sequences $(g_n.\mathrm{X}_k)_n$ and $(g_n.\mathrm{Y}_k)_n$). By the density of $E(\mathbb{X})$ in $\Lambda(\Gamma) \times \Lambda(\Gamma)$ (see Proposition 1.15) we find

$$(x,y) \in E(\mathbb{X}) \cap \big(g_n.I_k \times J_j\big)\,.$$

Then (B5) implies that the geodesic segment $(x,y)_{\mathbb{H}}$ intersects $\mathrm{bp}(\mathrm{C}_j)$ in some point, say z, and further intersects $g_n.\,\mathrm{bp}(\mathrm{C}_k)$ in some point, say w, and intersects $h_1.\,\mathrm{bp}(\mathrm{C}_{\ell_1})$ in some point, say u, with $u \in (z,w)_{\mathbb{H}}$. This contradicts (B7).

In turn, $h_1.I_{\ell_1} \not\subseteq I_j$. Then one endpoint of $h_1.I_{\ell_1}$, namely a, is contained in I_j, while the other endpoint is not. Convexity implies that $\overline{\mathrm{bp}(\mathrm{C}_j)} \cap h_1.\overline{\mathrm{bp}(\mathrm{C}_{\ell_1})} \neq \varnothing$ but $\overline{\mathrm{bp}(\mathrm{C}_j)} \neq h_1.\overline{\mathrm{bp}(\mathrm{C}_{\ell_1})}$. But this contradicts (B6). It follows that $\mathcal{C}$ is finitely ramified. $\square$

Proposition 4.26 indicates how we could turn an infinitely ramified set of branches into a finitely ramified one: if we find a way to augment the initial (infinitely ramified) set of branches with further branches such that all cusps of $\mathbb{X}$ are attached to the enlarged family of branches, then the ramification becomes finite. By comparing Figure 9 to Figure 10 in Example 4.21 above, one sees that this approach has been carried out successfully for the group Γ_2. Proposition 4.28 below states that this can always be done. We emphasize that its proof is constructive and provides an algorithm for the enlargement procedure of the set of branches.

For the proof of Proposition 4.28 we will take advantage of Ford fundamental domains and some of their specific properties, which the reader is therefore advised to recall from Section 1.10, in particular Lemma 1.37 and all the subsequent discussions, definitions, and results.

Lemma 4.27. *Suppose that $\pi(\infty)$ is a cusp of $\mathbb{X}$ and that the set of branches $\mathcal{C}$ satisfies $\mathrm{Att}_{\mathcal{C}}(\infty) = \varnothing$. Let $(j, g) \in A \times \Gamma$ be such that*

$$\max\left\{\operatorname{Im} z \;\middle|\; z \in \Gamma.\overline{\mathrm{bp}(\mathrm{C})}\right\} = \max\left\{\operatorname{Im} z \;\middle|\; z \in g.\overline{\mathrm{bp}(\mathrm{C}_j)}\right\}$$

(the existence of the pair (j, g) is guaranteed by Lemma 4.25). Pick $x \in \mathbb{R}$ such that $(g.\mathrm{X}_j, \infty) \subseteq \mathcal{F}_\infty(x)$. Then the following statements hold true.

(i) *We have $g.\overline{\mathrm{bp}(\mathrm{C}_j)} \cap \overline{\mathcal{K}} \neq \varnothing$. More precisely, the point of maximal height of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ is contained in $\overline{\mathcal{K}}$.*

(ii) *There exists $\varepsilon > 0$ such that*

$$(g.\mathrm{X}_j - \varepsilon, g.\mathrm{Y}_j - \varepsilon) \subseteq \operatorname{Re}\big(\mathcal{F}_\infty(x)\big)$$

or

$$(g.\mathrm{Y}_j + \varepsilon, g.\mathrm{X}_j + \varepsilon) \subseteq \operatorname{Re}\big(\mathcal{F}_\infty(x)\big)\,.$$

(iii) *There exists $x \in \mathbb{R}$ such that $(g.\mathrm{X}_j, \infty) \cap \mathcal{F}(x) \neq \varnothing$ and the point of maximal height of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ is contained in $\overline{\mathcal{F}(x)}$, where $\mathcal{F}(x) = \mathcal{F}_\infty(x) \cap \mathcal{K}$.*

Proof. Let z_0 denote the (unique) point of maximal height of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$. By the choice of (j, g), we have

$$\operatorname{Im} z_0 = \max\left\{\operatorname{Im} z \;\middle|\; z \in \Gamma.\overline{\mathrm{bp}(\mathrm{C})}\right\}\,. \tag{4.18}$$

In order to show (i), we assume for contradiction that $g.\overline{\mathrm{bp}(\mathrm{C}_j)} \cap \overline{\mathcal{K}} = \varnothing$. Then

$$g.\overline{\mathrm{bp}(\mathrm{C}_j)} \subseteq \bigcup_{h \in \Gamma \setminus \Gamma_\infty} \operatorname{int} \mathrm{I}(h)\,.$$

We fix $p = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma \setminus \Gamma_\infty$ such that $z_0 \in \operatorname{int} \mathrm{I}(p)$. Thus, $|cz_0 + d| < 1$, and it follows that

$$\operatorname{Im}(p.z_0) = \frac{\operatorname{Im} z_0}{|cz_0 + d|^2} > \operatorname{Im} z_0 \,,$$

which contradicts the choice of (j, g) and z_0 (see (4.18)). In turn,

$$g.\overline{\operatorname{bp}(\mathrm{C}_j)} \cap \overline{\mathcal{K}} \neq \varnothing \,.$$

For (ii), let $\lambda > 0$ be the cusp width of $\pi(\infty)$. Lemma 4.25 shows that the height of $g.\overline{\operatorname{bp}(\mathrm{C}_j)}$ is bounded from above by $\lambda/2$. Thus,

$$|g.\mathrm{X}_j - g.\mathrm{Y}_j| \leq \lambda \,.$$

Since $\mathcal{F}_\infty(x) = \operatorname{Re}|_{\mathbb{H}}^{-1}((x, x+\lambda))$ and $(g.\mathrm{X}_j, \infty) \in \mathcal{F}_\infty(x)$, the statement of (ii) follows immediately. Statement (iii) is an immediate consequence of (i) and (ii). □

With these preparations we can now provide and prove the enlargement procedure, in the proof of the following proposition.

Proposition 4.28. *Suppose that $\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$ is infinitely ramified and let m be the number of cusps of $\mathbb{X}$ not attached to $\mathcal{C}$. Then there exists a finitely ramified set of branches for $\widehat{\Phi}$ of the form*

$$\mathcal{C}' := \{\mathrm{C}_1, \ldots, \mathrm{C}_N, \mathrm{C}_{N+1}, \ldots, \mathrm{C}_{N+k}\}$$

for some $k \in \mathbb{N}$, $k \leq 2m$.

Proof. By Proposition 4.26, the hyperbolic orbisurface $\mathbb{X}$ has at least one cusp that is not attached to $\mathcal{C}$, say $\widehat{c}$. We will enlarge $\mathcal{C}$ to a set of branches $\mathcal{C}'$ to which $\widehat{c}$ is attached and which contains at most two branches more than $\mathcal{C}$. Since $\mathbb{X}$ has only finitely many cusps as a geometrically finite orbifold, a finite induction then yields the statement, including the counting bound.

Without loss of generality, we may suppose that $\widehat{c}$ is represented by ∞. If this is not the case, then we pick any representative of $\widehat{c}$, say c, and any $q \in \mathrm{PSL}_2(\mathbb{R})$ such that $q.c = \infty$, consider $\{q.\,\mathrm{C}_j \mid j \in A\}$ instead of $\mathcal{C}$, $q\Gamma q^{-1}$ instead of Γ, perform the enlargement as described in what follows and finally undo the transformation by applying q^{-1}. We distinguish the following two cases:

$$\operatorname{Att}_{\mathcal{C}}(\infty) \neq \varnothing \tag{I}$$

and

$$\operatorname{Att}_{\mathcal{C}}(\infty) = \varnothing \,. \tag{II}$$

In Case (I) we pick $(j,g) \in \mathrm{Att}_{\mathcal{C}}(\infty)$ and let

$$\mathrm{C}_{n+1} := \left\{ v \in \mathrm{S}\mathbb{H} \;\middle|\; \mathrm{bp}(v) \in g.\overline{\mathrm{bp}(\mathrm{C}_j)},\ \gamma_v(+\infty) \in g.J_j \right\}$$

be the set of unit tangent vectors that are based at $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ but point into the opposite direction as $g.\,\mathrm{C}_j$. We emphasize that we allow the whole set $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ as base points and do not restrict to those that lie on a geodesic connecting points in $\widehat{\mathbb{R}}_{\mathrm{st}}$. We set

$$\mathcal{C}' := \mathcal{C} \cup \{\mathrm{C}_{n+1}\} \qquad \text{and} \qquad A' := A \cup \{n+1\}\,.$$

In Case (II), Lemma 4.25 shows the existence of $(j,g) \in A \times \Gamma$ such that

$$\max \left\{ \operatorname{Im} z \;\middle|\; z \in g.\overline{\mathrm{bp}(\mathrm{C}_j)} \right\} = \max_{k \in A} \hbar_{\widehat{\mathcal{C}}}(k)\,.$$

We let

$$\mathrm{C}_{n+1} := \{ v \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(v) \in (g.\mathrm{X}_j, \infty)_{\mathbb{H}},\ \gamma_v(+\infty) \in (g.\mathrm{X}_j, +\infty) \}$$

and

$$\mathrm{C}_{n+2} := \{ v \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(v) \in (g.\mathrm{X}_j, \infty)_{\mathbb{H}},\ \gamma_v(+\infty) \in (-\infty, g.\mathrm{X}_j) \}$$

be the sets of unit tangent vectors based on the geodesic segment $(g.\mathrm{X}_j, \infty)_{\mathbb{H}}$ and pointing into one or the other of the associated half-spaces. We set

$$\mathcal{C}' := \mathcal{C} \cup \{\mathrm{C}_{n+1}, \mathrm{C}_{n+2}\}$$

and

$$A' := A \cup \{n+1, n+2\}\,.$$

In both cases we set

$$\mathrm{C}' := \bigcup_{j \in A'} \mathrm{C}_j\,.$$

To show ($\mathrm{B}_{\mathrm{Per}}$) for $\mathcal{C}'$, we note that $\mathcal{C}'$ is a superset of $\mathcal{C}$. Since $\mathcal{C}$ satisfies ($\mathrm{B}_{\mathrm{Per}}$), $\mathcal{C}'$ does so as well. This yields (B4) by virtue of Proposition 4.8.

The validity of (B2), (B3), and (B5) for $\mathcal{C}'$ is obvious from the construction of the additional branches. For each $j \in A'$ we define the sets I_j, J_j, $\mathrm{H}_+(j)$ and $\mathrm{H}_-(j)$ as in (B3). For $j \in A$ these sets obviously coincide with those related to $\mathcal{C}$. In Case (I) we have

$$I_{n+1} = g.J_j\,, \quad J_{n+1} = g.I_j \quad \text{and} \quad \mathrm{H}_\pm(n+1) = g.\mathrm{H}_\mp(j)\,.$$

In Case (II) we have

$$I_{n+1} = J_{n+2} = (g.\mathrm{X}_j, +\infty)\,, \quad J_{n+1} = I_{n+2} = (-\infty, g.\mathrm{X}_j)$$

and

$$\begin{aligned} \mathrm{H}_+(n+1) = \mathrm{H}_-(n+2) = \{z \in \mathbb{H} \mid \operatorname{Re} z > g.\mathrm{X}_j\} \\ \mathrm{H}_-(n+1) = \mathrm{H}_+(n+2) = \{z \in \mathbb{H} \mid \operatorname{Re} z < g.\mathrm{X}_j\}\,. \end{aligned}$$

We now prove (B1) for $\mathcal{C}'$. For $j \in A' \setminus A$ we pick $(x,y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$ and fix $\varepsilon > 0$ such that $\mathrm{B}_{\mathbb{R},\varepsilon}(x) \subseteq I_j$ and $\mathrm{B}_{\mathbb{R},\varepsilon}(y) \subseteq J_j$. (Note that $\infty \notin \widehat{\mathbb{R}}_{\mathrm{st}}$.) Then $(x,y) \in \Lambda(\Gamma) \times \Lambda(\Gamma)$. By Proposition 1.15 we find a geodesic γ on $\mathbb{H}$ that represents a periodic geodesic on $\mathbb{X}$ and satisfies

$$\gamma(+\infty) \in \mathrm{B}_{\mathbb{R},\varepsilon}(x) \qquad \text{and} \qquad \gamma(-\infty) \in \mathrm{B}_{\mathbb{R},\varepsilon}(y)\,.$$

The geodesic intersects C_j as can be seen directly from the definition of this set. This shows (B1).

In Case (I) property (B6) for $\mathcal{C}'$ follows immediately from

$$\overline{\mathrm{bp}(\mathrm{C}_{n+1})} = g.\overline{\mathrm{bp}(\mathrm{C}_j)} \qquad \text{and} \qquad \mathrm{H}_\pm(n+1) = g.\mathrm{H}_\mp(j)\,.$$

In order to establish (B6) for $\mathcal{C}'$ in Case (II), we let $a, b \in A'$, $h \in \Gamma$ be such that

$$\overline{\mathrm{bp}(\mathrm{C}_a)} \cap h.\overline{\mathrm{bp}(\mathrm{C}_b)} \neq \varnothing\,.$$

We consider first the case that $a = n+1$, $b = n+2$ and $\overline{\mathrm{bp}(\mathrm{C}_a)} = h.\overline{\mathrm{bp}(\mathrm{C}_b)}$. Recall that

$$\overline{\mathrm{bp}(\mathrm{C}_{n+1})} = \overline{\mathrm{bp}(\mathrm{C}_{n+2})} = (g.\mathrm{X}_j, \infty)_{\mathbb{H}}\,.$$

From $\mathrm{Att}_{\mathcal{C}}(\infty) = \varnothing$ it follows that $\pi(g.\mathrm{X}_j) \neq \pi(\infty)$ (because otherwise we would have $(j,g) \in \mathrm{Att}_{\mathcal{C}}(\infty)$, contradicting the assumption). Thus, h fixes both endpoints of $\overline{\mathrm{bp}(\mathrm{C}_{n+1})}$. Since ∞ is cuspidal, $h = \mathrm{id}$. Further, by construction, $\mathrm{H}_\pm(n+1) = \mathrm{H}_\mp(n+2)$. Hence, (B6) is satisfied in this case.

The previous case in combination with the fact that (B6) is satisfied for $\mathcal{C}$ allows us to restrict all further considerations to the case that $a = n+1$ and $b \in A \cup \{n+1\}$. We show first that necessarily $b = n+1$. To that end, in order to seek a contradiction, we assume that $b \in A$. Then $\overline{\mathrm{bp}(\mathrm{C}_a)} \neq h.\overline{\mathrm{bp}(\mathrm{C}_b)}$ as $\mathrm{Att}_{\mathcal{C}}(\infty) = \varnothing$, and hence the geodesic segments $\overline{\mathrm{bp}(\mathrm{C}_a)}$ and $h.\overline{\mathrm{bp}(\mathrm{C}_b)}$ intersect transversally. We recall the tuple $(j,g) \in A \times \Gamma$ from the construction of $\mathcal{C}'$. Since the geodesic segment $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ has maximal radius among all semi-circles in $\Gamma.\overline{\mathrm{bp}(\mathrm{C}_k)}$ with $k \in A$, and $g.\mathrm{X}_j$ is a joint endpoint of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ and $\overline{\mathrm{bp}(\mathrm{C}_{n+1})}$, the geodesic segment $h.\overline{\mathrm{bp}(\mathrm{C}_b)}$ intersects $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$. Since $\mathcal{C}$ satisfies (B6), it follows that $h.\overline{\mathrm{bp}(\mathrm{C}_b)} = g.\overline{\mathrm{bp}(\mathrm{C}_j)}$. But then $h.\overline{\mathrm{bp}(\mathrm{C}_b)}$ does not intersect $\overline{\mathrm{bp}(\mathrm{C}_{n+1})}$. In turn, this case is impossible.

It remains to consider the case that $a = b = n+1$ and

$$\overline{\mathrm{bp}(\mathrm{C}_{n+1})} \cap h.\overline{\mathrm{bp}(\mathrm{C}_{n+1})} \neq \varnothing$$

but

$$\overline{\mathrm{bp}(\mathrm{C}_{n+1})} \neq h.\overline{\mathrm{bp}(\mathrm{C}_{n+1})}\,.$$

We let $\beta := \overline{\mathrm{bp}(\mathrm{C}_{n+1})}$ and suppose without loss of generality that the endpoints of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ satisfy

$$g.\mathrm{X}_j < g.\mathrm{Y}_j\,.$$

(If $g.\mathrm{X}_j > g.\mathrm{Y}_j$, the argumentation in what follows applies with some changes of orderings.) Since $h.\beta$ is a non-vertical geodesic arc such that the real interval enclosed between its two endpoints contains $g.\mathrm{X}_j$, which is the common endpoint of β and $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$, the argumentation in the previous paragraph yields that

$$g.\overline{\mathrm{bp}(\mathrm{C}_j)} \subseteq \mathrm{conv}_{\mathrm{E}}(h.\beta) \setminus \partial\mathrm{conv}_{\mathrm{E}}(h.\beta)\,, \tag{4.19}$$

where, as in the proof of Lemma 4.25, $\mathrm{conv}_{\mathrm{E}}(M)$ denotes the convex hull of the set M in $\mathbb{C}$ with respect to the Euclidean metric.

We now fix $x \in \mathbb{R}$ such that $\beta \subseteq \mathcal{F}_\infty(x)$ and such that the point z_0 of maximal height of $g.\overline{\mathrm{bp}(\mathrm{C}_j)}$ is contained in $\overline{\mathcal{F}(x)}$ and hence in $\overline{\mathcal{K}}$. The choice of x is possible by Lemma 4.27. With (4.19), we obtain that

$$z_0 \in \mathrm{conv}_{\mathrm{E}}(h.\beta) \cap \overline{\mathcal{F}(x)}\,. \tag{4.20}$$

We consider the strip-shaped set

$$S := \left\{ w + \mathrm{i}t \;\middle|\; w \in g.\overline{\mathrm{bp}(\mathrm{C}_j)} \cap \overline{\mathcal{K}}\,,\ \mathrm{Re}\, w \in \mathrm{Re}(\mathcal{F}_\infty(x))\,,\ t > 0 \right\}\,.$$

The set S is convex due to the convexity of $\mathcal{K}$ and $\mathcal{F}_\infty(x)$ and the boundary structure of $\mathcal{K}$. Further,

$$S \subseteq \mathcal{F}(x) \tag{4.21}$$

and $(z_0, \infty)_{\mathbb{H}} \subseteq S$. The latter implies that $h.\beta \cap S \neq \varnothing$. We now define the two domains

$$L := \left(\mathbb{H} \setminus \mathrm{conv}_{\mathrm{E}}(g.\overline{\mathrm{bp}(\mathrm{C}_j)})\right) \cap \left\{ z \in \mathbb{H} \;\middle|\; \mathrm{Re}\, z < \inf_{w \in S} \mathrm{Re}\, w \right\}$$

and

$$R := \left(\mathbb{H} \setminus \mathrm{conv}_{\mathrm{E}}(g.\overline{\mathrm{bp}(\mathrm{C}_j)})\right) \cap \left\{ z \in \mathbb{H} \;\middle|\; \mathrm{Re}\, z > \sup_{w \in S} \mathrm{Re}\, w \right\}\,.$$

Because the sets L, R and $\overline{L} \cup \overline{R} \cup \overline{S}$ are convex, and $h.\beta$ intersects S, for the pair $(h.\infty, hg.\mathrm{X}_j)$ of the two endpoints of $h.\beta$ we obtain

$$(h.\infty, hg.\mathrm{X}_j) \in (\underset{\sim}{g}L \times \underset{\sim}{g}R) \cup (\underset{\sim}{g}R \times \underset{\sim}{g}L)\ . \tag{4.22}$$

Since $\beta \cap \mathcal{F}(x)$ is of the form $(b, \infty)_{\mathbb{H}}$ for some point $b \in \mathbb{H}$, and $\mathcal{F}(x)$ contains all subsets of the form $\{z \in \mathbb{H} \mid \operatorname{Re} z \in (b-\varepsilon, b+\varepsilon),\ \operatorname{Im} z > y_0\}$ for sufficiently small $\varepsilon > 0$ and sufficiently large $y_0 > 0$, (4.22) and the convexity of the sets L, R imply that we have

$$h.\mathcal{F}(x) \cap L \neq \varnothing \quad \text{if } \ h.\infty \in \underset{\sim}{g}L \tag{4.23}$$

and

$$h.\mathcal{F}(x) \cap R \neq \varnothing \quad \text{if } \ h.\infty \in \underset{\sim}{g}R\,. \tag{4.24}$$

We now aim to show that $h.\mathcal{F}(x)$ indeed intersects L and R. To that end we recall that we suppose that $g.\mathrm{X}_j < g.\mathrm{Y}_j$. If $h.\infty \in \underset{\sim}{g}L$, then $hg.\mathrm{X}_j \in \underset{\sim}{g}R$ and

$$\operatorname{Re}\left(hg.\overline{\mathrm{bp}(\mathrm{C}_j)}\right) \subseteq [hg.\mathrm{X}_j, +\infty)\ .$$

It follows that $h.z_0 \in h.\mathcal{F}(x) \cap R$. If $h.\infty \in \underset{\sim}{g}R$, then $hg.\mathrm{X}_j \in \underset{\sim}{g}L$ and, taking advantage of (B6) for $\mathcal{C}$, we find

$$\operatorname{Re}\left(hg.\overline{\mathrm{bp}(\mathrm{C}_j)}\right) \subseteq [hg.\mathrm{X}_j, g.\mathrm{X}_j]$$

and hence $h.z_0 \in h.\mathcal{F}(x) \cap L$. Combining this with (4.23) and (4.24), respectively, we find

$$h.\mathcal{F}(x) \cap L \neq \varnothing \qquad \text{and} \qquad h.\mathcal{F}(x) \cap R \neq \varnothing\,.$$

From the convexity of $h.\mathcal{F}(x)$ and the definitions of the sets L, R, and S it now follows that

$$h.\mathcal{F}(x) \cap S \neq \varnothing\,.$$

In combination with (4.21) this yields a contradiction. In turn, (B6) is valid for $\mathcal{C}'$.

In order to establish (B7) for $\mathcal{C}'$, we first show that the next and previous intersection times exist for all elements in $\mathrm{C}'_{\mathrm{st}}$. Thus, let $j \in A'$ and let $\nu \in \mathrm{C}_{j,\mathrm{st}}$. Using that $\Gamma.\mathcal{C}'$ is locally finite by Proposition 4.6, we see that if there exists any intersection between the geodesic γ_ν and $\Gamma.\mathrm{C}'$ at some time $t > 0$, then there exists a time-minimal one and hence $t^+_{\mathrm{C}'}(\nu)$ exists.

We suppose first that $j \in A$. Then $t^+_{\mathrm{C}}(\nu)$ exists by Proposition 4.15(ii). Thus, $t^+_{\mathrm{C}'}(\nu)$ exists as well. We suppose now that $j \in A' \setminus A$. Then

$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}\,.$$

As $\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \Lambda(\Gamma)$, Proposition 1.15 shows that for each $\varepsilon > 0$ we find a geodesic η_ε

on $\mathbb{H}$ that represents a periodic geodesic on $\mathbb{X}$ and whose endpoints satisfy

$$\eta_\varepsilon(\pm\infty) \in \mathrm{B}_{\mathbb{R},\varepsilon}(\gamma_\nu(\pm\infty))\,.$$

Since ($\mathrm{B}_{\mathrm{Per}}$) is valid for $\mathcal{C}$, we find $(k_\varepsilon, g_\varepsilon) \in A \times \Gamma$ such that η_ε intersects $g_\varepsilon.\,\mathrm{C}_{k_\varepsilon}$. Thus,

$$\big(\eta_\varepsilon(+\infty), \eta_\varepsilon(-\infty)\big) \in g_\varepsilon.I_{k_\varepsilon,\mathrm{st}} \times g_\varepsilon.J_{k_\varepsilon,\mathrm{st}}\,.$$

Since $g_\varepsilon.I_{k_\varepsilon}$ and $g_\varepsilon.J_{k_\varepsilon}$ are open and $\mathrm{X}_{k_\varepsilon}, \mathrm{Y}_{k_\varepsilon} \notin \widehat{\mathbb{R}}_{\mathrm{st}}$, we can choose ε so small that

$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in g_\varepsilon.I_{k_\varepsilon} \times g_\varepsilon.J_{k_\varepsilon}\,.$$

We fix such an ε and set $g := g_\varepsilon$, $k := k_\varepsilon$. Now

$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in g.I_{k,\mathrm{st}} \times g.J_{k,\mathrm{st}}$$

and (B5) for $\mathcal{C}$ show that $g^{-1}.\gamma_\nu$ intersects $\mathrm{C}_{k,\mathrm{st}}$, say in $\eta \in \mathrm{C}_{k,\mathrm{st}}$ at time t_0. We consider the system of iterated sequences of η with respect to $\mathcal{C}$, as defined in (4.9)–(4.15). Lemma 4.18 shows the existence of $n \in \mathbb{Z}$ such that

$$\mathrm{t}_{\mathrm{C},n}(\nu) > t_0\,,$$

which means that $g^{-1}.\gamma_\nu$ intersects $\mathrm{g}_{\mathrm{C},n}.\,\mathrm{C}_{\mathrm{k}_{\mathrm{C},n}(\nu)}$ at a time larger than t_0. Thus, there exists an intersection between γ_ν and C' at a positive time, and hence $t^+_{\mathrm{C}'}(\nu)$ exists. Analogously, we can show the existence of $t^-_{\mathrm{C}'}(\nu)$ in both cases.

For $j, k \in A'$ we set, motivated by Proposition 4.15(iii),

$$\mathcal{G}'(j,k) := \left\{ g \in \Gamma \;\middle|\; \exists\, \nu \in \mathrm{C}_{j,\mathrm{st}} \colon \gamma'_\nu(t^+_{\mathrm{C}'}(\nu)) \in g.\,\mathrm{C}_k \right\}\,.$$

We now show that $\mathcal{C}'$ satisfies (B7a) with the family $\{\mathcal{G}'(j,k)\}_{j,k\in A'}$ in place of $\{\mathcal{G}(a,b)\}_{a,b\in A'}$. Let $j,k \in A'$ and $g \in \mathcal{G}'(j,k)$. Thus, we find $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that $\gamma_\nu(t^+_{\mathrm{C}'}(\nu)) \in g.\,\mathrm{C}_k$ and hence

$$\gamma_\nu(+\infty) \in I_{j,\mathrm{st}} \cap g.I_k\,.$$

(B6) implies that $g.I_k \subseteq I_j$. It follows that

$$\bigcup_{k\in A'} \bigcup_{g\in\mathcal{G}'(j,k)} g.I_k \subseteq I_j \tag{4.25}$$

and

$$\bigcup_{k\in A'} \bigcup_{g\in\mathcal{G}'(j,k)} g.I_{k,\mathrm{st}} \subseteq I_{j,\mathrm{st}}\,. \tag{4.26}$$

The disjointness of these unions follows immediately from (B6). Further, for any $x \in I_{j,\mathrm{st}}$ we choose $y \in J_{j,\mathrm{st}}$. By (B5), the geodesic γ from y to x intersects $\mathrm{C}_{j,\mathrm{st}}$, say in ν. Now $t^+_{\mathrm{C}'}(\nu)$ exists as we have seen above. Thus,

$$\gamma_\nu(t^+_{\mathrm{C}'}(\nu)) \in \mathrm{g}_{\mathrm{C}',1}(\nu)\boldsymbol{.}\, \mathrm{C}_{\mathrm{k}_{\mathrm{C}',1}(\nu),\mathrm{st}}$$

and hence $\mathrm{g}_{\mathrm{C}',1}(\nu) \in \mathcal{G}'(j, \mathrm{k}_{\mathrm{C}',1}(\nu))$. Therefore

$$x = \gamma_\nu(+\infty) \in \mathrm{g}_{\mathrm{C}',1}(\nu)\boldsymbol{.} I_{\mathrm{k}_{\mathrm{C}',1}(\nu),\mathrm{st}}\,.$$

It follows that the inclusion in (4.26) is indeed an equality. This completes the proof of (B7a). The proof of (B7c) is analogous, using the existence of $t^-_{\mathrm{C}'}(\nu)$ for all $\nu \in \mathrm{C}'_{\mathrm{st}}$. (B7b) follows immediately from (B6) and the definition of the sets $\mathcal{G}'(j,k)$ for $j,k \in A'$. $\square$

4.5 Admissible Sets of Branches

This section is devoted to the proof that every set of branches can be rearranged into an admissible one (see (B8)). We further introduce an additional property of sets of branches, which will be needed later on (see Proposition 5.19 below) as a prerequisite in order to assure a non-collapsing behavior in the sense of (B9). Again, every set of branches can be rearranged to one with that property, and so that, simultaneously, admissibility is assured.

We retain all assumptions and notations from Section 4.3.

Definition 4.29. A set of branches $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ is called *weakly non-collapsing* if

($\mathrm{B_{col}}$) For every pair $(j,k) \in A \times A$, for every $\nu \in \mathrm{C}_j$ such that γ_ν intersects C_k at some time $t^* > 0$, the geodesic segment $\gamma_\nu((0,t^*))$ does not intersect $g\boldsymbol{.}\mathrm{C}$ for any $g \in \Gamma^*$.

Remark 4.30. Via contraposition it is easy to see that (B9) implies ($\mathrm{B_{col}}$): Assume that a given set of branches $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ is not weakly non-collapsing. Then we find $j,k,\ell \in A$, $g \in \Gamma^*$, $0 < t_1 < t_2$, and $\nu \in \mathbb{SH}$ such that

$$\gamma'_\nu(0) \in \mathrm{C}_j\,, \quad \gamma'_\nu(t_1) \in g\boldsymbol{.}\,\mathrm{C}_\ell\,, \quad \text{and} \quad \gamma'_\nu(t_2) \in \mathrm{C}_k\,.$$

Consider the system of iterated sequences $[(\mathrm{t}_{\mathrm{C},n}(\nu))_n, (\mathrm{k}_{\mathrm{C},n}(\nu))_n, (\mathrm{g}_{\mathrm{C},n}(\nu))_n]$ associated to ν by (4.15). By Proposition 4.19 there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\mathrm{t}_{\mathrm{C},n_1}(\nu) = t_1 \qquad \text{and} \qquad \mathrm{t}_{\mathrm{C},n_2}(\nu) = t_2\,.$$

The above then implies

$$\mathrm{g}_{\mathrm{C},1}(\nu)\mathrm{g}_{\mathrm{C},2}(\nu)\cdots\mathrm{g}_{\mathrm{C},n_2}(\nu) = \mathrm{id}\,.$$

Hence, $\mathcal{C}$ does not fulfill (B9). However, the properties (B9) and ($\mathrm{B}_{\mathrm{col}}$) are not equivalent, since ($\mathrm{B}_{\mathrm{col}}$) allows for elements $\nu \in \mathrm{C}$ whose induced geodesics have future intersections with C that are not immediate, while (B9) does not.

Property ($\mathrm{B}_{\mathrm{col}}$) demands that the set of branches is structured in a specific way, relative to the non-trivial Γ-translates of itself. Lemma 4.10 allows us to exchange branches with Γ-translates of themselves. In what follows, we describe a sorting algorithm which is based on that principle and which transforms any given set of branches into a weakly non-collapsing one.

For reasons of effectivity, we introduce so-called *branch trees*: Let $j \in A$. The root node (level 0) of the *branch tree relative to* j is $(j, \mathrm{id}) \in A \times \Gamma$. The nodes at level 1 are all tuples of the form $(\mathrm{k}_{\mathrm{C},1}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)) \in A \times \Gamma$, for $\nu \in \mathrm{C}_{j,\mathrm{st}}$. By virtue of (B7), the finiteness of A and the discreteness of Γ, there are at most countably many such tuples in level 1. The nodes at level r, for $r \in \mathbb{N}$, are all tuples of the form $(\mathrm{k}_{\mathrm{C},r}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},r}(\nu)) \in A \times \Gamma$, for $\nu \in \mathrm{C}_{j,\mathrm{st}}$. Two nodes are linked by an edge if and only if they are of the form

$$(\mathrm{k}_{\mathrm{C},r}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},r}(\nu)) \quad \text{and} \quad (\mathrm{k}_{\mathrm{C},r+1}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},r}(\nu)\mathrm{g}_{\mathrm{C},r+1}(\nu))$$

for *the same* $\nu \in \mathrm{C}_{j,\mathrm{st}}$. Hence, every path in the branch tree is of the form

$$(j, \mathrm{id}) \multimap (\mathrm{k}_{\mathrm{C},1}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)) \multimap (\mathrm{k}_{\mathrm{C},2}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\mathrm{g}_{\mathrm{C},2}(\nu)) \multimap \ldots \tag{4.27}$$

for *the same* $\nu \in \mathrm{C}_{j,\mathrm{st}}$ in each tuple, where $(k_1, h_1) \multimap (k_2, h_2)$ denotes an edge in the tree, for $k_1, k_2 \in A$, $h_1, h_2 \in \Gamma$. Hence, a path in a branch tree corresponds to the existence of a geodesic on $\mathbb{H}$ intersecting $h.\mathrm{C}_k$ for every node (k, h) on that path. We denote the branch tree with the root (j, id) by B_j. The collection of one or more branch trees is called a *branch forest*. In what follows we adopt a straightforward terminology of sub- and super-trees as sub- and super-graphs of trees. A sub- or super-tree is called *complete* if it contains all child nodes of its root node. The *level* of a node in a tree is the number of edges in the unique path joining the root node to it. Furthermore, we consider trees as directed graphs, with direction towards increasing level.

We define a left multiplication of elements $g \in \Gamma$ on the set $\{B_j \mid j \in A\}$ by defining gB_j to be the tree that arises from B_j by exchanging every node (k, h) in B_j for (k, gh). We emphasize that we construct further trees in this way. Then the complete sub-tree with the root $(\mathrm{k}_{\mathrm{C},r}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},r}(\nu))$ is given by $\mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},r}(\nu)B_{\mathrm{k}_{\mathrm{C},r}(\nu)}$, at every level $r \in \mathbb{N}$. Conversely, every complete sub-tree of B_j, $j \in A$, is of the form gB_k, for some $k \in A$ and $g \in \Gamma$. Every node (k, h), with $k \in A$ and $h \in \Gamma$, is unique within a branch tree B_j (but not necessarily within a branch forest). A complete sub-tree of some tree B_j, $j \in A$, is called Γ-*trivial* if it is of the form $\mathrm{id}B_k$ for some $k \in A$. This is obviously the case if and only if $\mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},r}(\nu) = \mathrm{id}$, with $r \in \mathbb{N}$ and $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that $k = \mathrm{k}_{\mathrm{C},r}(\nu)$. This fact implies the following characterization.

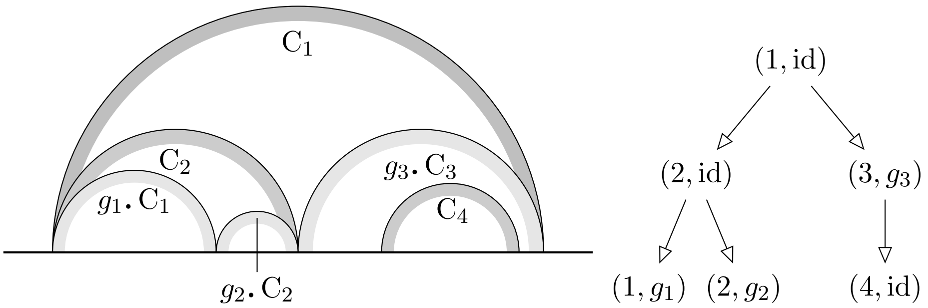


Figure 11: A schematic example of a local relationship in a set of branches and one of the branch trees emerging from it, up to level 2.

Lemma 4.31. *The set of branches $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ is weakly non-collapsing if and only if, for every $j \in A$, every complete super-tree of any Γ-trivial sub-tree of B_j is itself Γ-trivial.*

Using this characterization we can implement an algorithm to rearrange a set of branches in order to obtain a weakly non-collapsing structure. To that end, we first fix a convenient choice of root nodes given by initial branches: A branch C_j, $j \in A$, is called *initial* if

$$\forall\, k \in A \setminus \{j\} \colon \mathrm{H}_-(k) \not\subseteq \mathrm{H}_-(j)\,. \tag{4.28}$$

In particular this implies that

$$\mathrm{g}_{\mathrm{C},-1}(\nu)^{-1} \cdots \mathrm{g}_{\mathrm{C},-n}(\nu)^{-1} \neq \mathrm{id}$$

for every $\nu \in \mathrm{C}_j$ and every $n \in \mathbb{N}$. Hence, a branch C_j is initial if and only if B_j does not appear as a Γ-trivial sub-tree in any branch tree other than B_j. We set

$$D_{\mathrm{ini}} \coloneqq \{j \in A \mid \mathrm{C}_j \text{ is initial}\}\ . \tag{4.29}$$

Since A is finite, the set D_{ini} is nonempty. For every $k \in A$ there exists $j \in D_{\mathrm{ini}}$ such that either $k = j$, or $\mathrm{H}_+(k) \subseteq \mathrm{H}_+(j)$. In the latter case there exists $\nu \in \mathrm{C}_j$ and $n \in \mathbb{N}$ such that

$$(\mathrm{k}_{\mathrm{C},n}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu) \cdots \mathrm{g}_{\mathrm{C},n-1}(\nu)) = (k, \mathrm{id})\,.$$

Hence, every branch tree B_k, $k \in A$, is contained as a Γ-trivial sub-tree in some member of the branch forest $F_{\mathrm{ini}} \coloneqq \{B_j \mid j \in D_{\mathrm{ini}}\}$.

The following algorithm defines transformations $q_j \in \Gamma$ for every $j \in A$. In it we apply a notion of *cutting off* nodes (k, h) from the branch forest F_{ini}. By that we mean that, for the remainder of the algorithm, one is restricted from considering the complete sub-tree hB_k in *any* branch tree B_j, $j \in D_{\mathrm{ini}}$. The *remaining* nodes are all nodes not contained in a cut off sub-tree. For $r \in \mathbb{N}$ and $k \in A$ we define $L_r(k)$ to be the set of all nodes of the form (k, h), $h \in \Gamma$, at level r, anywhere in the branch forest F_{ini}.

Algorithm 4.32. The index r below starts at 1.

Step 0. Set $q_j := \mathrm{id}$ for every $j \in D_{\mathrm{ini}}$.
Cut off all nodes from F_{ini} for which the complete sub-tree with this node as root node does not contain a Γ-trivial sub-tree.
Carry out *Step 1*.

Step r. If no nodes remain at level r, the algorithm terminates.
Otherwise, cut off all nodes (j, g) from F_{ini} in level r with $j \in A$ for which q_j has already been defined.
For all $k \in A \setminus D_{\mathrm{ini}}$ for which $L_r(k)$ contains remaining nodes, choose such a node (k, h), set $q_k := h$, and cut off all the remaining nodes in $L_r(k) \setminus \{(k, h)\}$ from F_{ini}.
Carry out *Step r+1.* ◇

Lemma 4.33. *Algorithm 4.32 defines for every $j \in A$ a transformation $q_j \in \Gamma$, before terminating after at most $N + 2$ steps. (Recall that $\#A = N$.) The arising set $\mathcal{C}' := \{q_j . \mathrm{C}_j \mid j \in A\}$ is a weakly non-collapsing set of branches for the geodesic flow on $\mathbb{X}$.*

Proof. Since $\#A = N < +\infty$, the level at which nodes of the form (k, id), $k \in A$, may appear is bounded from above. Hence, from the cutting off of sub-trees that do not contain Γ-trivial sub-trees in *Step 0* onward, the (non-complete) sub-trees of remaining nodes are all finite. Thus, Algorithm 4.32 will eventually fail to encounter remaining nodes and will thus terminate after finitely many steps. Since in every step the algorithm either terminates or defines at least one transformation q_j which has not yet been defined, the number of steps is bounded by $N + 2$.

Let $r \in \mathbb{N}$ and $k \in A$, and assume that q_k has not yet been defined at the start of *Step r*. Since the branch tree B_k is contained in some member of F_{ini}, we find $j \in A$ and $g \in \Gamma$ such that q_j has been defined at *Step r−1* and gB_j contains the node (k, id). But then the sub-tree $q_j B_j$ contains the node $(k, q_j g^{-1})$, which is remaining at the start of *Step r*. Since this argument applies for every r for which Algorithm 4.32 does not terminate in *Step r*, and since no sub-tree containing B_k is cut off in *Step 0*, the algorithm must eventually encounter a node of the form (k, h) and thus define the transformation q_k.

Finally, the set $\mathcal{C}'$ is a set of branches for the geodesic flow on $\mathbb{X}$ by virtue of Lemma 4.10. After termination of Algorithm 4.32, every path in the sub-tree of remaining nodes in any member of F_{ini} is of the form

$$(k_1, q_{k_1}) \dashrightarrow (k_2, q_{k_2}) \dashrightarrow \ldots \dashrightarrow (k_n, q_{k_n}),$$

for some $n \in \mathbb{N}$ and $k_1, \ldots, k_n \in A$. Since the branch forest of all branch trees with respect to the set of branches $\mathcal{C}'$ is given by $\left\{ q_j^{-1} B_j \,\middle|\, j \in A \right\}$, this path then reads as

$$(k_1, \mathrm{id}) \twoheadrightarrow (k_2, \mathrm{id}) \twoheadrightarrow \ldots \twoheadrightarrow (k_n, \mathrm{id}) .$$

By Lemma 4.31, this implies that $\mathcal{C}'$ is weakly non-collapsing. $\square$

Example 4.34. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma \in \mathbb{N} \setminus \{1\}, \lambda > 2}$ from Example 1.46 and its set of branches $\mathcal{C}_\mathrm{P} = \{\mathrm{C}_{\mathrm{P},1}, \ldots, \mathrm{C}_{\mathrm{P},8}\}$ from Example 2.11 (see also Example 4.4). The set of transformations defined by Algorithm 4.32 for $\mathcal{C}_\mathrm{P}$ is given by

$$q_1 = q_2 = \ldots = q_7 = \mathrm{id} \qquad \text{and} \qquad q_8 = g_\sigma ,$$

meaning the only branch that gets swapped with one of its translates in order to obtain a weakly non-collapsing set of branches $\mathcal{C}'_\mathrm{P}$ is C_8.

We now tend to show that each set of branches can be turned into one that is simultaneously weakly non-collapsing and admissible. To that end a sorting of branches more restricted than what is provided by the branch trees is needed. To be more precise, we suppose that $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ is weakly non-collapsing and let $k \in A \setminus D_\mathrm{ini}$. Then there might be more than one $i \in D_\mathrm{ini}$ for which (k, id) is a node in B_i. To overcome this issue, we define inductively for every node an associated initial node. For $i \in D_\mathrm{ini}$ we set $\jmath(i) \coloneqq i$. For (k, id) a node at level 1 we pick one $i \in D_\mathrm{ini}$ with $(k, \mathrm{id}) \in B_i$ and set $\jmath(k) \coloneqq i$. Now let $r \in \mathbb{N}$ and assume that for all nodes (k, id) up to level r the index $\jmath(k)$ has already been defined. Then, for (k, id) a node at level $r + 1$, we pick one $i \in A$ with (i, id) a node at level r and (k, id) a node in B_i, and set $\jmath(k) \coloneqq \jmath(i)$. That way we obtain a map $\jmath\colon A \to D_\mathrm{ini}$. For $i \in D_\mathrm{ini}$ we define

$$D_i \coloneqq \{k \in A \mid \jmath(k) = i\} . \tag{4.30}$$

Then

$$A = \bigcup_{i \in D_\mathrm{ini}} D_i \tag{4.31}$$

and the union on the right hand side is disjoint.

Finally, recall the notion of (finite and infinite) branch ramification from Definition 4.20.

Proposition 4.35. *For every set of branches $\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$ there exist transformations $g_1, \ldots, g_N \in \Gamma$ such that $\widetilde{\mathcal{C}} \coloneqq \{g_1 . \mathrm{C}_1, \ldots, g_N . \mathrm{C}_N\}$ is admissible and weakly non-collapsing. If $\mathcal{C}$ is finitely ramified, then so is $\widetilde{\mathcal{C}}$.*

Proof. Because of Lemma 4.33 we may assume that $\mathcal{C}$ is weakly non-collapsing. Let $i \in D_\mathrm{ini}$ and let $k \in D_i$ be of maximal level in B_i, that is, $\mathrm{id} \notin \bigcup_{\ell \in A} \mathcal{G}(k, \ell)$.

Then, whenever $k \neq i$,

$$I_k \subsetneq I_i \qquad \text{and} \qquad J_i \subsetneq J_k\,.$$

Since I_k and J_i are both open and contain elements of $\Lambda(\Gamma)$ by virtue of (B1), Proposition 1.15 yields a hyperbolic transformation $h_i \in \Gamma$ such that

$$\big(\mathrm{f}_+(h_i), \mathrm{f}_-(h_i)\big) \in I_k \times J_i\,.$$

Then, because of (B5), the axis of h_i intersects each branch C_ℓ with $\ell \in D_i$. Thus, for every $\ell \in D_i$ we have

$$\big(\mathrm{f}_+(h_{\jmath(\ell)}), \mathrm{f}_-(h_{\jmath(\ell)})\big) \in I_\ell \times J_\ell \subseteq \widehat{\mathbb{R}} \setminus \{\mathrm{X}_\ell, \mathrm{Y}_\ell\} \times \widehat{\mathbb{R}} \setminus \{\mathrm{X}_\ell, \mathrm{Y}_\ell\}\,.$$

By Lemma 1.4 we therefore find

$$\lim_{n\to+\infty} h^n_{\jmath(\ell)}.\mathrm{X}_\ell = \lim_{n\to+\infty} h^n_{\jmath(\ell)}.\mathrm{Y}_\ell = \mathrm{f}_+(h_{\jmath(\ell)})\,,$$

meaning $h_{\jmath(\ell)}$ contracts the interval I_ℓ towards $\mathrm{f}_+(h_{\jmath(\ell)})$. Hence, for $\widetilde{n} \in \mathbb{N}$ sufficiently large,

$$\widehat{\mathbb{R}} \setminus \bigcup_{\ell\in A} h^{\widetilde{n}}_{\jmath(\ell)}.I_\ell$$

contains an open interval. In turn,

$$\widetilde{\mathcal{C}} \coloneqq \left\{ h^{\widetilde{n}}_{\jmath(\ell)}.\,\mathrm{C}_\ell \;\middle|\; \ell \in A \right\}$$

is an admissible set of branches (see also Lemma 4.10).

Now let $k \in A$ and denote the branch tree of k with respect to $\widetilde{\mathcal{C}}$ by $\widetilde{B}_k$. Let $\jmath(k) = i$. Then $\widetilde{B}_i$ contains the path

$$(i, \mathrm{id}) \longrightarrow (k_1, h_1) \longrightarrow \ldots \longrightarrow (k_n, h_n) \longrightarrow (k, \mathrm{id})\,, \tag{4.32}$$

with $k_1, \ldots, k_n \in A$ and $h_1, \ldots, h_n \in \Gamma$, $n \in \mathbb{N}$. Since $\jmath(k) = i$, by construction we have $\jmath(k_\iota) = i$ for all $\iota = 1, \ldots, n$. Therefore, the branch tree B_i in the branch forest of $\mathcal{C}$ contains the path (4.32) as well. Since $\mathcal{C}$ is weakly non-collapsing, we have $h_1 = \cdots = h_n = \mathrm{id}$. This shows that $\widetilde{\mathcal{C}}$ is also weakly non-collapsing. This finishes the proof of the first statement.

Finally, assume that $\mathcal{C}$ is finitely ramified. Then, by Proposition 4.26, every cusp of $\mathbb{X}$ is attached to $\mathcal{C}$, in the sense that for every cuspidal point c,

$$I(\mathrm{Att}_{\mathcal{C}}(c)) = \overline{\bigcup_{(j,h)\in\mathrm{Att}_{\mathcal{C}}(c)} h.I_j}$$

contains a full neighborhood of c in $\widehat{\mathbb{R}}$ (see Definition 4.24). But since the associated intervals transform as

$$I'_j = g_j . I_j \,,$$

for $j \in A$, we obtain

$$\overline{\bigcup_{(j,h)\in \mathrm{Att}_{\mathcal{C}}(c)} h.I_j} = \overline{\bigcup_{(j,h)\in \mathrm{Att}_{\mathcal{C}}(c)} hg_j^{-1}.I'_j} \,.$$

Hence, if we set

$$\mathrm{Att}'_{\mathcal{C}}(c) := \left\{ (j, hg_j^{-1}) \;\middle|\; (j,h) \in \mathrm{Att}_{\mathcal{C}}(c) \right\} \,,$$

then again $I(\mathrm{Att}'_{\mathcal{C}}(c))$ contains a full neighborhood of c. Applying again Proposition 4.26, this implies that $\mathcal{C}'$ is finitely ramified. □

4.6 Cross Sections From Sets of Branches

Throughout this section, let $\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$ be a set of branches for the geodesic flow $\widehat{\Phi}$ on $\mathbb{X}$, let $\mathrm{C} := \bigcup \mathcal{C}$ denote the branch union, and set

$$\widehat{\mathrm{C}} := \pi\,(\mathrm{C}) \;.$$

We now show that $\widehat{\mathrm{C}}$ is indeed a cross section for $\widehat{\Phi}$ with respect to certain measures and that the strong branch union C_{st} induces a strong cross section.

To that end let $\mathrm{Van}(\mathbb{X})$ denote the subset of $\mathscr{G}(\mathbb{X})$ of all geodesics for which there exists a lift on $\mathbb{H}$ having at least one endpoint in $\widehat{\mathbb{R}}\setminus\widehat{\mathbb{R}}_{\mathrm{st}}$ (and thus all of its lifts on $\mathbb{H}$ have this property). We note that $\mathscr{G}_{\mathrm{Per}}(\mathbb{X}) \subseteq \mathscr{G}(\mathbb{X}) \setminus \mathrm{Van}(\mathbb{X})$. We denote by $\mathcal{M}_{\mathrm{Van}(\mathbb{X})}$ the set of measures μ on (a σ-algebra on) $\mathscr{G}(\mathbb{X})$ with the property that $\mu(\mathrm{Van}(\mathbb{X})) = 0$. In particular, the counting measure of periodic geodesics belongs to $\mathcal{M}_{\mathrm{Van}(\mathbb{X})}$. Throughout we use the standard notation from the previous sections. In particular, we set $A = \{1, \ldots, N\}$ and define I_j, J_j, etc. as in (B3). Recall the properties (CS1)–(CS3) from Definition 1.47.

Proposition 4.36. *For each $\mu \in \mathcal{M}_{\mathrm{Van}(\mathbb{X})}$ the set $\widehat{\mathrm{C}}$ is a cross section for the geodesic flow $\widehat{\Phi}$ on $\mathbb{X}$ with respect to μ. In particular, each geodesic in $\mathscr{G}(\mathbb{X}) \setminus \mathrm{Van}(\mathbb{X})$ intersects $\widehat{\mathrm{C}}$ infinitely often in past and future.*

Proof. We start by establishing (CS1). To that end let $\widehat{\gamma} \in \mathscr{G}(\mathbb{X}) \setminus \mathrm{Van}(\mathbb{X})$ and let γ be any lift of $\widehat{\gamma}$ to $\mathbb{H}$. We first need to show that $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}$ or, equivalently, that γ intersects $\Gamma.\mathrm{C}$. Without loss of generality we may suppose that $\gamma(\pm\infty) \neq \infty$ (otherwise, we pick another representing geodesic for $\widehat{\gamma}$). Thus, the two endpoints $\gamma(\pm\infty)$ of γ are in $\mathbb{R}_{\mathrm{st}}$. In what follows we show (via proof by contradiction) that there exist $g \in \Gamma$ and $j \in A$ such that $\gamma(+\infty) \in g.I_j$ and $\gamma(-\infty) \in g.J_j$. Then γ intersects $g.\,\mathrm{C}_j$ by (B5), and hence $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}$.

In order to seek a contradiction to the existence of such elements in Γ and A, we assume that for all $g \in \Gamma$ and all $j \in A$ the two endpoints $\gamma(\pm\infty)$ of γ are either both in $g.I_j$ or both in $g.J_j$. Since $\gamma(\pm\infty) \in \mathbb{R}_{\mathrm{st}}$ and $\mathbb{R}_{\mathrm{st}}$ is contained in the limit set $\Lambda(\Gamma)$ of Γ, Proposition 1.15 implies that for each $\varepsilon > 0$ we find a geodesic η_ε on $\mathbb{H}$ such that $\pi(\eta_\varepsilon) \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$ and the endpoints

$$\big(x(\varepsilon), y(\varepsilon)\big) \coloneqq \big(\eta_\varepsilon(+\infty), \eta_\varepsilon(-\infty)\big)$$

of η_ε are ε-near to $\gamma(\pm\infty)$, respectively, i. e.,

$$|x(\varepsilon) - \gamma(+\infty)| < \varepsilon \quad \text{and} \quad |y(\varepsilon) - \gamma(-\infty)| < \varepsilon\,. \tag{4.33}$$

Again using Proposition 1.15, we may and shall suppose that $x(\varepsilon)$ and $y(\varepsilon)$ are exterior to the interval in $\mathbb{R}$ that is spanned by $\gamma(+\infty)$ and $\gamma(-\infty)$. By ($\mathrm{B}_{\mathrm{Per}}$) we find $h_\varepsilon \in \Gamma$ and $j_\varepsilon \in A$ such that $h_\varepsilon.\eta_\varepsilon$ intersects $\mathrm{C}_{j_\varepsilon}$. Thus,

$$\big(x(\varepsilon), y(\varepsilon)\big) \in h_\varepsilon^{-1}.I_{j_\varepsilon} \times h_\varepsilon^{-1}.J_{j_\varepsilon}\,.$$

By the assumption, either

$$\gamma(\pm\infty) \in h_\varepsilon^{-1}.I_{j_\varepsilon} \quad \text{or} \quad \gamma(\pm\infty) \in h_\varepsilon^{-1}.J_{j_\varepsilon}\,. \tag{4.34}$$

We consider h_ε and j_ε to be fixed once and for all for each $\varepsilon > 0$ separately.

We now construct inductively a sequence of "nested" translates of a complete geodesic segment as follows. Without loss of generality we suppose that $\gamma(+\infty) < \gamma(-\infty)$. We pick a (small) $\varepsilon_1 > 0$ and fix a geodesic η_{ε_1} on $\mathbb{H}$ with the properties as above with ε_1 in place of ε. We let $a_1, b_1 \in \widehat{\mathbb{R}}$ be the endpoints of $h_{\varepsilon_1}^{-1}.\overline{\mathrm{bp}(\mathrm{C}_{j_{\varepsilon_1}})}$, ordered such that

$$a_1 < \gamma(+\infty) < \gamma(-\infty) < b_1\,.$$

If $a_1 = \infty \in \widehat{\mathbb{R}}$, then the left hand part of this inequality is understood as $-\infty < \gamma(+\infty)$ in $\mathbb{R}$; analogously for the right hand part if $b_1 = \infty$. This configuration is the only one feasible under the condition that (4.33) and (4.34) remain both valid for a sufficiently small ε_1.

We set

$$\varepsilon_2 \coloneqq \min\left\{ \frac{|a_1 - \gamma(+\infty)|}{2}, \frac{|b_1 - \gamma(-\infty)|}{2} \right\}.$$

Then $\varepsilon_1 > \varepsilon_2$, and we repeat with ε_2 in place of ε_1. We emphasize that the chosen geodesic η_{ε_2} does not intersect $h_{\varepsilon_1}^{-1}.\,\mathrm{C}_{j_{\varepsilon_1}}$ as it is contained in either $h_{\varepsilon_1}^{-1}.\mathrm{H}_+(j_{\varepsilon_1})$ or $h_{\varepsilon_1}^{-1}.\mathrm{H}_-(j_{\varepsilon_1})$. Further, the endpoints (a_2, b_2) do not coincide with (a_1, b_1), and since $h_{\varepsilon_2}^{-1}.\overline{\mathrm{bp}(\mathrm{C}_{j_{\varepsilon_2}})}$ may not intersect $h_{\varepsilon_1}^{-1}.\overline{\mathrm{bp}(\mathrm{C}_{j_{\varepsilon_1}})}$ by (B6), we have $a_1 \le a_2$ and $b_2 \le b_1$ with at least one of the inequalities being strict.

We repeat inductively and obtain a sequence

$$h_{\varepsilon_1}^{-1}.\,\mathrm{C}_{j_{\varepsilon_1}}\,,\; h_{\varepsilon_2}^{-1}.\,\mathrm{C}_{j_{\varepsilon_2}}\,,\; h_{\varepsilon_3}^{-1}.\,\mathrm{C}_{j_{\varepsilon_3}}\,,\;\ldots$$

of certain Γ-translates of elements of the set of branches, and sequences

$$(a_n)_{n\in\mathbb{N}} \qquad \text{and} \qquad (b_n)_{n\in\mathbb{N}}$$

of the endpoints of the elements of the sequence $(h_{\varepsilon_n}^{-1}.\overline{\mathrm{bp}(\mathrm{C}_{j_{\varepsilon_n}})})_n$. Then the sequence $(a_n)_n$ is monotonically increasing and bounded from above by $\gamma(+\infty)$, and $(b_n)_n$ is monotonically decreasing and bounded from below by $\gamma(-\infty)$. Let

$$a \coloneqq \lim_{n\to\infty} a_n \qquad \text{and} \qquad b \coloneqq \lim_{n\to\infty} b_n\,.$$

We fix a point on the geodesic segment $(a,b)_{\mathbb{H}}$, say z. Then each neighborhood of z intersects infinitely many members of the family

$$\left\{ h_{\varepsilon_n}^{-1}.\overline{\mathrm{bp}(\mathrm{C}_{j_{\varepsilon_n}})} \;\middle|\; n\in\mathbb{N} \right\}\,,$$

which are pairwise disjoint. This contradicts Proposition 4.6. In turn, $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}$.

Without loss of generality we may suppose that $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}$ at time $t=0$. By Proposition 4.9(ii) there exists a unique lift η of $\widehat{\gamma}$ that intersects C at $t=0$. Let $\nu \coloneqq \eta'(0)$. Since $\widehat{\gamma}\notin \mathrm{Van}(\mathbb{X})$, we have $\{\eta(\pm\infty)\}\subseteq \widehat{\mathbb{R}}_{\mathrm{st}}$, and hence $\nu\in\mathrm{C}_{\mathrm{st}}$. Using Lemma 4.18 we now obtain (CS1) with $(t_n)_{n\in\mathbb{Z}} = (\mathrm{t}_{\mathrm{C},n}(\nu))_{n\in\mathbb{Z}}$.

In order to establish (CS2) we let $\widehat{\gamma}$ be any geodesic on $\mathbb{X}$ and pick any representing geodesic γ on $\mathbb{H}$. We note that the intersection times of $\widehat{\gamma}$ with $\widehat{\mathrm{C}}$ and of γ with $\Gamma.\mathrm{C}$ coincide (we picked geodesics with coinciding time parametrizations). The local finiteness of $\Gamma.\mathrm{C}$, as guaranteed by Proposition 4.6, immediately implies that the intersection times form a discrete subset of $\mathbb{R}$. $\square$

Corollary 4.37. *For each $\mu\in\mathcal{M}_{\mathrm{Van}(\mathbb{X})}$ the set $\widehat{\mathrm{C}}_{\mathrm{st}} \coloneqq \pi(\mathrm{C}_{\mathrm{st}})$ is a strong cross section for the geodesic flow $\widehat{\Phi}$ on $\mathbb{X}$ with respect to μ. Each geodesic in $\mathscr{G}(\mathbb{X})\setminus \mathrm{Van}(\mathbb{X})$ intersects $\widehat{\mathrm{C}}_{\mathrm{st}}$ infinitely often in past and future.*

Proof. The proof of Proposition 4.36 already establishes (CS1) for $\widehat{\mathrm{C}}_{\mathrm{st}}$ since it shows that any $\widehat{\gamma}\in\mathscr{G}(\mathbb{X})\setminus\mathrm{Van}(\mathbb{X})$ intersects $\widehat{\mathrm{C}}_{\mathrm{st}}$, not only $\widehat{\mathrm{C}}$, at least once and then infinitely often in past and future. Also the proof of (CS2) for $\widehat{\mathrm{C}}_{\mathrm{st}}$ can be taken directly from the proof of Proposition 4.36. For (CS3) we let $\widehat{\gamma}$ be any geodesic on $\mathbb{X}$ that intersects $\widehat{\mathrm{C}}_{\mathrm{st}}$ at least once. Then $\widehat{\gamma}\in\mathscr{G}(\mathbb{X})\setminus\mathrm{Van}(\mathbb{X})$ and it can be seen as in the proof of Proposition 4.36 that $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}_{\mathrm{st}}$ infinitely often in past and future. $\square$

4.7 Slow Transfer Operators

Let $A := \{1, \ldots, N\}$ and let $\mathcal{C} := \{\mathrm{C}_j \mid j \in A\}$ be a set of branches for the geodesic flow on $\mathbb{X}$. Because of Proposition 4.28 we may assume that $\mathcal{C}$ is finitely ramified. In this section we present the discrete dynamical system induced by $\mathcal{C}$ and the associated family of transfer operators. These transfer operators are the so-called *slow* transfer operators. The notion of *fast* transfer operators has been discussed in Section 3.3. We refer to Section 1.11 for the general notion of transfer operators we use.

As before we let $\mathrm{C} := \bigcup \mathcal{C}$ denote the branch union and resume the notation from (B3), (B7), and (4.4)–(4.5). For $\nu \in \mathrm{C}_{\mathrm{st}}$ we recall from (4.9)–(4.14) its system of iterated sequences

$$[(\mathrm{t}_{\mathrm{C},n}(\nu))_n, (\mathrm{k}_{\mathrm{C},n}(\nu))_n, (\mathrm{g}_{\mathrm{C},n}(\nu))_n] \,.$$

The first return map $\mathscr{R}\colon \mathrm{C}_{\mathrm{st}} \to \mathrm{C}_{\mathrm{st}}$ (cf. (1.88)) is given by

$$\mathscr{R}|_{\mathrm{C}_{j,\mathrm{st}}} \colon \begin{cases} \mathrm{C}_{j,\mathrm{st}} & \longrightarrow \quad \mathrm{C}_{\mathrm{k}_{\mathrm{C},1}(\nu),\mathrm{st}} \\ \nu & \longmapsto \quad \mathrm{g}_{\mathrm{C},1}(\nu)^{-1}.\gamma'_\nu(\mathrm{t}_{\mathrm{C},1}(\nu)) \end{cases} ,$$

for any $j \in A$. In order to present the discrete dynamical system induced by $\mathcal{C}$, as defined at the end of Section 1.11, we set for any $j, k \in A$ and $g \in \mathcal{G}(j,k)$,

$$D_{j,k,g} := g.I_{k,\mathrm{st}} \times \{j\} \,.$$

By (B7),

$$\bigcup_{j,k\in A} \bigcup_{g \in \mathcal{G}(j,k)} D_{j,k,g} = \bigcup_{j\in A} I_{j,\mathrm{st}} \times \{j\} \,.$$

Then

$$D := \bigcup_{j\in A} I_{j,\mathrm{st}} \times \{j\}$$

is the domain of the induced discrete dynamical system (D, F), where the map $F\colon D \to D$ decomposes into the *submaps* (local bijections)

$$F|_{D_{j,k,g}} \colon \begin{cases} g.I_{k,\mathrm{st}} \times \{j\} & \longrightarrow \quad I_{k,\mathrm{st}} \times \{k\} \\ (x, j) & \longmapsto \quad (g^{-1}.x, k) \end{cases} ,$$

for $j, k \in A$ and $g \in \mathcal{G}(j,k)$. One easily checks that the diagram

$$\begin{array}{ccc} \mathrm{C}_{\mathrm{st}} & \xrightarrow{\mathscr{R}} & \mathrm{C}_{\mathrm{st}} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ D & \xrightarrow{F} & D \end{array}$$

indeed commutes, where the surjection $\iota\colon \mathrm{C}_{\mathrm{st}} \mapsto D$ is piecewise defined by

$$\iota|_{\mathrm{C}_{j,\mathrm{st}}}\colon \nu \longmapsto \left(\gamma_\nu(+\infty), j\right),$$

with $j \in A$.

Let V be a finite-dimensional complex vector space and let

$$\mathrm{Fct}(D; V) := \{f\colon D \to V \mid f \text{ function}\}$$

denote the space of V-valued functions on D. Further let $\chi\colon \Gamma \to \mathrm{GL}(V)$ be a representation of Γ on V. We define an associated *weight function* ω on D by

$$\omega|_{D_{j,k,g}}\colon (x, j) \longmapsto \chi(g^{-1}),$$

for any $j, k \in A$ and $g \in \mathcal{G}(j,k)$. The *(slow) transfer operator* $\mathcal{L}_s$ with parameter $s \in \mathbb{C}$ and weight ω associated to the map F is (initially only formally) defined as an operator on $\mathrm{Fct}(D; V)$ by

$$\mathcal{L}_s f((x,k)) := \sum_{(y,j)\in F^{-1}((x,k))} \omega((y,j)) \left|F'(y,j)\right|^{-s} f((y,j)),$$

for any $(x,k) \in D$ and $f \in \mathrm{Fct}(D; V)$. The space of functions which is used as domain of the transfer operator and on which it then defines an actual operator depends a lot on the intended application. It typically is a subset of $\mathrm{Fct}(D; V)$ or a closely related space of functions with complex domains. For the transfer-operator based interpretations of Laplace eigenfunctions, which motivate the research culminating in this thesis and which we briefly surveyed in Section 1.12, the function spaces of choice are subspaces of $\mathrm{Fct}(V; D)$ consisting of highly regular functions. We omit any further discussion and refer to [44, 57, 56, 60, 15, 64] for details. However, we note that if all transition sets $\mathcal{G}(j,k)$, $j, k \in A$, are finite, which we may assume by virtue of Proposition 4.28, then $\mathcal{L}_s$ is already well-defined as an operator on $\mathrm{Fct}(D; V)$. For infinite transition sets, questions of convergence would need to be considered.

We end this section with two examples, the first of which illustrating the structure of the slow transfer operators in the case of Schottky surfaces. For other Fuchsian groups, the structure is similar. For Schottky surfaces, transfer operators are classically defined using a Koebe–Morse coding for the geodesic flow (see, e. g., [10]). This example also shows that the approach via sets of branches reproduces the classical transfer operators and generalizes the classical construction.

Example 4.38. Let Γ_{S} be a Schottky group with Schottky data $(r, \{\mathcal{D}_{\pm j}\}_{j=1}^{r}, \{s_{\pm j}\}_{j=1}^{r})$, and recall the set of branches $\{\mathrm{C}_{\pm 1}, \dots, \mathrm{C}_{\pm r}\}$ from Example 4.3. Let

$$\mathrm{C}_{\mathrm{S}} := \bigcup_{j=1}^{r} \mathrm{C}_j \cup \bigcup_{k=1}^{r} \mathrm{C}_{-k}\,.$$

For $j \in \{\pm 1, \ldots, \pm r\}$ consider the subspace $A^2(\mathcal{D}_j)$ of $L^2(\mathcal{D}_j)$ of holomorphic functions (Bergman space with $p = 2$). Then the direct sum

$$\mathcal{H} := \bigoplus_{j=1}^{r} A^2(\mathcal{D}_j) \oplus \bigoplus_{k=1}^{r} A^2(\mathcal{D}_{-k})$$

is a Hilbert space. If we identify functions $f \in \mathcal{H}$ with the function vectors

$$\bigoplus_{j=1}^{r} f_j \oplus \bigoplus_{k=1}^{r} f_{-k}\,, \quad \text{with} \quad f_l \in A^2(\mathcal{D}_l)\,, \quad l \in \{\pm 1, \ldots, \pm r\}\,,$$

then the slow transfer operator for $\mathrm{C_S}$ with parameter $s \in \mathbb{C}$ and constant weight $\omega \equiv 1$ (trivial representation) takes the form

$$\mathcal{L}_s = \begin{pmatrix} \tau_s(s_1) & \tau_s(s_2) & \ldots & \tau_s(s_r) & 0 & \tau_s(s_{-2}) & \ldots & \tau_s(s_{-r}) \\ \tau_s(s_1) & \tau_s(s_2) & \ldots & \tau_s(s_r) & \tau_s(s_{-1}) & 0 & \ldots & \tau_s(s_{-r}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tau_s(s_1) & \tau_s(s_2) & \ldots & \tau_s(s_r) & \tau_s(s_{-1}) & \tau_s(s_{-2}) & \ldots & 0 \\ 0 & \tau_s(s_2) & \ldots & \tau_s(s_r) & \tau_s(s_{-1}) & \tau_s(s_{-2}) & \ldots & \tau_s(s_{-r}) \\ \tau_s(s_1) & 0 & \ldots & \tau_s(s_r) & \tau_s(s_{-1}) & \tau_s(s_{-2}) & \ldots & \tau_s(s_{-r}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tau_s(s_1) & \tau_s(s_2) & \ldots & 0 & \tau_s(s_{-1}) & \tau_s(s_{-2}) & \ldots & \tau_s(s_{-r}) \end{pmatrix},$$

where

$$\tau_s(g^{-1})f(x) := \big(g'(x)\big)^s f(g.x)$$

for $f\colon U \to \mathbb{C}$, $x \in U$, $g \in \Gamma_\mathrm{S}$.

Example 4.39. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ from Example 1.46 and the two sets of branches $\mathcal{C}_\mathrm{P}$ and $\mathcal{C}_\mathrm{P}'$ from Example 2.11 and Example 4.34. As in Example 4.38 above, we use the trivial representation realized by the constant weight $\omega \equiv 1$, and the left action of $\Gamma_{\sigma,\lambda}$ on $\mathrm{Fct}(D;V)$ realized by τ_s. Note that $\tau_s(\mathrm{id}) = \mathrm{id}$. For $i = 1, \ldots, 8$ we write

$$f_i := f \circ \mathbb{1}_{D_i}\,,$$

where $D_i := I_{i,\mathrm{st}} \times \{i\}$, with $I_1, \ldots, I_8$ as in Example 2.11, and where $\mathbb{1}_M$ denotes the characteristic function of a set M. Then the map

$$\begin{cases} \mathrm{Fct}(D;V) & \longrightarrow & \bigoplus_{i=1}^{8} \mathrm{Fct}(D_i;V) \\ f & \longmapsto & (f_1, \ldots, f_8)^\top \end{cases}$$

is an isomorphism. Utilizing this, the transfer operators $\mathcal{L}_s$ for $\mathcal{C}_\mathrm{P}$ and $\mathcal{L}_s'$ for $\mathcal{C}_\mathrm{P}'$

with parameter $s \in \mathbb{C}$, $\operatorname{Re} s \gg 1$, admit the matrix representations

$$\mathcal{L}_s = \begin{pmatrix} 0 & 0 & 0 & 0 & \tau_s(\mathrm{t}_\lambda^{-1}) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \tau_s(g_\sigma^{-1}) & 0 \\ 0 & 0 & \displaystyle\sum_{k=0}^{\sigma-2} \tau_s(g_\sigma^{-k}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_s(g_\sigma) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_s(\mathrm{t}_\lambda) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tau_s(g_\sigma^{-1}) & 0 & 0 & \tau_s(g_\sigma^{-1}) & 0 \end{pmatrix}$$

and

$$\mathcal{L}'_s = \begin{pmatrix} 0 & 0 & 0 & 0 & \tau_s(\mathrm{t}_\lambda^{-1}) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \tau_s(g_\sigma^{-1}) & 0 \\ 0 & 0 & \displaystyle\sum_{k=0}^{\sigma-2} \tau_s(g_\sigma^{-k}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_s(g_\sigma) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_s(\mathrm{t}_\lambda g_\sigma^{-1}) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Chapter 5

Algorithms for Branch Reduction

Let Γ be a geometrically finite Fuchsian group containing hyperbolic elements. Denote by $\mathbb{X} = \Gamma\backslash\mathbb{H}$ the associated hyperbolic orbisurface and assume that $\mathbb{X}$ has hyperbolic ends. Let $N \in \mathbb{N}$, $A := \{1, \ldots, N\}$, and let

$$\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$$

be a set of branches for the geodesic flow on $\mathbb{X}$ in the sense of Definition 4.1. Because of the Propositions 4.28 and 4.35 we may assume that $\mathcal{C}$ is finitely ramified, admissible, and weakly non-collapsing. Denote by $\mathrm{C} = \bigcup \mathcal{C}$ the branch union and let $\widehat{\mathrm{C}} := \pi(\mathrm{C})$, where $\pi\colon\ \mathrm{S}\mathbb{H} \to \mathrm{S}\mathbb{X}$ is the canonical quotient map from (1.32).

In general, the cross section $\widehat{\mathrm{C}}$ and the set of branches $\mathcal{C}$ do not yet give rise to a strict transfer operator approach. More precisely, using the notation from Section 3.1, if we attempt to use the family of intervals $\{I_a\}_{a\in A}$ as part of a structure tuple and form, for $a, b \in A$, the sets $P_{a,b}, C_{a,b}$ and $\{g_p\}_{p\in P_{a,b}}$ of elements of Γ such that the associated discrete dynamical system (D, F) (see the discussion right after Property 1) coincides with the discrete dynamical system associated to $\mathcal{C}$ (see Section 4.7), then Properties 1–5 from Section 3.1 are typically not satisfied. For the associated transfer operators that means that we typically cannot find a Banach space on which they act as nuclear operators of order 0 and have a well-defined Fredholm determinant (even ignoring the requirement that it should be related to the Selberg zeta function). This issue, if present, originates from (D, F) not being uniformly expanding. The non-uniform expansiveness of (D, F) can have the following two reasons:

(a) The identity element of Γ is among the action elements of F^n for some $n \in \mathbb{N}$. That is, some iterate of the map F has a submap of the form

$$\begin{cases} \widetilde{I}_{a,\mathrm{st}} \times \{b\} & \longrightarrow & \widetilde{I}_{a,\mathrm{st}} \times \{a\} \\ (x, b) & \longmapsto & (x, a) \end{cases},$$

for some subinterval $\widetilde{I}_{a,\mathrm{st}}$ of $I_{a,\mathrm{st}}$.

(b) Some iterate of the map F has a submap expressing the action of a parabolic element of Γ, and the fixed point of this element is an inexhaustible source for iterations. That is, there exists $n \in \mathbb{N}$ such that F^n has a submap conjugate to

$$\begin{cases} (1,\infty)_{\mathrm{st}} \times \{a\} & \longrightarrow & (0,\infty)_{\mathrm{st}} \times \{a\} \\ (x,a) & \longmapsto & (x-1,a) \end{cases} .$$

Then any iterate of F^n has a submap of this form, and hence a "big part" in which no expansion takes place.

For the set of branches $\mathcal{C}$, issue (a) means that $\mathcal{C}$ contains a branch which contains an element, say ν, such that the associated geodesic γ_ν intersects another branch in $\mathcal{C}$. Issue (b) is present if $\mathbb{X}$ has cusps. The cross section $\widehat{\mathrm{C}}$ detects every winding of a geodesic around a cusp as a separate event, and hence the set of branches $\mathcal{C}$ and the associated discrete dynamical system (D, F) encode each single one of them separately.

To overcome these issues, we require an appropriate acceleration of the dynamics, which translates to a reduction procedure of the branches. This will be done in three separate steps, which we call *branch reduction*, *identity elimination* and *cuspidal acceleration*. We start with the branch reduction in Section 5.1. The identity elimination is discussed in Section 5.2, the cuspidal acceleration in Section 5.3. In Section 5.4 we will then study the structure of the so called *accelerated system* that emerges, and thereby lay the groundwork for the explicit definition of the structure tuple for the strict transfer operator approach.

5.1 Branch Reduction

The branch reduction, which we present in this section (Algorithms 5.4 and 5.5), aims at simplifying the constructions by reducing the number of branches to a "minimum." Albeit not being absolutely necessary, in many cases the branch reduction considerably reduces the complexity of the situation. In addition, it provides several intermediate "reduced slow transfer operator families," which are typically useful for other applications as well.

5.1.1 Return Graphs

We associate to the set of branches $\mathcal{C}$ a directed graph RG with weighted edges, called *return graph*, which encodes the next intersection properties among the branches in $\mathcal{C}$ in just the right way for an efficient presentation and discussion of the branch reduction algorithm. In contrast to the branch trees from Section 4.5, the return graph is a weighted, directed graph with nodes in A. Hence, we obtain one graph encapsulating the whole dynamic of transition within a given set of branches.

The return graph RG associated to $\mathcal{C}$ is defined as follows:

- The set of nodes or vertices of RG is A.
- The set of edges of RG from the node j to the node k is bijective to the forward transition set $\mathcal{G}(j,k)$. For each $g \in \mathcal{G}(j,k)$, the graph RG contains an edge from j to k with weight g.

Let j, k be nodes of RG. We call k a *successor* of j if RG has an edge from j to k, and a *predecessor* of j if RG has an edge from k to j. For an edge from j to k weighted by g we will often write $j \xrightarrow{g} k$. A *path* in RG of length $m \in \mathbb{N}$ is a sequence of consecutive edges of the form

$$j_0 \xrightarrow{g_1} j_1 \xrightarrow{g_2} \dots \xrightarrow{g_m} j_m\,. \tag{5.1}$$

If $j_m = j_0$, then we also call this path a *cycle*. See Example 5.3 and the Figures 12 and 13 below for an example of a return graph.

We recall from Proposition 4.15(iii) that for any $j, k \in A$, the forward transition set is

$$\mathcal{G}(j,k) = \left\{ g \in \Gamma \;\middle|\; \exists \nu \in \mathrm{C}_j \colon \gamma'_\nu(t^+_{\mathrm{C}}(\nu)) \in g\boldsymbol{.}\,\mathrm{C}_k \right\}\,.$$

Its elements determine exactly the translates of the branch C_k on which the next intersections of geodesics starting on C_j are located (see Section 4.3). We consider weights of paths as multiplicative. In other words, the path

$$j \xrightarrow{g} k \xrightarrow{h} \ell \tag{5.2}$$

gives rise to the total weight gh. Any path in RG of the form as in (5.2) indicates that there is (at least) one geodesic starting on the branch C_j, traversing $g\boldsymbol{.}\,\mathrm{C}_k$ and then intersecting $gh\boldsymbol{.}\,\mathrm{C}_\ell$, and not intersecting any other translates of branches inbetween. The following lemma shows that this interpretation of paths is indeed correct, and that all paths in RG arise in this way. We emphasize that the vector ν in the second part of the following lemma is not necessarily unique.

Lemma 5.1. *The paths in* RG *are fully characterized by the systems of iterated sequences from Section 4.3 of the elements in* C. *More precisely:*

(i) *For all $\nu \in \mathrm{C}_{\mathrm{st}}$ and all $n \in \mathbb{N}$, the return graph* RG *contains the path*

$$\mathrm{k}_{\mathrm{C},0}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},1}(\nu)} \mathrm{k}_{\mathrm{C},1}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},2}(\nu)} \dots \xrightarrow{\mathrm{g}_{\mathrm{C},n}(\nu)} \mathrm{k}_{\mathrm{C},n}(\nu)\,.$$

(ii) *Let $m \in \mathbb{N}$ and suppose that*

$$k_0 \xrightarrow{g_1} k_1 \xrightarrow{g_2} \dots \xrightarrow{g_{m-1}} k_{m-1} \xrightarrow{g_m} k_m$$

is a path in the return graph RG. *Then there exists* $\nu \in \mathrm{C}_{k_0,\mathrm{st}}$ *such that*

$$k_j = \mathrm{k}_{\mathrm{C},j}(\nu) \qquad \textit{for } j \in \{0, \ldots, m\}$$

and

$$g_j = \mathrm{g}_{\mathrm{C},j}(\nu) \qquad \textit{for } j \in \{1, \ldots, m\}\,.$$

Proof. To prove (i) let $\nu \in \mathrm{C}_{\mathrm{st}}$ and set $\nu_m := \gamma'_\nu(\mathrm{t}_{\mathrm{C},m}(\nu))$ for all $m \in \mathbb{N}_0$ (recall from Section 4.3 that $\mathrm{t}_{\mathrm{C},m}(\nu)$ is well-defined for all $m \in \mathbb{N}_0$). For each $m \in \mathbb{N}_0$ we have $\mathrm{g}_{\mathrm{C},1}(\nu_m) \in \mathcal{G}(\mathrm{k}_{\mathrm{C},0}(\nu_m), \mathrm{k}_{\mathrm{C},1}(\nu_m))$. Hence, the return graph RG contains the edge

$$\mathrm{k}_{\mathrm{C},0}(\nu_m) \xrightarrow{\mathrm{g}_{\mathrm{C},1}(\nu_m)} \mathrm{k}_{\mathrm{C},1}(\nu_m)\,. \tag{5.3}$$

Since

$$\mathrm{g}_{\mathrm{C},1}(\nu_m) = \mathrm{g}_{\mathrm{C},m+1}(\nu)\,, \quad \text{and} \quad \mathrm{k}_{\mathrm{C},\iota}(\nu_m) = \mathrm{k}_{\mathrm{C},m+\iota}(\nu)\,,$$

for $\iota \in \{0, 1\}$, the edge in (5.3) equals

$$\mathrm{k}_{\mathrm{C},m}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},m+1}(\nu)} \mathrm{k}_{\mathrm{C},m+1}(\nu)\,.$$

Letting m run through $\{0, \ldots, n-1\}$, we now obtain that the path

$$\mathrm{k}_{\mathrm{C},0}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},1}(\nu)} \mathrm{k}_{\mathrm{C},1}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},2}(\nu)} \ldots \xrightarrow{\mathrm{g}_{\mathrm{C},n}(\nu)} \mathrm{k}_{\mathrm{C},n}(\nu)$$

is contained in RG. This proves (i).

We now show (ii). Let $j \in \{1, \ldots, m\}$. Since

$$k_{j-1} \xrightarrow{g_j} k_j$$

is an edge in RG by hypothesis, we have $g_j \in \mathcal{G}(k_{j-1}, k_j)$. Lemma 4.12 shows

$$g_j.\mathrm{H}_+(k_j) \subsetneq \mathrm{H}_+(k_{j-1})$$

and hence

$$g_j.I_{k_j,\mathrm{st}} \subseteq I_{k_{j-1},\mathrm{st}}\,. \tag{5.4}$$

Set $g := g_1 \cdots g_m$. Applying repeatedly these inclusion considerations we obtain

$$g.I_{k_m,\mathrm{st}} \subseteq I_{k_0,\mathrm{st}}\,.$$

We pick

$$(x, y) \in g.I_{k_m,\mathrm{st}} \times J_{k_0,\mathrm{st}}\,.$$

By (B5) and Remark 4.11 we find (a unique) $\nu \in \mathrm{C}_{k_0,\mathrm{st}}$ such that

$$(x,y) = \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big)\,.$$

In order to show that ν satisfies the claimed properties we proceed inductively. From $g.I_{k_m,\mathrm{st}} \subseteq g_1.I_{k_1,\mathrm{st}}$ (which is seen by iteration of (5.4)), we obtain that $\gamma_\nu(+\infty) \in g_1.I_{k_1,\mathrm{st}}$. From this and $g_1 \in \mathcal{G}(k_0,k_1)$ it follows that

$$\gamma_\nu'(t^+_{\mathrm{C},1}(\nu)) \in g_1.\mathrm{C}_{k_1,\mathrm{st}}\,,$$

by Corollary 4.17 and Remark 4.11. Then, by definition,

$$k_0 = \mathrm{k}_{\mathrm{C},0}(\nu)\,, \quad k_1 = \mathrm{k}_{\mathrm{C},1}(\nu) \quad \text{and} \quad g_1 = \mathrm{g}_{\mathrm{C},1}(\nu)\,.$$

Suppose now that for some $j_0 \in \{2,\ldots,m-1\}$ we have already established that

$$\gamma_\nu'(t^+_{\mathrm{C},j}(\nu)) \in g_j \cdots g_1.\mathrm{C}_{k_j,\mathrm{st}} \qquad \text{for } j \in \{1,\ldots,j_0-1\}$$

as well as

$$k_j = \mathrm{k}_{\mathrm{C},j}(\nu) \qquad \text{for } j \in \{0,\ldots,j_0-1\}$$

and

$$g_j = \mathrm{g}_{\mathrm{C},j}(\nu) \qquad \text{for } j \in \{1,\ldots,j_0-1\}.$$

Then

$$\widetilde{\nu} \coloneqq g_1^{-1}\cdots g_{j_0-1}^{-1}.\gamma_\nu'(t^+_{\mathrm{C},j_0-1}(\nu)) \in \mathrm{C}_{k_{j_0-1},\mathrm{st}}$$

and the associated geodesic $\gamma_{\widetilde{\nu}}$ is given by

$$\gamma_{\widetilde{\nu}}(t) = g_1^{-1}\cdots g_{j_0-1}^{-1}.\gamma_\nu(t + t^+_{\mathrm{C},j_0-1}(\nu)) \qquad \text{for all } t \in \mathbb{R}.$$

Since

$$\gamma_\nu(+\infty) \in g_1 \cdots g_{j_0}.I_{k_{j_0},\mathrm{st}}\,,$$

we have

$$\gamma_{\widetilde{\nu}}(+\infty) = g_1^{-1}\cdots g_{j_0-1}^{-1}.\gamma_\nu(+\infty) \in g_{j_0}.I_{k_{j_0},\mathrm{st}}\,.$$

Together with $g_{j_0} \in \mathcal{G}(k_{j_0-1}, k_{j_0})$ this yields that

$$\gamma_{\widetilde{\nu}}'(t^+_{\mathrm{C},1}(\widetilde{\nu})) \in g_{j_0}^{-1}.\mathrm{C}_{k_{j_0},\mathrm{st}}\,,$$

using Corollary 4.17 and Remark 4.11. Therefore,

$$\begin{aligned} k_{j_0-1} &= \mathrm{k}_{\mathrm{C},0}(\widetilde{\nu}) = \mathrm{k}_{\mathrm{C},j_0-1}(\nu)\,, \\ k_{j_0} &= \mathrm{k}_{\mathrm{C},1}(\widetilde{\nu}) = \mathrm{k}_{\mathrm{C},j_0}(\nu)\,, \\ g_{j_0} &= \mathrm{g}_{\mathrm{C},1}(\widetilde{\nu}) = \mathrm{g}_{\mathrm{C},j_0}(\nu)\,, \end{aligned}$$

as well as

$$\gamma'_{\widetilde{\nu}}(t^+_{\mathrm{C},1}(\widetilde{\nu})) = \gamma'_\nu(t^+_{\mathrm{C},j_0}(\nu))\,.$$

This completes the proof of (ii). □

The return graph RG is highly connected and weighted paths are essentially unique as the following proposition proves. These properties are crucial for the proof of the correctness of the branch reduction algorithm presented below (Algorithms 5.4 and 5.5). See Proposition 5.7.

Proposition 5.2. *The paths in the return graph* RG *obey the following structures:*

(i) *Every node in* RG *is contained in a cycle.*

(ii) *Let $j, k \in A$ and suppose that*

$$j \xrightarrow{g_1} p_1 \xrightarrow{g_2} \dots \xrightarrow{g_{m-1}} p_{m-1} \xrightarrow{g_m} k$$

and

$$j \xrightarrow{h_1} q_1 \xrightarrow{h_2} \dots \xrightarrow{h_{n-1}} q_{n-1} \xrightarrow{h_n} k$$

are paths in RG *such that $g_1 g_2 \cdots g_m = h_1 h_2 \cdots h_n$. Then $m = n$ and for all $i \in \{1, \dots, m-1\}$ and $\ell \in \{1, \dots, m\}$ we have $p_i = q_i$ and $g_\ell = h_\ell$.*

(iii) *Let $j, k \in A$. If* RG *contains a path from j to k, then it also contains a path from k to j.*

Proof. In order to prove (i) we fix $j \in A$. Because of (B1) and Remark 4.11 we find $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that $\widehat{\gamma}_\nu := \pi(\gamma_\nu)$ is a periodic geodesic on $\mathbb{X}$. Thus, we find $t_0 \in (0, \infty)$ such that $\widehat{\gamma}'_\nu(t_0) = \widehat{\gamma}'_\nu(0)$. Consequently, $\eta := \gamma'_\nu(t_0) \in g\,\mathbf{.}\,\mathrm{C}_j$ for some $g \in \Gamma$, by Proposition 4.9. Proposition 4.19 now shows that there exists a unique $n \in \mathbb{N}$ (we note that $t_0 > 0$) such that $t_0 = \mathrm{t}_{\mathrm{C},n}(\nu)$, $j = \mathrm{k}_{\mathrm{C},n}(\nu)$ and $g = \mathrm{g}_{\mathrm{C},1}(\nu) \cdots \mathrm{g}_{\mathrm{C},n}(\nu)$. By Lemma 5.1(i), RG contains the path

$$j = \mathrm{k}_{\mathrm{C},0}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},1}(\nu)} \mathrm{k}_{\mathrm{C},1}(\nu) \xrightarrow{\mathrm{g}_{\mathrm{C},2}(\nu)} \dots \xrightarrow{\mathrm{g}_{\mathrm{C},n}(\nu)} \mathrm{k}_{\mathrm{C},n}(\nu) = j\,.$$

Hence, j is contained in a cycle of RG, which establishes (i).

For the proof of (ii) we pick, as seen to be possible by Lemma 5.1(ii), elements $\nu_1, \nu_2 \in \mathrm{C}_{j,\mathrm{st}}$ such that the path generated by ν_1 is

$$j \xrightarrow{g_1} p_1 \xrightarrow{g_2} \ldots \xrightarrow{g_{m-1}} p_{m-1} \xrightarrow{g_m} k$$

and the path generated by ν_2 is

$$j \xrightarrow{h_1} q_1 \xrightarrow{h_2} \ldots \xrightarrow{h_{n-1}} q_{n-1} \xrightarrow{h_n} k\,.$$

This means that

$$j = \mathrm{k}_{\mathrm{C},0}(\nu_1) = \mathrm{k}_{\mathrm{C},0}(\nu_2)$$

and

$$\begin{aligned} p_\ell &= \mathrm{k}_{\mathrm{C},\ell}(\nu_1) && \text{for } \ell \in \{1,\ldots,m-1\} \\ g_\ell &= \mathrm{g}_{\mathrm{C},\ell}(\nu_1) && \text{for } \ell \in \{1,\ldots,m\} \\ q_\ell &= \mathrm{k}_{\mathrm{C},\ell}(\nu_2) && \text{for } \ell \in \{1,\ldots,n-1\} \\ h_\ell &= \mathrm{g}_{\mathrm{C},\ell}(\nu_2) && \text{for } \ell \in \{1,\ldots,n\}\,. \end{aligned}$$

Let

$$g \coloneqq g_1 \cdots g_m = h_1 \cdots h_n\,.$$

By combining Lemma 5.1, Corollary 4.17 (recall (4.12)), and (B7) we find, considering ν_1,

$$g.I_k = g_1 \cdots g_m.I_k \subseteq g_1 \ldots g_{m-1}.I_{p_{m-1}} \subseteq \ldots \subseteq g_1 g_2.I_{p_2} \subseteq g_1.I_{p_1} \subseteq I_j\,.$$

Considering ν_2, we obtain

$$g.I_k = h_1 \cdots h_n.I_k \subseteq h_1 \cdots h_{n-1}.I_{q_{n-1}} \subseteq \ldots \subseteq h_1 h_2.I_{q_2} \subseteq h_1.I_{q_1} \subseteq I_j\,.$$

The disjointness of the unions in (B7) yields that $g_1.I_{p_1} \cap h_1.I_{q_1} = \varnothing$ whenever $(g_1,p_1) \neq (h_1,q_1)$. However, since $g_1.I_{p_1}$ and $h_1.I_{q_1}$ both contain $g.I_k$ (which is nonempty), we obtain $(g_1,p_1) = (h_1,q_1)$. Applying this argument iteratively, we find $n = m$, as well as $g_\ell = h_\ell$, $p_i = q_i$ for all $\ell \in \{1,\ldots,n\}$ and all $i \in \{1,\ldots,n-1\}$.

In order to prove (iii) let

$$j \xrightarrow{g_1} p_1 \xrightarrow{g_2} \ldots \xrightarrow{g_{m-1}} p_{m-1} \xrightarrow{g_m} k \tag{5.5}$$

be a path in the return graph RG of length $m \in \mathbb{N}$. Lemma 5.1(ii) shows that there exists $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that the first m elements of the system of iterated sequences of ν produce the path (5.5). The choice of ν is not unique. In what follows, we show that ν can be chosen such that $\widehat{\gamma}_\nu$ is a periodic geodesic on $\mathbb{X}$.

As in the proof of (ii) we see that

$$g_1 \cdots g_m.I_k \subseteq g_1 \cdots g_{m-1}.I_{p_{m-1}} \subseteq \ldots \subseteq g_1 g_2.I_{p_2} \subseteq g_1.I_{p_1} \subseteq I_j \,. \tag{5.6}$$

Using Remark 4.2(d), we then obtain

$$g_1 \cdots g_m.J_k \supseteq g_1 \cdots g_{m-1}.J_{p_{m-1}} \supseteq \ldots \supseteq g_1 g_2.J_{p_2} \supseteq g_1.J_{p_1} \supseteq J_j \,. \tag{5.7}$$

Let $g \coloneqq g_1 \cdots g_m$ and recall from (1.41) the set $E(\mathbb{X})$ of endpoint pairs of representatives of periodic geodesics on $\mathbb{X}$. Since $E(\mathbb{X})$ is dense in $\Lambda(\Gamma) \times \Lambda(\Gamma)$ by virtue of Proposition (1.15) and $\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \Lambda(\Gamma)$, we find

$$(x, y) \in g.I_{k,\mathrm{st}} \times J_{j,\mathrm{st}} \,.$$

Let γ be a geodesic on $\mathbb{H}$ with $(\gamma(+\infty), \gamma(-\infty)) = (x, y)$. By (B5) and Remark 4.11, γ intersects $\mathrm{C}_{j,\mathrm{st}}$, say in ν. Iterated application of Corollary 4.17 shows that ν produces the path (5.5), i. e.,

$$\begin{aligned} j = \mathrm{k}_{\mathrm{C},0}(\nu) &\xrightarrow{g_1 = \mathrm{g}_{\mathrm{C},1}(\nu)} p_1 = \mathrm{g}_{\mathrm{C},1}(\nu) \xrightarrow{g_2 = \mathrm{g}_{\mathrm{C},2}(\nu)} \ldots \\ &\xrightarrow{g_{m-1} = \mathrm{g}_{\mathrm{C},m-1}(\nu)} p_{m-1} = \mathrm{k}_{\mathrm{C},m-1}(\nu) \xrightarrow{g_m = \mathrm{g}_{\mathrm{C},m}(\nu)} k = \mathrm{k}_{\mathrm{C},m}(\nu) \,. \end{aligned}$$

Since γ_ν represents a periodic geodesic on $\mathbb{X}$ as being a reparametrization of γ, the system of iterated sequences of ν is periodic and hence the (infinite) path in RG determined by ν (see Lemma 5.1) contains a subpath from k to j. This completes the proof. $\square$

Example 5.3. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma \in \mathbb{N}\setminus\{1\}, \lambda > 2}$ from Example 1.46 and its sets of branches $\mathcal{C}_{\mathrm{P}}$ from Example 2.11 and $\mathcal{C}'_{\mathrm{P}}$ from Example 4.34. The return graphs associated to $\mathcal{C}_{\mathrm{P}}$ and to $\mathcal{C}'_{\mathrm{P}}$ can easily be read off from Figure 5 and are given in Figures 12 and 13, respectively. In either graph the "double edge" from 3 to 4 is supposed to indicate a multitude of edges weighted by g_σ^n, for $n = 0, \ldots, \sigma - 2$, respectively.

5.1.2 Algorithms for Branch Reduction

We recall that $\mathcal{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_N\}$ is a (fixed, given) set of branches for the geodesic flow on $\mathbb{X}$, and that $\mathrm{C} = \bigcup \mathcal{C}$ denotes its branch union and $\widehat{\mathrm{C}} = \pi(\mathrm{C})$. We set

$$A_0 \coloneqq A = \{1, \ldots, N\}\,, \quad \mathcal{G}_0(j,k) \coloneqq \mathcal{G}(j,k)$$

for all $j, k \in A_0$, and

$$H_0(\ell) \coloneqq \{k \in A_0 \mid \mathcal{G}_0(\ell, k) \neq \varnothing\}$$

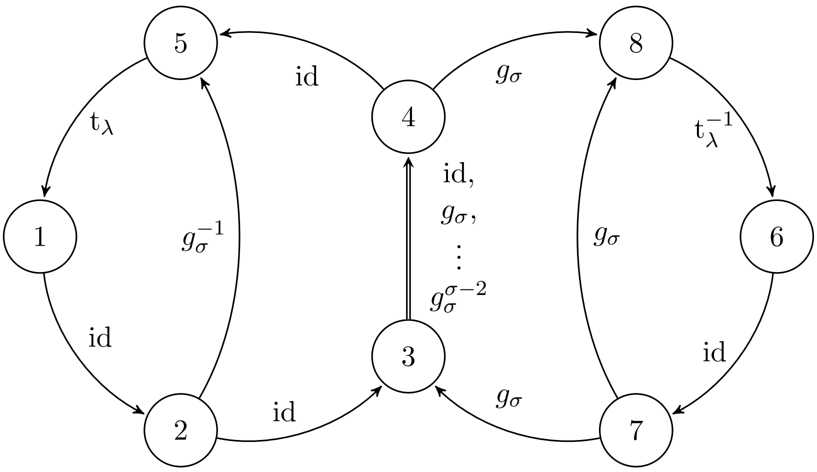


Figure 12: The return graph RG_0 for the set of branches $\mathcal{C}_\mathrm{P}$ for $\Gamma_{\sigma,\lambda}$.

for all $\ell \in A_0$. In what follows, we present the *branch reduction algorithm*, split into the two Algorithms 5.4 and 5.5, that reduces the return graph by constructing a (finite) cascade of subsets A_r of A_0 and related sets $H_r(j)$ and $\mathcal{G}_r(j,k)$, for $r = 1, 2, \ldots$, until we have achieved that $j \in H_r(j)$ for all $j \in A_r$. The algorithm includes choices and, depending on the group Γ, the cardinality of the set of remaining nodes might vary for different choices. (We refrain from fixing these choices in any artificial way and hence slightly abuse the notion of "algorithm" here.) This phenomenon potentially leads to different families of transfer operators, reflecting the inherent ambiguity of discretization of flows on quotient spaces. Fast transfer operators with the same spectral parameter arising from different such choices need not be mutually conjugate. However, their Fredholm determinants will coincide, as is guaranteed by the combination of the Theorems 3.1 and 6.1 below, and hence the 1-eigenfunctions of the two families of fast transfer operators are closely linked. Therefore, we observe independence of these choices in this aspect.

As mentioned, the branch reduction algorithm naturally splits into two standalone parts, each of which we present below as separate procedures. The first part, presented as Algorithm 5.4, removes those nodes of the return graph that only have a single outgoing edge and the edge does not loop back to the same node. The second part, presented as Algorithm 5.5, successively deletes nodes which are not among their own successors.

Algorithm 5.4 (Branch reduction, part I)**.** The index r starts at 1.
Step r. Set

$$R_r := \{j \in A_{r-1} \,|\, \#H_{r-1}(j) = 1 \wedge H_{r-1}(j) \neq \{j\} \wedge \#\mathcal{G}_{r-1}(j,.) = 1\}\,.$$

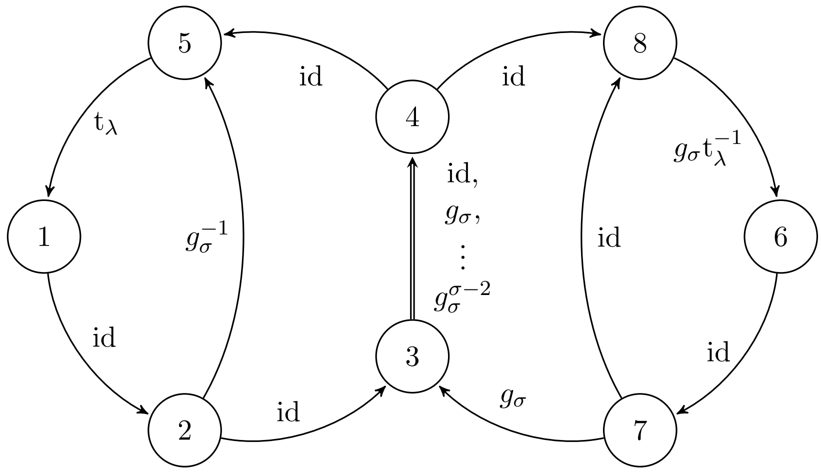


Figure 13: The return graph RG_0 for the set of branches $\mathcal{C}'_{\mathrm{P}}$ for $\Gamma_{\sigma,\lambda}$.

If $R_r = \varnothing$, then the algorithm terminates. Otherwise choose $j \in R_r$, set

$$A_r := A_{r-1} \setminus \{j\}\,,$$

and define for all $i, k \in A_r$

$$\mathcal{G}_r(i,k) := \mathcal{G}_{r-1}(i,k) \cup \bigcup_{g_1 \in \mathcal{G}_{r-1}(i,j)} \bigcup_{g_2 \in \mathcal{G}_{r-1}(j,k)} \{g_1 g_2\}$$

and

$$H_r(i) := \{\ell \in A_r \mid \mathcal{G}_r(i,\ell) \neq \varnothing\}\ .$$

Carry out *Step r+1.* ◇

In each step of Algorithm 5.4 a node, say j, is chosen and deleted (in the sense that $j \in A_{r-1}$ but $j \notin A_r$). Subsequently, each pair of an incoming edge and an outgoing edge of j is combined to a new edge, thereby "bridging" above j. More precisely, suppose that there is an edge from i to j weighted by g_1 and an edge from j to k weighted by g_2 for some $i, k \in A_{r-1} \setminus \{j\}$ and $g_1, g_2 \in \Gamma$, then we combine these to an edge from i to k weighted by $g_1 g_2$. We note that if, for $j, k \in A_{r-1}$, we have $k \notin H_{r-1}(j)$, then $\mathcal{G}_{r-1}(j,k) = \varnothing$ and hence $\mathcal{G}_r(i,k) = \mathcal{G}_{r-1}(i,k)$ for all $i \in A_r$.

Let $\kappa_1 \in \mathbb{N}_0$ be the unique number for which $R_{\kappa_1} \neq \varnothing$ but $R_{\kappa_1+1} = \varnothing$. In other words, $\kappa_1 + 1$ is the step in which Algorithm 5.4 terminates. (See Proposition 5.7 for its existence.) Then Algorithm 5.4 constructed the sets A_r, $H_r(j)$, and $\mathcal{G}_r(j,k)$ for all $r \in \{1, \ldots, \kappa_1\}$ and $j, k \in A_r$. (We emphasize that in the case $\kappa_1 = 0$, Algorithm 5.4 is void and does not construct any new sets.)

The second part of the branch reduction algorithm, Algorithm 5.5 below, now aims at reducing the number of branches even further, by successively deleting all nodes that are not themselves among their respective successors. In form, it is almost identical to Algorithm 5.4. The only but crucial difference is the base set from which the nodes are chosen in each step.

Algorithm 5.5 (Branch reduction, part II)**.** The index r starts at κ_1.
Step r. Define $P_r := \{j \in A_{r-1} \mid j \notin H_{r-1}(j)\}$. If $P_r = \varnothing$, the algorithm terminates. Otherwise choose $j \in P_r$, set

$$A_r := A_{r-1} \setminus \{j\},$$

and define for all $i, k \in A_r$

$$\mathcal{G}_r(i,k) := \mathcal{G}_{r-1}(i,k) \cup \bigcup_{g_1 \in \mathcal{G}_{r-1}(i,j)} \bigcup_{g_2 \in \mathcal{G}_{r-1}(j,k)} \{g_1 g_2\}$$

and

$$H_r(i) := \{\ell \in A_r \mid \mathcal{G}_r(i,\ell) \neq \varnothing\} \ .$$

Carry out *Step r+1.* ◇

Let $\kappa_2 \in \mathbb{N}_0$ be defined analogously to κ_1 but with respect to Algorithm 5.5. That is, $\kappa_2 + 1$ shall be the step in which Algorithm 5.5 terminates. In other words, κ_2 is the unique number larger than or equal to κ_1 such that $P_{\kappa_2} \neq \varnothing$ but $P_{\kappa_2+1} = \varnothing$. In Proposition 5.7 we will show that κ_2 is indeed well-defined.

For each $r \in \{1, \ldots, \kappa_2\}$ we define the *return graph of level r*, RG_r, analogously to $\mathrm{RG}_0 = \mathrm{RG}$, with A_r being the set of nodes, and edges and edge weights resulting from the transition sets $\mathcal{G}_r(j,k)$, $j, k \in A_r$. We emphasize that it may happen that Algorithm 5.4 or 5.5 is void, or even both, and consequently $\kappa_1 = 0$ or $\kappa_2 = \kappa_1$, or both. See Example 5.6.

Example 5.6. Consider the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The two elements

$$s := \mathrm{s}_{\frac{3\pi}{2}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad t := \mathrm{t}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

form a complete set of generators for Γ. A well-known cross section for the geodesic flow on the modular surface $\Gamma\diagdown\mathbb{H}$ is given by the representative

$$\mathrm{C}_1 := \{\nu \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(\nu) \in (0,\infty)_{\mathbb{H}},\ \gamma_\nu(+\infty) \in (0,+\infty)_{\mathbb{R}}\}$$

(see Figure 14). The set $\{\mathrm{C}_1\}$ has the structure of a set of branches. From Figure 14 we read off that the return graph RG_0 of Γ w.r.t. $\{\mathrm{C}_1\}$ consists solely of the two loops

$$1 \xrightarrow{t} 1 \quad \text{and} \quad 1 \xrightarrow{st^{-1}s} 1 \,.$$

Therefore, $H_0(1) = \{1\}$ and the sets R_1 and P_1 from the Algorithms 5.4 and 5.5 are empty. Consequently, we find $0 = \kappa_1 = \kappa_2$.

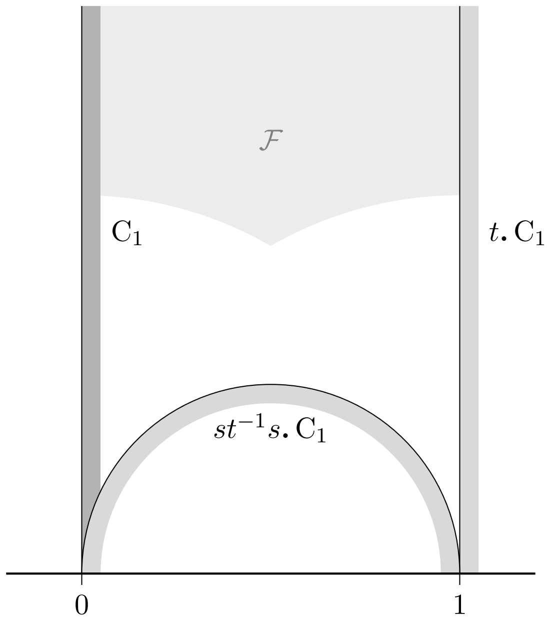


Figure 14: A fundamental domain for the modular group alongside the representative C_1 for a cross section on the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.

With these preparations we can now show that Algorithms 5.4 and 5.5 are indeed correct and provide sets of branches. For any $r \in \{0, \ldots, \kappa_2\}$ we set

$$\mathcal{C}_r := \{\mathrm{C}_j \mid j \in A_r\}$$

and

$$\mathrm{C}^{(r)} := \bigcup \mathcal{C}_r = \bigcup_{j \in A_r} \mathrm{C}_j \,.$$

Proposition 5.7. *Algorithms 5.4 and 5.5 terminate after finitely many steps without reducing the set of nodes to the empty set. Further, for each $r \in \{0, \ldots, \kappa_2\}$ the family $\mathcal{C}_r$ is a set of branches for the geodesic flow on $\mathbb{X}$. The family $\{\mathcal{G}_r(j,k)\}_{j,k \in A_r}$ is the family of forward transition sets in (B7). If $\mathcal{C}_0$ is admissible, then $\mathcal{C}_r$ is admissible as well.*

Proof. In each step of Algorithms 5.4 and 5.5, one element of the set A_0 gets eliminated, resulting in the decreasing cascade of subsets

$$\ldots \subsetneq A_3 \subsetneq A_2 \subsetneq A_1 \subsetneq A_0 \,.$$

Therefore, the number of steps in both algorithms is bounded from above by $\#A_0 < +\infty$. In turn, both algorithms terminate (after finitely many steps) and hence κ_1 and κ_2 are well-defined. We first show that $A_{\kappa_1} \neq \varnothing$. If $\kappa_1 = 0$, then $A_{\kappa_1} = A_0 \neq \varnothing$. Thus, suppose now that $\kappa_1 \geq 1$. To seek a contradiction, we assume that $A_{\kappa_1} = \varnothing$. Thus $A_{\kappa_1 - 1}$ contains exactly one elements, say j_0. Then either $\# H_{\kappa_1 - 1}(j_0) = 1$ and hence $H_{\kappa_1 - 1}(j_0) = \{j_0\}$, or $\# H_{\kappa_1 - 1}(j_0) = 0$ and hence $\# \mathcal{G}_{\kappa_1 - 1}(j_0, j_0) = 0$. In either case, $R_{\kappa_1} = \varnothing$. This contradicts the definition of κ_1, and hence $A_{\kappa_1} \neq \varnothing$. We now show that $A_{\kappa_2} \neq \varnothing$. If $\kappa_2 = \kappa_1$, then $A_{\kappa_2} = A_{\kappa_1} \neq \varnothing$. Thus, we suppose now that $\kappa_2 > \kappa_1$. As before, to seek a contradiction, we assume that $A_{\kappa_2} = \varnothing$. Again, $A_{\kappa_2 - 1}$ contains exactly one element, say k_0. By Proposition 5.2 we find a cycle of RG_0 that contains k_0. In each step of the node-elimination-processes of Algorithms 5.4 and 5.5, at most one node (other than k_0) of this cycle gets eliminated. If an elimination of a node in the cycle happens, then the two adjacent nodes of the eliminated node in the cycle get connected by a new edge that combines the two old ones. Thus, the cycle is "preserved" but shortened and has changed weights. Thus, after the step $\kappa_2 - 1$, the node k_0 is contained in a cycle, which is just a loop at k_0. In turn, $P_{\kappa_2} = \varnothing$. This contradicts the definition of κ_2. Hence, $A_{\kappa_2} \neq \varnothing$.

We now show that the families $\mathcal{C}_r$ are sets of branches, for any choice of $r \in \{0, \dots, \kappa_2\}$. Let $r \in \{0, \dots, \kappa_2\}$. Then the family $\mathcal{C}_r = \{\mathrm{C}_j \mid j \in A_r\}$ is a subset of the original set of branches $\mathcal{C} = \mathcal{C}_0 = \{\mathrm{C}_1, \dots, \mathrm{C}_N\}$. Hence most of the properties that we impose for a set of branches are immediate from those of $\mathcal{C}$. Indeed, the only properties which remain to be proven for $\mathcal{C}_r$ are (B4) and (B7), where for the former we take advantage of Proposition 4.8 and prove ($\mathrm{B}_{\mathrm{Per}}$) instead. For both properties we proceed by an inductive argument and note that they are already known to be valid for $\mathcal{C}_0 = \mathcal{C}$.

Let $r \in \{0, \dots, \kappa_2 - 1\}$ be such that (B7) is already established for $\mathcal{C}_r$ by using, for all $j, k \in A_r$, the set $\mathcal{G}_r(j,k)$ for the forward transition set in (B7). Let $j \in A_{r+1}$ and suppose that $A_r \setminus A_{r+1} = \{p\}$. If $p \notin H_r(j)$, then

$$\mathcal{G}_{r+1}(j,k) = \mathcal{G}_r(j,k)$$

for all $k \in H_{r+1}(j)$. In this case, (B7a) for r and $r+1$ are identical statements for the considered index j and hence, (B7a) holds for $r+1$. If $p \in H_r(j)$, then

$$H_{r+1}(j) = (H_r(j) \setminus \{p\}) \cup H_r(p)\,,$$

where necessarily $p \notin H_r(p)$. Taking advantage of the inductive hypothesis for the first equality below, we find

$$\begin{aligned}
I_{j,\mathrm{st}} &= \bigcup_{k\in H_r(j)} \bigcup_{g\in\mathcal{G}_r(j,k)} g.I_{k,\mathrm{st}} \\
&= \bigcup_{k\in H_r(j)\setminus\{p\}} \bigcup_{g\in\mathcal{G}_r(j,k)} g.I_{k,\mathrm{st}} \cup \bigcup_{h\in\mathcal{G}_r(j,p)} h.I_{p,\mathrm{st}} \\
&= \bigcup_{k\in H_r(j)\setminus\{p\}} \bigcup_{g\in\mathcal{G}_r(j,k)} g.I_{k,\mathrm{st}} \cup \bigcup_{h\in\mathcal{G}_r(j,p)} \bigcup_{q\in H_r(p)} \bigcup_{w\in\mathcal{G}_r(p,q)} hw.I_{q,\mathrm{st}} \\
&= \bigcup_{k\in H_r(j)\setminus\{p\}} \bigcup_{g\in\mathcal{G}_r(j,k)} g.I_{k,\mathrm{st}} \cup \bigcup_{q\in H_r(p)} \bigcup_{h\in\mathcal{G}_r(j,p)} \bigcup_{w\in\mathcal{G}_r(p,q)} hw.I_{q,\mathrm{st}} \\
&= \bigcup_{k\in H_{r+1}(j)} \bigcup_{g\in\mathcal{G}_{r+1}(j,k)} g.I_{k,\mathrm{st}}\,.
\end{aligned}$$

Since $\mathcal{C}_r$ is known to satisfy (B7), the unions in all steps are disjoint. This establishes the second part of (B7a) for $\mathcal{C}_{r+1}$. The first part as well as (B7c) follow analogously. Further, (B7b) is immediate by the construction of $\mathcal{C}_{r+1}$.

Now let $r \in \{0,\ldots,\kappa_2-1\}$ be such that ($\mathrm{B}_{\mathrm{Per}}$) is established for $\mathcal{C}_r$. Together with the previous discussion this then already shows that $\mathcal{C}_r$ is a set of branches. Suppose that $\widehat{\gamma}$ is a periodic geodesic on $\mathbb{X}$. By hypothesis, $\widehat{\gamma}$ has a lift to $\mathbb{H}$ which intersects $\mathrm{C}^{(r)}$. In order to show ($\mathrm{B}_{\mathrm{Per}}$) for $\mathcal{C}_{r+1}$, it remains to show that there is also such a lift that intersects $\mathrm{C}^{(r+1)}$. To seek a contradiction, we assume that all lifts of $\widehat{\gamma}$ intersect $\Gamma.\mathrm{C}^{(r)}$ only on $\Gamma.\mathrm{C}_p$. Then Proposition 4.9 implies that $p \in H_r(p)$, which contradicts $p \in A_r \setminus A_{r+1}$ (note that $\mathcal{C}_r$ is already known to be a set of branches). In turn, $\mathcal{C}_{r+1}$ satisfies ($\mathrm{B}_{\mathrm{Per}}$).

Finally, the claim that the set of branches $\mathcal{C}_r$ retains admissibility from $\mathcal{C}_0$ for all $r \in \{0,\ldots,\kappa_2\}$ follows immediately from $A_r \subseteq A_0$, for this implies

$$\bigcup_{j\in A_r} I_{j,\mathrm{st}} \subseteq \bigcup_{j\in A_0} I_{j,\mathrm{st}}\,. \qquad \square$$

Example 5.8. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ from Example 1.46 and its sets of branches $\mathcal{C}_{\mathrm{P}}$ from Example 2.11 and $\mathcal{C}'_{\mathrm{P}}$ from Example 4.34. A complete reduction procedure for $\Gamma_{\sigma,\lambda}$ takes 6 steps in total and leads to the return graph RG_6 depicted in Figure 15. In this example, it so happens that every possible sequence of choices for the Algorithms 5.4 and 5.5 leads to the same return graph RG_6, regardless of whether one starts out with $\mathcal{C}_{\mathrm{P}}$ or with $\mathcal{C}'_{\mathrm{P}}$. The arising set of branches $\{\mathrm{C}_{\mathrm{P},2},\mathrm{C}_{\mathrm{P},7}\}$ is easily seen to be non-collapsing in either case, for $\{2,7\} = D_{\mathrm{ini}}$.

Example 5.8 shows that, for some configurations, a (complete) branch reduction renders a formerly not weakly non-collapsing set of branches into a weakly non-collapsing one. However, this is not always the case. Conversely, the weakly non-collapsing property is retained via branch reduction, as the following result

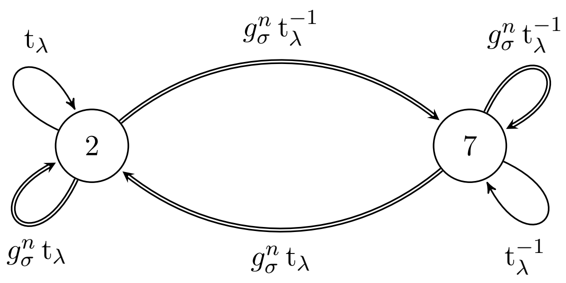


Figure 15: The maximally reduced return graph RG_6 for $\mathcal{C}_\mathrm{P}$. The double edges indicate multiple edges for $n = 1, \dots, \sigma - 1$, respectively.

shows. It further shows that finiteness of ramification is preserved.

Proposition 5.9. *Let $r \in \{0, \dots, \kappa_2\}$.*

(i) *If the set of branches $\mathcal{C}$ is weakly non-collapsing, then $\mathcal{C}_r$ is weakly non-collapsing.*

(ii) *If the set of branches $\mathcal{C}$ is finitely ramified, then $\mathcal{C}_r$ is finitely ramified.*

Proof. Let $r \in \{0, \dots, \kappa_2\}$ and let $j \in A_r \setminus A_{r+1}$. Let $i, k \in A_{r+1}$, $g_1 \in \mathcal{G}_r(i, j)$, and $g_2 \in \mathcal{G}_r(j, k)$, i. e., the return graph RG_r contains the path

$$i \xrightarrow{g_1} j \xrightarrow{g_2} k\,.$$

By construction, the return graph RG_{r+1} contains the path

$$i \xrightarrow{g_1 g_2} k\,.$$

Now, if $\mathcal{C}_r$ is weakly non-collapsing, then $g_1 g_2$ cannot be the identity, unless $g_1 = g_2 = \mathrm{id}$. But this already implies that $\mathcal{C}_{r+1}$ is weakly non-collapsing, for all paths not containing j are unaffected. This yields (i).

If $\mathbb{X}$ has a no cusps, then (ii) holds by Lemma 4.23. We now suppose that $\mathbb{X}$ has cusps, that $\mathcal{C} = \mathcal{C}_0$ is finitely ramified and that $\kappa_2 \geq 1$. It suffices to show that $\mathcal{C}_1$ is finitely ramified as the remaining claims then follow immediately by finite induction. By Proposition 4.26 it further suffices to show that each cusp of $\mathbb{X}$ is attached to $\mathcal{C}_1$.

Let $\widehat{c}$ be a cusp of $\mathbb{X}$, represented by $c \in \widehat{\mathbb{R}}$. By hypothesis and Proposition 4.26, $\widehat{c}$ is attached to $\mathcal{C}_0$. Thus,

$$I(\mathrm{Att}_{\mathcal{C}_0}(c)) := \overline{\bigcup_{(j,h) \in \mathrm{Att}_{\mathcal{C}_0}(c)} h.I_j}\,,$$

where $\mathrm{Att}_{\mathcal{C}_0}(c) := \{(j, h) \in A_0 \times \Gamma \mid c \in h.g\,\overline{\mathrm{bp}(\mathrm{C}_j)}\}$, is a neighborhood of c in $\widehat{\mathbb{R}}$. If $\mathrm{Att}_{\mathcal{C}_1}(c) = \mathrm{Att}_{\mathcal{C}_0}(c)$, then $\widehat{c}$ is also attached to $\mathcal{C}_1$. It remains to consider the case that $\mathrm{Att}_{\mathcal{C}_1}(c) \neq \mathrm{Att}_{\mathcal{C}_0}(c)$. Then there exists $(j_0, h) \in \mathrm{Att}_{\mathcal{C}_0}(c)$ that is not contained in $\mathrm{Att}_{\mathcal{C}_1}(c)$. The index j_0 is the (unique) node of RG_0 that gets eliminated in Algorithm 5.4 or 5.5. Let x_0 be the unique endpoint of $\overline{\mathrm{bp}(\mathrm{C}_{j_0})}$, and hence of I_{j_0}, such that $h.x_0 = c$. Since each cusp representative is an accumulation point of $\widehat{\mathbb{R}}_{\mathrm{st}}$, x_0 is also an endpoint of $I_{j_0,\mathrm{st}}$. By (B7a) we find a (unique) pair $(k, g) \in A_0 \times \mathcal{G}_0(j_0, k)$ such that $hg.I_{k,\mathrm{st}} \subseteq h.I_{j_0,\mathrm{st}}$ and c is an endpoint of both, $hg.I_{k,\mathrm{st}}$ and $hg.\overline{\mathrm{bp}(\mathrm{C}_k)}$. It follows that

$$\left(I(\mathrm{Att}_{\mathcal{C}_0}(c)) \setminus \overline{h.I_{j_0}}\right) \cup \overline{hg.I_k}$$

is also a neighborhood of c in $\widehat{\mathbb{R}}$. Further, $k \neq j_0$ because Algorithms 5.4 and 5.5 require $j_0 \notin H_0(j_0)$ for elimination of j_0. Therefore, $(k, hg) \in \mathrm{Att}_{\mathcal{C}_1}(c)$. Substituting each appearance of j_0 in $\mathrm{Att}_{\mathcal{C}_0}(c)$ in this way we obtain $\mathrm{Att}_{\mathcal{C}_1}(c)$ and find that $I(\mathrm{Att}_{\mathcal{C}_1}(c))$ is a neighborhood of c in $\widehat{\mathbb{R}}$. Thus, $\widehat{c}$ is attached to $\mathcal{C}_1$. This shows (ii). □

Example 5.10. The branch reduction algorithm facilitates a simple case study about how perturbations by elliptic transformations affect transfer operators.

Recall the Schottky group Γ_{S} and its Schottky data and set of branches,

$$\left(r, \{\mathcal{D}_j, \mathcal{D}_{-j}\}_{j=1}^{r}, \{s_j, s_{-j}\}_{j=1}^{r}\right) \quad \text{and} \quad \mathcal{C}_{\mathrm{S}} \coloneqq \{\mathrm{C}_{\pm 1}, \ldots, \mathrm{C}_{\pm r}\},$$

from Example 4.3. Recall further its family of transfer operators $\{\mathcal{L}_s\}_{s\in\mathbb{C}}$ from Example 4.38. From this family it can be seen that the set of branches $\mathcal{C}_{\mathrm{S}}$ is already minimal in terms of the branch reduction algorithm, for every entry on the diagonal of the transfer operator $\mathcal{L}_s$ is non-zero (i. e., not the operator mapping to the zero function) for every $s \in \mathbb{C}$, implying that every node in the return graph is its own successor.

We construct a new group $\Gamma_{\mathrm{S},\sigma}$ from Γ_{S} by introducing a single elliptic point. This results in a single conical singularity in the orbit space, thus rendering it a orbisurface rather than a proper hyperbolic surface. We do so by expanding the set of generators $\{s_{\pm j}\}_{j=1}^{r}$ by an elliptic transformation s_0 chosen as follows: By assumption, the sets $\mathcal{D}_j$ are mutually disjoint open Euclidean disks in $\mathbb{C}$. Choose indices $j, k \in \{\pm 1, \ldots, \pm r\}$ so that $\mathcal{D}_j$ and $\mathcal{D}_k$ are adjacent, that is, for $x_j \in \overline{\mathcal{D}_j}$ and $x_k \in \overline{\mathcal{D}_k}$ such that $|x_j - x_k|$ equals the (Euclidean) distance of $\mathcal{D}_j$ and $\mathcal{D}_k$ (then, necessarily, $x_j, x_k \in \mathbb{R}$) the interval $(\min\{x_j, x_k\}, \max\{x_j, x_k\})$ does not intersect any disk $\mathcal{D}_i$, $i \in \{\pm 1, \ldots, \pm r\}$. Without loss of generality we may assume $x_j < x_k$. Let $\sigma \in \mathbb{N}\setminus\{1\}$ and let $s_0 \in \mathrm{PSL}_2(\mathbb{R})$ be an elliptic transformation of order σ such that

$$\mathrm{Re}(\mathrm{I}(s_0)) \cup \mathrm{Re}(\mathrm{I}(s_0^{-1})) \subseteq (x_j, x_k)\,.$$

This is always feasible. For instance, let

$$\phi := \begin{bmatrix} \frac{\sqrt{x_k - x_j}}{2} & \frac{x_j + x_k}{\sqrt{x_k - x_j}} \\ 0 & \frac{2}{\sqrt{x_k - x_j}} \end{bmatrix}$$

and let g_σ be as in Example 1.46. Then

$$s_0 = \phi \cdot g_\sigma \cdot \phi^{-1}$$

fulfills these conditions. Let

$$\mathcal{F}_{\mathrm{S},\sigma} := \mathbb{H} \setminus \left(\bigcup_{j=1}^{r} (\overline{\mathcal{D}_j} \cup \overline{\mathcal{D}_{-j}}) \cup \overline{\operatorname{int} \mathrm{I}(s_0)} \cup \overline{\operatorname{int} \mathrm{I}(s_0^{-1})} \right).$$

Then $\mathcal{F}_{\mathrm{S},\sigma}$ is a convex polygon in $\mathbb{H}$ with side-pairing $\{s_{-r}, \ldots, s_r\} \subseteq \mathrm{PSL}_2(\mathbb{R})$ fulfilling all demands of the Poincaré theorem (Proposition 1.36). Hence,

$$\Gamma_{\mathrm{S},\sigma} = \langle s_0, \ldots, s_r \,|\, s_0^\sigma = \mathrm{id} \rangle$$

is a geometrically finite Fuchsian group. We call $\Gamma_{\mathrm{S},\sigma}$ a *perturbed Schottky group of order σ*.

We can further augment the set of branches $\mathcal{C}_\mathrm{S}$ for Γ_S in order to obtain a set of branches for $\Gamma_{\mathrm{S},\sigma}$. To that end, recall that $I_0 := \mathrm{Re}(\mathrm{I}(s_0)) \cup \mathrm{Re}(\mathrm{I}(s_0)^{-1})$ is an open interval in $\mathbb{R}$ by virtue of Lemma 1.21(ii), let $x, y \in \mathbb{R}$ be such that $(x, y) = I_0$, set $\beta_0 := (x, y)_\mathbb{H}$, and define

$$\mathrm{C}_0 := \{\nu \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(\nu) \in \beta_0\,,\ \gamma_\nu(+\infty) \in I_0\}\ .$$

Then one checks that $\mathcal{C}_{\mathrm{S},\sigma} := \{\mathrm{C}_{-r}, \ldots, \mathrm{C}_r\}$ is a set of branches for the geodesic flow on the orbit space of $\Gamma_{\mathrm{S},\sigma}$. Indeed, one finds

$$\mathcal{G}_0(0, j) := \begin{cases} \{s_0^{-1}, \ldots, s_0^{1-\sigma}\} & \text{if } j \neq 0 \\ \varnothing & \text{if } j = 0 \end{cases} \quad \text{and} \quad \mathcal{G}_0(j, 0) = \{s_j^{-1}\}\,,$$

for all $j \in \{\pm 1, \ldots, \pm r\}$. Therefore, contrary to the set of branches $\mathcal{C}_\mathrm{S}$, the set of branches $\mathcal{C}_{\mathrm{S},\sigma}$ can be reduced, for C_0 turns out to be dispensable. Algorithm 5.5 reduces $\mathcal{C}_{\mathrm{S},\sigma}$ back down to $\mathcal{C}_\mathrm{S}$, which, because of Proposition 5.7, is thus a set of branches for $\Gamma_{\mathrm{S},\sigma}$ as well, albeit with a different family of transition sets. These transition sets now take the form

$$\mathcal{G}_1(j, k) = \begin{cases} \left\{ s_j^{-1} s_0^{-n} \,\middle|\, n = 0, \ldots, \sigma - 1 \right\} & \text{if } k \neq -j \\ \left\{ s_j^{-1} s_0^{-m} \,\middle|\, m = 1, \ldots, \sigma - 1 \right\} & \text{if } k = -j \end{cases}.$$

Hence, with the same setup and notations as in Example 4.38, the transfer operator with parameter $s \in \mathbb{C}$ for $\Gamma_{\mathrm{S},\sigma}$ now takes the form

$$\mathcal{L}_s = \begin{pmatrix} S_s(s_1) & S_s(s_2) & \dots & S_s(s_r) & T_s(s_{-1}) & S_s(s_{-2}) & \dots & S_s(s_{-r}) \\ S_s(s_1) & S_s(s_2) & \dots & S_s(s_r) & S_s(s_{-1}) & T_s(s_{-2}) & \dots & S_s(s_{-r}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_s(s_1) & S_s(s_2) & \dots & S_s(s_r) & S_s(s_{-1}) & S_s(s_{-2}) & \dots & T_s(s_{-r}) \\ T_s(s_1) & S_s(s_2) & \dots & S_s(s_r) & S_s(s_{-1}) & S_s(s_{-2}) & \dots & S_s(s_{-r}) \\ S_s(s_1) & T_s(s_2) & \dots & S_s(s_r) & S_s(s_{-1}) & S_s(s_{-2}) & \dots & S_s(s_{-r}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_s(s_1) & S_s(s_2) & \dots & T_s(s_r) & S_s(s_{-1}) & S_s(s_{-2}) & \dots & S_s(s_{-r}) \end{pmatrix},$$

where

$$S_s(s_j) := \sum_{m=0}^{\sigma-1} \tau_s(s_0^m s_j) \quad \text{and} \quad T_s(s_j) := \sum_{n=1}^{\sigma-1} \tau_s(s_0^n s_j)$$

for $j \in \{\pm 1, \dots, \pm r\}$.

5.2 Identity Elimination and Reduced Sets of Branches

In order to fulfill all requirements of a strict transfer operator approach, it is essential to ensure a unique coding of periodic geodesics in terms of the chosen set of generators for the underlying Fuchsian group. This property, which we call the *non-collapsing property*, is codified in (B9). In terms of the return graph of the considered set of branches, it states that the weights along edge sequences never combine to the identity. Sets of branches that initially do not satisfy (B9) can be remodeled via a reduction procedure that removes such identity transformations from the system. In this section we present, discuss, and prove such a procedure, which we call *identity elimination.*

Let $\mathcal{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_N\}$ be a set of branches for the geodesic flow on $\mathbb{X}$. The Propositions 4.35 and 5.9(i) allow us to suppose $\mathcal{C}$ to be weakly non-collapsing (see ($\mathrm{B}_{\mathrm{col}}$)) without shrinking the realm of applicability of the identity elimination algorithm. This additional hypothesis eliminates some technical subleties. It assures that all identity transformations present in the system are *visible* to the algorithm as no identities are concealed by Γ-copies of $\mathcal{C}$.

Let κ_2, A_r, $H_r(j)$, and $\mathcal{G}_r(j,k)$ for $j, k \in A_r$ and $r \in \{1, \dots, \kappa_2\}$ be as in Section 5.1. The procedure discussed in this section is uniform in the "level of reduction," meaning uniform with respect to $r \in \{0, \dots, \kappa_2\}$. For that reason we omit the subscript r throughout. However, we remark that a sufficient level of reduction might already render the emerging set of branches non-collapsing, as can be observed, e. g., in Example 5.8. However, this is not always the case, not even in the weakly non-collapsing case.

Recall the subset D_{ini} of A of indices of initial branches from (4.29). Recall further the branch trees B_j, $j \in A$, and the branch forest F_{ini} for $\mathcal{C}$ from Section 4.3. For every $i \in D_{\mathrm{ini}}$ consider all nodes in B_i of the form $(*, \mathrm{id})$. Since $\mathcal{C}$ is weakly non-collapsing, these nodes (and their connecting edges) form a sub-tree B_i' of B_i of finite depth. Furthermore, for the same reason, for every $k \in A$ there exists $j \in D_{\mathrm{ini}}$ such that B_j' contains the node (k, id). Denote by F_{ini}' the forest of sub-trees B_i', $i \in D_{\mathrm{ini}}$. The forest F_{ini}' can be seen as a disconnected, directed graph of which each connected component is a tree of nodes of the form $(*, \mathrm{id})$. Therefore, each (directed) path in F_{ini}' can be indexed by a tuple consisting of the index (the element in A) of the root node of its super-tree and the index of its end node. (It is necessary to consider the root node in this indexing as well, because end nodes for different paths in F_{ini}' may coincide.) We denote by $\Delta_{\mathrm{ini}} \subseteq D_{\mathrm{ini}} \times A$ the set of these indices. By construction, $\#\Delta_{\mathrm{ini}} < +\infty$ and for every $i \in D_{\mathrm{ini}}$ the set Δ_{ini} contains an element of the form $(i, *)$. To each path $(i, k) \in \Delta_{\mathrm{ini}}$ we assign its length $\ell_{(i,k)} \in \mathbb{N}_0$, that is the level of (k, id) in B_i'. Then $\ell_{(i,k)}$ is the unique integer for which there exists $\nu \in \mathrm{C}_i$ such that

$$\big(\mathrm{k}_{\mathrm{C},\ell_{(i,k)}}(\nu), \mathrm{g}_{\mathrm{C},\ell_{(i,k)}}(\nu)\big) = (k, \mathrm{id}),$$

with $\mathrm{k}_{\mathrm{C},*}(\nu)$ as in (4.12) and $\mathrm{g}_{\mathrm{C},*}(\nu)$ as in (4.13) and (4.14). We further assign a sequence $(a_n^\delta)_{n=0}^{\ell_\delta}$ in A to each $\delta = (i, k) \in \Delta_{\mathrm{ini}}$ by imposing that (i, k) indexes the path

$$\begin{aligned}(i, \mathrm{id}) = \big(a^{(i,k)}_{\ell_{(i,k)}}, \mathrm{id}\big) \rightharpoonup \big(a^{(i,k)}_{\ell_{(i,k)}-1}, \mathrm{id}\big) \rightharpoonup \ldots \\ \ldots \rightharpoonup \big(a^{(i,k)}_1, \mathrm{id}\big) \rightharpoonup \big(a^{(i,k)}_0, \mathrm{id}\big) = (k, \mathrm{id}).\end{aligned}$$

The following algorithm (Algorithm (5.11) below) will traverse paths in backwards direction. That is what motivates the counter-intuitive numbering of the members of the (finite) sequence (a_n^δ). The possibility of multiple occurrences of a single node (k, id) in F_{ini}' necessitates an iterative approach, where certain branches and transition sets might be redefined several times. We therefore initialize the procedure by setting

$$\ell_{\delta_0} := 0, \qquad \mathcal{G}^{(0)}_{\ell_{\delta_0}}(j,k) := \mathcal{G}(j,k) \qquad \text{and} \qquad \mathrm{C}_j^{(0,0)} := \mathrm{C}_j$$

for all $j, k \in A$. Further, we fix an (arbitrary) enumeration of Δ_{ini} and write

$$\Delta_{\mathrm{ini}} = \{\delta_1, \ldots, \delta_\eta\}, \tag{5.8}$$

with $\eta := \#\Delta_{\mathrm{ini}}$.

Algorithm 5.11 (Identity elimination)**.** The index r runs from 1 to η.
Step r. Set

$$\mathcal{G}_0^{(r)}(\lambda,\kappa) := \mathcal{G}_{\ell_{\delta_{r-1}}}^{(r-1)}(\lambda,\kappa) \qquad \text{and} \qquad \mathrm{C}_\lambda^{(r,0)} := \mathrm{C}_\lambda^{(r-1,\ell_{\delta_{r-1}})}$$

for all $\lambda,\kappa \in A$. The index s_r runs from 1 to ℓ_{δ_r}.

Substep s_r. For all $j \in A$ define

$$\mathcal{G}_{s_r}^{(r)}(j, a_{s_r-\ell}^{\delta_r}) := \mathcal{G}_{s_r-1}^{(r)}(j, a_{s_r-\ell}^{\delta_r}) \setminus \{\mathrm{id}\} \cup \mathcal{G}_{s_r-1}^{(r)}(j, a_{s_r}^{\delta_r})\,,$$

for $\ell = 1,\dots,s_r$, as well as

$$\mathcal{G}_{s_r}^{(r)}(j,\kappa) := \mathcal{G}_{s_r-1}^{(r)}(j,\kappa)\,,$$

for all $\kappa \in A \setminus \{a_0^{\delta_r},\dots,a_{s_r-1}^{\delta_r}\}$. Further set

$$V_{s_r}^{(r)} := \left\{ \nu \in \mathrm{C}_{a_{s_r}^{\delta_r}}^{(r,s_r-1)} \;\middle|\; \big(\mathrm{k}_{\mathrm{C},1}(\nu), \mathrm{g}_{\mathrm{C},1}(\nu)\big) = \big(a_{s_r-1}^{\delta_r}, \mathrm{id}\big) \right\}$$

and define

$$\mathrm{C}_j^{(r,s_r)} := \mathrm{C}_j^{(r,s_r-1)} \setminus V_{s_r}^{(r)}\,. \qquad \diamond$$

Let $r,r' \in \{1,\dots,\eta\}$ and $s_r \in \{1,\dots,\ell_{\delta_r}\}$, $s_{r'} \in \{1,\dots,\ell_{\delta_{r'}}\}$. We define a relation "$\leq$" on the set of pairs (r,s_r) by setting

$$(r,s_r) \leq (r',s_{r'}) \quad :\Longleftrightarrow \quad r < r' \;\vee\; \big(r = r' \;\wedge\; s_r \leq s_{r'}\big)\,. \tag{5.9}$$

Then "$\leq$" is a total order.

Lemma 5.12. *For $j \in A$, $r,r' \in \{1,\dots,\eta\}$ and $s_r \in \{1,\dots,\ell_{\delta_r}\}$ and $s_{r'} \in \{1,\dots,\ell_{\delta_{r'}}\}$ we have*

(i) $V_{s_r}^{(r)} \cap V_{s_{r'}}^{(r')} \neq \varnothing$ *if and only if* $V_{s_r}^{(r)} = V_{s_{r'}}^{(r')}$,

(ii) $\mathrm{C}_j^{(r,s_r)} = \mathrm{C}_j \setminus \left(\bigcup_{p=1}^{r-1} \bigcup_{\ell=1}^{\ell_{\delta_p}} V_\ell^{(p)} \cup \bigcup_{i=1}^{s_r} V_i^{(r)} \right)$,

(iii) $\mathrm{C}_j^{(r',s_{r'})} \subseteq \mathrm{C}_j^{(r,s_r)} \subseteq \mathrm{C}_j$ *if and only if* $(r,s_r) \leq (r',s_{r'})$.

Proof. Statement (i) is immediate from the definition of the sets $V_{s_r}^{(r)}$ in Algorithm 5.11 and the uniqueness of the system of iterated sequences from (4.15) for any given $\nu \in \mathrm{C}$. Statement (ii) follows by straightforward, recursive application of the definition of $\mathrm{C}_j^{(r,s_r)}$ in Algorithm 5.11. From this presentation of $\mathrm{C}_j^{(r,s_r)}$ we obtain that, if $r < r'$,

$$\begin{aligned}
\mathrm{C}_j \setminus \mathrm{C}_j^{(r',s_{r'})} &= \bigcup_{p=1}^{r'-1} \bigcup_{\ell=1}^{\ell_{\delta_p}} V_\ell^{(p)} \cup \bigcup_{i=1}^{s_{r'}} V_i^{(r')} \\
&= \bigcup_{p=1}^{r-1} \bigcup_{\ell=1}^{\ell_{\delta_p}} V_\ell^{(p)} \cup \bigcup_{i=1}^{s_r} V_i^{(r)} \cup \bigcup_{i=s_r+1}^{\ell_{\delta_r}} V_i^{(r)} \cup \bigcup_{p=r+1}^{r'-1} \bigcup_{\ell=1}^{\ell_{\delta_p}} V_\ell^{(p)} \cup \bigcup_{i=1}^{s_{r'}} V_i^{(r')} \\
&\supseteq \mathrm{C}_j \setminus \mathrm{C}_j^{(r,s_r)} ,
\end{aligned}$$

and, if $r = r'$ and $s_r \leq s_{r'}$,

$$\mathrm{C}_j \setminus \mathrm{C}_j^{(r',s_{r'})} = \bigcup_{p=1}^{r'-1} \bigcup_{\ell=1}^{\ell_{\delta_p}} V_\ell^{(p)} \cup \bigcup_{i=1}^{s_r} V_i^{(r)} \cup \bigcup_{i=s_r+1}^{s_{r'}} V_i^{(r')} \supseteq \mathrm{C}_j \setminus \mathrm{C}_j^{(r,s_r)} .$$

This immediately yields (iii). □

From Lemma 5.12(ii) we obtain that, for every $j \in A$, Algorithm 5.11 ultimately defines the set of unit tangent vectors

$$\widetilde{\mathrm{C}}_j := \mathrm{C}_j^{(\eta,\ell_{\delta_\eta})} = \mathrm{C}_j \setminus \bigcup_{r=1}^{\eta} \bigcup_{s=1}^{\ell_{\delta_\eta}} V_s^{(r)} . \tag{5.10}$$

Accordingly, we set

$$\mathcal{I}_j := I_j \setminus \bigcup_{r=1}^{\eta} \bigcup_{s=1}^{\ell_{\delta_\eta}} \left\{ \gamma_\nu(+\infty) \;\middle|\; \nu \in V_s^{(r)} \right\} . \tag{5.11}$$

Depending on the initial set of branches $\mathcal{C}$ and the level of reduction, Algorithm 5.11 might render branches essentially empty, in the sense that $\widetilde{\mathrm{C}}_{j,\mathrm{st}} = \varnothing$. We account for this possibility by updating the index set to be

$$\widetilde{A} := \left\{ k \in A \;\middle|\; \widetilde{\mathrm{C}}_{k,\mathrm{st}} \neq \varnothing \right\} . \tag{5.12}$$

From (5.10) and the definition of the sets $V_*^{(*)}$ we read off that the branch C_j can only be rendered essentially empty by Algorithm 5.11 if $\bigcup_{k\in A} \mathcal{G}(j,k) = \{\mathrm{id}\}$. This implies in particular that

$$\{j \in A \mid (*,j) \in \Delta_{\mathrm{ini}}\} \subseteq \widetilde{A} .$$

Hence, $\widetilde{A} \neq \varnothing$. We further define

$$\widetilde{\mathcal{G}}(j,k) := \mathcal{G}^{(\eta)}_{\ell_{\delta_\eta}}(j,k)\,, \tag{5.13}$$

for $j,k \in \widetilde{A}$, as well as

$$\widetilde{\mathcal{C}} := \left\{ \widetilde{\mathrm{C}}_j \;\middle|\; j \in \widetilde{A} \right\} \qquad \text{and} \qquad \widetilde{\mathrm{C}} := \bigcup \widetilde{\mathcal{C}} = \bigcup_{j \in \widetilde{A}} \widetilde{\mathrm{C}}_j\,. \tag{5.14}$$

From (5.11) it is apparent that the sets $\widetilde{\mathrm{C}}_j$, $j \in A$, are not necessarily branches in the sense of Definition 4.1 anymore, due to possible violation of (B5). We account for that by introducing the notion of a *reduced set of branches* in Definition 5.17 below.

We now analyze the structure and interrelation of the sets of unit tangent vectors and transformations successively defined by Algorithm 5.11. To that end, we let $r \in \{0,\ldots,\eta\}$, $s_r \in \{0,\ldots,\ell_{\delta_r}\}$, and set

$$\mathcal{C}^{(r)}_{s_r} := \left\{ \mathrm{C}^{(r,s_r)}_j \;\middle|\; j \in \widetilde{A} \right\} \qquad \text{and} \qquad \mathrm{C}^{(r)}_{s_r} := \bigcup \mathcal{C}^{(r)}_{s_r}\,.$$

Then, obviously,

$$\mathrm{C}^{(r)}_{s_r} = \mathrm{C}^{(r)}_{s_r-1} \setminus V^{(r)}_{s_r}\,. \tag{5.15}$$

For $\nu \in \mathrm{C}^{(r)}_{s_r}$ we define a system of sequences

$$\left[\left(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r},n}(\nu)\right)_n, \left(\mathrm{k}_{\mathrm{C}^{(r)}_{s_r},n}(\nu)\right)_n, \left(\mathrm{g}_{\mathrm{C}^{(r)}_{s_r},n}(\nu)\right)_n \right] \tag{5.16}$$

as in (4.15), with $\mathrm{C}^{(r)}_{s_r}$ in place of C.

Lemma 5.13. *Let $r \in \{0,\ldots,\eta\}$ and $s_r \in \{0,\ldots,\ell_{\delta_r}\}$. For $\nu \in \mathrm{C}^{(r)}_{s_r}$ the system of sequences in (5.16) is well-defined. Furthermore, the set $\left\{ \mathcal{G}^{(r)}_{s_r}(j,k) \;\middle|\; j,k \in \widetilde{A} \right\}$ is a full set of transition sets for $\mathcal{C}^{(r)}_{s_r}$ up to identities, in the sense that*

(i) $\forall\, j,k \in A\; \forall\, \nu \in \mathrm{C}^{(r,s_r)}_j, \mathrm{k}_{\mathrm{C}^{(r)}_{s_r},1}(\nu) = k\colon \mathrm{g}_{\mathrm{C}^{(r)}_{s_r},1}(\nu) \in \mathcal{G}^{(r)}_{s_r}(j,k) \cup \{\mathrm{id}\}\,,$

(ii) $\forall\, j,k \in A\; \forall\, g \in \mathcal{G}^{(r)}_{s_r}(j,k)\; \exists\, \nu \in \mathrm{C}^{(r,s_r)}_j\colon (\mathrm{k}_{\mathrm{C}^{(r)}_{s_r},1}(\nu), \mathrm{g}_{\mathrm{C}^{(r)}_{s_r},1}(\nu)) = (k,g)\,.$

Proof. We argue by induction over the totally ordered set of pairs (r,s_r), $r \in \{1,\ldots,\eta\}$, $s_r \in \{1,\ldots,\ell_{\delta_r}\}$. For $r = s_r = 0$ there is nothing to show. We suppose that all claims have already been proven for $(r, s_r - 1)$ for some $r \geq 0$. Let $\nu \in \mathrm{C}^{(r)}_{s_r}$. From (5.15) we read off that

$$\mathrm{t}_{\mathrm{C}^{(r)}_{s_r},n}(\nu) = \mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},m_n}(\nu)\,, \quad \mathrm{k}_{\mathrm{C}^{(r)}_{s_r},n}(\nu) = \mathrm{k}_{\mathrm{C}^{(r)}_{s_r-1},m_n}(\nu)\,,$$

and

$$\mathrm{g}_{\mathrm{C}^{(r)}_{s_r},n}(\nu) = \mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},n}(\nu) \cdots \mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},m_n}(\nu)\,, \tag{5.17}$$

where, *a priori*, $m_0 := 0$ and

$$m_n := \begin{cases} \min\left\{ m \geq n \;\middle|\; \gamma'_\nu\big(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},m}(\nu)\big) \notin \Gamma.V^{(r)}_{s_r} \right\} & \text{for } n > 0\,, \\ \max\left\{ m \leq n \;\middle|\; \gamma'_\nu\big(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},m}(\nu)\big) \notin \Gamma.V^{(r)}_{s_r} \right\} & \text{for } n < 0\,. \end{cases}$$

But since each node (j, id) appears at most once in the path δ_r, we have $a^{\delta_r}_{s_r} \neq a^{\delta_r}_{s_r-1}$ and hence

$$m_n = \begin{cases} n & \text{if } \gamma'_\nu\big(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},m}(\nu)\big) \notin \Gamma.V^{(r)}_{s_r}\,, \\ n+1 & \text{if } \gamma'_\nu\big(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},m}(\nu)\big) \in \Gamma.V^{(r)}_{s_r} \text{ and } n > 0\,, \\ n-1 & \text{if } \gamma'_\nu\big(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},m}(\nu)\big) \in \Gamma.V^{(r)}_{s_r} \text{ and } n < 0\,, \end{cases} \tag{5.18}$$

Since all objects exist by hypothesis, the system of sequences from (5.16) is well-defined. We set

$$j := \mathrm{k}_{\mathrm{C}^{(r)}_{s_r},0}(\nu)\,, \quad k := \mathrm{k}_{\mathrm{C}^{(r)}_{s_r},1}(\nu)\,, \quad \text{and} \quad g := \mathrm{g}_{\mathrm{C}^{(r)}_{s_r},1}(\nu)\,.$$

Concerning statement (i) we have to show that

$$g \in \mathcal{G}^{(r)}_{s_r}(j,k) \cup \{\mathrm{id}\}\,. \tag{5.19}$$

To that end note that

$$\mathcal{G}^{(r)}_{s_r-1}(j,k) \subseteq \mathcal{G}^{(r)}_{s_r}(j,k) \cup \{\mathrm{id}\}\,.$$

If $k \neq a^{\delta_r}_{s_r}$, then

$$\gamma'_\nu\big(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},1}(\nu)\big) \in \mathrm{C}^{(r)}_{s_r-1} \setminus \mathrm{C}^{(r,s_r-1)}_{a^{\delta_r}_{s_r}} \subseteq \mathrm{C}^{(r)}_{s_r-1} \setminus V^{(r)}_{s_r}\,.$$

Hence, by the discussion above and the hypothesis,

$$g = \mathrm{g}_{\mathrm{C}^{(r)}_{s_r},1}(\nu) = \mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},1}(\nu) \in \mathcal{G}^{(r)}_{s_r-1}(j,k) \subseteq \mathcal{G}^{(r)}_{s_r}(j,k) \cup \{\mathrm{id}\}\,.$$

Now let $k = a^{\delta_r}_{s_r}$. If $\gamma'_\nu(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},1}(\nu)) \notin \Gamma.V^{(r)}_{s_r}$, then we may argue as before. It thus remains to consider the case that

$$\gamma'_\nu(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r-1},1}(\nu)) \in \Gamma.V^{(r)}_{s_r}\,.$$

Then, from the definition of $V_{s_r}^{(r)}$ one obtains

$$\mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},2}(\nu) = \mathrm{id}\ .$$

From that, (5.17), (5.18), and the hypothesis we obtain that

$$\begin{aligned} g = \mathrm{g}_{\mathrm{C}^{(r)}_{s_r},1}(\nu) &= \mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},1}(\nu)\cdot \mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},2}(\nu) \\ &= \mathrm{g}_{\mathrm{C}^{(r)}_{s_r-1},1}(\nu) \in \mathcal{G}^{(r)}_{s_r-1}(j,k) \subseteq \mathcal{G}^{(r)}_{s_r}(j,k)\,. \end{aligned}$$

This yields (i).

Concerning (ii) let $j \in A$ and $k \in \{a_0^{\delta_r},\dots,a_{s_r-1}^{\delta_r}\}$, for in all other cases the claim is immediate from the hypothesis. Let $s \in \{0,\dots,s_r-1\}$ be such that $k = a_s^{\delta_r}$. If $\mathcal{G}^{(r)}_{s_r}(j,k) = \varnothing$, then there is nothing to show. Thus, we consider the case that

$$\mathcal{G}^{(r)}_{s_r}(j,k) \neq \varnothing\,.$$

For $g \in \mathcal{G}^{(r)}_{s_r-1}(j,k)$ the claim is immediate from the hypothesis. If we have $\mathcal{G}^{(r)}_{s_r-1}(j,a^{\delta_r}_{s_r}) = \varnothing$, then we are finished. Thus, we suppose that

$$\mathcal{G}^{(r)}_{s_r-1}(j,a^{\delta_r}_{s_r}) \neq \varnothing$$

and pick $g \in \mathcal{G}^{(r)}_{s_r-1}(j,a^{\delta_r}_{s_r})$. We show that there exists $\nu \in \mathrm{C}_j^{(r,s_r)}$ such that

$$\gamma'_\nu(\mathrm{t}_{\mathrm{C},s_r-s+1}(\nu)) \in \mathrm{C}_k^{(r,s)} \quad \text{and} \quad \gamma'_\nu(\mathrm{t}_{\mathrm{C},\ell}(\nu)) \in V^{(r)}_{s_r-\ell+1} \tag{5.20}$$

for all $\ell \in \{1,\dots,s_r-s\}$. First note that, by definition of the path δ_r, we have $V_i^{(r)} \neq \varnothing$ for all $i \in \{1,\dots,\ell_{\delta_r}\}$, and because of $k \in \widetilde{A}$ and Lemma 5.12(iii) we have $\mathrm{C}^{(r,s)}_{k,\mathrm{st}} \neq \varnothing$. Because of $g \in \mathcal{G}^{(r)}_{s_r-1}(j,a^{\delta_r}_{s_r})$, the hypothesis, and the structure of the path δ_r, we have

$$g.\mathrm{H}_+(a^{\delta_r}_\ell) \subsetneq g.\mathrm{H}_+(a^{\delta_r}_{s_r}) \subsetneq \mathrm{H}_+(j)\,,$$

for all $\ell \in \{0,\dots,s_r-1\}$. Hence, in particular there exists $\nu \in \mathrm{C}_j^{(r,s_r-1)}$ such that

$$\big(\gamma_\nu(+\infty),\gamma_\nu(-\infty)\big) \in g.\mathcal{I}_{k,\mathrm{st}} \times J_{j,\mathrm{st}}\,.$$

Then also $\nu \in \mathrm{C}_j^{(r,s_r)}$, for otherwise $j = a^{\delta_r}_{s_r}$ and $\nu \in V^{(r)}_{s_r}$. But then $g = \mathrm{id}$ and $a^{\delta_r}_{s_r-1} = a^{\delta_r}_{s_r}$ by the definition of $V^{(r)}_{s_r}$, which contradicts the structure of the path δ_r. Lemma 5.12(iii) now implies that

$$\gamma_\nu(+\infty) \in g.\mathcal{I}_{k,\mathrm{st}} \subseteq g.\left\{\gamma_\mu(+\infty) \,\middle|\, \mu \in \mathrm{C}^{(r,s)}_{k,\mathrm{st}}\right\}\,.$$

Since $J_{j,\mathrm{st}} \subseteq J_{k,\mathrm{st}}$, this together with (B5) and again Lemma 5.12(iii) imply that

$$\gamma'_\nu(0,+\infty) \cap \mathrm{C}_k^{(r,s)} \neq \varnothing\,, \tag{5.21}$$

and by counting intersections with the initial set of branches we see that ν fulfills (5.20). From Algorithm 5.11 we now see that, at *Substep* s_r, all sets $V_\ell^{(r)}$ for $\ell \in \{1,\ldots,s_r\}$ have already been removed from their respective branch. Hence, while γ_ν does intersect each of the branches $g\boldsymbol{.}\,\mathrm{C}_{a^{\delta_r}_{s_r-\ell}}$, it does not intersect $g\boldsymbol{.}\,\mathrm{C}^{(r,s_r)}_{a^{\delta_r}_{s_r-\ell}}$, for $\ell \in \{0,\ldots,s_r-s\}$. From this and (5.21) we conclude

$$\mathrm{k}_{\mathrm{C}^{(r)}_{s_r},1}(\nu) = k$$

and

$$\mathrm{g}_{\mathrm{C}^{(r)}_{s_r},1}(\nu) = \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},s_r-s+1}(\nu) = g\cdot\mathrm{id}\cdots\mathrm{id} = g\,.$$

This shows (ii) and thereby finishes the proof. □

Remark 5.14. In part (i) of Lemma 5.13 it is indeed necessary to include the identity transformation, for, depending on the enumeration of Δ_{ini} and whether or not $\mathcal{G}^{(r)}_{s_r-1}(j,a^{\delta_r}_{s_r})$ contains the identity, $\mathcal{G}^{(r)}_{s_r}(j,k)$ might end up differing from the actual transition set for $j,k \in A$ with respect to $\mathrm{C}^{(r)}_{s_r}$ by lacking exactly the identity transformation. This is due to a slight imprecision in Algorithm 5.11 in the handling of such situations, which we accepted in favor of clarity. Simply put, Algorithm 5.11 might remove certain identities "too soon." But since *all* identity transformations are removed in the end (see Proposition 5.19 below), this deviation does not affect the final transition sets.

Proposition 5.15. *The sets $\widetilde{\mathrm{C}}_j$ and $\widetilde{\mathcal{G}}(j,k)$, $j,k \in \widetilde{A}$, are independent of the enumeration chosen in* (5.8)*, and we have*

$$\bigcup_{j,k\in\widetilde{A}} \widetilde{\mathcal{G}}(j,k) \subseteq \Gamma^*\,.$$

Proof. Let $j \in \widetilde{A}$. If $\mathrm{id} \notin \bigcup_{k\in A}\mathcal{G}(j,k)$, then there is nothing to show, since every set $\mathcal{G}^{(r)}_{s_r}(j,*)$ emerges as the union of two sets of the form $\mathcal{G}^{(r)}_{s_r-1}(j,*)$, for all r and all s_r, and hence cannot introduce identity transformations. Thus, suppose that $\mathrm{id} \in \bigcup_{k\in A}\mathcal{G}(j,k)$. For every $k \in \widetilde{A}$ for which $\mathrm{id} \in \mathcal{G}(j,k)$ there exists $r \in \{1,\ldots,\eta\}$ such that $(j,\mathrm{id}) \rightharpoonup (k,\mathrm{id})$ is a sub-path of δ_r. This means there exists $s \in \{1,\ldots,\ell_{\delta_r}\}$ such that $j = a^{\delta_r}_s$ and $k = a^{\delta_r}_{s-1}$. We set $a := a^{\delta_r}_{\ell_{\delta_r}}$.

Recursive application of the definition of the sets $\mathcal{G}^{(r)}_*(j,*)$ yields

$$\mathcal{G}^{(r)}_{\ell_{\delta_r}}(j,k) = \mathcal{G}^{(r)}_{s-1}(j,k) \setminus \{\mathrm{id}\} \cup \bigcup_{\ell=s-1}^{\ell_{\delta_r}-1} \mathcal{G}^{(r)}_{\ell}(j,a^{\delta_r}_{\ell+1}) \setminus \{\mathrm{id}\} \cup \mathcal{G}^{(r)}_{\ell_{\delta_r}-1}(j,a)\,.$$

Hence, in order to conclude that $\mathrm{id} \notin \mathcal{G}^{(r)}_{\ell_{\delta_r}}(j,k)$ it suffices to show that $\mathrm{id} \notin \mathcal{G}^{(r)}_{\ell_{\delta_r}-1}(j,a)$. Assume for contradiction that

$$\mathrm{id} \in \mathcal{G}^{(r)}_{\ell_{\delta_r}-1}(j,a)\,.$$

By Lemma 5.13(ii) we find $\nu \in \mathrm{C}_j^{(r,\ell_{\delta_r}-1)}$ such that

$$\big(\mathrm{k}_{\mathrm{C}^{(r)}_{\ell_{\delta_r}-1},1}(\nu), \mathrm{g}_{\mathrm{C}^{(r)}_{\ell_{\delta_r}-1},1}(\nu)\big) = (a,\mathrm{id})\,.$$

By Lemma 5.12(iii) also $\nu \in \mathrm{C}_j$ and thus there exists $n \in \mathbb{N}$ such that

$$\big(\mathrm{k}_{\mathrm{C},n}(\nu), \mathrm{g}_{\mathrm{C},n}(\nu)\big) = (a,\mathrm{id})\,.$$

This means that the return graph RG_0 contains a non-degenerate path from j to a with accumulated weight id. By choice of j, the tuple (j,id) is a node in the tree B'_a, which means either $j=a$, or RG_0 contains a non-degenerate path from a to j with accumulated weight id. In either case we obtain a proper loop in RG_0 with weight id, which is contradictory. Hence, $\mathrm{id} \notin \mathcal{G}^{(r)}_{\ell_{\delta_r}-1}(j,a)$, and therefore $\mathrm{id} \notin \mathcal{G}^{(r)}_{\ell_{\delta_r}}(j,k)$. Since this argument applies for all $r \in \{1,\dots,\eta\}$ for which (j,id) is contained in the path δ_r, we conclude

$$\mathrm{id} \notin \bigcup_{k\in\widetilde{A}} \widetilde{\mathcal{G}}(j,k)\,,$$

which yields the second claim.

From (5.10) it is immediately clear that $\widetilde{\mathrm{C}}_j$ does not depend on the enumeration of Δ_{ini}. (We emphasize that the definition of the sets $V^{(r)}_{s_r}$ for $r \in \{1,\dots,\eta\}$ does not depend on the specific enumeration.) Lemma 5.13 implies that

$$\widetilde{\mathcal{G}}(j,k) = \mathcal{G}^{(\eta)}_{\ell_{\delta_\eta}}(j,k)$$

is a full transition set for $j,k \in \widetilde{A}$ up to identities, which in turn necessitates that $\widetilde{\mathcal{G}}(j,k) \cup \{\mathrm{id}\}$ also does not depend on the enumeration of Δ_{ini}. Since we have $\mathrm{id} \notin \widetilde{\mathcal{G}}(j,k)$ by the first part of the proof, this implies that $\widetilde{\mathcal{G}}(j,k)$ itself is independent of the enumeration of Δ_{ini}. Hence, the first claim follows and the proof is finished. $\square$

Corollary 5.16. *For all $j \in \widetilde{A}$ we have*

$$\widetilde{\mathrm{C}}_j = \mathrm{C}_j \setminus \bigcup_{k \in A} \left\{ \nu \in \mathrm{C}_j \mid \gamma'_\nu(\mathrm{t}_{\mathrm{C},1}(\nu)) \in \mathrm{C}_k \right\} .$$

Definition 5.17. A set $\widetilde{\mathcal{C}} = \left\{ \widetilde{\mathrm{C}}_j \,\middle|\, j \in \widetilde{A} \right\}$ of subsets of $\mathrm{S}\mathbb{H}$ is called a *reduced set of branches for the geodesic flow on* $\mathbb{X}$ if it satisfies the properties (B1), (B2), (B3), and (B6) from Definition 4.1, the property ($\mathrm{B}_{\mathrm{Per}}$) from Proposition 4.8, as well as the following three properties:

($\mathrm{B5}_{\mathrm{red}}\mathrm{I}$) For each $j \in \widetilde{A}$ and each pair $(x,y) \in \mathcal{I}_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$ there exists a (unique) vector $\nu \in \widetilde{\mathrm{C}}_j$ such that

$$(x,y) = \left(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\right) .$$

($\mathrm{B5}_{\mathrm{red}}\mathrm{II}$) For each $j \in \widetilde{A}$ and each pair $(x,y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$ there exist $k \in \widetilde{A}$ and a (unique) vector $\nu \in \widetilde{\mathrm{C}}_k$ such that

$$(x,y) = \left(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\right) .$$

($\mathrm{B7}_{\mathrm{red}}$) For each pair $(a,b) \in \widetilde{A} \times \widetilde{A}$ there exists a (possibly empty) subset $\widetilde{\mathcal{G}}(a,b)$ of Γ such that

(a) for all $j \in \widetilde{A}$ we have

$$\bigcup_{k \in \widetilde{A}} \bigcup_{g \in \widetilde{\mathcal{G}}(j,k)} g.\mathcal{I}_k \subseteq \mathcal{I}_j$$

and

$$\bigcup_{k \in \widetilde{A}} \bigcup_{g \in \widetilde{\mathcal{G}}(j,k)} g.\mathcal{I}_{k,\mathrm{st}} = \mathcal{I}_{j,\mathrm{st}} ,$$

and these unions are disjoint,

(b) for each pair $(j,k) \in \widetilde{A} \times \widetilde{A}$, each $g \in \widetilde{\mathcal{G}}(j,k)$, and each pair of points $(z,w) \in \mathrm{bp}(\widetilde{\mathrm{C}}_j) \times g.\,\mathrm{bp}(\widetilde{\mathrm{C}}_k)$, the geodesic segment $(z,w)_{\mathbb{H}}$ is nonempty, is contained in $\mathrm{H}_+(j)$ and does not intersect $\Gamma.\widetilde{\mathrm{C}}$, where

$$\widetilde{\mathrm{C}} := \bigcup_{j \in \widetilde{A}} \widetilde{\mathrm{C}}_j ,$$

(c) for all $j \in \widetilde{A}$ we have

$$J_{j,\mathrm{st}} \subseteq \bigcup_{k\in\widetilde{A}} \bigcup_{h\in\widetilde{\mathcal{G}}(k,j)} h^{-1}.J_{k,\mathrm{st}}\,.$$

A reduced set of branches is called *admissible* if it satisfies property (B8), and it is called *non-collapsing* if it satisfies property (B9) from Definition 4.1.

Remark 5.18. Depending on the underlying Fuchsian group, a non-collapsing behavior and (B5) are often incompatible to each other for the explicit algorithmic construction procedures of sets of branches we use (namely, the cusp expansion algorithm from Chapter 2). But non-collapsing reduced sets of branches will suffice for the purpose of all the following discussions and constructions. The approach via sets of branches that get adequately reduced to ensure non-collapsing behavior at the cost of the strong property (B5) has been chosen over an axiomatic approach in order to mimic the algorithmic process of constructing these sets, and to simplify the verification whether a given family of subsets of $\mathrm{S}\mathbb{H}$ is a set of branches or not. Consistently, Proposition 5.19 below shows that for every Fuchsian group for which a set of branches exists, we obtain a non-collapsing reduced set of branches via the above procedure.

Proposition 5.19. *The set $\widetilde{\mathcal{C}}$ is a non-collapsing reduced set of branches for the geodesic flow on $\mathbb{X}$ with associated forward transition sets given by $\widetilde{\mathcal{G}}(j,k)$ for any choice of $j,k \in \widetilde{A}$. If $\mathcal{C}$ is admissible, then so is $\widetilde{\mathcal{C}}$.*

Proof. In order to distinguish the application of the defining properties (B1)–(B8) for $\mathcal{C}$ from those for $\widetilde{\mathcal{C}}$ we aim to prove, we denote the latter ones by ($\mathrm{B1}_{\mathrm{red}}$)–($\mathrm{B3}_{\mathrm{red}}$) and ($\mathrm{B6}_{\mathrm{red}}$)–($\mathrm{B8}_{\mathrm{red}}$), respectively. We emphasize again that $\widetilde{\mathcal{C}}$ is not required to satisfy (B5).

Property ($\mathrm{B1}_{\mathrm{red}}$) is immediate from (B1) and the definition of $\widetilde{A}$. Further, Property ($\mathrm{B2}_{\mathrm{red}}$) is immediate from (B2) and the fact that Algorithm 5.11 does not interfere with the sets J_j and $J_{j,\mathrm{st}}$ for $j \in \widetilde{A}$. Since $\widetilde{A} \subseteq A$ and $\widetilde{\mathrm{C}}_j \subseteq \mathrm{C}_j$ for every $j \in \widetilde{A}$, the properties ($\mathrm{B3}_{\mathrm{red}}$) and ($\mathrm{B6}_{\mathrm{red}}$) are direct consequences of (B3) and (B6), respectively. And since $\mathcal{I}_j \subseteq I_j$ for all $j \in \widetilde{A}$ by virtue of (5.11), property ($\mathrm{B8}_{\mathrm{red}}$) is immediate from (B8).

Let $j \in \widetilde{A}$ and $\nu \in \mathrm{C}_{j,\mathrm{st}}$. Set $x := \gamma_\nu(+\infty)$ and $y := \gamma_\nu(-\infty)$. Then we have $(x,y) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$, and $\nu \in \widetilde{\mathrm{C}}_j$ if and only if $x \in \mathcal{I}_j$. This together with (B5) already yields ($\mathrm{B5}_{\mathrm{red}}\mathrm{I}$). Property ($\mathrm{B5}_{\mathrm{red}}\mathrm{II}$) follows immediately from ($\mathrm{B5}_{\mathrm{red}}\mathrm{I}$) in this case. Assume now that $\nu \notin \widetilde{\mathrm{C}}_j$. Then, by (5.11), there exists $k_1 \in H(j)$ such that $x \in I_{k_1}$. If $x \notin \mathcal{I}_{k_1}$, then, again by (5.11), there exists $k_2 \in H(k_1)$ such that $x \in I_{k_2}$. Iterating this argument is equivalent to traveling down a path in Δ_{ini}. Or in other words, there exist $\delta_r \in \Delta_{\mathrm{ini}}$ and $p \in \{1,\dots,\ell_{\delta_r}\}$ such that

$$j = a_p^{\delta_r} \quad \text{and} \quad k_\iota = a_{p-\iota}^{\delta_r} \quad \text{for } \iota = 1,2,\dots\,.$$

Hence, we obtain $k_p = a_0^{\delta_r}$ by virtue of Algorithm 5.11, and thus

$$\widetilde{\mathrm{C}}_{k_p} = \mathrm{C}_{k_p} \qquad \text{and} \qquad x \in I_{k_p} = \mathcal{I}_{k_p}\,.$$

Since $\mathrm{bp}(\mathrm{C}_{k_p}) \subseteq \mathrm{H}_+(\mathrm{C}_j)$, we find $y \in J_j \subseteq J_{k_p}$. Since $(x,y) \in \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}}$, by (B5) we find a unique $\nu' \in \mathrm{C}_{k_p}$ such that $(\gamma_{\nu'}(+\infty), \gamma_{\nu'}(-\infty)) = (x,y)$. This yields (B5$_{\mathrm{red}}$II).

Let $\widehat{\gamma} \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$. By Proposition 4.8 there exists $\gamma \in \mathscr{G}(\mathbb{H})$, $\pi(\gamma) = \widehat{\gamma}$, such that γ intersects C. Let $j \in A$ be such that $\gamma'(t) \in \mathrm{C}_j$ for some $t \in \mathbb{R}$. Because of Corollary 5.16 we may assume $j \in \widetilde{A}$. Then

$$\big(\gamma(+\infty), \gamma(-\infty)\big) \in I_{j,\mathrm{st}} \times J_{j,\mathrm{st}}\,.$$

If $\gamma(+\infty) \in \mathcal{I}_{j,\mathrm{st}}$, then $\gamma'(t) \in \widetilde{\mathrm{C}}_j$ by (B5$_{\mathrm{red}}$I). Otherwise, by (B5$_{\mathrm{red}}$II) we find $t' \in \mathbb{R}$ and $k \in \widetilde{A}$ such that $\gamma'(t') \in \widetilde{\mathrm{C}}_k$. Hence, in either case γ intersects $\widetilde{\mathrm{C}}$, which implies that $\widetilde{\mathcal{C}}$ fulfills (B$_{\mathrm{Per}}$).

Let again $j \in \widetilde{A}$. Since $\widetilde{\mathrm{C}}_j \neq \varnothing$, we have $\mathcal{I}_{j,\mathrm{st}} \neq \varnothing$. Hence, there exists $\nu \in \widetilde{\mathrm{C}}_j$ such that

$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) \in \mathcal{I}_{j,\mathrm{st}} \times J_{j,\mathrm{st}} \subseteq \widehat{\mathbb{R}}_{\mathrm{st}} \times \widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \Lambda(\Gamma) \times \Lambda(\Gamma)\,.$$

In particular, $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) \in \mathcal{I}_j \times J_j$, which is an open set in $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$. Therefore, there exists $\varepsilon > 0$ such that

$$\mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(\gamma_\nu(+\infty)) \times \mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(\gamma_\nu(-\infty)) \subseteq \mathcal{I}_j \times J_j\,,$$

where $\mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(x)$ is as in (1.16) for $x \in \widehat{\mathbb{R}}$. By Proposition 1.15 we find a representative γ of some periodic geodesic on $\mathbb{X}$ such that

$$\big(\gamma(+\infty), \gamma(-\infty)\big) \in \mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(\gamma_\nu(+\infty)) \times \mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(\gamma_\nu(-\infty))\,.$$

By construction,

$$\big(\gamma(+\infty), \gamma(-\infty)\big) \in \mathcal{I}_{j,\mathrm{st}} \times J_{j,\mathrm{st}}\,.$$

Combining this with (B5$_{\mathrm{red}}$I) yields (B1$_{\mathrm{red}}$). Finally, all statements of (B7$_{\mathrm{red}}$) follow from the combination of (B7) with (5.11), (B5$_{\mathrm{red}}$I), and (B5$_{\mathrm{red}}$II). □

Let $\nu \in \widetilde{\mathrm{C}}$ and define the *system of iterated sequences of* ν *with respect to* $\widetilde{\mathrm{C}}$ as

$$[(\mathrm{t}_{\widetilde{\mathrm{C}},n}(\nu))_n, (\mathrm{k}_{\widetilde{\mathrm{C}},n}(\nu))_n, (\mathrm{g}_{\widetilde{\mathrm{C}},n}(\nu))_n]\,,$$

where

$$\mathrm{t}_{\widetilde{\mathrm{C}},n}(\nu) := \mathrm{t}_{\mathrm{C}^{(\eta)}_{\ell_{\delta_\eta}},n}(\nu)\,, \quad \mathrm{k}_{\widetilde{\mathrm{C}},n}(\nu) := \mathrm{k}_{\mathrm{C}^{(\eta)}_{\ell_{\delta_\eta}},n}(\nu)\,,$$

and

$$\mathrm{g}_{\widetilde{\mathrm{C}},n}(\nu) := \mathrm{g}_{\mathrm{C}^{(\eta)}_{\ell_{\delta_\eta}},n}(\nu)\,,$$

for all $n \in \mathbb{Z}$, with $\left[\left(\mathrm{t}_{\mathrm{C}^{(r)}_{s_r},n}(\nu)\right)_n, \left(\mathrm{k}_{\mathrm{C}^{(r)}_{s_r},n}(\nu)\right)_n, \left(\mathrm{g}_{\mathrm{C}^{(r)}_{s_r},n}(\nu)\right)_n\right]$ as in (5.16) for $r \in \{1,\dots,\eta\}$ and $s_r \in \{1,\dots,\ell_{\delta_r}\}$. Because of Proposition 5.19 this system of sequences is independent of the enumeration of Δ_{ini}. We further obtain the following analogue of Proposition 4.19.

Corollary 5.20. *Let $\nu \in \widetilde{\mathrm{C}}_{\mathrm{st}}$, $k \in \widetilde{A}$, $t \in \mathbb{R}$ and $g \in \Gamma$ be such that*

$$\gamma'_\nu(t) \in g.\widetilde{\mathrm{C}}_k\,.$$

Then there exists a unique element $n \in \mathbb{Z}$ such that $\mathrm{sgn}(n) = \mathrm{sgn}(t)$ and

$$k = \mathrm{k}_{\widetilde{\mathrm{C}},n}(\nu)\,,\quad t = \mathrm{t}_{\widetilde{\mathrm{C}},n}(\nu)\,,\quad \textit{and}\quad g = \mathrm{g}_{\widetilde{\mathrm{C}},\mathrm{sgn}(t)}(\nu)\mathrm{g}_{\widetilde{\mathrm{C}},2\,\mathrm{sgn}(t)}(\nu)\cdots\mathrm{g}_{\widetilde{\mathrm{C}},n}(\nu)\,.$$

A reduced set of branches is called *finitely ramified* if $\#\widetilde{\mathcal{G}}(j,k) < +\infty$ for all $j,k \in \widetilde{A}$. The following result shows that Algorithm 5.11 does not negate the efforts of Section 4.4.

Proposition 5.21. *If $\mathcal{C}$ is finitely ramified, then so is $\widetilde{\mathcal{C}}$.*

Proof. Let $j,k \in \widetilde{A}$. By hypothesis,

$$\#\mathcal{G}^{(0)}_{\ell_{\delta_0}}(j,k) = \#\mathcal{G}(j,k) < +\infty\,.$$

In every step of Algorithm 5.11, a new transition set $\mathcal{G}^{(r)}_0(j,k)$ emerges as the union of at most two sets of the type $\mathcal{G}^{(r-1)}_*(j,k)$, which are seen to be of finite cardinality by recursive application of this argument. Since Algorithm 5.11 terminates after finitely many steps, this yields the set $\widetilde{\mathcal{G}}(j,k)$ as a finite union of finite subsets of Γ. □

Example 5.22. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ from Example 1.46 as well as its weakly non-collapsing set of branches $\mathcal{C}'_{\mathrm{P}}$ from Example 4.34. The completely reduced set of branches $\{\mathrm{C}_{\mathrm{P},2}, \mathrm{C}_{\mathrm{P},7}\}$ is already non-collapsing, as has been seen in Example 5.8. So there is no need to apply Algorithm 5.11 in this case. But since branch reduction is optional, we might investigate the outcome of the identity elimination if applied to $\mathcal{C}'_{\mathrm{P}}$. We immediately find $D_{\mathrm{ini}} = \{1,6\}$ and from the forest F'_{ini}, which is depicted in Figure 16, we see that $\eta = 3$. Furthermore, from Figure 13 it can be seen that $\bigcup_{k\in A}\mathcal{G}(j,k) = \{\mathrm{id}\}$ for $j = 1,4,6$, wherefore we obtain $\widetilde{A} = \{2,3,5,7,8\}$. In order to display the emerging reduced set of branches $\widetilde{\mathcal{C}}_{\mathrm{P}} = \left\{\widetilde{\mathrm{C}}_{\mathrm{P},j} \,\middle|\, j \in \widetilde{A}\right\}$ we provide a "return graph" (Figure 17), a depiction of the reduced branches themselves (Figure 18), as well as a list of the sets $\mathcal{I}_j$, $j \in \widetilde{A}$: We have

$$B'_1 :\qquad\qquad\qquad B'_6 :$$

$$\begin{tikzcd}[column sep=small]
& (1,\mathrm{id}) \arrow[d] & \\
& (2,\mathrm{id}) \arrow[d] & \\
& (3,\mathrm{id}) \arrow[d] & \\
& (4,\mathrm{id}) \arrow[dl] \arrow[dr] & \\
(5,\mathrm{id}) & & (8,\mathrm{id})
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(6,\mathrm{id}) \arrow[d] \\
(7,\mathrm{id}) \arrow[d] \\
(8,\mathrm{id})
\end{tikzcd}$$

Figure 16: The subtrees B'_1 and B'_6 of F'_{ini} for the set of branches $\mathcal{C}'_{\mathrm{P}}$ for $\Gamma_{\sigma,\lambda}$.

$$\begin{aligned}
&\mathcal{I}_2 = \big(-1, c(g_\sigma)\big)\,, \quad \mathcal{I}_3 = \big(c(g_\sigma), c(g_\sigma^{-1})\big)\,, \quad \mathcal{I}_5 = \big(1, +\infty\big)\,,\\
&\mathcal{I}_7 = \big(-\infty, c(g_\sigma^{-1})\big)\,, \quad \text{and} \quad \mathcal{I}_8 = \big(c(g_\sigma^{-1}), 1\big)\,.
\end{aligned}$$

5.3 Cuspidal Acceleration

In this section we present the *cuspidal acceleration algorithm*, which is the main step in our construction of strict transfer operator approaches. This algorithm ultimately yields the passage from a non-uniformly expanding discrete dynamical system to a uniformly expanding one, which then guarantees nuclearity of the arising fast transfer operators. As the naming suggests, this algorithm only affects hyperbolic orbisurfaces with cusps.

As the algorithms of branch reduction and identity elimination, also the cuspidal acceleration algorithm works by modifying a given set of representatives for a cross section. Here, we start with a cross section and a set of representatives for which the induced discrete dynamical system is typically not uniformly expanding. (If the induced system is already uniformly expanding, then the cuspidal acceleration algorithm is void and does not modify the cross section.) The non-uniformity in the expansion rate originates from the property of the initial cross section to encode each single winding of a geodesic around a cusp as a separate intersection event. To achieve uniformity, successive windings around a cusp should be merged into one (somewhat collective) event. The cuspidal acceleration algorithm achieves exactly this by a careful elimination of certain unit tangent vectors in the set of representatives for the initial cross section. We re-

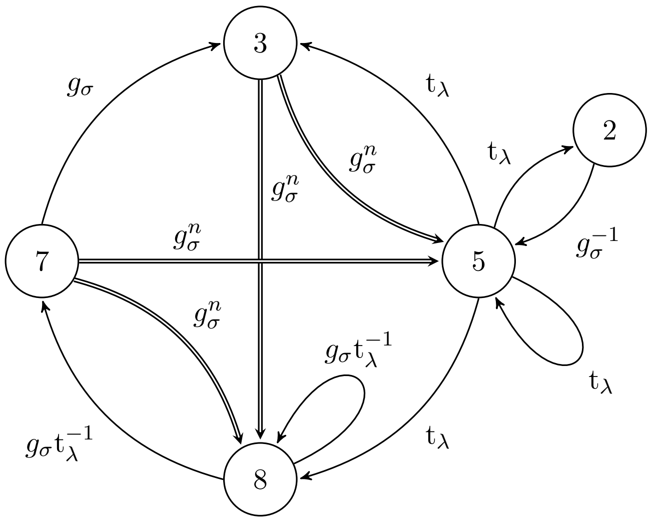


Figure 17: A return-style graph for the reduced set of branches $\widetilde{\mathcal{C}}_{\mathrm{P}}$ for $\Gamma_{\sigma,\lambda}$. Again, double edges indicate multiple edges, this time for $n = 1, \ldots, \sigma - 2$.

fer to Remark 5.27 below for a more detailed explanation. The algorithm itself consists indeed of a single (simultaneous) elimination step, for which reason it is stated as a definition, namely Definition 5.26, in which the *accelerated* set of representatives of the *accelerated* cross section is defined. The remaining section is then dedicated to the proof that this set is indeed a cross section. The following sections mostly discuss how this cross section and the set of representatives give rise (in a natural way) to a strict transfer operator approach.

Throughout this section let

$$\mathcal{C}^{(\mathrm{i})} := \left\{ \mathrm{C}_j^{(\mathrm{i})} \,\middle|\, j \in A^{(\mathrm{i})} \right\}$$

be a set of branches with $A^{(\mathrm{i})} := \{1, \ldots, N\}$ (with "(i)" indicating *initial*). We emphasize that the considerations in what follows do *not* require that the set of branches $\mathcal{C}^{(\mathrm{i})}$ is branch reduced. I. e., it is not required that the branch reduction algorithm from Section 5.1 has been applied to $\mathcal{C}^{(\mathrm{i})}$. We further let $\widetilde{A}$, $\widetilde{\mathcal{C}}$, $\widetilde{\mathrm{C}}$, $\mathcal{I}_j$ and $\widetilde{\mathcal{G}}(j,k)$ for $j, k \in \widetilde{A}$, and

$$[(\mathrm{t}_{\widetilde{\mathrm{C}},n}(\nu))_n, (\mathrm{k}_{\widetilde{\mathrm{C}},n}(\nu))_n, (\mathrm{g}_{\widetilde{\mathrm{C}},n}(\nu))_n]$$

for $\nu \in \widetilde{\mathrm{C}}_j$, $j \in \widetilde{A}$, be as in Section 5.2. That is, $\widetilde{\mathcal{C}}$ is a reduced set of branches (see Definition 5.17). Because of the Propositions 5.19 and 5.21 we may and shall assume that $\widetilde{\mathcal{C}}$ is non-collapsing and finitely ramified. For $j \in \widetilde{A}$ we further define

$$\widetilde{H}(j) := \left\{ k \in \widetilde{A} \,\middle|\, \widetilde{\mathcal{G}}(j,k) \neq \varnothing \right\} .$$

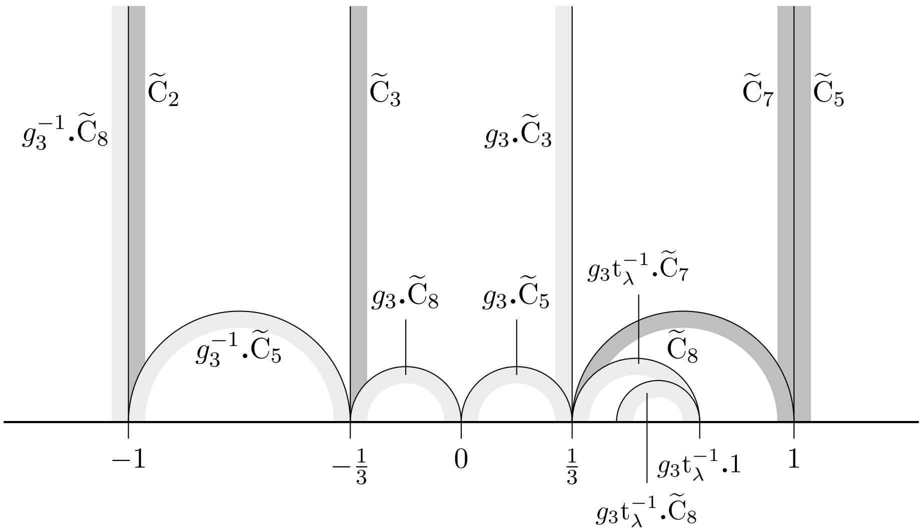


Figure 18: The reduced set of branches $\widetilde{\mathcal{C}}_{\mathrm{P}}$ for $\Gamma_{3,\lambda}$ emerging from $\mathcal{C}'_{\mathrm{P}}$ via Algorithm 5.11.

> **Convention**
> From now on we omit all tildes ($\widetilde{\ }$) from the notation. Thus, throughout this section, $\mathcal{C}$ denotes a *reduced* set of branches in the sense of Definition 5.17. We caution that this notation is not fully consistent with the one of the previous sections but preferred here in favor of avoiding overloaded notation. We further assume that $\widehat{\mathrm{C}}$ is a *strong* cross section for $\widehat{\Phi}$ (see Section 1.11).

We will take advantage of a certain cyclic behavior of Γ-translates of $\mathcal{C}$ at cusps. By this we refer to the following property: Let $j \in A$ and recall the endpoints X_j and Y_j of $\overline{\mathrm{bp}(\mathrm{C}_j)}$ from Remark 4.2(d). Recall further that

$$\mathcal{I}_j \subseteq I_j = (\mathrm{X}_j, \mathrm{Y}_j)_c \,.$$

There are two possibilities for $\mathrm{Z}_j \in \{\mathrm{X}_j, \mathrm{Y}_j\}$ in regard to $\mathcal{I}_j$:

(a) either Z_j is a boundary point of $\mathcal{I}_j$ in the $\widehat{\mathbb{R}}$-topology, or

(b) there exists $\varepsilon > 0$ such that $\mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(\mathrm{Z}_j) \cap \mathcal{I}_j = \varnothing$, with $\mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(.)$ as in (1.16).

We suppose that, say, X_j is a boundary point of $\mathcal{I}_j$ and represents a cusp of $\mathbb{X}$, say $\widehat{c}$ (see (B2)). Those two assumptions are not mutually exclusive, because, given the latter, if the former were not the case, by Corollary 5.16 we would find $j' \in A$ such that $\mathcal{I}_{j'} \subseteq I_{j'} \subseteq I_j$ and X_j is a boundary point of $\mathcal{I}_{j'}$. Then $\mathrm{X}_{j'} = \mathrm{X}_j$, and thus we may proceed with j' instead of j. Now, due to (B7$_{\mathrm{red}}$) we find $k \in H(j)$ and a transformation $g \in \mathcal{G}(j,k)$ such that $\mathrm{X}_j = g\boldsymbol{.}\mathrm{X}_k$ and X_k is an endpoint of $\mathcal{I}_k$. The tuple (k,g) is uniquely determined. Clearly, X_k is again a representative of $\widehat{c}$. By iterating this argument we are led back, after finitely many steps, to some Γ-translate of X_j, from where on the cycle repeats (see Lemma 5.24 below). This yields the notion of X-*cycles*, which is made more rigorous by the following definition. We may argue analogously if Y_j is a boundary point of $\mathcal{I}_j$ and represents a cusp of $\mathbb{X}$.

Definition 5.23. Let $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ and let

$$A_{\mathrm{Z}} \coloneqq \{ j \in A \mid \mathrm{Z}_j \text{ is cuspidal and a boundary point of } \mathcal{I}_j \}$$

be the subset of elements $j \in A$ for which the endpoint Z_j of the geodesic segment $\overline{\mathrm{bp}(\mathrm{C}_j)}$ represents a cusp of $\mathbb{X}$ and coincides with a boundary point of $\mathcal{I}_j$ in the $\widehat{\mathbb{R}}$-topology. By the discussion right before this definition, for each $j \in A_{\mathrm{Z}}$ there exists a (unique) pair $(k_j, g_j) \in H(j) \times \mathcal{G}(j, k_j)$ that satisfies $\mathrm{Z}_j = g_j\boldsymbol{.}\mathrm{Z}_{k_j}$ and $k_j \in A_{\mathrm{Z}}$. We call the pair (k_j, g_j) the Z-*tuple of* j. Further, we define the maps

$$\psi_{\mathrm{Z}} \colon \begin{cases} A_{\mathrm{Z}} & \longrightarrow & A_{\mathrm{Z}} \\ j & \longmapsto & k_j \end{cases} \qquad \text{and} \qquad g_{\mathrm{Z}} \colon \begin{cases} A_{\mathrm{Z}} & \longrightarrow & \Gamma \\ j & \longmapsto & g_j \end{cases} .$$

For each $j \in A_{\mathrm{Z}}$, iterated application of ψ_{Z} leads to the sequence

$$\mathrm{cyc}_{\mathrm{Z}}(j) \coloneqq (\psi_{\mathrm{Z}}^r(j))_{r \in \mathbb{N}_0} ,$$

which we call the Z-*cycle of* j.

Lemma 5.24. *Let* $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ *and* $j \in A_{\mathrm{Z}}$. *Then the sequence* $\mathrm{cyc}_{\mathrm{Z}}(j)$ *is periodic with (minimal) period length*

$$\min \{ n \in \mathbb{N} \mid \exists\, \nu \in \mathrm{C}_j \colon \mathrm{k}_{\mathrm{C},n}(\nu) = j \wedge \mathrm{g}_{\mathrm{C},1}(\nu) \cdots \mathrm{g}_{\mathrm{C},n}(\nu)\boldsymbol{.}\mathrm{Z}_j = \mathrm{Z}_j \} .$$

Proof. Let $\mathcal{F}$ be a Ford fundamental domain for the action of Γ on $\mathbb{H}$. Then each cusp $\widehat{c}$ of $\mathbb{X}$ has at least one representative $c \in \widehat{\mathbb{R}}$ such that c is an infinite vertex of $\mathcal{F}$ and each sufficiently small geodesic segment on $\mathbb{X}$ with endpoint $\widehat{c}$ (i. e., contained in a sufficiently small horoball centered at $\widehat{c}$) has a representing geodesic segment on $\mathbb{H}$ with endpoint c that is contained in $\overline{\mathcal{F}}$ (see Proposition 1.43 and the proof of Proposition 4.6). Consequently, there exists $h \in \Gamma$ such that Z_j is an infinite vertex of $h\boldsymbol{.}\mathcal{F}$ and the geodesic segment $\overline{\mathrm{bp}(\mathrm{C}_j)}$ intersects $h\boldsymbol{.}\overline{\mathcal{F}}$ in any small horoball centered at Z_j. By the Poincaré theorem on fundamental polyhedra (Proposition 1.36), the (conjugate) primitive vertex cycle transformation of Z_j, say p, is parabolic, fixes Z_j and is a generator of the stabilizer group of Z_j in Γ. Thus, either $p\boldsymbol{.}\mathcal{I}_{j,\mathrm{st}} \subseteq \mathcal{I}_{j,\mathrm{st}}$ or $p^{-1}\boldsymbol{.}\mathcal{I}_{j,\mathrm{st}} \subseteq \mathcal{I}_{j,\mathrm{st}}$, where we may suppose the former without loss of generality. Because of Lemma 1.8, for all $\varepsilon > 0$ we have

$$\mathcal{I}_{j,\mathrm{st}} \cap \mathrm{B}_{\widehat{\mathbb{R}},\varepsilon}(\mathrm{Z}_j) \neq \varnothing\,. \tag{5.22}$$

Let $\nu \in \mathrm{C}_j$ be such that $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) \in p\boldsymbol{.}\mathcal{I}_{j,\mathrm{st}} \times J_{j,\mathrm{st}}$. The combination of Lemma 4.13(ii) and Proposition 4.19 yields a unique element $n \in \mathbb{N}$ only depending on j such that for all $0 \le m \le n$ we have

$$\gamma_\nu'(\mathrm{t}_{\mathrm{C},m}(\nu)) \in \mathrm{g}_{\mathrm{C},1}(\nu)\cdots \mathrm{g}_{\mathrm{C},m}(\nu)\boldsymbol{.}\,\mathrm{C}_{\mathrm{k}_{\mathrm{C},m}(\nu)}$$

with $\mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},n}(\nu) = p$ and $\mathrm{k}_{\mathrm{C},n}(\nu) = j$. Since p fixes Z_j and

$$\mathrm{g}_{\mathrm{C},m+1}(\nu)\boldsymbol{.}\,\mathcal{I}_{\mathrm{k}_{\mathrm{C},m+1}(\nu),\mathrm{st}} \subseteq \mathcal{I}_{\mathrm{k}_{\mathrm{C},m}(\nu),\mathrm{st}}$$

for all $0 \le m < n$, (5.22) implies

$$\mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},m}(\nu)\boldsymbol{.}\mathrm{Z}_{\mathrm{k}_{\mathrm{C},m}(\nu)} = \mathrm{Z}_j$$

for all such m. Moreover, since $\overline{\mathrm{bp}(\mathrm{C}_j)}$ intersects $h\boldsymbol{.}\overline{\mathcal{F}}$ in any small horoball centered at Z_j, the part of $\overline{\mathrm{bp}(\mathrm{C}_j)}$ sufficiently near Z_j is contained in the fundamental domain $h\boldsymbol{.}\mathcal{F}$ or in its boundary. The fundamental domains neighboring $h\boldsymbol{.}\mathcal{F}$ at Z_j are $ph\boldsymbol{.}\mathcal{F}$ and $p^{-1}h\boldsymbol{.}\mathcal{F}$. In turn, the indices $\mathrm{k}_{\mathrm{C},m}$ of the iterated intersection branches of ν are not equal to j for $m \in \{1,\ldots,n-1\}$. Further, since $\mathrm{g}_{\mathrm{C},m}(\nu) \in \mathcal{G}(\mathrm{k}_{\mathrm{C},m}(\nu), \mathrm{k}_{\mathrm{C},m+1}(\nu))$ for all $m \in \{0,\ldots,n\}$, we obtain

$$\mathrm{cyc}_{\mathrm{Z}}(j)_m = \psi_{\mathrm{Z}}^m(j) = \mathrm{k}_{\mathrm{C},m}(\nu)\,.$$

Set $\nu_r := p^{r-1}\boldsymbol{.}\nu$ for $r \ge 1$. Then we find $\mathrm{k}_{\mathrm{C},m}(\nu_r) = \mathrm{k}_{\mathrm{C},m}(\nu)$ for every choice of $m \in \mathbb{N}_0$, and the translates $\mathrm{g}_{\mathrm{C},m}(\nu_r)\boldsymbol{.}\,\mathcal{I}_{\mathrm{k}_{\mathrm{C},m}(\nu_r),\mathrm{st}}$ fulfill the same conditions as before. This yields the periodicity of $\mathrm{cyc}_{\mathrm{Z}}(j)$ with period length n, which is indeed the minimal period length as seen from the generator properties of p. This completes the proof. □

For any $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$, the set A_{Z} from Definition 5.23 decomposes into finite cycles under the map ψ_{Z}, as shown in Lemma 5.24. For $j \in A_{\mathrm{Z}}$, we denote the (minimal) period length of the ψ_{Z}-cycle of j by $\sigma_{\mathrm{Z}}(j)$, thus

$$\begin{aligned}\sigma_{\mathrm{Z}}(j) &= \min\{r \in \mathbb{N} \mid \psi_{\mathrm{Z}}^r(j) = j\}\\ &= \min\{n \in \mathbb{N} \mid \exists \nu \in \mathrm{C}_j \colon \mathrm{k}_{\mathrm{C},n}(\nu) = j \wedge \mathrm{g}_{\mathrm{C},1}(\nu)\cdots\mathrm{g}_{\mathrm{C},n}(\nu)\boldsymbol{.}\mathrm{Z}_j = \mathrm{Z}_j\}\,,\end{aligned}$$

and we set

$$u_{j,\mathrm{Z}} := g_{\mathrm{Z}}(j)g_{\mathrm{Z}}(\psi_{\mathrm{Z}}(j))\cdots g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{\sigma_{\mathrm{Z}}(j)-1}(j))\,, \tag{5.23}$$

with g_{Z} being the map from Definition 5.23. As seen in the proof of Lemma 5.24, the element $u_{j,\mathrm{Z}}$ is a generator of the stabilizer subgroup of Z_j in Γ and hence

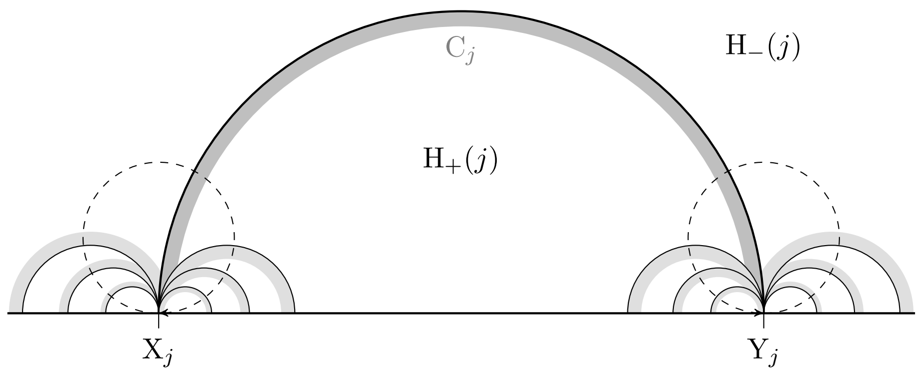


Figure 19: Branch cycles for a single branch C_j. The X-cycles circle clockwise, Y-cycles circle counterclockwise.

parabolic. In particular, $u_{j,\mathrm{Z}}.\mathrm{Z}_j = \mathrm{Z}_j$. The latter can also be deduced immediately from the property that $g_{\mathrm{Z}}(k)^{-1}.\mathrm{Z}_k = \mathrm{Z}_{\psi_{\mathrm{Z}}(k)}$ for all $k \in A_{\mathrm{Z}}$ by observing that

$$\begin{aligned} u_{j,\mathrm{Z}}^{-1}.\mathrm{Z}_j &= g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{\sigma_{\mathrm{Z}}(j)-1}(j))^{-1} \cdots g_{\mathrm{Z}}(j)^{-1}.\mathrm{Z}_j = g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{\sigma_{\mathrm{Z}}(j)-1}(j))^{-1}.\mathrm{Z}_{\psi_{\mathrm{Z}}^{\sigma_{\mathrm{Z}}(j)-1}(j)} \\ &= \mathrm{Z}_{\psi_{\mathrm{Z}}\left(\psi_{\mathrm{Z}}^{\sigma_{\mathrm{Z}}(j)-1}(j)\right)} = \mathrm{Z}_{\psi_{\mathrm{Z}}^{\sigma_{\mathrm{Z}}(j)}(j)} = \mathrm{Z}_j\,. \end{aligned}$$

For any $j \in A$ we set

$$\mathrm{Cyc}_{j,\mathrm{Z}} := \begin{cases} \{\psi_{\mathrm{Z}}^r(j) \mid r \in \{0, \ldots, \sigma_{\mathrm{Z}}(j) - 1\}\} & \text{if } j \in A_{\mathrm{Z}}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus, for $j \in A_{\mathrm{Z}}$, the set $\mathrm{Cyc}_{j,\mathrm{Z}}$ contains exactly the elements of the Z-cycle of j. For any $k \in \mathrm{Cyc}_{j,\mathrm{Z}}$, the sequence $\mathrm{cyc}_{\mathrm{Z}}(k)$ is a shift of the sequence $\mathrm{cyc}_{\mathrm{Z}}(j)$, the transformation $u_{k,\mathrm{Z}}$ is conjugate to $u_{j,\mathrm{Z}}$, and $\mathrm{Cyc}_{k,\mathrm{Z}} = \mathrm{Cyc}_{j,\mathrm{Z}}$. From another point of view, the sets $\mathrm{Cyc}_{j,\mathrm{Z}}$, $j \in A_{\mathrm{Z}}$, are the equivalence classes for the equivalence relation

$$j \sim k \quad :\Longleftrightarrow \quad \exists\, r \in \mathbb{N} \colon \psi_{\mathrm{Z}}^r(j) = k$$

on A_{Z}. The number of equivalence classes depends on the number of cusps of $\mathbb{X}$.

Example 5.25. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ from Example 1.46 and the return graph for its weakly non-collapsing set of branches $\mathcal{C}'_{\mathrm{P}}$ (Figure 13) from Example 5.3, as well as its reduced return graph RG_6 for the set of branches $\{\mathrm{C}_{\mathrm{P},2}, \mathrm{C}_{\mathrm{P},7}\}$ from Example 5.8. As we have seen, both sets are non-collapsing. There is one X-cycle and one Y-cycle in RG_6 (Figure 15) given by

$$7 \xrightarrow{\mathrm{t}_\lambda^{-1}} 7 \qquad \text{and} \qquad 2 \xrightarrow{\mathrm{t}_\lambda} 2\,,$$

respectively. For the reduced set of branches $\widetilde{\mathcal{C}}_{\mathrm{P}}$ (Figure 17) we retrieve these cycles as

$$7 \xrightarrow{g_\sigma} 3 \xrightarrow{g_\sigma^{\sigma-2}} 8 \xrightarrow{g_\sigma \mathrm{t}_\lambda^{-1}} 7 \qquad \text{and} \qquad 5 \xrightarrow{\mathrm{t}_\lambda} 5\,,$$

respectively. Hence, in the former setting we obtain the sets

$$A_{\mathrm{X}} = \{7\} \qquad \text{and} \qquad A_{\mathrm{Y}} = \{2\}$$

as well as the transformations

$$u_{7,\mathrm{X}} = g_{\mathrm{X}}(7) = \mathrm{t}_\lambda^{-1} \qquad \text{and} \qquad u_{2,\mathrm{Y}} = g_{\mathrm{Y}}(2) = \mathrm{t}_\lambda\,.$$

On the other hand, in the latter setting we obtain

$$A_{\mathrm{X}} = \{3,7,8\} \qquad \text{and} \qquad A_{\mathrm{Y}} = \{5\}$$

and the transformations

$$g_{\mathrm{X}}(3) = g_\sigma^{\sigma-2}\,, \quad g_{\mathrm{X}}(7) = g_\sigma\,, \quad g_{\mathrm{X}}(8) = g_\sigma \mathrm{t}_\lambda^{-1}\,, \quad \text{and} \quad u_{5,\mathrm{Y}} = g_{\mathrm{Y}}(5) = \mathrm{t}_\lambda\,.$$

Hence, we find

$$u_{7,\mathrm{X}} = g_{\mathrm{X}}(7) \cdot g_{\mathrm{X}}(3) \cdot g_{\mathrm{X}}(8) = g_\sigma \cdot g_\sigma^{\sigma-2} \cdot g_\sigma \mathrm{t}_\lambda^{-1} = g_\sigma^\sigma \mathrm{t}_\lambda^{-1} = \mathrm{t}_\lambda^{-1}\,,$$

and, since $g_\sigma^{\sigma-1} = g_\sigma^{-1}$,

$$u_{3,\mathrm{X}} = g_\sigma^{-1} \mathrm{t}_\lambda^{-1} g_\sigma \qquad \text{and} \qquad u_{8,\mathrm{X}} = g_\sigma \mathrm{t}_\lambda^{-1} g_\sigma^{-1}\,.$$

We now introduce the acceleration procedure mentioned above. Again, the process is presented in geometric terms, by a deletion of certain subsets of unit tangent vectors from the reduced branches. The emerging system gives rise to a "faster" symbolic dynamics arising from a new cross section for the geodesic flow (see Proposition 5.31 below). The emphasis lies on branches that, even after the identity elimination, are still attached to cuspidal points in the sense of Definition 5.23. We therefore call the procedure *cuspidal acceleration* or *cuspidal acceleration algorithm.*

Definition 5.26 (Cuspidal acceleration)**.** For $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$ define the sets

$$K_{\mathrm{Z}}(j) \coloneqq \begin{cases} \left\{\nu \in \mathrm{C}_{j,\mathrm{st}} \;\middle|\; \gamma_\nu(+\infty) \in g_{\mathrm{Z}}(j).\mathcal{I}_{\psi_{\mathrm{Z}}(j),\mathrm{st}}\right\} & \text{if } j \in A_{\mathrm{Z}},\\ \varnothing & \text{otherwise,}\end{cases}$$

and

$$M_{\mathrm{Z}}(j) \coloneqq \begin{cases} \left\{\nu \in \mathrm{C}_{j,\mathrm{st}} \;\middle|\; \gamma_\nu(-\infty) \in g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}.J_{\psi_{\mathrm{Z}}^{-1}(j),\mathrm{st}}\right\} & \text{if } j \in A_{\mathrm{Z}},\\ \varnothing & \text{otherwise.}\end{cases}$$

We call $K_{\mathrm{Z}}(j)$ the *forward* and $M_{\mathrm{Z}}(j)$ the *backward* Z*-elimination set* of j. For $j \in A$ we call

$$\mathrm{C}_{j,\mathrm{acc}} := \bigcap_{\mathrm{Z}\in\{\mathrm{X},\mathrm{Y}\}} \mathrm{C}_{j,\mathrm{st}} \setminus \bigl(K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)\bigr)$$

the *acceleration* of C_j and

$$\mathrm{C}_{\mathrm{acc}} := \bigcup_{j\in A} \mathrm{C}_{j,\mathrm{acc}}$$

the *acceleration* of C.

Remark 5.27. We comment on the motivation for Definition 5.26. Let $\widehat{\gamma} \in \mathscr{G}(\mathbb{X})$ and let $\widehat{c}$ be a cusp of $\mathbb{X}$. We say that the set of branches $\mathcal{C}$ *detects that* $\widehat{\gamma}$ *travels towards* $\widehat{c}$ if there exists a representing geodesic γ of $\widehat{\gamma}$ on $\mathbb{H}$ and $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ and $j \in A_{\mathrm{Z}}$ such that Z_j represents $\widehat{c}$, the geodesic γ intersects C_j at some time, say t_0, and $\gamma(+\infty) \in g_{\mathrm{Z}}(j)\boldsymbol{.}\mathcal{I}_{\psi_{\mathrm{Z}}(j)}$. In such a case, the next intersection (after time t_0) between γ and $\Gamma\boldsymbol{.}\mathrm{C}$ is on $g_{\mathrm{Z}}(j)\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}(j)}$ at, say, time t_1. Further next and previous intersections of γ and $\Gamma\boldsymbol{.}\mathrm{C}$ might be “near” Z_j, thus given by the Z-cycle of j. More precisely, it might happen that γ intersects $g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}^{-1}(j)}$, in which case the previous intersection of γ and $\Gamma\boldsymbol{.}\mathrm{C}$ is indeed on $g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}^{-1}(j)}$, as can easily be seen from the definition of the Z-cycle of j. Likewise, the next intersection after time t_1 might be on $g_{\mathrm{Z}}(j)g_{\mathrm{Z}}(\psi_{\mathrm{Z}}(j))\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}^2(j)}$. Let us suppose that γ intersects

$$\begin{aligned}
&g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1} g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-2}(j))^{-1} \cdots g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-k_1}(j))^{-1}\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}^{-k_1}(j)}\,, \ \ldots\,, \qquad (5.24)\\
&g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}^{-1}(j)}\,, \mathrm{C}_j\,, g_{\mathrm{Z}}(j)\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}(j)}\,, \ \ldots\,,\\
&g_{\mathrm{Z}}(j) g_{\mathrm{Z}}(\psi_{\mathrm{Z}}(j)) \cdots g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{k_2-1}(j))\boldsymbol{.}\mathrm{C}_{\psi_{\mathrm{Z}}^{k_2}(j)}
\end{aligned}$$

with $k_1, k_2 \in \mathbb{N}_0$ maximal. Large values for k_1, k_2 indicate that $\widehat{\gamma}$ stays “near” the cusp $\widehat{c}$ for a rather long time, and larger values for k_1, k_2 translate to deeper cusp excursions. We call the part of $\widehat{\gamma}$ corresponding to (5.24) a *maximal cusp excursion* into the cusp region of $\widehat{c}$ or a *sojourn* of $\widehat{\gamma}$ near $\widehat{c}$. We emphasize that $\widehat{\gamma}$ can experience several disjoint sojourns at the same cusp, each one separated from the others by some time spend “far away” from the cusp.

Each sojourn of $\widehat{\gamma}$ near $\widehat{c}$ typically contains several windings around the cusp (region of) $\widehat{c}$, expressed by a high power of the parabolic element $u_{j,\mathrm{Z}}$ from (5.23), as explained further below. Using the (reduced) set of branches $\mathcal{C}$ for the coding of the geodesic flow on $\mathbb{X}$, as done for the development of slow transfer operators, leads to separate coding of each single cusp winding (and also of all the intermediate intersections). It is exactly this detailed (“slow”) coding of cusp windings that cause slow transfer operators typically to be non-nuclear. To enforce nuclearity, the idea is to encode each sojourn by a single step in the coding (“fast” coding) or, in other words, to induce on the cusp excursions, at the expense of constructing an infinitely branched discrete dynamical system (and a symbolic dynamics with an infinite alphabet). Technically, this acceleration will be achieved by omitting those tangent vectors from the branches in $\mathcal{C}$ that are internal to a Z-cycle.

To be more precise, we now express the intersection properties in (5.24) in terms of the endpoints of γ. To facilitate notation, we set, for any $k \in \mathrm{Cyc}_{j,\mathrm{Z}}$,

$$g_{\mathrm{Z}}(j,k) := g_{\mathrm{Z}}(j) g_{\mathrm{Z}}(\psi_{\mathrm{Z}}(j)) \cdots g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{r-1}(j)) \tag{5.25}$$

with $r := \min\left\{\ell \in \mathbb{N} \;\middle|\; \psi_{\mathrm{Z}}^{\ell}(j) = k\right\}$. Then

$$g_{\mathrm{Z}}(j,\psi_{\mathrm{Z}}(j)) = g_{\mathrm{Z}}(j) \quad \text{and} \quad g_{\mathrm{Z}}(j,j) = u_{j,\mathrm{Z}}.$$

If $\mathrm{Z} = \mathrm{X}$, then the interval $g_{\mathrm{X}}(j).I_{\psi_{\mathrm{X}}(j)}$ decomposes as

$$\bigcup_{n\in\mathbb{N}_0} \bigcup_{k\in\mathrm{Cyc}_{j,\mathrm{X}}} \left(u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k) g_{\mathrm{X}}(k).\mathrm{Y}_{\psi_{\mathrm{X}}(k)},\; u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k).\mathrm{Y}_k\right)$$

and thus the set $g_{\mathrm{X}}(j).\mathcal{I}_{\psi_{\mathrm{X}}(j)}$ decomposes as

$$\bigcup_{n\in\mathbb{N}_0} \bigcup_{k\in\mathrm{Cyc}_{j,\mathrm{X}}} \left(u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k) g_{\mathrm{X}}(k).\mathrm{Y}_{\psi_{\mathrm{X}}(k)},\; u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k).\mathrm{Y}_k\right) \cap u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k).\mathcal{I}_k\,.$$

On the other hand, the interval $g_{\mathrm{X}}(\psi_{\mathrm{X}}^{-1}(j))^{-1}.J_{\psi_{\mathrm{X}}^{-1}(j)}$ decomposes as

$$\bigcup_{n\in\mathbb{N}_0} \bigcup_{k\in\mathrm{Cyc}_{j,\mathrm{X}}} \left(u_{j,\mathrm{X}}^{-n} g_{\mathrm{X}}(k,j)^{-1}.\mathrm{Y}_k, u_{j,\mathrm{X}}^{-n} g_{\mathrm{X}}(k,j)^{-1} g_{\mathrm{X}}(\psi_{\mathrm{X}}^{-1}(k))^{-1}.\mathrm{Y}_{\psi_{\mathrm{X}}^{-1}(k)}\right)\,.$$

If $\mathrm{Z} = \mathrm{Y}$, then the decomposition is analogous, with the roles of X and Y interchanged and the order of the interval boundaries switched. See Figure 20. For any choice of $n \in \mathbb{N}_0$ and $k \in \mathrm{Cyc}_{j,\mathrm{Z}}$, $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$, we set

$$\begin{aligned}
D_{n,\mathrm{X}}^{+}(j,k) &:= \left(u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k) g_{\mathrm{X}}(k).\mathrm{Y}_{\psi_{\mathrm{X}}(k)},\; u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k).\mathrm{Y}_k\right)\,,\\
D_{n,\mathrm{Y}}^{+}(j,k) &:= \left(u_{j,\mathrm{Y}}^n g_{\mathrm{Y}}(j,k).\mathrm{X}_k,\; u_{j,\mathrm{Y}}^n g_{\mathrm{Y}}(j,k) g_{\mathrm{Y}}(k).\mathrm{X}_{\psi_{\mathrm{Y}}(k)}\right)\,,\\
D_{n,\mathrm{X}}^{-}(j,k) &:= \left(u_{j,\mathrm{X}}^{-n} g_{\mathrm{X}}(k,j)^{-1}.\mathrm{Y}_k, u_{j,\mathrm{X}}^{-n} g_{\mathrm{X}}(k,j)^{-1} g_{\mathrm{X}}(\psi_{\mathrm{X}}^{-1}(k))^{-1}.\mathrm{Y}_{\psi_{\mathrm{X}}^{-1}(k)}\right)\,,
\end{aligned}$$

and

$$D_{n,\mathrm{Y}}^{-}(j,k) := \left(u_{j,\mathrm{Y}}^{-n} g_{\mathrm{Y}}(k,j)^{-1} g_{\mathrm{Y}}(\psi_{\mathrm{Y}}^{-1}(k))^{-1}.\mathrm{X}_{\psi_{\mathrm{Y}}^{-1}(k)}, u_{j,\mathrm{Y}}^{-n} g_{\mathrm{Y}}(k,j)^{-1}.\mathrm{X}_k\right)\,.$$

Based on these intervals we further set

$$\mathcal{D}_{n,\mathrm{X}}^{+}(j,k) := D_{n,\mathrm{X}}^{+}(j,k) \cap u_{j,\mathrm{X}}^n g_{\mathrm{X}}(j,k).\mathcal{I}_k$$

and

$$\mathcal{D}^+_{n,\mathrm{Y}}(j,k) := D^+_{n,\mathrm{Y}}(j,k) \cap u^n_{j,\mathrm{Y}} g_{\mathrm{Y}}(j,k)\boldsymbol{.}\mathcal{I}_k\,.$$

Then the geodesic γ (with the properties as in (5.24)) satisfies $\gamma(+\infty) \in \mathcal{D}^+_{n,\mathrm{Z}}(j,k)$ if and only if $k_2 = n+r$ with r as in (5.25). It satisfies $\gamma(-\infty) \in D^-_{m,\mathrm{Z}}(j,k)$ if and only if $k_1 = m + s$ with $s = \min\left\{\ell \in \mathbb{N} \;\middle|\; \psi^\ell_{\mathrm{Z}}(k) = j\right\}$. The sum of the values of n for k_1 and k_2 is the number of full windings around the cusp (region of) $\widehat{c}$ of this sojourn of $\widehat{\gamma}$ near $\widehat{c}$.

For the acceleration we now want to eliminate from the branches all those tangent vectors that cause intersections within a sojourn, or, in other words, within a Z-cycle. This elimination process is of a local nature; we only need to ask for the nature of the next and the previous intersection, and not of any further intersections. For the branch C_j it means that we need to eliminate all those vectors $\nu \in \mathrm{C}_j$ for which

$$\gamma_\nu(+\infty) \in g_{\mathrm{Z}}(j)\boldsymbol{.}\mathcal{I}_{\psi_{\mathrm{Z}}(j),\mathrm{st}} \quad \text{and} \quad \gamma_\nu(-\infty) \in g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}\boldsymbol{.}J_{\psi_{\mathrm{Z}}^{-1}(j),\mathrm{st}}\,.$$

Thus, we need to eliminate from C_j exactly the set $K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)$ for the acceleration.

In Proposition 5.28(i) below we show that the sets $K_{\mathrm{X}}(j)$ and $K_{\mathrm{Y}}(j)$ as well as $M_{\mathrm{X}}(j)$ and $M_{\mathrm{Y}}(j)$ do not intersect, thus, there is no interference between different cycles during the elimination or acceleration procedure. In the remainder of this section we show that this heuristics on the necessary modifications of the set of branches indeed leads to the desired results.

We recall that $\{\mathrm{C}_j \mid j \in A\}$ is a reduced set of branches for the geodesic flow on $\mathbb{X}$. In the case that $\mathbb{X}$ does not have cusps, it is consistent with Definition 5.26 to set $\mathrm{C}_{j,\mathrm{acc}} := \mathrm{C}_{j,\mathrm{st}}$ for all $j \in A$. For $M \subseteq S\mathbb{H}$ we define

$$I(M) := \{\gamma_\nu(+\infty) \mid \nu \in M\} \quad \text{and} \quad J(M) := \{\gamma_\nu(-\infty) \mid \nu \in M\}\,. \tag{5.26}$$

We set

$$A^* := \{j \in A \mid \mathrm{C}_{j,\mathrm{acc}} \neq \varnothing\}\,. \tag{5.27}$$

We further set $\widehat{\mathrm{C}}_{\mathrm{acc}} := \pi(\mathrm{C}_{\mathrm{acc}})$.

Proposition 5.28. *For any $j \in A$ and $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$, the elimination sets $K_{\mathrm{Z}}(j)$ and $M_{\mathrm{Z}}(j)$ and the acceleration $\mathrm{C}_{j,\mathrm{acc}}$ of C_j satisfy the following properties:*

(i) *We have $K_{\mathrm{X}}(j) \cap K_{\mathrm{Y}}(j) = \varnothing$ and $M_{\mathrm{X}}(j) \cap M_{\mathrm{Y}}(j) = \varnothing$.*

(ii) *The set $\mathrm{C}_{j,\mathrm{acc}}$ is empty if and only if there exists $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$ such that $j \in A_{\mathrm{Z}}$*

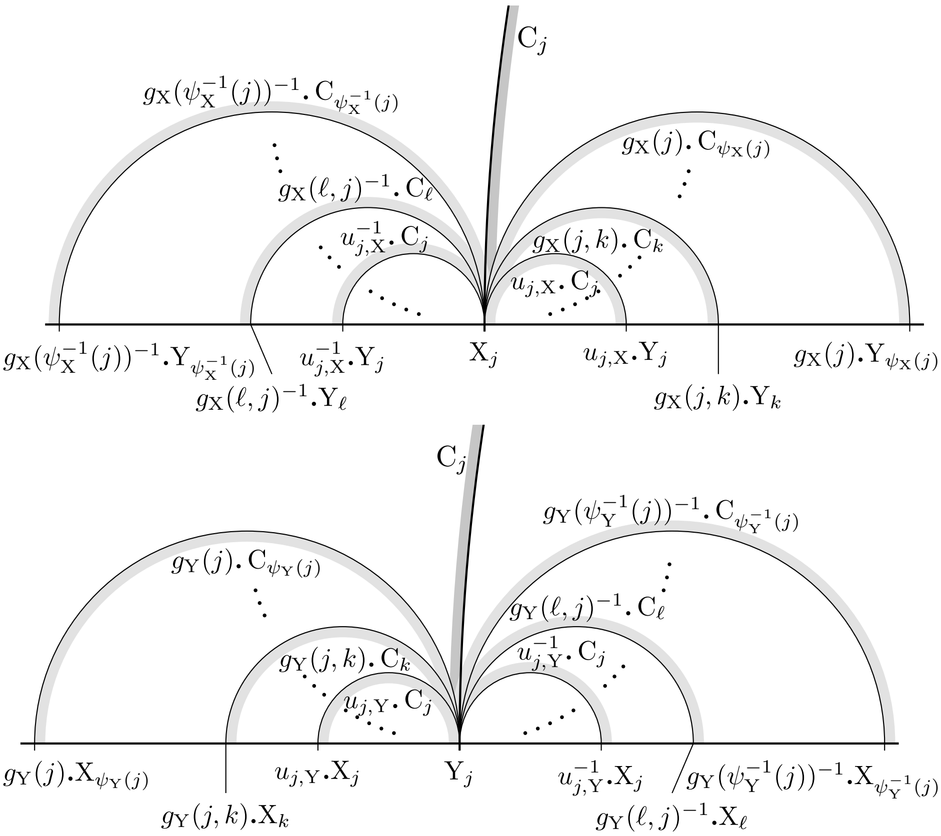


Figure 20: The situation for an X-cycle (above) and a Y-cycle (below).

and

$$\mathcal{I}_{j,\mathrm{st}} = g_{\mathrm{Z}}(j)\mathbf{.}\mathcal{I}_{\psi_{\mathrm{Z}}(j),\mathrm{st}} \quad \textit{and} \quad J_{j,\mathrm{st}} = g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}\mathbf{.}J_{\psi_{\mathrm{Z}}^{-1}(j),\mathrm{st}}\,.$$

(iii) *If* $j \in A_{\mathrm{X}} \cap A_{\mathrm{Y}}$, *then* $\mathrm{C}_{j,\mathrm{acc}} \neq \varnothing$.

Proof. For the proof of (i) we suppose that $j \in A_{\mathrm{X}} \cap A_{\mathrm{Y}}$ (because otherwise there is nothing to show) and assume, in order to seek a contradiction, that the sets $K_{\mathrm{X}}(j)$ and $K_{\mathrm{Y}}(j)$ are *not* disjoint. Then there exists $\nu \in \mathrm{C}_{j,\mathrm{st}}$ such that

$$\begin{aligned}\gamma_\nu(+\infty) &\in g_{\mathrm{X}}(j)\mathbf{.}\mathcal{I}_{\psi_{\mathrm{X}}(j),\mathrm{st}} \cap g_{\mathrm{Y}}(j)\mathbf{.}\mathcal{I}_{\psi_{\mathrm{Y}}(j),\mathrm{st}} \\ &\qquad \subseteq g_{\mathrm{X}}(j)\mathbf{.}I_{\psi_{\mathrm{X}}(j),\mathrm{st}} \cap g_{\mathrm{Y}}(j)\mathbf{.}I_{\psi_{\mathrm{Y}}(j),\mathrm{st}}\,.\end{aligned} \tag{5.28}$$

For any $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ we have

$$g_{\mathrm{Z}}(j)\mathbf{.}\mathrm{Z}_{\psi_{\mathrm{Z}}(j)} = \mathrm{Z}_j \qquad \text{and} \qquad g_{\mathrm{Z}}(j)\mathbf{.}\overline{\mathrm{bp}(\mathrm{C}_{\psi_{\mathrm{Z}}(j)})} \neq \overline{\mathrm{bp}(\mathrm{C}_j)}\,. \tag{5.29}$$

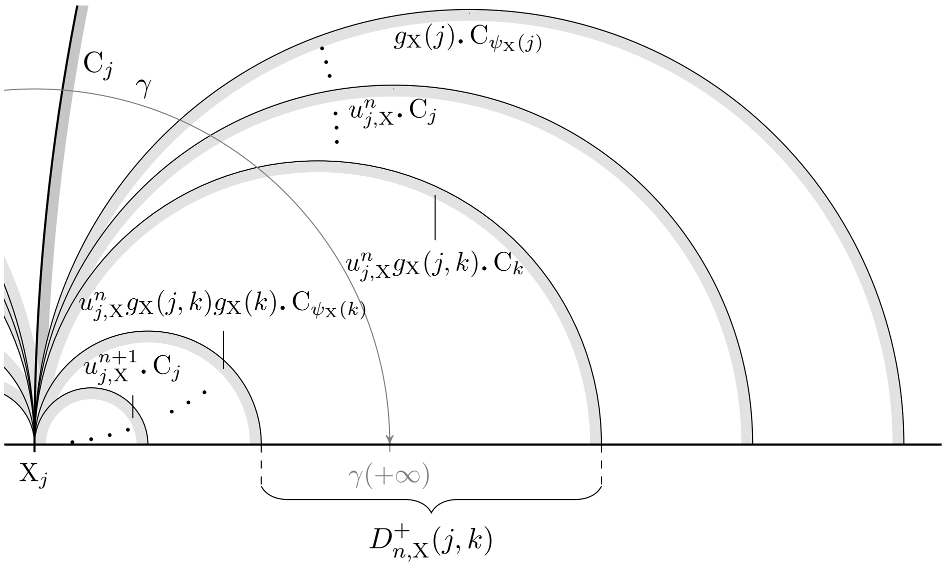


Figure 21: A representative γ of a geodesic with sojourn near a cusp detected by an X-cycle in forward time.

Thus, (5.28) implies

$$g_{\mathrm{X}}(j).\overline{\mathrm{bp}(\mathrm{C}_{\psi_{\mathrm{X}}(j)})} \cap g_{\mathrm{Y}}(j).\overline{\mathrm{bp}(\mathrm{C}_{\psi_{\mathrm{Y}}(j)})} \neq \varnothing\,.$$

From (B6) it now follows that

$$g_{\mathrm{X}}(j) = g_{\mathrm{Y}}(j) =: g \qquad \text{and} \qquad \psi_{\mathrm{X}}(j) = \psi_{\mathrm{Y}}(j) =: k\,.$$

With (5.29) we obtain $g.\mathrm{Z}_k = \mathrm{Z}_j$, and hence $g.\overline{\mathrm{bp}(\mathrm{C}_k)} = \overline{\mathrm{bp}(\mathrm{C}_j)}$, which contradicts (5.29). In turn, $K_{\mathrm{X}}(j) \cap K_{\mathrm{Y}}(j) = \varnothing$. The proof of $M_{\mathrm{X}}(j) \cap M_{\mathrm{Y}}(j) = \varnothing$ is analogous.

For the proof of (ii) we let $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ and note that the equalities

$$\mathcal{I}_{j,\mathrm{st}} = g_{\mathrm{Z}}(j).\mathcal{I}_{\psi_{\mathrm{Z}}(j),\mathrm{st}} \qquad \text{and} \qquad J_{j,\mathrm{st}} = g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}.J_{\psi_{\mathrm{Z}}^{-1}(j),\mathrm{st}}$$

are equivalent to $K_{\mathrm{Z}}(j) = \mathrm{C}_{j,\mathrm{st}}$ and $M_{\mathrm{Z}}(j) = \mathrm{C}_{j,\mathrm{st}}$, respectively. Hence, if these equalities are satisfied for the considered index $j \in A_{\mathrm{Z}}$, then

$$\mathrm{C}_{j,\mathrm{acc}} \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)\big) = \mathrm{C}_{j,\mathrm{st}} \setminus \mathrm{C}_{j,\mathrm{st}} = \varnothing\,.$$

In order to prove the converse implication, we suppose that $\mathrm{C}_{j,\mathrm{acc}} = \varnothing$. Then we find $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ such that $j \in A_{\mathrm{Z}}$ because otherwise $\mathrm{C}_{j,\mathrm{acc}} = \mathrm{C}_{j,\mathrm{st}} \neq \varnothing$. Thus, we know already (see the discussion before this proposition) that

$$K_{\mathrm{Z}}(j) \neq \varnothing \neq M_{\mathrm{Z}}(j)$$

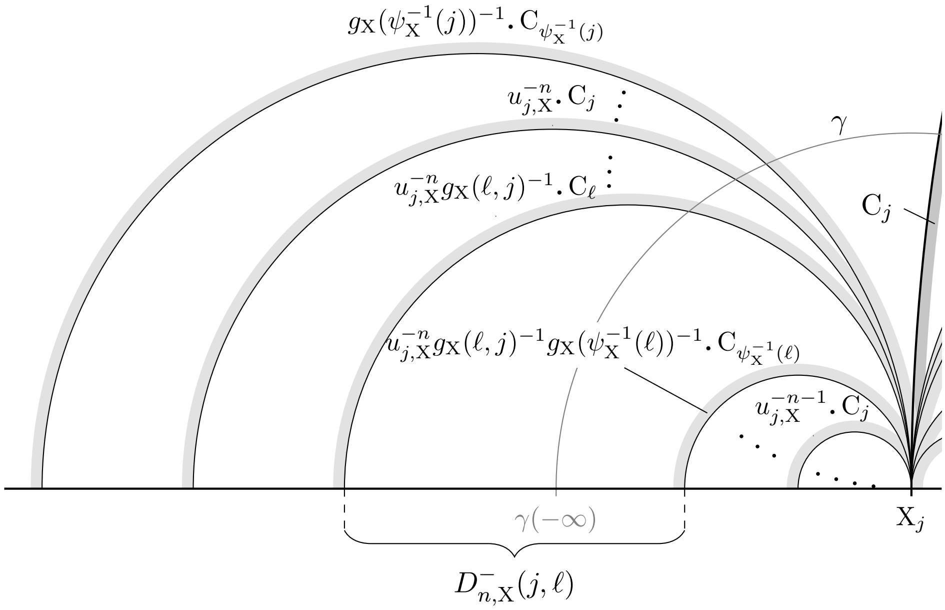


Figure 22: A representative γ of a geodesic with sojourn near a cusp detected by an X-cycle in backward time.

and will now show that $K_{\mathrm{Z}}(j) = \mathrm{C}_{j,\mathrm{st}} = M_{\mathrm{Z}}(j)$. From the definition of $\mathrm{C}_{j,\mathrm{acc}}$ it follows immediately that

$$\mathrm{C}_{j,\mathrm{st}} = \big(K_{\mathrm{X}}(j) \cap M_{\mathrm{X}}(j)\big) \cup \big(K_{\mathrm{Y}}(j) \cap M_{\mathrm{Y}}(j)\big)\,.$$

Let $\mathrm{Z}' \in \{\mathrm{X},\mathrm{Y}\}$ be such that $\{\mathrm{Z},\mathrm{Z}'\} = \{\mathrm{X},\mathrm{Y}\}$. From (i) we obtain the inclusions

$$K_{\mathrm{Z}'}(j) \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Z}}(j) \qquad \text{and} \qquad M_{\mathrm{Z}'}(j) \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus M_{\mathrm{Z}}(j)\,.$$

Thus, $K_{\mathrm{Z}'}(j) \cap M_{\mathrm{Z}'}(j) \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus (K_{\mathrm{Z}}(j) \cup M_{\mathrm{Z}}(j))$ and hence

$$\mathrm{C}_{j,\mathrm{st}} = \big(K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)\big) \cup \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{Z}}(j) \cup M_{\mathrm{Z}}(j)\big)\,.$$

It follows that

$$\begin{aligned}
\varnothing &= \mathrm{C}_{j,\mathrm{st}} \setminus \Big(\big(K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)\big) \cup \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{Z}}(j) \cup M_{\mathrm{Z}}(j)\big)\Big) \\
&= \big((\mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Z}}(j)) \cup (\mathrm{C}_{j,\mathrm{st}} \setminus M_{\mathrm{Z}}(j))\big) \cap \big(K_{\mathrm{Z}}(j) \cup M_{\mathrm{Z}}(j)\big) \\
&= \big((\mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Z}}(j)) \cap M_{\mathrm{Z}}(j)\big) \cup \big((\mathrm{C}_{j,\mathrm{st}} \setminus M_{\mathrm{Z}}(j)) \cap K_{\mathrm{Z}}(j)\big)\,.
\end{aligned}$$

Therefore

$$(\mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Z}}(j)) \cap M_{\mathrm{Z}}(j) = \varnothing \quad \text{and} \quad (\mathrm{C}_{j,\mathrm{st}} \setminus M_{\mathrm{Z}}(j)) \cap K_{\mathrm{Z}}(j) = \varnothing\,. \tag{5.30}$$

In order to seek a contradiction we assume that $\mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Z}}(j) \neq \varnothing$. Then we find $x \in \mathcal{I}_{j,\mathrm{st}} \setminus g_{\mathrm{Z}}(j).\mathcal{I}_{\psi_{\mathrm{Z}}(j),\mathrm{st}}$ and $y \in g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}.J_{\psi_{\mathrm{Z}}^{-1}(j),\mathrm{st}}$. By (B5$_{\mathrm{red}}$I) there exists $\nu \in \mathrm{C}_{j,\mathrm{st}}$ with $(\gamma_\nu(+\infty), \gamma_\nu(-\infty)) = (x, y)$. Thus,

$$\nu \in M_{\mathrm{Z}}(j) \cap (\mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Z}}(j)) \,,$$

which yields a contradiction to (5.30). In turn, $K_{\mathrm{Z}}(j) = \mathrm{C}_{j,\mathrm{st}}$ and, by an analogous argument, also $M_{\mathrm{Z}}(j) = \mathrm{C}_{j,\mathrm{st}}$.

For the proof of (iii) we suppose that $j \in A_{\mathrm{X}} \cap A_{\mathrm{Y}}$. Then we find

$$\nu \in K_{\mathrm{X}}(j) \cap M_{\mathrm{Y}}(j) \,,$$

due to (B5$_{\mathrm{red}}$I) and the nonemptiness of $K_{\mathrm{X}}(j)$ and $M_{\mathrm{Y}}(j)$. Using that

$$K_{\mathrm{X}}(j) \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus K_{\mathrm{Y}}(j) \qquad \text{and} \qquad M_{\mathrm{Y}}(j) \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus M_{\mathrm{X}}(j)$$

by (i), we obtain

$$\begin{aligned}
\varnothing \neq\ & K_{\mathrm{X}}(j) \cap M_{\mathrm{Y}}(j) \\
& \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{Y}}(j) \cup M_{\mathrm{X}}(j)\big) \\
& \subseteq \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{Y}}(j) \cup M_{\mathrm{X}}(j)\big) \cup \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{Y}}(j) \cup K_{\mathrm{X}}(j)\big) \\
& \quad \cup \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{X}}(j) \cup M_{\mathrm{Y}}(j)\big) \cup \mathrm{C}_{j,\mathrm{st}} \setminus \big(M_{\mathrm{X}}(j) \cup M_{\mathrm{Y}}(j)\big) \\
& = \mathrm{C}_{j,\mathrm{st}} \setminus \Big(\big(K_{\mathrm{Y}}(j) \cup M_{\mathrm{X}}(j)\big) \cap \big(K_{\mathrm{Y}}(j) \cup K_{\mathrm{X}}(j)\big) \\
& \qquad\qquad \cap \big(K_{\mathrm{X}}(j) \cup M_{\mathrm{Y}}(j)\big) \cap \big(M_{\mathrm{X}}(j) \cup M_{\mathrm{Y}}(j)\big)\Big) \\
& = \mathrm{C}_{j,\mathrm{st}} \setminus \bigcup_{\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}} \big(K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)\big) \\
& = \mathrm{C}_{j,\mathrm{acc}} \,.
\end{aligned}$$

This completes the proof. ☐

Remark 5.29. Let $\mathrm{Z}, \mathrm{Z}' \in \{\mathrm{X}, \mathrm{Y}\}$, $\mathrm{Z} \neq \mathrm{Z}'$, and $j \in A_{\mathrm{Z}}$. The conditions

$$\mathcal{I}_{j,\mathrm{st}} = g_{\mathrm{Z}}(j).\mathcal{I}_{\psi_{\mathrm{Z}}(j),\mathrm{st}} \quad \text{and} \quad J_{j,\mathrm{st}} = g_{\mathrm{Z}}(\psi_{\mathrm{Z}}^{-1}(j))^{-1}.J_{\psi_{\mathrm{Z}}^{-1}(j),\mathrm{st}}$$

in Proposition 5.28(ii) imply that Z'_j is an inner point of a representative interval of some funnel of $\mathbb{X}$. In particular, because of (B7$_{\mathrm{red}}$a), the structure of $\mathcal{I}_{j,\mathrm{st}}$ implies

$$H(j) = \{\psi_{\mathrm{Z}}(j)\} \quad \text{and} \quad \mathcal{G}(j, \psi_{\mathrm{Z}}(j)) = \{g_{\mathrm{Z}}(j)\} \,.$$

In this case $j \neq \psi_{\mathrm{Z}}(j)$, because otherwise $\mathcal{I}_{j,\mathrm{st}}$ would be empty. Algorithm 5.4 removes all branches of that type from the set of branches. Hence, if the level of reduction before applying the cuspidal acceleration is at least κ_1, with κ_1 as in Section 5.1, then we have $A^* = A$.

Proposition 5.30. *Let γ be a geodesic on $\mathbb{H}$ that intersects $\Gamma.\mathrm{C}_{\mathrm{st}}$. Then γ intersects $\Gamma.\mathrm{C}_{\mathrm{acc}}$. More precisely, if γ intersects $\Gamma.\mathrm{C}_{\mathrm{st}}$ at time t^*, then there exist $t_+^*, t_-^* \in \mathbb{R}$ with $t_+^* \geq t^* \geq t_-^*$ such that γ intersects $\Gamma.\mathrm{C}_{\mathrm{acc}}$ at time t_+^* and at time t_-^*.*

Proof. Without loss of generality we may suppose that the intersection between the geodesic γ and $\Gamma.\mathrm{C}_{\mathrm{st}}$ at time t^* is on C_{st}, say at $\nu = \gamma'(t^*) \in \mathrm{C}_{j,\mathrm{st}}$ with $j \in A$. We may further suppose that ν is an element of

$$\mathrm{C}_{j,\mathrm{st}} \setminus \mathrm{C}_{j,\mathrm{acc}} = \bigcup_{\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}} \left(K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j) \right),$$

as otherwise there is nothing to prove. Thus, there is $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ such that $\nu \in K_{\mathrm{Z}}(j) \cap M_{\mathrm{Z}}(j)$ and $j \in A_{\mathrm{Z}}$, and Z is unique by Proposition 5.28(i). By the discussion in Remark 5.27, we find $n \in \mathbb{N}_0$ and $k \in \mathrm{Cyc}_{j,\mathrm{Z}}$ such that

$$\gamma(+\infty) \in D^+_{n,\mathrm{Z}}(j,k)_{\mathrm{st}},$$

with the set $D^+_{n,\mathrm{Z}}(j,k)$ as defined in Remark 5.27. Since

$$g_{\mathrm{Z}}(j,k)^{-1} u_{j,\mathrm{Z}}^{-n}.D^+_{n,\mathrm{Z}}(j,k)_{\mathrm{st}} = I_{k,\mathrm{st}} \setminus g_{\mathrm{Z}}(k).I_{\psi_{\mathrm{Z}}(k),\mathrm{st}}, \tag{5.31}$$

Lemma 4.13 shows that γ intersects $u_{j,\mathrm{Z}}^{n} g_{\mathrm{Z}}(j,k).\mathrm{C}_k$ at some time $t_+^* > t^*$. More precisely, using Remark 4.11 and the full extent of the equality (5.31), we obtain that

$$g_{\mathrm{Z}}(j,k)^{-1} u_{j,\mathrm{Z}}^{-n}.\gamma'(t_+^*) \in \mathrm{C}_{k,\mathrm{st}} \setminus K_{\mathrm{Z}}(k).$$

By construction, $g_{\mathrm{Z}}(j,k)^{-1} u_{j,\mathrm{Z}}^{-n}.\gamma'(t_+^*) \in M_{\mathrm{Z}}(k)$. Since $M_{\mathrm{X}}(k) \cap M_{\mathrm{Y}}(k) = \varnothing$ by Proposition 5.28(i), it follows that

$$g_{\mathrm{Z}}(j,k)^{-1} u_{j,\mathrm{Z}}^{-n}.\gamma'(t_+^*) \in \mathrm{C}_{k,\mathrm{st}} \setminus \bigcup_{\mathrm{W} \in \{\mathrm{X},\mathrm{Y}\}} \left(K_{\mathrm{W}}(k) \cap M_{\mathrm{W}}(k) \right) = \mathrm{C}_{k,\mathrm{acc}}.$$

Thus, γ intersects $\Gamma.\mathrm{C}_{\mathrm{acc}}$ at time t_+^*. The proof of the existence of an intersection time $t_-^* \leq t^*$ is analogous, using the set $D^-_{n,\mathrm{Z}}(j,k)$ (from Remark 5.27) for suitable $n \in \mathbb{N}_0$, $k \in \mathrm{Cyc}_{j,\mathrm{Z}}$ instead of $D^-_{n,\mathrm{Z}}(j,k)$, as well as Lemma 4.14 instead of Lemma 4.13. $\square$

Proposition 5.31. *For each $\mu \in \mathcal{M}_{\mathrm{Van}(\mathbb{X})}$ the set $\widehat{\mathrm{C}}_{\mathrm{acc}}$ is a strong cross section for the geodesic flow on $\mathbb{X}$ with respect to μ. Each geodesic in $\mathscr{G}(\mathbb{X}) \setminus \mathrm{Van}(\mathbb{X})$ intersects $\widehat{\mathrm{C}}_{\mathrm{acc}}$ infinitely often in past and future.*

Proof. We recall from Corollary 4.37 that $\widehat{\mathrm{C}}_{\mathrm{st}}$ is a strong cross section with respect to μ and that each geodesic in $\mathscr{G}(\mathbb{X}) \setminus \mathrm{Van}(\mathbb{X})$ intersects $\widehat{\mathrm{C}}_{\mathrm{st}}$ infinitely often in past and future. Therefore, as $\widehat{\mathrm{C}}_{\mathrm{acc}} \subseteq \widehat{\mathrm{C}}_{\mathrm{st}}$, the validity of (CS2) for $\widehat{\mathrm{C}}_{\mathrm{acc}}$ is immediate from its validity for $\widehat{\mathrm{C}}_{\mathrm{st}}$. In order to establish the conditions (CS1) and (CS3) for $\widehat{\mathrm{C}}_{\mathrm{acc}}$ we let $\widehat{\gamma}$ be a geodesic on $\mathbb{X}$ that intersects $\widehat{\mathrm{C}}_{\mathrm{st}}$ (which is true for all geodesics in $\mathscr{G}(\mathbb{X}) \setminus \mathrm{Van}(\mathbb{X})$ and hence for μ-almost geodesics on $\mathbb{X}$) and let $(t_n)_{n\in\mathbb{Z}}$ be the bi-infinite sequence of intersection times. From Proposition 5.30 it follows that the sequence of intersection times of $\widehat{\gamma}$ with $\widehat{\mathrm{C}}_{\mathrm{acc}}$ is a *bi-infinite* subsequence of $(t_n)_{n\in\mathbb{Z}}$, showing that $\widehat{\gamma}$ intersects $\widehat{\mathrm{C}}_{\mathrm{acc}}$ infinitely often in past and future. This completes the proof. □

5.4 Structure of Accelerated Systems

In this section we discuss the structure of the acceleration of a reduced set of branches. In particular, we provide a partition of the set of representatives that is better suited for a coding of geodesics (or, equivalently, the passage to a discrete dynamical system) than the family immediate from Definition 5.26. These results will be crucial for the discussion in Chapter 6, where we establish the strict transfer operator approach. We resume the notation from Section 5.3. Thus, $\mathcal{C}$ denotes a *reduced* set of branches for the geodesic flow on $\mathbb{X}$.

The cuspidal acceleration procedure effectively dissects each branch into up to three mutually disjoint pieces, as the following lemma shows.

Lemma 5.32. *For every $j \in A$ we have*

$$\mathrm{C}_{j,\mathrm{acc}} = \big(\mathrm{C}_{j,\mathrm{st}} \setminus (K_{\mathrm{X}}(j) \cup K_{\mathrm{Y}}(j))\big) \cup \big(K_{\mathrm{X}}(j) \setminus M_{\mathrm{X}}(j)\big) \cup \big(K_{\mathrm{Y}}(j) \setminus M_{\mathrm{Y}}(j)\big)$$

and the union on the right hand side is disjoint.

Proof. Let $j \in A$. For the sake of improved readability we use the abbreviations

$$\mathrm{C} := \mathrm{C}_{j,\mathrm{st}}\,, \qquad K_{\mathrm{Z}} := K_{\mathrm{Z}}(j)\,, \qquad M_{\mathrm{Z}} := M_{\mathrm{Z}}(j)$$

for $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$. Since $K_{\mathrm{X}} \cap K_{\mathrm{Y}} = \varnothing$ by Proposition 5.28(i), the claimed union is disjoint. In order to show the claimed equality, we recall from Definition 5.26 that

$$\mathrm{C}_{j,\mathrm{acc}} = \mathrm{C} \setminus \big((K_{\mathrm{X}} \cap M_{\mathrm{X}}) \cup (K_{\mathrm{Y}} \cap M_{\mathrm{Y}})\big)\,,$$

which clearly contains the set $\mathrm{C} \setminus (K_{\mathrm{X}} \cup K_{\mathrm{Y}})$. Using again that $K_{\mathrm{X}} \cap K_{\mathrm{Y}} = \varnothing$, we see that also $K_{\mathrm{X}} \setminus M_{\mathrm{X}}$ and $K_{\mathrm{Y}} \setminus M_{\mathrm{Y}}$ are subsets of $\mathrm{C}_{j,\mathrm{acc}}$. Thus,

$$\big(\mathrm{C} \setminus (K_{\mathrm{X}} \cup K_{\mathrm{Y}})\big) \cup \big(K_{\mathrm{X}} \setminus M_{\mathrm{X}}\big) \cup \big(K_{\mathrm{Y}} \setminus M_{\mathrm{Y}}\big) \subseteq \mathrm{C}_{j,\mathrm{acc}}\,. \tag{5.32}$$

It remains to establish the converse inclusion relation. To that end we consider any element $\nu \in \mathrm{C}_{j,\mathrm{acc}}$. If $\nu \notin K_{\mathrm{X}} \cup K_{\mathrm{Y}}$, then ν is obviously contained in the union on the left hand side of (5.32). If $\nu \in K_{\mathrm{X}}$, then $\nu \notin M_{\mathrm{X}}$ since otherwise we would have $\nu \in K_{\mathrm{X}} \cap M_{\mathrm{X}}$, which contradicts to $\nu \in \mathrm{C}_{j,\mathrm{acc}}$. Analogously, for $\nu \in K_{\mathrm{Y}}$. In turn, the inclusion relation in (5.32) is indeed an equality. □

Recall the index set A^* from (5.27). For $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$ we set

$$A_{\mathrm{Z}}^* := A^* \cap A_{\mathrm{Z}} \,. \tag{5.33}$$

The following definition is motivated by Lemma 5.32. We recall from Definition 5.26 that $K_{\mathrm{Z}}(j) = M_{\mathrm{Z}}(j) = \varnothing$ whenever $j \in A^* \setminus A_{\mathrm{Z}}^*$.

Definition 5.33. For $j \in A^*$ we set

$$\mathrm{C}_{(j,\mathrm{R})}^{\mathrm{acc}} := \mathrm{C}_{j,\mathrm{st}} \setminus \big(K_{\mathrm{X}}(j) \cup K_{\mathrm{Y}}(j)\big) \,,$$

and, for $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$,

$$\mathrm{C}_{(j,\mathrm{Z})}^{\mathrm{acc}} := K_{\mathrm{Z}}(j) \setminus M_{\mathrm{Z}}(j) \,.$$

We further set

$$\widehat{A} := \left\{ (j, V) \in A^* \times \{\mathrm{X}, \mathrm{R}, \mathrm{Y}\} \;\middle|\; \mathrm{C}_{(j,V)}^{\mathrm{acc}} \neq \varnothing \right\} .$$

For $a \in \widehat{A}$ we call $\mathrm{C}_a^{\mathrm{acc}}$ an *accelerated* (or *induced*) *branch* and denote by

$$\mathcal{C}_{\mathrm{acc}} := \left\{ \mathrm{C}_a^{\mathrm{acc}} \;\middle|\; a \in \widehat{A} \right\}$$

the *set of all accelerated branches* or the *accelerated system*. For any $j \in A^*$ we set

$$I_{(j,V)} := \begin{cases} g_V(j).\mathcal{I}_{\psi_V(j)} & \text{if } V \in \{\mathrm{X}, \mathrm{Y}\} \,, \\ \mathcal{I}_j \setminus \big(g_{\mathrm{X}}(j).\mathcal{I}_{\psi_{\mathrm{X}}(j)} \cup g_{\mathrm{Y}}(j).\mathcal{I}_{\psi_{\mathrm{Y}}(j)}\big) & \text{if } V = \mathrm{R} \,, \end{cases}$$

as well as

$$J_{(j,V)} := \begin{cases} J_j \setminus g_V(\psi_V^{-1}(j))^{-1}.J_{\psi_V^{-1}(j)} \,, & \text{if } V \in \{\mathrm{X}, \mathrm{Y}\} \,, \\ J_j \,, & \text{if } V = \mathrm{R} \,, \end{cases}$$

where, for $j \in A^* \setminus A_{\mathrm{Z}}^*$, we set $\mathcal{I}_{\psi_{\mathrm{Z}}(j)} := \varnothing$ and $J_{\psi_{\mathrm{Z}}^{-1}(j)} := \varnothing$ for $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$.

We note that for each $j \in A^*$, the set $\mathrm{C}_{j,\mathrm{acc}}$ decomposes into the disjoint union

$$\mathrm{C}_{j,\mathrm{acc}} = \mathrm{C}_{(j,\mathrm{R})}^{\mathrm{acc}} \cup \mathrm{C}_{(j,\mathrm{X})}^{\mathrm{acc}} \cup \mathrm{C}_{(j,\mathrm{Y})}^{\mathrm{acc}}$$

by Lemma 5.32, an observation that is needed for the proof of the following lemma.

Lemma 5.34. *The accelerated system $\mathcal{C}_{\mathrm{acc}}$ satisfies the following properties:*

(i) *We have*

$$\bigcup_{a \in \widehat{A}} \mathrm{C}_a^{\mathrm{acc}} = \bigcup_{j \in A^*} \mathrm{C}_{j,\mathrm{acc}} = \mathrm{C}_{\mathrm{acc}} \,.$$

(ii) *For all $j \in A^*$ we have $\mathcal{I}_j = I_{(j,\mathrm{X})} \cup I_{(j,\mathrm{R})} \cup I_{(j,\mathrm{Y})}$. This union is disjoint.*

(iii) *For all $a \in \widehat{A}$ we have $I_{a,\mathrm{st}} = I(\mathrm{C}_a^{\mathrm{acc}})$.*

(iv) *For all $a \in \widehat{A}$ none of the boundary points of I_a and J_a are hyperbolic fixed points.*

Proof. The statement of (i) follows directly from Lemma 5.32. Further, (ii) is obviously true because, for $V \in \{\mathrm{X}, \mathrm{Y}\}$, we have $I_{(j,V)} = g_V(j).\mathcal{I}_{\psi_V(j)}$. In order to establish (iii) we let $a := (j, V) \in \widehat{A}$ and suppose first that $V \in \{\mathrm{X}, \mathrm{Y}\}$. The definition of $\mathrm{C}_a^{\mathrm{acc}}$ immediately shows that

$$I(\mathrm{C}_a^{\mathrm{acc}}) \subseteq I(K_V(j)) = I_{(j,V),\mathrm{st}} = I_{a,\mathrm{st}}\,.$$

For the converse inclusion relation we pick any point

$$y \in J_{j,\mathrm{st}} \setminus g_V(\psi_V^{-1}(j))^{-1}.J_{\psi_V^{-1}(j)}\,.$$

Its existence follows from $K_V(j) \setminus M_V(j) = \mathrm{C}_a^{\mathrm{acc}} \neq \varnothing$. By (B5$_{\mathrm{red}}$I), we find for each $x \in I(K_V(j))$ an element $\nu \in \mathrm{C}_j$ such that

$$\big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big) = (x, y)\,.$$

Each such element ν is in $K_V(j) \setminus M_V(j) = \mathrm{C}_a^{\mathrm{acc}}$. Thus, $I(K_V(j)) \subseteq I(\mathrm{C}_a^{\mathrm{acc}})$. We suppose now that $V = \mathrm{R}$. Then

$$\begin{aligned} I(\mathrm{C}_a^{\mathrm{acc}}) &= \mathcal{I}_{j,\mathrm{st}} \setminus \big(I(K_\mathrm{X}(j)) \cup I(K_\mathrm{Y}(j))\big) \\ &= \mathcal{I}_{j,\mathrm{st}} \setminus \big(g_\mathrm{X}(j).\mathcal{I}_{\psi_\mathrm{X}(j),\mathrm{st}} \cup g_\mathrm{Y}(j).\mathcal{I}_{\psi_\mathrm{Y}(j),\mathrm{st}}\big) \\ &= I_{a,\mathrm{st}}\,, \end{aligned}$$

where we set $\mathcal{I}_{\psi_\mathrm{Z}(j)} := \varnothing$ whenever $(j, \mathrm{Z}) \notin \widehat{A}$, for $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$. This shows (iii).

Finally, we observe that, for every $a \in \widehat{A}$, the boundary points of I_a and J_a in the $\widehat{\mathbb{R}}$-topology emerge as Γ-translates of the endpoints of the geodesic segments $\overline{\mathrm{bp}(\mathrm{C}_j)}$ for $j \in A$. In other words, they are contained in the set

$$\Gamma.\{\mathrm{X}_j, \mathrm{Y}_j \mid j \in A\}\,.$$

Hence, (iv) is a consequence of (B2). □

Our next goal is to "update" the transition sets according to the accelerated branches. In other words, for $a, b \in \widehat{A}$ we search for a characterization of the set

$$\widehat{\mathcal{G}}(a, b) := \big\{g \in \Gamma \;\big|\; \exists\, \nu \in \mathrm{C}_a^{\mathrm{acc}} \colon \gamma'(\mathrm{t}_{\mathrm{acc},1}(\nu)) \in g.\,\mathrm{C}_b^{\mathrm{acc}}\big\} \tag{5.34}$$

in terms of the transition sets for $\mathcal{C}$, where $\mathrm{t}_{\mathrm{acc},1}(\nu)$ denotes the first return time with respect to $\mathrm{C}_{\mathrm{acc}}$. This will enable us to prove that the first return time $\mathrm{t}_{\mathrm{acc},1}(\nu)$

does indeed exist for all $\nu \in \mathrm{C}_{\mathrm{acc}}$, and thus the sets $\mathcal{G}_{\mathrm{acc}}(a,b)$ belonging to the accelerated system are well-defined (see (5.38) below). Further, it is a crucial step in the determination of the transformations that constitute the structure tuple for the strict transfer operator approach.

We start by updating the cycles, cycle transformations, and cycle sets from Section 5.3 to the situation of the accelerated branches.

Definition 5.35. Let $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$ and let $j \in A$ be such that $(j,\mathrm{Z}) \in \widehat{A}$. We call the subsequence $\mathrm{cyc}^*_{\mathrm{Z}}(j)$ of $\mathrm{cyc}_{\mathrm{Z}}(j)$ of all k that are elements of A^* the *induced* Z-*cycle* of j. We further define the map $\psi^*_{\mathrm{Z}}\colon A^*_{\mathrm{Z}} \to A^*_{\mathrm{Z}}$ by

$$\psi^*_{\mathrm{Z}}(\mathrm{cyc}^*_{\mathrm{Z}}(j)_n) \coloneqq \mathrm{cyc}^*_{\mathrm{Z}}(j)_{n+1}$$

for all $n \in \mathbb{N}$. We also define the transformations

$$g^*((j,\mathrm{Z})) \coloneqq g_{\mathrm{Z}}(j,\psi^*_{\mathrm{Z}}(j))$$

and set as before

$$g^*((j,\mathrm{Z}),(k,\mathrm{Z})) \coloneqq g^*((j,\mathrm{Z}))\cdot g^*((\psi^*_{\mathrm{Z}}(j),\mathrm{Z}))\cdots g^*(((\psi^*_{\mathrm{Z}})^{r_{j,k}-1}(j),\mathrm{Z}))\,,$$

where $r_{j,k} \coloneqq \min\left\{\ell \in \mathbb{N} \;\middle|\; (\psi^*_{\mathrm{Z}})^{\ell}(j) = k\right\}$, as well as

$$u_{(j,\mathrm{Z})} \coloneqq g^*((j,\mathrm{Z}),(j,\mathrm{Z}))\,.$$

Finally, we define the *induced cycle set* of (j,Z) to be

$$\mathrm{Cyc}^*_{(j,\mathrm{Z})} \coloneqq \left\{(j,\mathrm{Z}),(\psi^*_{\mathrm{Z}}(j),\mathrm{Z}),\ldots,((\psi^*_{\mathrm{Z}})^{r_{j,j}-1}(j),\mathrm{Z})\right\}.$$

Remark 5.36. Note that the updated transformations $g^*((j,V))$ already make up for the loss of branches during the acceleration procedure. Thus, the cycle transformation remains unaltered, meaning that

$$u_{(j,V)} = u_{j,V}\,, \tag{5.35}$$

for every $(j,V) \in A^* \times \{\mathrm{X},\mathrm{Y}\}$. Note further that the induced Z-cycles are allowed to contain members k for which (k,Z) is not an element of $\widehat{A}$. This is necessitated by the following eventuality: Let $j \in A$ be such that $\mathrm{C}_{j,\mathrm{st}} = M_{\mathrm{Z}}(j)$ for some $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$, but $\mathrm{C}_{j,\mathrm{st}} \neq K_{\mathrm{Z}}(j)$. Then $j \in A^*_{\mathrm{Z}}$, but $\mathrm{C}^{\mathrm{acc}}_{(j,\mathrm{Z})} = \varnothing$, thus $(j,\mathrm{Z}) \notin \widehat{A}$. But $\widehat{A}$ will include the index (j,R). This distinction proves necessary in the following construction of induced transition sets: j must be included in the induced cycle in order to allow the other induced branches of that cycle to "see" (j,R). But (j,Z) is excluded from $\widehat{A}$ and hence we do not construct transition sets for it. If the level of reduction is at least κ_1, with κ_1 as in Section 5.1, then this eventuality does not occur and, consequentially, $(k, \mathrm{Z}) \in \widehat{A}$ for every member k of any induced Z-cycle.

We are now fully prepared to determine the *induced transition sets* $\mathcal{G}_{\mathrm{acc}}(a,b)$ for $a, b \in \widehat{A}$. Let $a = (j, V) \in \widehat{A}$ and suppose first that $V \in \{\mathrm{X}, \mathrm{Y}\}$. In Remark 5.27 we have derived the decomposition

$$I_{(j,V)} = g_V(j).\mathcal{I}_{\psi_V(j)} = \bigcup_{n\in\mathbb{N}_0} \bigcup_{k\in\mathrm{Cyc}_{j,V}} \mathcal{D}^+_{n,V}(j,k)\,.$$

We can now rewrite

$$\begin{aligned} D^+_{n,V}(j,k) &= u^n_{j,V} g_V(j,k).I_k \setminus u^n_{j,V} g_V(j,k) g_V(k).I_{\psi_V(k)} \\ &= u^n_{j,V} g_V(j,k).\big(I_k \setminus I_{(k,V)}\big)\,. \end{aligned}$$

Hence,

$$\mathcal{D}^+_{n,V}(j,k) = D^+_{n,V}(j,k) \cap u^n_{j,V} g_V(j,k).\mathcal{I}_k = u^n_{j,V} g_V(j,k).\big(\mathcal{I}_k \setminus I_{(k,V)}\big)\,.$$

Therefore, by passing to "st"-sets we obtain

$$\begin{aligned} I_{(j,V),\mathrm{st}} &= \bigcup_{n\in\mathbb{N}_0} \bigcup_{k\in\mathrm{Cyc}_{j,V}} u^n_{j,V} g_V(j,k).\big(\mathcal{I}_{k,\mathrm{st}} \setminus I(K_V(k))\big) \\ &= \bigcup_{n\in\mathbb{N}_0} \bigcup_{k\in\mathrm{Cyc}_{j,V}} u^n_{j,V} g_V(j,k).\big(I_{(k,\mathrm{R}),\mathrm{st}} \cup I_{(k,V'),\mathrm{st}}\big)\,, \end{aligned} \tag{5.36}$$

with V' such that $\{V, V'\} = \{\mathrm{X}, \mathrm{Y}\}$ and all unions being disjoint. By taking Remark 5.36 into account, Proposition 5.28(ii) allows us to pass to $(k, V) \in \mathrm{Cyc}^*_{(j,V)}$ in the second union. The transformations $g_V(j,k)$ then need to be substituted by $g^*((j,V),(k,V))$. Hence we obtain the disjoint union

$$I_{(j,V),\mathrm{st}} = \bigcup_{n\in\mathbb{N}_0} \bigcup_{(k,V)\in\mathrm{Cyc}^*_{(j,V)}} u^n_{(j,V)} g^*((j,V),(k,V)).\big(I_{(k,\mathrm{R}),\mathrm{st}} \cup I_{(k,V'),\mathrm{st}}\big)\,. \tag{5.37}$$

For any $b \in \widehat{A}$, $b = (k, W)$ (and $a = (j, V)$, $V \in \{\mathrm{X}, \mathrm{Y}\}$) we therefore define

$$\mathcal{G}_{\mathrm{acc}}(a,b) := \begin{cases} \bigcup_{n\in\mathbb{N}_0} \{u^n_a g^*(a,\widetilde{b})\} & \text{if } W \neq V \text{ and } \widetilde{b} := (k,V) \in \mathrm{Cyc}^*_a\,, \\ \varnothing & \text{otherwise.} \end{cases} \tag{5.38}$$

We suppose now that $V = \mathrm{R}$, thus $a = (j, \mathrm{R})$. We consider $(k, g) \in A \times \Gamma$ such that $g \in \mathcal{G}(j,k)$, pick $\nu \in \mathrm{C}^{\mathrm{acc}}_a$ such that

$$\gamma_\nu(+\infty) \notin g_{\mathrm{X}}(j).\mathcal{I}_{\psi_{\mathrm{X}}(j),\mathrm{st}} \cup g_{\mathrm{Y}}(j).\mathcal{I}_{\psi_{\mathrm{Y}}(j),\mathrm{st}}\,,$$

and let η denote the intersection vector of γ_ν with $g.\,\mathrm{C}_k$. In what follows we argue that η is contained in $g\boldsymbol{.}\mathrm{C}_b^{\mathrm{acc}}$ for some $b \in \widehat{A}$ of the form $b = (k, W)$. To that end, we first note that

$$I_{(j,\mathrm{R}),\mathrm{st}} = \bigcup_{\substack{k\in A\\ k\notin\{\psi_{\mathrm{X}}(j),\psi_{\mathrm{Y}}(j)\}}} \bigcup_{g\in\mathcal{G}(j,k)} g\boldsymbol{.}\mathcal{I}_{k,\mathrm{st}} \quad\cup \bigcup_{\substack{g\in\mathcal{G}(j,\psi_{\mathrm{X}}(j))\\ g\neq g_{\mathrm{X}}(j)}} g\boldsymbol{.}\mathcal{I}_{\psi_{\mathrm{X}}(j),\mathrm{st}}$$
$$\cup \bigcup_{\substack{g\in\mathcal{G}(j,\psi_{\mathrm{Y}}(j))\\ g\neq g_{\mathrm{Y}}(j)}} g\boldsymbol{.}\mathcal{I}_{\psi_{\mathrm{Y}}(j),\mathrm{st}}$$

by (B7$_{\mathrm{red}}$a), Definition 5.33, and Lemma 5.34(ii). Therefore the hypotheses on ν imply that $g\boldsymbol{.}\mathcal{I}_k \subseteq I_{(j,\mathrm{R})}$. Then, for any $\mathrm{Z} \in \{\mathrm{X}, \mathrm{Y}\}$, we have $J_j \neq g\boldsymbol{.}J(M_{\mathrm{Z}}(k))$, as follows immediately from the definition of $M_{\mathrm{Z}}(k)$. Thus,

$$J_j \cap g\boldsymbol{.}J(M_{\mathrm{Z}}(k)) = \varnothing$$

by Proposition 4.5(i). It follows that $\eta \in \mathrm{C}_{k,\mathrm{acc}}$ and, by Lemma 5.32, $\eta \in \mathrm{C}_b^{\mathrm{acc}}$ for some $b = (k, W) \in \widehat{A}$. Therefore, for any $b = (k, W) \in \widehat{A}$ (and $a = (j, \mathrm{R})$) we define

$$\mathcal{G}_{\mathrm{acc}}(a, b) := \begin{cases} \mathcal{G}(j, k) \setminus \{g_{\mathrm{X}}(j)\} & \text{if } k = \psi_{\mathrm{X}}(j)\,, \\ \mathcal{G}(j, k) \setminus \{g_{\mathrm{Y}}(j)\} & \text{if } k = \psi_{\mathrm{Y}}(j)\,, \\ \mathcal{G}(j, k) & \text{otherwise.} \end{cases} \tag{5.39}$$

In the following proposition we collect the adapted “set of branches”-style properties satisfied by $\mathcal{C}_{\mathrm{acc}}$. Each of the statements follows, in a straightforward way, from its respective counterpart in Definition 5.17, the properties collected in Lemma 5.34, and the constructions above, for which reason we omit a detailed proof.

Proposition 5.37. *The accelerated system $\mathcal{C}_{\mathrm{acc}}$ satisfies the following properties:*

(B1$_{\mathrm{acc}}$) *For each $a \in \widehat{A}$ there exists $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ such that $\pi(\gamma_\nu) \in \mathscr{G}_{\mathrm{Per}}(\mathbb{X})$.*

(B2$_{\mathrm{acc}}$) *For each $a = (j, V) \in \widehat{A}$ the set $\mathrm{bp}(\mathrm{C}_a^{\mathrm{acc}})$ is contained in a complete geodesic segment b_a in $\mathbb{H}$ with endpoints in $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$. This segment is given by*

$$\mathrm{b}_a = \overline{\mathrm{bp}(\mathrm{C}_j)}\,.$$

(B3$_{\mathrm{acc}}$) *For each $a \in \widehat{A}$ all elements of $\mathrm{C}_a^{\mathrm{acc}}$ point into the same open half-space relative to b_a.*

(B4$_{\mathrm{acc}}$) *The Γ-translates of $\left\{ I_a \;\middle|\; a \in \widehat{A} \right\}$ cover $\widehat{\mathbb{R}}_{\mathrm{st}}$.*

(B5$_{\mathrm{acc}}$) *For each $a \in \widehat{A}$ and each pair $(x, y) \in I_{a,\mathrm{st}} \times J_{a,\mathrm{st}}$ there exists a unique* *element* $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ *such that*

$$(x,y) = \big(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\big)\,.$$

($\mathrm{B}6_{\mathrm{acc}}$) *If* $\mathrm{b}_a \cap g\boldsymbol{.}\mathrm{b}_b \neq \varnothing$ *for some* $a = (j,V), b = (k,W) \in \widehat{A}$ *and some* $g \in \Gamma$, *then either* $j = k$ *and* $g = \mathrm{id}$, *or* $\mathrm{H}_\pm(j) = g\boldsymbol{.}\mathrm{H}_\mp(k)$.

($\mathrm{B}7_{\mathrm{acc}}$) *Let* $\mathcal{G}_{\mathrm{acc}}(a,b)$ *be defined as in* (5.38) *and* (5.39) *and let* $\widehat{\mathcal{G}}(a,b)$ *be defined as in* (5.34) *for* $a, b \in \widehat{A}$.

(i) *For every* $a \in \widehat{A}$ *we have*

$$I_{a,\mathrm{st}} = \bigcup_{b\in\widehat{A}} \bigcup_{g\in\mathcal{G}_{\mathrm{acc}}(a,b)} g\boldsymbol{.}I_{b,\mathrm{st}}\,.$$

This union is disjoint.

(ii) *For every* $a \in \widehat{A}$ *and every* $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ *there exists* $t \in (0,+\infty)$ *such that* $\gamma_\nu(t) \in \Gamma\boldsymbol{.}\mathrm{C}_{\mathrm{acc}}$.

(iii) *For every pair* $(a,b) \in \widehat{A} \times \widehat{A}$ *we have*

$$\mathcal{G}_{\mathrm{acc}}(a,b) = \widehat{\mathcal{G}}(a,b)\,.$$

($\mathrm{B}8_{\mathrm{acc}}$) *If* $\mathcal{C}$ *is admissible, then* $\mathcal{C}_{\mathrm{acc}}$ *is admissible in the sense that there exist* $q \in \widehat{\mathbb{R}}$ *and an open neighborhood* $\mathcal{U}$ *of* q *in* $\widehat{\mathbb{R}}$ *such that*

$$\mathcal{U} \cap \bigcup_{a\in\widehat{A}} I_{a,\mathrm{st}} = \varnothing \qquad \text{and} \qquad q \notin I_a\,,$$

for every $a \in \widehat{A}$.

Remark 5.38. Let $a, b \in \widehat{A}$. By combining part (iii) of ($\mathrm{B}7_{\mathrm{acc}}$) with (5.34), the definition of $\mathrm{C}_a^{\mathrm{acc}}$, and Algorithm 5.11 we see that every element $g \in \mathcal{G}_{\mathrm{acc}}(a,b)$ emerges as the product of transition set elements. This means we find an integer $n \in \mathbb{N}$, indices $k_1, \ldots, k_{n+1} \in A$, and transformations $h_i \in \mathcal{G}(k_i, k_{i+1})$ such that

$$g = h_1 h_2 \cdots h_n\,.$$

Corollary 5.39. *Let* $\nu \in \mathrm{C}_{\mathrm{acc}}$. *Then there exist uniquely determined sequences*

$$(\mathrm{t}_{\mathrm{acc},n}(\nu))_{n\in\mathbb{Z}} \text{ in } \mathbb{R}\,, \quad (\mathrm{k}_{\mathrm{acc},n}(\nu))_{n\in\mathbb{Z}} \text{ in } \widehat{A}\,, \quad \text{and} \quad (\mathrm{g}_{\mathrm{acc},n}(\nu))_{n\in\mathbb{Z}} \text{ in } \Gamma\,,$$

which satisfy the following properties:

(i) *The sequence* $(\mathrm{t}_{\mathrm{acc},n}(\nu))_{n\in\mathbb{Z}}$ *is a subsequence of* $(\mathrm{t}_{\mathrm{C},n}(\nu))_{n\in\mathbb{Z}}$. *It satisfies*

$$\mathrm{t}_{\mathrm{acc},0}(\nu) = 0$$

and

$$\mathrm{t}_{\mathrm{acc},n}(\nu) = \begin{cases} \min\left\{t > \mathrm{t}_{\mathrm{acc},n-1}(\nu) \mid \gamma'_\nu(t) \in \Gamma.\mathrm{C}_{\mathrm{acc}}\right\} & \text{for } n \geq 1\,, \\ \max\left\{t < \mathrm{t}_{\mathrm{acc},n+1}(\nu) \mid \gamma'_\nu(t) \in \Gamma.\mathrm{C}_{\mathrm{acc}}\right\} & \text{for } n \leq -1\,. \end{cases}$$

(ii) *For all $n \in \mathbb{Z}$ we have*

$$\mathrm{g}_{\mathrm{acc},n}(\nu) \in \mathcal{G}_{\mathrm{acc}}(\mathrm{k}_{\mathrm{acc},n-1}(\nu), \mathrm{k}_{\mathrm{acc},n}(\nu))\,.$$

(iii) *Let $a \in \widehat{A}$, $t \in \mathbb{R}$, and $g \in \Gamma$ be such that $\gamma'_\nu(t) \in g\,.\,\mathrm{C}_a^{\mathrm{acc}}$. Then there exists exactly one index $n \in \mathbb{Z}$ such that*

$$a = \mathrm{k}_{\mathrm{acc},n}(\nu)\,, \qquad t = \mathrm{t}_{\mathrm{acc},n}(\nu)\,,$$

and

$$g = \mathrm{g}_{\mathrm{acc},\operatorname{sgn}(t)}(\nu) \cdot \mathrm{g}_{\mathrm{acc},2\operatorname{sgn}(t)}(\nu) \cdots \mathrm{g}_{\mathrm{acc},n}(\nu)\,.$$

The following result on the relation between elements of the set of representatives of the accelerated cross section and finite sequences of transition elements should be considered an "accelerated variant" of Lemma 5.1.

Corollary 5.40. *Let $m \in \mathbb{N}$ and suppose that $a_0, \ldots, a_m \in \widehat{A}$ and $g_1, \ldots, g_m \in \Gamma$ are such that*

$$g_j \in \mathcal{G}_{\mathrm{acc}}(a_{j-1}, a_j) \qquad \text{for } j \in \{1, \ldots, m\}\,.$$

Then there exists $\nu \in \mathrm{C}_{a_0}^{\mathrm{acc}}$ such that

$$a_j = \mathrm{k}_{\mathrm{acc},j}(\nu) \qquad \text{for } j \in \{0, \ldots, m\}$$

and

$$g_j = \mathrm{g}_{\mathrm{acc},j}(\nu) \qquad \text{for } j \in \{1, \ldots, m\}\,.$$

Furthermore, the subset of $\mathrm{C}_{a_0}^{\mathrm{acc}}$ of all vectors with that property is given by

$$\mathrm{C}_{a_0}^{\mathrm{acc}}|_{h_m.\mathrm{C}_{a_m}^{\mathrm{acc}}} := \left\{\nu \in \mathrm{C}_{a_0}^{\mathrm{acc}} \;\middle|\; \exists t^* > 0 : \gamma_\nu(t^*) \in h_m.\mathrm{C}_{a_m}^{\mathrm{acc}}\right\}\,, \tag{5.40}$$

where $h_m := g_1 \cdots g_m$.

Proof. The assumptions imply that

$$g_j.I_{a_j} \subseteq I_{a_{j-1}} \quad \text{and} \quad J_{a_{j-1}} \subseteq g_j.J_{a_j}\,,$$

for all $j \in \{1, \ldots, n\}$. The combination of (B5$_{\mathrm{acc}}$) and (B7$_{\mathrm{acc}}$) therefore implies

$$\varnothing \neq \mathrm{C}_{a_0}^{\mathrm{acc}}|_{g_1\cdots g_m.\mathrm{C}_{a_m}^{\mathrm{acc}}} \subseteq \mathrm{C}_{a_0}^{\mathrm{acc}}|_{g_1\cdots g_{m-1}.\mathrm{C}_{a_{m-1}}^{\mathrm{acc}}} \subseteq \ldots \subseteq \mathrm{C}_{a_0}^{\mathrm{acc}}|_{g_1.\mathrm{C}_{a_1}^{\mathrm{acc}}}\,.$$

Hence, we may choose $\nu_0 \in \mathrm{C}^{\mathrm{acc}}_{a_0}|_{h_m.\mathrm{C}^{\mathrm{acc}}_{a_m}}$, with h_m as above, and consider the system of sequences $[(\mathrm{t}_{\mathrm{acc},n}(\nu_0))_n, (\mathrm{k}_{\mathrm{acc},n}(\nu_0))_n, (\mathrm{g}_{\mathrm{acc},n}(\nu_0))_n]$ associated to it by Corollary 5.39. Part (iii) of Property ($\mathrm{B7}_{\mathrm{acc}}$) together with (5.34) and the uniqueness of the associated sequences immediately implies

$$a_1 = \mathrm{k}_{\mathrm{acc},1}(\nu_0) \quad \text{and} \quad g_1 = \mathrm{g}_{\mathrm{acc},1}(\nu_0)\,.$$

The first assertion now follows inductively by setting $\nu_j := \gamma'_{\nu_{j-1}}(\mathrm{t}_{\mathrm{acc},1}(\nu_{j-1}))$ for $j = 1, \ldots, m$ and repeating this argument.

The second assertion is immediate from Corollary 5.39. $\square$

In the next section we will establish the structure tuple for the strict transfer operator approach. For that goal, several sets of transformations are needed, all of which being required to be finite. The following result, which is a straightforward consequence of ($\mathrm{B7}_{\mathrm{acc}}$iii) and (5.39), plays a crucial part in assuring finiteness for many of these sets.

Corollary 5.41. *Suppose that the set of branches $\mathcal{C}$ is finitely ramified. Then for all $a, b \in \widehat{A}$ with $a = (j, \mathrm{R})$ for some $j \in A^*$ we have*

$$\#\mathcal{G}_{\mathrm{acc}}(a, b) < +\infty\,.$$

Chapter 6

Existence of Strict Transfer Operator Approaches

Let Γ be a geometrically finite Fuchsian group with hyperbolic elements that admits the construction of a set of branches. In this section we state and prove that a set of branches for the geodesic flow on the orbit space of Γ gives rise to a strict transfer operator approach (Section 3.1). See Theorem 6.1. In order to do so, we discuss how any given set of branches defines a structure tuple. This is the objective of Section 6.1. The proof of Theorem 6.1 is then split into the Sections 6.2–6.6.

6.1 Structure Tuple and First Main Result

By Proposition 4.35, every set of branches for Γ (in the sense of Definition 4.1) can be turned into one that is admissible. By Proposition 4.28, it can further be extended to one that is finitely ramified, preserving the property of admissibility. Moreover, Proposition 5.19 guarantees that it can then be reduced to a reduced set of branches (in the sense of Definition 5.17) that is non-collapsing. Proposition 5.21 shows that the finite ramification property remains preserved. Therefore, we may and shall suppose that $\mathcal{C} = \{\mathrm{C}_j \mid j \in A\}$ is a reduced, admissible, non-collapsing and finitely ramified set of branches for Γ.

We resume the notations from Section 5.4 and recall, in particular, the index set $\widehat{A}$ as well as the sets Cyc^*_a, I_a, and J_a, and the transformations $g^*(a)$, u_a, and $g^*(a,b)$ for $a, b \in \widehat{A}$. For $a = (j, \mathrm{Z}) \in \widehat{A}$ we denote by

$$\pi_A\colon \begin{cases} \widehat{A} \longrightarrow A \\ a \longmapsto j \end{cases} \qquad \text{and by} \qquad \pi_{\mathrm{Z}}\colon \begin{cases} \widehat{A} \longrightarrow \{\mathrm{X}, \mathrm{R}, \mathrm{Y}\} \\ a \longmapsto \mathrm{Z} \end{cases}$$

the projection onto the first and second component, respectively. Hence, for every $a \in \widehat{A}$ we have

$$a = \big(\pi_A(a), \pi_{\mathrm{Z}}(a)\big)\,.$$

For each $a, b \in \widehat{A}$, we define

$$\widetilde{a}_b := \big(\pi_A(a), \pi_{\mathrm{Z}}(b)\big)\,.$$

Using that we set

$$P_{a,b} := \begin{cases} \{u_b^{-1}\} & \text{if } \pi_{\mathrm{Z}}(a) \neq \pi_{\mathrm{Z}}(b) \text{ and } \widetilde{a}_b \in \mathrm{Cyc}_b^*\,,\\ \varnothing & \text{otherwise,} \end{cases}$$

where $\mathrm{Cyc}^*_{(\cdot,\mathrm{R})} := \varnothing$. For $p \in P_{a,b}$ we further define

$$g_p := g^*(b, \widetilde{a})^{-1}\,. \tag{6.1}$$

Furthermore, we define

$$C_{a,b} := \begin{cases} \mathcal{G}_{\mathrm{acc}}(b,a)^{-1} & \text{if } \pi_{\mathrm{Z}}(b) = \mathrm{R}\,,\\ \{g^*(b,\widetilde{a})^{-1}\} & \text{if } \pi_{\mathrm{Z}}(b) \notin \{\pi_{\mathrm{Z}}(a), \mathrm{R}\} \text{ and } \widetilde{a}_b \in \mathrm{Cyc}_b^*\,,\\ \varnothing & \text{otherwise,} \end{cases}$$

where for $G \subseteq \Gamma$ we denote $G^{-1} := \{g^{-1} \mid g \in G\}$. We emphasize that the requirement "$(\pi_A(a), \pi_{\mathrm{Z}}(b)) \in \mathrm{Cyc}_b^*$" in the conditions of these definitions is indeed correct and should not read "$(\pi_A(a), \pi_{\mathrm{Z}}(a)) \in \mathrm{Cyc}_b^*$." Further we note that for $a, b \in \widehat{A}$ such that $\pi_A(a) = \pi_A(b)$ and $P_{a,b} \neq \varnothing$ we could define g_p in (6.1) (for the unique $p \in P_{a,b}$) to be the identity element in Γ and then also define $C_{a,b}$ to be $\{\mathrm{id}\}$. However, the chosen definition has a slight advantage in the proofs of what follows. Finally, since $\mathcal{C}$ is admissible, $\widehat{\mathbb{R}} \setminus \bigcup_{j\in A} I_j$ has inner points. Let ξ be such an inner point and let $q_\xi \in \mathrm{PSL}_2(\mathbb{R})$ be such that $q_\xi . \xi = \infty$. Hence, for instance, a possible choice is

$$q_\xi := \begin{bmatrix} \xi & -1-\xi^2 \\ 1 & -\xi \end{bmatrix}\,.$$

Then ∞ is an inner point of $q_\xi.(\widehat{\mathbb{R}} \setminus \bigcup_{j\in A} I_j)$, meaning that each of the sets $q_\xi . I_j$ is an interval in $\mathbb{R}$. Since

$$I_a \subseteq \mathcal{I}_{\pi_A(a)} \subseteq I_{\pi_A(a)}$$

for every $a \in \widehat{A}$, the convex hull of $q_\xi.\overline{I_{a,\mathrm{st}}}$ in $\mathbb{R}$, which we may denote by $\mathrm{conv}(q_\xi.\overline{I_{a,\mathrm{st}}})$, is an interval in $\mathbb{R}$ as well. Note that

$$\mathrm{conv}(q_\xi.\overline{I_{a,\mathrm{st}}}) = \overline{\mathrm{conv}(q_\xi.I_{a,\mathrm{st}})}\,. \tag{6.2}$$

We define for every $a \in \widehat{A}$,

$$\widehat{I}_a := q_\xi^{-1}.\mathrm{conv}(q_\xi.\overline{I_{a,\mathrm{st}}})\,. \tag{6.3}$$

Because of (6.2) and the continuity of q_ξ, this definition is independent of the choice of ξ and q_ξ.

With these preparations we are now ready to formulate our first main result, which assures existence of strict transfer operator approaches given a set of branches.

Theorem 6.1. *Let Γ be a geometrically finite Fuchsian group that contains hyperbolic elements and admits the construction of a set of branches for the geodesic flow on its orbit space $\mathbb{X}$. Then Γ admits a strict transfer operator approach with structure tuple given by*

$$\mathcal{S} := \left(\widehat{A}, \{\widehat{I}_a\}_{a\in\widehat{A}}, \{P_{a,b}\}_{a,b\in\widehat{A}}, \{C_{a,b}\}_{a,b\in\widehat{A}}, \{\{g_p\}_{p\in P_{a,b}}\}_{a,b\in\widehat{A}}\right).$$

Before we discuss a proof of Theorem 6.1, we finish our series of examples concerning the family $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ by providing structure tuples and the associated transfer operator families (see also Section 3.3).

Example 6.2. Recall the family of Fuchsian groups $\{\Gamma_{\sigma,\lambda}\}_{\sigma\in\mathbb{N}\setminus\{1\},\lambda>2}$ and recall further its reduced set of branches $\{\mathrm{C}_{\mathrm{P},2}, \mathrm{C}_{\mathrm{P},7}\}$ from Example 5.8. As discussed before, this set is non-collapsing. In Example 5.25 we identified the X- and Y-cycle

$$7 \xrightarrow{\mathrm{t}_\lambda^{-1}} 7 \qquad \text{and} \qquad 2 \xrightarrow{\mathrm{t}_\lambda} 2\,,$$

from which we obtain the index set

$$\widehat{A} = \{(2,\mathrm{R}), (2,\mathrm{Y}), (7,\mathrm{X}), (7,\mathrm{R})\}\,,$$

with

$$\mathrm{Cyc}^*_{(2,\mathrm{Y})} = \{(2,\mathrm{Y})\}\,, \qquad \mathrm{Cyc}^*_{(7,\mathrm{X})} = \{(7,\mathrm{X})\}\,,$$

and

$$\mathrm{Cyc}^*_{(2,\mathrm{R})} = \mathrm{Cyc}^*_{(7,\mathrm{R})} = \varnothing\,.$$

From Figure 15 we read off the transition sets

$$\begin{aligned}
\mathcal{G}_{\mathrm{acc}}((2,\mathrm{R}),(2,\mathrm{R})) &= \mathcal{G}_{\mathrm{acc}}((2,\mathrm{R}),(2,\mathrm{Y})) = \mathcal{G}_{\mathrm{acc}}((7,\mathrm{R}),(2,\mathrm{R})) \\
&= \mathcal{G}_{\mathrm{acc}}((7,\mathrm{R}),(2,\mathrm{Y})) = \{g_\sigma^n \mathrm{t}_\lambda \mid n = 1,\ldots,\sigma-1\}\,, \\
\mathcal{G}_{\mathrm{acc}}((2,\mathrm{R}),(7,\mathrm{X})) &= \mathcal{G}_{\mathrm{acc}}((2,\mathrm{R}),(7,\mathrm{R})) = \mathcal{G}_{\mathrm{acc}}((7,\mathrm{R}),(7,\mathrm{X})) \\
&= \mathcal{G}_{\mathrm{acc}}((7,\mathrm{R}),(7,\mathrm{R})) = \left\{g_\sigma^n \mathrm{t}_\lambda^{-1} \;\middle|\; n = 1,\ldots,\sigma-1\right\}\,,
\end{aligned}$$

as well as

$$\mathcal{G}_{\mathrm{acc}}((2,\mathrm{Y}),(2,\mathrm{R})) = \{t_\lambda^n \mid n\in\mathbb{N}\}\,, \quad \mathcal{G}_{\mathrm{acc}}((7,\mathrm{X}),(7,\mathrm{R})) = \left\{t_\lambda^{-n} \;\middle|\; n\in\mathbb{N}\right\}\,,$$

and $\mathcal{G}_{\mathrm{acc}}(a,b) = \varnothing$ for all other choices of $(a,b)\in\widehat{A}\times\widehat{A}$. From that we deduce the structure tuple for $\Gamma_{\sigma,\lambda}$, consisting of the index set $\widehat{A}$ and the following quantities:

- the family of intervals $\{\widehat{I}_a\}_{a\in\widehat{A}}$ consisting of

$$\widehat{I}_{(2,\mathrm{R})} = [-1, \lambda - 1]\,, \qquad \widehat{I}_{(2,\mathrm{Y})} = [\lambda - 1, +\infty]\,,$$
$$\widehat{I}_{(7,\mathrm{X})} = [-\infty, 1 - \lambda]\,, \qquad \widehat{I}_{(7,\mathrm{R})} = [1 - \lambda, 1]\,,$$

- the families of transformation sets consisting of

$$P_{(2,\mathrm{R}),(2,\mathrm{Y})} = \{\mathrm{t}_\lambda^{-1}\}\,, \quad P_{(7,\mathrm{R}),(7,\mathrm{X})} = \{\mathrm{t}_\lambda\}\,,$$

  and $P_{a,b} = \varnothing$ for every other choice of $(a,b) \in \widehat{A} \times \widehat{A}$, and

$$\begin{aligned} C_{(2,\mathrm{R}),(2,\mathrm{R})} &= C_{(2,\mathrm{R}),(7,\mathrm{R})} = C_{(2,\mathrm{Y}),(2,\mathrm{R})} = C_{(2,\mathrm{Y}),(7,\mathrm{R})} \\ &= \left\{\mathrm{t}_\lambda^{-1} g_\sigma^n \;\middle|\; n = 1, \ldots, \sigma - 1\right\}\,, \\ C_{(7,\mathrm{X}),(2,\mathrm{R})} &= C_{(7,\mathrm{X}),(7,\mathrm{R})} = C_{(7,\mathrm{R}),(2,\mathrm{R})} = C_{(7,\mathrm{R}),(7,\mathrm{R})} \\ &= \{\mathrm{t}_\lambda g_\sigma^n \mid n = 1, \ldots, \sigma - 1\}\,, \end{aligned}$$

  as well as

$$C_{(2,\mathrm{R}),(2,\mathrm{Y})} = \{\mathrm{t}_\lambda^{-1}\} \quad \text{and} \quad C_{(7,\mathrm{R}),(7,\mathrm{X})} = \{\mathrm{t}_\lambda\}\,,$$

  and $C_{a,b} = \varnothing$ for all other choices of $(a,b) \in \widehat{A} \times \widehat{A}$,

- and the transformations

$$\begin{aligned} g_p &= \mathrm{t}_\lambda^{-1} \quad \text{for } p \in P_{(2,\mathrm{R}),(2,\mathrm{Y})}\,, \\ g_p &= \mathrm{t}_\lambda \quad \text{for } p \in P_{(7,\mathrm{R}),(7,\mathrm{X})}\,. \end{aligned}$$

The associated (formal) fast transfer operator with parameter $s \in \mathbb{C}$, $\operatorname{Re} s \gg 1$, admits the matrix representation

$$\widetilde{\mathcal{L}}_s = \begin{pmatrix} \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda^{-1} g_\sigma^n) & \sum_{k=1}^{\infty} \alpha_s(\mathrm{t}_\lambda^{-k}) & 0 & \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda^{-1} g_\sigma^n) \\ \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda^{-1} g_\sigma^n) & 0 & 0 & \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda^{-1} g_\sigma^n) \\ \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda g_\sigma^n) & 0 & 0 & \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda g_\sigma^n) \\ \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda g_\sigma^n) & 0 & \sum_{k=1}^{\infty} \alpha_s(\mathrm{t}_\lambda^{k}) & \sum_{n=1}^{\sigma-1} \alpha_s(\mathrm{t}_\lambda g_\sigma^n) \end{pmatrix}\,,$$

where α_s is as in (3.4).

Now recall the reduced set of branches $\widetilde{\mathcal{C}}_{\mathrm{P}}$ from Example 5.22. The structure tuple for this group and induced cross section is composed of the following quantities:

- the index set

$$\widehat{A} = \{(2,\mathrm{R}),(3,\mathrm{X}),(3,\mathrm{R}),(5,\mathrm{R}),(5,\mathrm{Y}),(7,\mathrm{X}),(7,\mathrm{R}),(8,\mathrm{X}),(8,\mathrm{R})\}\,,$$

- the family of intervals $\{\widehat{I}_a\}_{a\in\widehat{A}}$ consisting of

$$\begin{aligned}
&\widehat{I}_{(2,\mathrm{R})} = [-1, c(g_\sigma)]\,, \\
&\widehat{I}_{(3,\mathrm{X})} = [c(g_\sigma), g_\sigma^{\sigma-2}.1]\,, && \widehat{I}_{(3,\mathrm{R})} = [g_\sigma^{\sigma-2}.1, 1]\,, \\
&\widehat{I}_{(5,\mathrm{R})} = [1, \lambda+1]\,, && \widehat{I}_{(5,\mathrm{Y})} = [\lambda+1, +\infty]\,, \\
&\widehat{I}_{(7,\mathrm{X})} = [-\infty, c(g_\sigma)]\,, && \widehat{I}_{(7,\mathrm{R})} = [c(g_\sigma), c(g_\sigma^{-1})]\,, \\
&\widehat{I}_{(8,\mathrm{X})} = [c(g_\sigma^{-1}), g_\sigma \mathrm{t}_\lambda^{-1}.1]\,, && \widehat{I}_{(8,\mathrm{R})} = [g_\sigma \mathrm{t}_\lambda^{-1}.1, 1]\,,
\end{aligned}$$

- the families of transformation sets and associated transformations g_p, for $p \in P_{a,b}$, consisting of

$$\begin{aligned}
&P_{(5,\mathrm{R}),(5,\mathrm{Y})} = \{\mathrm{t}_\lambda^{-1}\}\,, && g_p = \mathrm{t}_\lambda^{-1}\,, \\
&P_{(3,\mathrm{R}),(3,\mathrm{X})} = \{g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma\}\,, && g_p = g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma\,, \\
&P_{(3,\mathrm{R}),(7,\mathrm{X})} = \{\mathrm{t}_\lambda\}\,, && g_p = g_\sigma^{-1}\,, \\
&P_{(3,\mathrm{R}),(8,\mathrm{X})} = \{g_\sigma\mathrm{t}_\lambda g_\sigma^{-1}\}\,, && g_p = g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma^{-1}\,, \\
&P_{(7,\mathrm{R}),(3,\mathrm{X})} = \{g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma\}\,, && g_p = \mathrm{t}_\lambda g_\sigma\,, \\
&P_{(7,\mathrm{R}),(7,\mathrm{X})} = \{\mathrm{t}_\lambda\}\,, && g_p = \mathrm{t}_\lambda\,, \\
&P_{(7,\mathrm{R}),(8,\mathrm{X})} = \{g_\sigma\mathrm{t}_\lambda g_\sigma^{-1}\}\,, && g_p = \mathrm{t}_\lambda g_\sigma^{-1}\,, \\
&P_{(8,\mathrm{R}),(3,\mathrm{X})} = \{g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma\}\,, && g_p = g_\sigma^{2}\,, \\
&P_{(8,\mathrm{R}),(7,\mathrm{X})} = \{\mathrm{t}_\lambda\}\,, && g_p = g_\sigma\,, \\
&P_{(8,\mathrm{R}),(8,\mathrm{X})} = \{g_\sigma\mathrm{t}_\lambda g_\sigma^{-1}\}\,, && g_p = g_\sigma\mathrm{t}_\lambda g_\sigma^{-1}\,,
\end{aligned}$$

$$
\begin{aligned}
C_{(2,\mathrm{R}),(5,\mathrm{R})} &= C_{(3,\mathrm{X}),(5,\mathrm{R})} = C_{(3,\mathrm{R}),(5,\mathrm{R})} = C_{(5,\mathrm{R}),(5,\mathrm{Y})} \\
&= C_{(8,\mathrm{X}),(5,\mathrm{R})} = C_{(8,\mathrm{R}),(5,\mathrm{R})} = \{\mathrm{t}_\lambda^{-1}\}\,, \\
C_{(5,\mathrm{R}),(3,\mathrm{R})} &= C_{(5,\mathrm{R}),(7,\mathrm{R})} = C_{(5,\mathrm{Y}),(3,\mathrm{R})} = C_{(5,\mathrm{Y}),(7,\mathrm{R})} \\
&= C_{(8,\mathrm{X}),(7,\mathrm{R})} = C_{(8,\mathrm{R}),(7,\mathrm{R})} = \left\{g_\sigma^{-m} \;\middle|\; m = 1, \ldots, \sigma - 2\right\}\,, \\
C_{(5,\mathrm{R}),(2,\mathrm{R})} &= C_{(5,\mathrm{Y}),(2,\mathrm{R})} = C_{(8,\mathrm{R}),(7,\mathrm{X})} = \{g_\sigma\}\,, \\
C_{(8,\mathrm{X}),(3,\mathrm{R})} &= C_{(8,\mathrm{R}),(3,\mathrm{R})} = \left\{g_\sigma^{-m} \;\middle|\; m = 1, \ldots, \sigma - 3\right\}\,, \\
C_{(7,\mathrm{R}),(8,\mathrm{X})} &= C_{(8,\mathrm{X}),(8,\mathrm{R})} = C_{(8,\mathrm{R}),(8,\mathrm{R})} = \{\mathrm{t}_\lambda g_\sigma^{-1}\}\,, \\
C_{(3,\mathrm{R}),(3,\mathrm{X})} &= \{g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma\}\,, \qquad C_{(3,\mathrm{R}),(7,\mathrm{X})} = \{g_\sigma^{-1}\}\,, \\
C_{(3,\mathrm{R}),(8,\mathrm{X})} &= \{g_\sigma^{-1}\mathrm{t}_\lambda g_\sigma^{-1}\}\,, \quad C_{(7,\mathrm{R}),(3,\mathrm{X})} = \{\mathrm{t}_\lambda g_\sigma\}\,, \\
C_{(7,\mathrm{R}),(7,\mathrm{X})} &= \{\mathrm{t}_\lambda\}\,, \qquad\qquad C_{(8,\mathrm{R}),(3,\mathrm{X})} = \{g_\sigma^2\}\,, \\
C_{(8,\mathrm{R}),(8,\mathrm{X})} &= \{g_\sigma \mathrm{t}_\lambda g_\sigma^{-1}\}\,,
\end{aligned}
$$

and $P_{a,b} = C_{a,b} = \varnothing$ for any other choice of $(a,b) \in \widehat{A} \times \widehat{A}$.

The associated fast transfer operator with parameter $s \in \mathbb{C}$ takes the form

$$\widetilde{\mathcal{L}}_s = \begin{pmatrix}
0 & 0 & 0 & \alpha_s(\mathrm{t}_\lambda^{-1}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_s(\mathrm{t}_\lambda^{-1}) & 0 & 0 & 0 & 0 & 0 \\
0 & \sum\limits_{n=1}^{\infty}\alpha_s(g_\sigma^{-1}\mathrm{t}_\lambda^n g_\sigma) & 0 & \alpha_s(\mathrm{t}_\lambda^{-1}) & 0 & \sum\limits_{n=0}^{\infty}\alpha_s(g_\sigma^{-1}\mathrm{t}_\lambda^n) & 0 & \sum\limits_{n=1}^{\infty}\alpha_s(g_\sigma^{-1}\mathrm{t}_\lambda^n g_\sigma^{-1}) & 0 \\
\alpha_s(g_\sigma) & 0 & \sum\limits_{m=1}^{\sigma-2}\alpha_s(g_\sigma^{-m}) & 0 & \sum\limits_{n=1}^{\infty}\alpha_s(\mathrm{t}_\lambda^{-n}) & 0 & \sum\limits_{m=1}^{\sigma-2}\alpha_s(g_\sigma^{-m}) & 0 & 0 \\
\alpha_s(g_\sigma) & 0 & \sum\limits_{m=1}^{\sigma-2}\alpha_s(g_\sigma^{-m}) & 0 & 0 & 0 & \sum\limits_{m=1}^{\sigma-2}\alpha_s(g_\sigma^{-m}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sum\limits_{n=1}^{\infty}\alpha_s(\mathrm{t}_\lambda^n g_\sigma) & 0 & 0 & 0 & \sum\limits_{n=1}^{\infty}\alpha_s(\mathrm{t}_\lambda^n) & 0 & \sum\limits_{n=1}^{\infty}\alpha_s(\mathrm{t}_\lambda^n g_\sigma^{-1}) & 0 \\
0 & 0 & \sum\limits_{m=1}^{\sigma-3}\alpha_s(g_\sigma^{-m}) & \alpha_s(\mathrm{t}_\lambda^{-1}) & 0 & 0 & \sum\limits_{m=1}^{\sigma-2}\alpha_s(g_\sigma^{-m}) & 0 & \alpha_s(\mathrm{t}_\lambda g_\sigma^{-1}) \\
0 & \sum\limits_{n=0}^{\infty}\alpha_s(g_\sigma\mathrm{t}_\lambda^n g_\sigma) & \sum\limits_{m=1}^{\sigma-3}\alpha_s(g_\sigma^{-m}) & \alpha_s(\mathrm{t}_\lambda^{-1}) & 0 & \sum\limits_{n=0}^{\infty}\alpha_s(g_\sigma\mathrm{t}_\lambda^n) & \sum\limits_{m=1}^{\sigma-2}\alpha_s(g_\sigma^{-m}) & \sum\limits_{n=1}^{\infty}\alpha_s(g_\sigma\mathrm{t}_\lambda^n g_\sigma^{-1}) & \alpha_s(\mathrm{t}_\lambda g_\sigma^{-1})
\end{pmatrix}.$$

Remark 6.3. Let $a \in \widehat{A}$. Then

$$\widehat{I}_{a,\mathrm{st}} = \widehat{I}_a \cap \widehat{\mathbb{R}}_{\mathrm{st}} = q_\xi^{-1}.\big(\mathrm{conv}(q_\xi.I_{a,\mathrm{st}}) \cap \widehat{\mathbb{R}}_{\mathrm{st}}\big) = q_\xi^{-1}.\big(q_\xi.I_{a,\mathrm{st}}\big) = I_{a,\mathrm{st}}\,.$$

Therefore, we may drop the $\widehat{\ }$ whenever "st"-sets are concerned. The intervals $\widehat{I}_a$ were introduced solely to fit the needs of a strict transfer operator approach *verbatim.*

As mentioned above, the bulk of the proof of Theorem 6.1 is split into the following five sections. In the remainder of this section we verify the initial requirements for a strict transfer operator approach (the list of demands before Property 1 in Section 3.1). Indeed, the finiteness of the set $\widehat{A}$ is obvious from its construction. Let $a, b \in \widehat{A}$. From Definition 5.35 and Remark 5.38 we see immediately that each of the sets $P_{a,b}$ and $C_{a,b}$ consists completely of elements of Γ. Consequentially, $g_p \in \Gamma$ for every $p \in P_{a,b}$. The finiteness of $P_{a,b}$ is obvious from its definition. The same is true for $C_{a,b}$ whenever $\pi_{\mathrm{Z}}(b) \neq \mathrm{R}$. In the case $\pi_{\mathrm{Z}}(b) = \mathrm{R}$, since $\mathcal{C}$ is finitely ramified, we obtain from Corollary 5.41 that

$$\#C_{a,b} = \#\mathcal{G}_{\mathrm{acc}}(b,a) < +\infty\,.$$

Finally, since $P_{a,b} = \{u_b^{-1}\}$ whenever it is non-empty, $u_b = u_{\pi_A(b),\pi_{\mathrm{Z}}(b)}$ by (5.35), and $u_{j,\mathrm{Z}}$ is parabolic for every choice of $(j,\mathrm{Z}) \in A_{\mathrm{Z}} \times \{\mathrm{X},\mathrm{Y}\}$ by the discussion right after (5.23), every set $P_{a,b}$ consists solely of parabolic elements.

6.2 Property 1

By a straightforward inspection we observe that

$$C_{a,b}^{-1} \cup \big\{p^{-n}g_p^{-1} \;\big|\; p \in P_{a,b},\ n \in \mathbb{N}\big\} = \mathcal{G}_{\mathrm{acc}}(b,a)$$

for all $a, b \in \widehat{A}$. This union is disjoint. In the second set of the union on the left hand side no element gets constructed twice. Therefore, the first part of Property 1(I) and all of (II) and of (III) follow immediately from (B7$_{\mathrm{acc}}$i). For the second part of (I) we let $a, b \in \widehat{A}$ be such that $P_{a,b} \neq \varnothing$. Thus, $P_{a,b} = \{u_b^{-1}\}$. Since the element u_b^{-1} is parabolic (as we showed in Section 5.3), by Lemma 1.1(ii) we find a transformation $q \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$u_b^{-1} = q \cdot \mathrm{t}_\kappa \cdot q^{-1}\,,$$

for some $\kappa \in \mathbb{R} \setminus \{0\}$ and t_κ as in (1.7) . Thus, $u_b^{-n} = u_b^{-1}$ for some $n \in \mathbb{N}$ would imply $\mathrm{t}_\kappa^{n-1} = \mathrm{id}$, and hence $n = 1$, since $\mathrm{t}_\kappa^m = \mathrm{t}_{m\kappa}$ for all $m \in \mathbb{Z}$ and $\kappa \neq 0$. In turn, $u_b^{-n} \neq u_b^{-1}$ for $n \geq 2$, which establishes the second part of (I).

6.3 Property 2

For each $n \in \mathbb{N}$, the set Per_n with respect to $\mathcal{S}$ consists of all $g \in \Gamma$ for which there exists an $a \in \widehat{A}$ such that

$$\begin{cases} g^{-1}.I_{a,\mathrm{st}} \times \{a\} & \longrightarrow & I_{a,\mathrm{st}} \times \{a\} \\ (x,a) & \longmapsto & (g.x,a) \end{cases}$$

is a submap of F^n. These submaps correspond to loops in the "return-style" graph associated to the accelerated system $\mathcal{C}_{\mathrm{acc}}$, wherefore $\mathrm{id} \notin \mathrm{Per}_n$. Property 2 asserts the union

$$\mathrm{Per} := \bigcup_{n=1}^{\infty} \mathrm{Per}_n$$

to be disjoint. In order to prove this assertion we decompose, for each $n \in \mathbb{N}$, the set Per_n into the sets

$$\mathrm{Per}_{a,n} := \left\{ g \in \Gamma \;\middle|\; \begin{cases} g^{-1}.I_{a,\mathrm{st}} \times \{a\} & \to & I_{a,\mathrm{st}} \times \{a\} \\ (x,a) & \mapsto & (g.x,a) \end{cases} \text{is a submap of } F^n \right\}.$$

for $a \in \widehat{A}$. We emphasize that the union $\mathrm{Per}_n = \bigcup_{a \in \widehat{A}} \mathrm{Per}_{a,n}$ is not necessarily disjoint. Further, for any $n, m \in \mathbb{N}$, we have $\mathrm{Per}_n \cap \mathrm{Per}_m \neq \varnothing$ if and only if there are $a, b \in \widehat{A}$ such that $\mathrm{Per}_{a,n} \cap \mathrm{Per}_{b,m} \neq \varnothing$. Therefore, the assertion of Property 2 is equivalent to $\mathrm{Per}_{a,n} \cap \mathrm{Per}_{b,m} \neq \varnothing$ implying $n = m$ for all (not necessarily distinct) $a, b \in \widehat{A}$.

The following result equips us with all the necessary information regarding the elements of the sets $\mathrm{Per}_{a,n}$.

Proposition 6.4. *Let $a \in \widehat{A}$, $n \in \mathbb{N}$, and $g \in \mathrm{Per}_{a,n}$. Then g is hyperbolic. Its repelling fixed point $\mathrm{f}_-(g)$ is an inner point of $\widehat{I}_a$. Its attracting fixed point $\mathrm{f}_+(g)$ is an element of $J_{\pi_A(a)}$.*

Proof. The definitions of the sets $P_{a,b}$, $C_{a,b}$, and $\mathrm{Per}_{a,n}$ for any $a, b \in \widehat{A}$ and $n \in \mathbb{N}$, together with Corollary 5.40 imply the existence of $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ such that

$$g^{-1} = \mathrm{g}_{\mathrm{acc},1}(\nu) \cdot \mathrm{g}_{\mathrm{acc},2}(\nu) \cdots \mathrm{g}_{\mathrm{acc},n}(\nu)\,. \tag{6.4}$$

Therefore,

$$g^{-1}.I_a \subseteq g^{-1}.I_{\pi_A(a)} \subseteq I_a \subseteq I_{\pi_A(a)}\,. \tag{6.5}$$

Hence, for all $k \in \mathbb{N}$, we have

$$g^{-k-1}.\overline{\mathrm{bp}(\mathrm{C}_{\pi_A(a)})} \subseteq g^{-k}.\mathrm{H}_+(\pi_A(a))\,.$$

Since Γ-translates of branches do not accumulate in $\mathbb{H}$ (see Proposition 4.6), the "limits" of the set sequences $(g^{-k}.\overline{I_a})_{k \in \mathbb{N}}$ and $(g^{-k}.\overline{I_{\pi_A(a)}})_{k \in \mathbb{N}}$ are equal and a singleton in $\widehat{\mathbb{R}}$, that is, the intersection

$$\bigcap_{k\in\mathbb{N}} g^{-k}.\overline{I_a} = \bigcap_{k\in\mathbb{N}} g^{-k}.\overline{I_{\pi_A(a)}} \tag{6.6}$$

is a singleton in $\widehat{\mathbb{R}}$, consisting of a fixed point of g^{-1}. Therefore g is either hyperbolic or parabolic.

We will now show that g is not parabolic, by means of a proof by contradiction. To that end we assume that g is parabolic. Then the singleton from (6.6) consists of the unique fixed point $\mathrm{f}(g)$ of g. We recall that in *small* neighborhoods of $\mathrm{f}(g)$ in $\widehat{\mathbb{R}}$, the action of g, as being a parabolic element, is attracting to $\mathrm{f}(g)$ on one of the sides of $\mathrm{f}(g)$ and repelling on the other. Thus, for any interval I in $\widehat{\mathbb{R}}$ with $\mathrm{f}(g)$ in the interior of I and I not being all of $\widehat{\mathbb{R}}$, we have $g^{-1}.I \nsubseteq I$. Therefore (6.5) implies that $\mathrm{f}(g)$ is a boundary point of I_a and also of $I_{\pi_A(a)}$. This implies that $\pi_{\mathrm{Z}}(a) \in \{\mathrm{X}, \mathrm{Y}\}$ and hence, by a slight abuse of notation,

$$\mathrm{f}(g) = \pi_{\mathrm{Z}}(a)_{\pi_A(a)}$$

(i. e., for $a = (j, \mathrm{Z})$ we have $\mathrm{f}(g) = \mathrm{Z}_j$). Further we see that for the vector ν from (6.4) we find exactly one pair $(b, r) \in \mathrm{Cyc}_a^* \times \mathbb{N}_0$ where b is of the form $(k, \pi_{\mathrm{Z}}(a)) \in \widehat{A}$ and such that

$$\gamma_\nu(+\infty) \in u_a^r g^*(a,b).\big(I_{(\pi_A(b),\mathrm{R}),\mathrm{st}} \cup I_{(\pi_A(b),\pi_{\mathrm{Z}}(a)'),\mathrm{st}}\big)$$

(see (5.37)), where $\pi_{\mathrm{Z}}(a)'$ is such that $\{\pi_{\mathrm{Z}}(a), \pi_{\mathrm{Z}}(a)'\} = \{\mathrm{X}, \mathrm{Y}\}$. Hence,

$$\mathrm{g}_{\mathrm{acc},1} = u_a^r g^*(a,b)$$

and

$$g^{-1}.I_a \subseteq \mathrm{g}_{\mathrm{acc},1}^{-1} g^{-1}.I_a \subseteq I_{(\pi_A(b),\mathrm{R})} \cup I_{(\pi_A(b),\pi_{\mathrm{Z}}(a)')} \,. \tag{6.7}$$

Since $b \in \mathrm{Cyc}_a^*$, the set $I_b = I_{(\pi_A(b),\pi_{\mathrm{Z}}(a))}$ does not vanish and, moreover, $\mathrm{f}(g)$ is a boundary point of I_b. Comparing with (6.7), we see that $\mathrm{f}(g)$ cannot be contained in $g^{-1}.\overline{I_a}$, which is a contradiction to (6.6) being the singleton $\{\mathrm{f}(g)\}$. In turn, the element g is not parabolic.

We obtain that the element g is hyperbolic. The singleton (6.6) consists of the fixed point $\mathrm{f}_+(g^{-1}) = \mathrm{f}_-(g)$ (attracting for g^{-1}, repelling for g), as follows immediately from its definition (6.6). Further, we clearly have

$$\mathrm{f}_-(g) \in \overline{I_a} \subseteq \overline{I_{\pi_A(a)}} \,.$$

Since, by construction, the boundary points of I_a in $\widehat{\mathbb{R}}$ are elements of $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$, it follows that $\mathrm{f}_-(g)$ is an inner point of I_a. Finally, we can apply the same line of reasoning with the roles of g and g^{-1} interchanged to show that $\mathrm{f}_+(g) = \mathrm{f}_-(g^{-1})$ is an element of $J_{\pi_A(a)}$. ◻

In what follows we extend on the initial argument in the proof of Proposition 6.4 to find an equivalent description of the set $\mathrm{Per}_{a,n}$ in terms of iterated intersections of induced geodesics with $\Gamma.\mathrm{C}_{\mathrm{acc}}$. Together with an inspection of the relationship between $\mathrm{C}_a^{\mathrm{acc}}$ and $\mathrm{C}_b^{\mathrm{acc}}$ in view of Proposition 6.4, this will enable us to effectively compare the quantities m and n.

For $v, w \in \widehat{A}$ and $h \in \Gamma$ we define $\mathrm{C}_v^{\mathrm{acc}}|_{h.\,\mathrm{C}_w^{\mathrm{acc}}}$ to be the subset of $\mathrm{C}_v^{\mathrm{acc}}$ of all vectors ν for which γ_ν eventually intersects $h.\,\mathrm{C}_w^{\mathrm{acc}}$. To be more precise, we set

$$\mathrm{C}_v^{\mathrm{acc}}|_{h.\,\mathrm{C}_w^{\mathrm{acc}}} := \left\{\nu \in \mathrm{C}_v^{\mathrm{acc}} \;\middle|\; \exists\, t^* > 0 : \gamma_\nu'(t^*) \in h.\,\mathrm{C}_v^{\mathrm{acc}}\right\} \tag{6.8}$$

(see also (5.40)). Since, for every $\nu \in \mathrm{C}_v^{\mathrm{acc}}|_{h.\,\mathrm{C}_w^{\mathrm{acc}}}$, the intersection time t^* is uniquely determined by the quantities v, w, h, and ν, the value

$$\varphi(v, w, h, \nu) := \#\left\{t \in (0, t^*] \;\middle|\; \gamma_\nu'(t) \in \Gamma.\mathrm{C}_{\mathrm{acc}}\right\} \tag{6.9}$$

is well-defined.

The following result is an immediate consequence of Corollary 5.40.

Lemma 6.5. *Let $v, w \in \widehat{A}$ and $h \in \Gamma$ be such that $\mathrm{C}_v^{\mathrm{acc}}|_{h.\,\mathrm{C}_w^{\mathrm{acc}}} \neq \varnothing$. Then for every choice of $\nu, \eta \in \mathrm{C}_v^{\mathrm{acc}}|_{h.\,\mathrm{C}_w^{\mathrm{acc}}}$ we have*

$$\varphi(v, w, h, \nu) = \varphi(v, w, h, \eta)\,.$$

Because of Lemma 6.5, for every two pairs $(v, p), (w, q) \in \widehat{A} \times \Gamma$ we can define the *intersection count*

$$\varphi((v, p), (w, q)) := \begin{cases} \varphi(v, w, p^{-1}q, \nu) & \text{if } \mathrm{C}_v^{\mathrm{acc}}|_{p^{-1}q.\,\mathrm{C}_w^{\mathrm{acc}}} \neq \varnothing\,, \\ 0 & \text{otherwise,} \end{cases} \tag{6.10}$$

with an arbitrary choice of $\nu \in \mathrm{C}_v^{\mathrm{acc}}|_{p^{-1}q.\,\mathrm{C}_w^{\mathrm{acc}}}$ in the former case.

Lemma 6.6. *The intersection count has the following properties:*

(i) *For all $(v, p), (w, q) \in \widehat{A} \times \Gamma$ the intersection count $\varphi((v, p), (w, q))$ is invariant under Γ in the sense that*

$$\forall h \in \Gamma : \varphi((v, hp), (w, hq)) = \varphi((v, p), (w, q))\,.$$

(ii) *Let $(v, p), (w, q), (u, h) \in \widehat{A} \times \Gamma$ be such that*

$$\mathrm{C}_v^{\mathrm{acc}}|_{p^{-1}q.\,\mathrm{C}_w^{\mathrm{acc}}} \neq \varnothing \qquad \textit{and} \qquad \mathrm{C}_w^{\mathrm{acc}}|_{q^{-1}h.\,\mathrm{C}_u^{\mathrm{acc}}} \neq \varnothing\,.$$

Then

$$\varphi((v, p), (u, h)) = \varphi((v, p), (w, q)) + \varphi((w, q), (u, h))\,.$$

(iii) *For all* $v \in \widehat{A}$, $n \in \mathbb{N}$, *and* $h \in \mathrm{Per}_{v,n}$ *we have*

$$\varphi((v, \mathrm{id}), (v, h^{-1})) = n\,.$$

Proof. The statements (i) and (iii) are immediately clear from the definitions of the intersection count and the sets involved. Regarding statement (ii), the hypotheses imply that

$$p\boldsymbol{.}\, \mathrm{C}_v^{\mathrm{acc}}\, |_{q\boldsymbol{.}\,\mathrm{C}_w^{\mathrm{acc}}} \neq \varnothing \qquad \text{and} \qquad q\boldsymbol{.}\, \mathrm{C}_w^{\mathrm{acc}}\, |_{h\boldsymbol{.}\,\mathrm{C}_u^{\mathrm{acc}}} \neq \varnothing\,.$$

Therefore, we have

$$\mathrm{C}_v^{\mathrm{acc}}\, |_{p^{-1}h\boldsymbol{.}\,\mathrm{C}_u^{\mathrm{acc}}} \neq \varnothing\,.$$

Hence all intersection counts involved are non-zero. We pick $\nu \in \mathrm{C}_v^{\mathrm{acc}}\, |_{p^{-1}h\boldsymbol{.}\,\mathrm{C}_u^{\mathrm{acc}}}$. Then ν is also an element of $\mathrm{C}_v^{\mathrm{acc}}\, |_{p^{-1}q\boldsymbol{.}\,\mathrm{C}_w^{\mathrm{acc}}}$ and hence there exist uniquely determined $t_1^*, t_2^* \in (0, +\infty)$, $t_1^* < t_2^*$, such that

$$\gamma_\nu'(t_1^*) \in p^{-1}q\boldsymbol{.}\, \mathrm{C}_w^{\mathrm{acc}} \qquad \text{and} \qquad \gamma_\nu'(t_2^*) \in p^{-1}h\boldsymbol{.}\, \mathrm{C}_u^{\mathrm{acc}}\,.$$

With that we calculate

$$\begin{aligned}
\varphi((v,p),(u,h)) &= \varphi(v, u, p^{-1}h, \nu) \\
&= \#\left\{t \in (0, t_2^*] \;\middle|\; \gamma_\nu'(t) \in \Gamma\boldsymbol{.}\mathrm{C}_{\mathrm{acc}}\right\} \\
&= \#\left\{t \in (0, t_1^*] \;\middle|\; \gamma_\nu'(t) \in \Gamma\boldsymbol{.}\mathrm{C}_{\mathrm{acc}}\right\} \\
&\quad + \#\left\{t \in (t_1^*, t_2^*] \;\middle|\; \gamma_\nu'(t) \in \Gamma\boldsymbol{.}\mathrm{C}_{\mathrm{acc}}\right\} \\
&= \varphi(v, w, p^{-1}q, \nu) + \varphi(w, u, q^{-1}pp^{-1}h, \nu) \\
&= \varphi((v,p),(w,q)) + \varphi((w,q),(u,h))\,.
\end{aligned}$$ ☐

Let $a, b \in \widehat{A}$ and $m, n \in \mathbb{N}$ be such that $\mathrm{Per}_{a,n} \cap \mathrm{Per}_{b,m} \neq \varnothing$. Denote

$$j := \pi_A(a) \qquad \text{and} \qquad k := \pi_A(b)\,.$$

Further let $g \in \mathrm{Per}_{a,n} \cap \mathrm{Per}_{b,m}$. Because of Proposition 6.4 the transformation g is hyperbolic with

$$\big(\mathrm{f}_-(g), \mathrm{f}_+(g)\big) \in \big(I_a \cap I_b\big) \times \big(J_j \cap J_k\big)\,.$$

This implies in particular that

$$I_{a,\mathrm{st}} \cap I_{b,\mathrm{st}} \neq \varnothing \qquad \text{and} \qquad J_{j,\mathrm{st}} \cap J_{k,\mathrm{st}} \neq \varnothing\,.$$

The combination of those two shows that either

$$\mathrm{H}_+(j) \subseteq \mathrm{H}_+(k) \qquad \text{or} \qquad \mathrm{H}_+(k) \subseteq \mathrm{H}_+(j)\,,$$

where we suppose the former without loss of generality. Furthermore, there exists $N \in \mathbb{N}_0$ such that

$$g^{-N}.\mathrm{H}_+(j) \subseteq \mathrm{H}_+(k) \subseteq g^{N+2}.\mathrm{H}_+(j)$$

and

$$\mathrm{C}_b^{\mathrm{acc}}|_{g^{N+2}.\mathrm{C}_a^{\mathrm{acc}}} \neq \varnothing \qquad \text{and} \qquad \mathrm{C}_a^{\mathrm{acc}}|_{g^{-N}.\mathrm{C}_b^{\mathrm{acc}}} \neq \varnothing\,. \tag{6.11}$$

This is due to the fact (see Proposition 6.4) that the set sequences $(g^r.\overline{\mathrm{bp}(\mathrm{C}_j)})_{r\in\mathbb{Z}}$ and $(g^r.\overline{\mathrm{bp}(\mathrm{C}_k)})_{r\in\mathbb{Z}}$ converge to the singleton $\{\mathrm{f}_\pm(g)\}$ for $k \to \pm\infty$ and that the fixed points $\mathrm{f}_-(g)$ and $\mathrm{f}_+(g)$ are inner points of $I_j \cap I_k$ and $J_j \cap J_k$, respectively. The 2 in the exponent of g instead of a 1 accounts for the possibility that k equals $\psi_{\mathrm{Z}}^*(j)$ or j equals $\psi_{\mathrm{Z}}^*(k)$ for either $\mathrm{Z} \in \{\mathrm{X},\mathrm{Y}\}$. We note that (6.11) in combination with

$$\mathrm{C}_a^{\mathrm{acc}}|_{g^{-1}.\mathrm{C}_a^{\mathrm{acc}}} \neq \varnothing$$

(which is due to $g \in \mathrm{Per}_{a,n}$) yields that

$$\mathrm{C}_a^{\mathrm{acc}}|_{g^{-(N+2)}.\mathrm{C}_b^{\mathrm{acc}}} \neq \varnothing \qquad \text{and} \qquad \mathrm{C}_b^{\mathrm{acc}}|_{g^{N}.\mathrm{C}_a^{\mathrm{acc}}} \neq \varnothing\,.$$

With Lemma 6.6 we now obtain

$$\begin{aligned}
n &= \varphi((a,\mathrm{id}),(a,g^{-1})) = \varphi((a,g^{N+1}),(a,g^N))\\
&= \frac{1}{2}\Big(\varphi((a,g^{N+2}),(a,g^{N+1})) + \varphi((a,g^{N+1}),(a,g^N))\Big)\\
&= \frac{1}{2}\varphi((a,g^{N+2}),(a,g^N))\\
&= \frac{1}{2}\Big(\varphi((a,g^{N+2}),(b,\mathrm{id})) + \varphi((b,\mathrm{id}),(a,g^N))\Big)\\
&= \frac{1}{2}\Big(\varphi((a,g^{N+2}),(b,\mathrm{id})) + \varphi((b,g^2),(a,g^{N+2}))\Big)\\
&= \frac{1}{2}\varphi((b,g^2),(b,\mathrm{id})) = \varphi((b,\mathrm{id}),(b,g^{-1})) = m\,.
\end{aligned}$$

This completes the proof of Property 2.

6.4 Property 3

Part (I) of Property 3 has already been shown to hold in Proposition 6.4. For the proof of the parts (II) and (III) we start with some preparations.

For the first step we recall the set $\mathrm{Per}_{a,n}$ for $a \in \widehat{A}$ and $n \in \mathbb{N}$ from the proof of Property 2 (Section 6.3) and further that each element in $\mathrm{Per}_{a,n}$ is hyperbolic by Proposition 6.4. Therefore, for any $g \in \mathrm{Per}_{a,n}$, its two fixed points

$$\mathrm{f}_\pm(g^{-1}) = \mathrm{f}_\mp(g)$$

are elements of $\widehat{\mathbb{R}}_{\mathrm{st}}$ and each geodesic γ on $\mathbb{H}$ with $\gamma(\pm\infty) = \mathrm{f}_\pm(g^{-1})$ is a representative of the axis $\alpha(g^{-1})$ of g^{-1}. The geodesic on $\mathbb{X}$ associated to γ is periodic with period length $\ell(g^{-1})$. See Section 1.7 and, in particular, Lemma 1.11.

Lemma 6.7. *Let $a \in \widehat{A}$, $n \in \mathbb{N}$ and $g \in \mathrm{Per}_{a,n}$. Let γ be a representative of $\alpha(g^{-1})$. Then γ intersects $\mathrm{C}_a^{\mathrm{acc}}$.*

Proof. By Proposition 6.4,

$$\mathrm{f}_+(g^{-1}) \in I_{a,\mathrm{st}} \qquad \text{and} \qquad \mathrm{f}_-(g^{-1}) \in J_{\pi_A(a),\mathrm{st}}\,.$$

From (B5) and Remark 4.11 we obtain that γ intersects $\mathrm{C}_{\pi_A(a),\mathrm{st}}$, and hence γ intersects $\mathrm{C}_{\pi_A(a),\mathrm{acc}}$. By combining the Lemmas 5.32, 5.34(ii), and 5.34(iii), we see that γ intersects $\mathrm{C}_a^{\mathrm{acc}}$. $\square$

Definition 6.8. Let $a \in \widehat{A}$, $g \in \Gamma$ be hyperbolic, and suppose that γ is a geodesic on $\mathbb{H}$ satisfying $\gamma(\pm\infty) = \mathrm{f}_\pm(g)$ and $\gamma'(0) \in \mathrm{C}_a^{\mathrm{acc}}$. We set

$$\nu(a,g) := \gamma'(0)\,.$$

The vector $\nu(a,g)$ is well-defined whenever $(\mathrm{f}_+(g), \mathrm{f}_-(g)) \in I_{a,\mathrm{st}} \times J_{\pi_A(a),\mathrm{st}}$ by virtue of Lemma 6.7, hence, in particular in the case that $g^{-1} \in \mathrm{Per}_{a,n}$ for any $n \in \mathbb{N}$. The geodesic γ in Definition 6.8 is then a representative of $\alpha(g)$ and equals $\gamma_{\nu(a,g)}$.

Lemma 6.9. *Let $a \in \widehat{A}$, $g \in \Gamma$ be hyperbolic such that*

$$\big(\mathrm{f}_+(g^{-1}), \mathrm{f}_-(g^{-1})\big) \in I_{a,\mathrm{st}} \times J_{\pi_A(a),\mathrm{st}}\,,$$

and set $\widetilde{\nu} := \nu(a, g^{-1})$.

(i) *For each $m \in \mathbb{Z}$ set*

$$h_m := \mathrm{g}_{\mathrm{acc},\mathrm{sgn}(m)}(\widetilde{\nu}) \cdots \mathrm{g}_{\mathrm{acc},m}(\widetilde{\nu})\,.$$

Then we have, for each $m \in \mathbb{Z}$,

$$\nu(\mathrm{k}_{\mathrm{acc},m}(\widetilde{\nu}), h_m g h_m^{-1}) = h_m^{-1}.\gamma'_{\widetilde{\nu}}(\mathrm{t}_{\mathrm{acc},m}(\widetilde{\nu}))\,.$$

(ii) *For all $m \in \mathbb{N}$ we have $\widetilde{\nu} = \nu(a,g) = \nu(a,g^m)$.*

(iii) *The sequences*

$$\big((\mathrm{k}_{\mathrm{acc},n}(\widetilde{\nu}), \mathrm{g}_{\mathrm{acc},n}(\widetilde{\nu}))\big)_{n\in\mathbb{N}} \quad \text{and} \quad \big((\mathrm{k}_{\mathrm{acc},-n}(\widetilde{\nu}), \mathrm{g}_{\mathrm{acc},-n}(\widetilde{\nu}))\big)_{n\in\mathbb{N}}$$

in $\widehat{A} \times \Gamma$ are periodic with period length

$$\ell := \varphi((a,\mathrm{id}),(a,g^{-1}))\,.$$

For all $m \in \mathbb{N}_0$ we have

$$\mathrm{g}_{\mathrm{acc},m\ell+1}(\widetilde{\nu}) \cdots \mathrm{g}_{\mathrm{acc},(m+1)\ell}(\widetilde{\nu}) = g^{-1}$$

and

$$\mathrm{g}_{\mathrm{acc},-m\ell-1}(\widetilde{\nu}) \cdots \mathrm{g}_{\mathrm{acc},-(m+1)\ell}(\widetilde{\nu}) = g\,.$$

In particular we obtain

$$\mathrm{g}_{\mathrm{acc},-n}(\widetilde{\nu}) = \mathrm{g}_{\mathrm{acc},n\ell-n+1}(\widetilde{\nu})^{-1}$$

for all $n \in \mathbb{N}$.

Proof. Statements (i) and (ii) are straightforward consequences of Lemma 1.11(ii) and (iii), respectively. In order to verify (iii) consider the geodesic $\gamma := \gamma_{\widetilde{\nu}}$. Then γ is a representative of $\alpha(g^{-1})$. Thus, $\gamma(\mathbb{R})$ is fixed by g and g^{-1}. By Lemma 6.7, γ intersects $g^m\mathbf{.}\,\mathrm{C}_a^{\mathrm{acc}}$ for every $m \in \mathbb{Z}$. Let $t_1 > 0$ be such that $\gamma'(t_1) = g^{-1}\mathbf{.}\widetilde{\nu}$. By (6.9) and (6.10), the number of intersections of the geodesic γ with $\Gamma\mathbf{.}\mathrm{C}_{\mathrm{acc}}$ at times $t \in (0, t_1]$ is given by $\varphi((a, \mathrm{id}), (a, g^{-1}))$. Applying (i) with $m = 1$ now shows the periodicity of the first sequence $((\mathrm{k}_{\mathrm{acc},n}(\widetilde{\nu}), \mathrm{g}_{\mathrm{acc},n}(\widetilde{\nu})))_n$ with the claimed period length, as well as

$$\mathrm{g}_{\mathrm{acc},m\ell+1}(\widetilde{\nu}) \cdots \mathrm{g}_{\mathrm{acc},(m+1)\ell}(\widetilde{\nu}) = g^{-1}$$

for every $m \in \mathbb{N}$. Because of Lemma 6.6(i) the remaining statements follow analogously, by considering g instead of g^{-1} and $t_2 < 0$ such that $\gamma'(t_2) = g\mathbf{.}\widetilde{\nu}$. $\square$

Proposition 6.10. *Let $a \in \widehat{A}$, $n \in \mathbb{N}$ and $g \in \Gamma$. Then $g \in \mathrm{Per}_{a,n}$ if and only if there exists $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ and $t^* > 0$ such that*

$$\#\left\{t \in (0, t^*] \;\middle|\; \gamma_\nu'(t) \in \Gamma\mathbf{.}\mathrm{C}_{\mathrm{acc}}\right\} = n$$

and

$$\gamma_\nu'(t^*) = g^{-1}\mathbf{.}\nu\,.$$

In this case, t^* is the displacement length $\ell(g^{-1})$ of g^{-1}.

Proof. We suppose first that $g \in \mathrm{Per}_{a,n}$. By Lemma 6.6(iii),

$$\varphi((a, \mathrm{id}), (a, g^{-1})) = n\,. \tag{6.12}$$

By (6.10) this implies (since $n \neq 0$) that $\mathrm{C}_a^{\mathrm{acc}}|_{g^{-1}\mathbf{.}\,\mathrm{C}_a^{\mathrm{acc}}}$ is not empty, and further that

$$\mathrm{f}_-(g^{-1}) \in J_{\pi_A(a),\mathrm{st}} \subseteq g^{-1}\mathbf{.}J_{\pi_A(a),\mathrm{st}}$$

and

$$\mathrm{f}_+(g^{-1}) \in I_{a,\mathrm{st}} \cap g^{-1}\mathbf{.}I_{a,\mathrm{st}}\,.$$

Let $\widetilde{\nu}$ denote the unique element of $\mathrm{C}_a^{\mathrm{acc}}$ satisfying

$$\big(\gamma_{\widetilde{\nu}}(+\infty), \gamma_{\widetilde{\nu}}(-\infty)\big) = \big(\mathrm{f}_+(g^{-1}), \mathrm{f}_-(g^{-1})\big)$$

and let t^* denote the intersection time of $\gamma_{\widetilde{\nu}}$ with $g^{-1}.\,\mathrm{C}_a^{\mathrm{acc}}$. Then

$$\#\left\{t \in (0,t^*] \;\middle|\; \gamma_{\widetilde{\nu}}'(t) \in \Gamma.\mathrm{C}_{\mathrm{acc}}\right\} = n$$

by (6.10) and (6.12). Further let $\xi := \gamma_{\widetilde{\nu}}'(t^*)$ denote the intersection vector of $\gamma_{\widetilde{\nu}}$ with $g^{-1}.\,\mathrm{C}_a^{\mathrm{acc}}$. Then $g.\xi \in \mathrm{C}_a^{\mathrm{acc}}$ and

$$\gamma_{g.\xi}(\pm\infty) = g.\gamma_{\widetilde{\nu}}(\pm\infty) = \mathrm{f}_\pm(g^{-1})\,.$$

Hence,

$$\gamma_{\widetilde{\nu}}'(t^*) = \xi = g^{-1}.\widetilde{\nu}\,.$$

This proves the first claimed implication. For the converse implication we suppose that $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ and $t^* > 0$ are chosen such that

$$\#\left\{t \in (0,t^*] \;\middle|\; \gamma_{\nu}'(t) \in \Gamma.\mathrm{C}_{\mathrm{acc}}\right\} = n \tag{6.13}$$

and $\gamma_\nu'(t^*) = g^{-1}.\nu$. From Corollary 5.39(iii) and (6.13) we obtain that

$$g^{-1} = \mathrm{g}_{\mathrm{acc},1}(\nu)\cdots\mathrm{g}_{\mathrm{acc},n}(\nu)\,.$$

The definition of the map F now immediately implies that $g \in \mathrm{Per}_{a,n}$. This completes the proof of the converse implication. The equality $t^* = \ell(g^{-1})$ subsequently follows from the definition of the displacement length and geodesics being parameterized by arc length. ☐

The following result implies Property 3(II).

Proposition 6.11. *Let $a \in \widehat{A}$, $n \in \mathbb{N}$, and $h \in \mathrm{Per}_{a,n}$. Let $h_0 \in \Gamma$ be primitive such that $h_0^m = h$. Then $h_0 \in \mathrm{Per}_{a,\frac{n}{m}}$.*

Proof. Since $h \in \mathrm{Per}_{a,n}$, we find $\nu \in \mathrm{C}_a^{\mathrm{acc}}$ such that, with $t^* := \ell(h^{-1})$ (the displacement length of h^{-1}), we have

$$\gamma_\nu'(t^*) = h^{-1}.\nu \tag{6.14}$$

and

$$\#\left\{t \in (0,t^*] \;\middle|\; \gamma_{\nu}'(t) \in \Gamma.\,\mathrm{C}_{\mathrm{acc}}\right\} = n \tag{6.15}$$

by Proposition 6.10. From (6.14) it follows that γ_ν is a representative of the axis of h^{-1} and hence also of h_0^{-1} (cf. Section 1.7), and further that $\nu = \nu(a, h^{-1})$ and that $\nu(a, h_0^{-1})$ exists and equals ν, the latter by Lemma 6.9(ii). Therefore, for $\widetilde{t} := \ell(h_0^{-1})$ we obtain

$$\gamma_\nu'(\widetilde{t}) = h_0^{-1}.\nu\,.$$

The system of accelerated iterated sequences of any element in $\mathrm{C}_{\mathrm{acc}}$ depends only on this considered element (and of course the choice of $\mathcal{C}_{\mathrm{acc}}$) and it is equivariant under the action of elements in Γ, as can be observed from Corollary 5.39. See also Lemmas 6.6(i) and 6.9(iii). Thus, the accelerated sequence of intersection times of $h_0^{-1}.\nu$ is only shifted against the accelerated sequence of intersection times of ν, and in particular, the sequence of differences

$$\left(\mathrm{t}_{\mathrm{acc},n+1}(\nu) - \mathrm{t}_{\mathrm{acc},n}(\nu)\right)_n = \left(\mathrm{t}_{\mathrm{acc},n+1}(h_0^{-1}.\nu) - \mathrm{t}_{\mathrm{acc},n}(h_0^{-1}.\nu)\right)_n \tag{6.16}$$

is periodic. The relation $\ell(h^{-1}) = m\ell(h_0^{-1})$ between the displacement lengths given by Lemma 1.11(iii) implies that m (from $h_0^m = h$) is a period length of the sequence (6.16). From this and (6.15) it follows that

$$\#\left\{t \in (0,\tilde{t}] \mid \gamma_\nu'(t) \in \Gamma.\,\mathrm{C}_{\mathrm{acc}}\right\} = \frac{1}{m}\#\left\{t \in (0,t^*] \mid \gamma_\nu'(t) \in \Gamma.\,\mathrm{C}_{\mathrm{acc}}\right\} = \frac{n}{m}\,.$$

By Proposition 6.10, $h_0 \in \mathrm{Per}_{a,\frac{n}{m}}$. ☐

In order to establish part (III) of Property 3 we let $[g] \in [\Gamma]_{\mathrm{h}}$ and consider the equivalence class of periodic geodesics $\varrho([g^{-1}])$ on $\mathbb{X}$, where ϱ is the map from (1.39). The combination of ($\mathrm{B}_{\mathrm{Per}}$) (see Proposition 4.8) with Proposition 5.31 (or Proposition 5.30) imply that we find $a \in \widehat{A}$ and a geodesic γ on $\mathbb{H}$ such that γ intersects $\mathrm{C}_a^{\mathrm{acc}}$. By Lemma 1.11(ii) there exists a representative h of $[g]$ such that γ represents the axis of h, thus, $[\gamma] = \alpha(h^{-1})$. Applying Proposition 6.10 with $t^* = \ell(g^{-1}) = \ell(h^{-1})$ and ν being the intersection vector of γ with $\mathrm{C}_a^{\mathrm{acc}}$ yields that $h \in \mathrm{Per}_{a,n}$ for some $n \in \mathbb{N}$. Property 2 now shows uniqueness of n, which completes the proof of (III).

6.5 Property 4

Let $g \in \mathrm{Per}$. Recall the word length $\omega(g)$ of g from (3.1). Let $a \in \widehat{A}$ and $n \in \mathbb{N}$ be such that $g \in \mathrm{Per}_{a,n}$. Let $g_0 \in \Gamma$ be the primitive of g and recall the objects $m(g)$ and $p(g)$ from (3.2) and before. By applying Proposition 6.11 we observe

$$\omega(g_0) = \frac{m(g)\omega(g_0)}{m(g)} = \frac{\omega(g)}{m(g)} = p(g)\,. \tag{6.17}$$

Recall further the vector $\nu(b,h)$ from Definition 6.8, which is uniquely determined for all $b \in \widehat{A}$ and $h^{-1} \in \mathrm{Per}_{b,m}$, $m \in \mathbb{N}$. We will make extensive use of the following abbreviation: For $b \in \widehat{A}$, $h^{-1} \in \bigcup_{k\in\mathbb{N}} \mathrm{Per}_{b,k}$, and $m \in \mathbb{Z}$ we write

$$\mathrm{g}_m(b,h) := \left(\mathrm{g}_{\mathrm{acc},\mathrm{sgn}(m)}(\nu(b,h)) \cdots \mathrm{g}_{\mathrm{acc},m}(\nu(b,h))\right)^{\mathrm{sgn}(m)}\,.$$

Note that, because of Lemma 6.9 and Proposition 6.11, we have

$$\mathrm{g}_{k\cdot\omega(h_0)}(b,h) = h_0^k$$

for all $k \in \mathbb{N}_0$, where h_0 denotes the primitive of h in Γ. It is further in line with this definition to set $\mathrm{g}_0(b,h) := \mathrm{id}$ for every possible choice of b and h.

We recall that the reduced set of branches $\mathcal{C}$ giving rise to the accelerated system $\mathcal{C}_{\mathrm{acc}}$ shall be and is assumed to be non-collapsing by virtue of Proposition 5.19.

Lemma 6.12. *Let b and h be as before and let $k, \ell \in \mathbb{N}$, $k \neq \ell$. Then*

$$\mathrm{g}_k(b,h)^{-1}\mathrm{g}_\ell(b,h) \neq \mathrm{id}\ .$$

Proof. Without loss of generality, we may suppose that $k < \ell$. By Remark 5.38, for every $j \in \{k+1, \ldots, \ell\}$ we find $m_j \in \mathbb{N}$, $k_{j,0}, \ldots, k_{j,m_j} \in A$, and transformations $h_{j,i} \in \mathcal{G}(k_{j,i-1}, k_{j,i})$ for every $i \in \{1, \ldots, m_j\}$ such that

$$\mathrm{g}_{\mathrm{acc},j}(\nu(b,h)) = h_{j,1} \cdots h_{j,m_j}\ .$$

By construction, $\mathrm{C}^{\mathrm{acc}}_{\mathrm{k}_{\mathrm{acc},j}(\nu(b,h))} \subseteq \mathrm{C}_{k_{j,m_j}}$, for all j. Thereby, $k_{j,m_j} = k_{j+1,0}$. From this we obtain

$$\begin{aligned}
\mathrm{g}_k(b,h)^{-1}\mathrm{g}_\ell(b,h) &= \mathrm{g}_{\mathrm{acc},k+1}(\nu(b,h)) \cdots \mathrm{g}_{\mathrm{acc},\ell}(\nu(b,h)) \\
&= h_{k+1,1} \cdots h_{k+1,m_{k+1}} h_{k+2,1} \cdots \\
&\quad \cdots h_{k+2,m_{k+2}} \cdots\cdots h_{\ell,1} \cdots h_{\ell,m_\ell} \\
&\neq \mathrm{id}\,,
\end{aligned}$$

where the final relation is due to (B9) (see also Definition 5.17). □

Lemma 6.13. *Let g, a, and n be as before. Let $q \in \Gamma$ be such that $qgq^{-1} \in \mathrm{Per}_n$. Then there exists exactly one $\kappa = \kappa(g,a,n) \in \mathbb{N}$, $\kappa \leq \omega(g_0)$, such that*

$$\mathrm{g}_\kappa(a, g^{-1}) \cdot q \cdot g = g \cdot \mathrm{g}_\kappa(a, g^{-1}) \cdot q\,. \tag{6.18}$$

Or, equivalently, there exists a unique $\kappa \in \{1, \ldots, \omega(g_0)\}$ such that

$$q \in Q_\kappa := \left\{ \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g_0^\ell \;\middle|\; \ell \in \mathbb{Z} \right\}\,, \tag{6.19}$$

where g_0 denotes the primitive of g in Γ.

Proof. The equivalence of the relations (6.18) and (6.19) is an immediate consequence of Lemma 1.6.

Let $b \in \widehat{A}$ be such that $h := qgq^{-1} \in \mathrm{Per}_{b,n}$. In the case $q = \mathrm{id}$ we have

$$\mathrm{g}_{\omega(g_0)}(a, g^{-1}) \cdot g = g \cdot \mathrm{g}_{\omega(g_0)}(a, g^{-1})$$

by Lemma 6.9 and Proposition 6.11, because $\mathrm{g}_{\omega(g_0)}(a, g^{-1}) = g_0^{-1}$. See also the argumentation right before Lemma 6.12. Therefore, the identity (6.18) is valid for $\kappa = \omega(g_0)$. It remains to show that (6.18) is not valid for any smaller value for κ in $\mathbb{N}$. To that end we consider (6.18) as the quest for elements commuting with g, given that $q = \mathrm{id}$. By applying Lemma 1.6 we see that any solution $\mathrm{g}_\kappa(a, g^{-1})$ of (6.18) must be a non-trivial power of g_0. Within $\{1, \ldots, \omega(g_0)\}$ only $\kappa = \omega(g_0)$ yields such an element in Γ.

We consider now the case that $q \neq \mathrm{id}$. By Lemma 1.11(ii) the axis of the transformation $h^{-1} = qg^{-1}q^{-1}$ is given by $q.\alpha(g^{-1})$. All of the representatives of $\alpha(h^{-1})$ intersect $\mathrm{C}_b^{\mathrm{acc}}$ by Lemma 6.7. Hence, $\gamma_{\nu(a,g^{-1})}$, which is a representative of $\alpha(g^{-1})$, intersects $q^{-1}.\mathrm{C}_b^{\mathrm{acc}}$. This implies the existence of $r \in \mathbb{Z} \setminus \{0\}$ such that

$$q = \mathrm{g}_r(a, g^{-1})^{-\operatorname{sgn}(r)}\,.$$

There is a unique way to write

$$r = \lambda \cdot \omega(g_0) + \kappa\,,$$

with $\lambda \in \mathbb{Z}$ and $\kappa \in (0, \omega(g_0)] \cap \mathbb{Z}$. Using Lemma 6.9(iii) we calculate for $r > 0$,

$$\begin{aligned} qgq^{-1} &= \mathrm{g}_{\lambda\omega(g_0)+\kappa}(a, g^{-1})^{-1} \cdot g_0^{m(g)} \cdot \mathrm{g}_{\lambda\omega(g_0)+\kappa}(a, g^{-1}) \\ &= \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g_0^{-\lambda} \cdot g_0^{m(g)} \cdot g_0^{\lambda} \cdot \mathrm{g}_\kappa(a, g^{-1}) \\ &= \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g \cdot \mathrm{g}_\kappa(a, g^{-1})\,. \end{aligned}$$

For $r < 0$, we calculate, again by applying Lemma 6.9(iii),

$$\begin{aligned} \mathrm{g}_r(a, g^{-1})^{-1} &= \mathrm{g}_{\mathrm{acc},-1} \cdots \mathrm{g}_{\mathrm{acc},r} \\ &= \mathrm{g}_{\mathrm{acc},\omega(g_0)}^{-1} \mathrm{g}_{\mathrm{acc},2\omega(g_0)-1}^{-1} \cdots \mathrm{g}_{\mathrm{acc},-r\omega(g_0)-(r-1)}^{-1} \\ &= \left(\mathrm{g}_{\mathrm{acc},\omega(g_0)}^{-1} \mathrm{g}_{\mathrm{acc},2\omega(g_0)-1}^{-1} \cdots \mathrm{g}_{\mathrm{acc},1}^{-1}\right)^{-\lambda} \mathrm{g}_{\mathrm{acc},2\omega(g_0)}^{-1} \cdots \mathrm{g}_{\mathrm{acc},\kappa+1}^{-1} \\ &\qquad \cdot \mathrm{g}_{\mathrm{acc},\kappa}^{-1} \cdots \mathrm{g}_{\mathrm{acc},1}^{-1} \mathrm{g}_{\mathrm{acc},1} \cdots \mathrm{g}_{\mathrm{acc},\kappa} \\ &= \left(\mathrm{g}_{\mathrm{acc},\omega(g_0)}^{-1} \mathrm{g}_{\mathrm{acc},2\omega(g_0)-1}^{-1} \cdots \mathrm{g}_{\mathrm{acc},1}^{-1}\right)^{-\lambda+1} \mathrm{g}_\kappa(a, g^{-1}) \\ &= g_0^{-\lambda+1} \mathrm{g}_\kappa(a, g^{-1})\,, \end{aligned}$$

where every transformation $\mathrm{g}_{\mathrm{acc},j}$, $j \in \mathbb{Z}$, is to be understood with respect to the vector $\nu(a, g^{-1})$. Thus, we can proceed as in the case $r > 0$ to obtain

$$qgq^{-1} = \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g \cdot \mathrm{g}_\kappa(a, g^{-1})\,.$$

It remains to show that κ is unique in $\{1, \ldots, \omega(g_0)\}$ with

$$q = \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g_0^{\ell}$$

for some $\ell \in \mathbb{Z}$. To that end, we suppose that there exist $\vartheta \in \{1, \ldots, \omega(g_0)\}$ and $m \in \mathbb{Z}$ such that

$$q = \mathrm{g}_\vartheta(a, g^{-1})^{-1} \cdot g_0^m .$$

Then

$$\mathrm{g}_\kappa(a, g^{-1})\mathrm{g}_\vartheta(a, g^{-1})^{-1} = g_0^p \tag{6.20}$$

for some $p \in \mathbb{Z}$. For $p \in \mathbb{N}_0$, this is equivalent to

$$g_0^{-p}\mathrm{g}_\kappa(a, g^{-1})\mathrm{g}_\vartheta(a, g^{-1})^{-1} = \mathrm{id} . \tag{6.21}$$

From Lemma 6.9(iii) we obtain

$$g_0^{-p}\mathrm{g}_\kappa(a, g^{-1}) = \mathrm{g}_{p\omega(g_0)+\kappa}(a, g^{-1}) .$$

Using this identity in (6.21) and combining with Lemma 6.12 we find

$$p\omega(g_0) + \kappa = \vartheta ,$$

thus $p = 0$ and

$$\kappa = \vartheta .$$

For the case that in (6.20), p is a nonpositive integer, we convert (6.20) into

$$\mathrm{g}_\kappa(a, g^{-1})\mathrm{g}_\vartheta(a, g^{-1})^{-1}g_0^{-p} = \mathrm{id} .$$

Again using Lemma 6.9(iii) we obtain

$$\mathrm{g}_\vartheta(a, g^{-1})^{-1}g_0^{-p} = \left(g_0^p\mathrm{g}_\vartheta(a, g^{-1})\right)^{-1} = \mathrm{g}_{-p\omega(g_0)+\vartheta}(a, g^{-1})^{-1}$$

With Lemma 6.12 we find

$$\kappa = -p\omega(g_0) + \vartheta .$$

Therefore, $p = 0$ and

$$\kappa = \vartheta .$$

This completes the proof. □

We are now ready to establish Property 4. Consider the sets Q_κ from (6.19) for $\kappa = \kappa(g, a, n)$ given by Lemma 6.13. Since $g_0^{m(g)} = g$, we have

$$qgq^{-1} = \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g \cdot \mathrm{g}_\kappa(a, g^{-1}) ,$$

for every $q \in Q_\kappa$. Therefore, because of the uniqueness of κ from Lemma 6.13, the number $\#([g] \cap \mathrm{Per}_n)$ is bounded from above by

$$\#\{1, \ldots, \omega(g_0)\} = \omega(g_0) \overset{(6.17)}{=} p(g) .$$

Hence, it remains to show that

$$h_\kappa \coloneqq \mathrm{g}_\kappa(a, g^{-1})^{-1} \cdot g \cdot \mathrm{g}_\kappa(a, g^{-1}) \in \mathrm{Per}_n \tag{6.22}$$

for all $\kappa \in \{1, \ldots, \omega(g_0)\}$. This is achieved via the same argument as in the proof of Lemma 6.13: By Lemma 1.11(ii) we have

$$\alpha(h_\kappa^{-1}) = \mathrm{g}_\kappa(a, g^{-1})^{-1} \boldsymbol{.} \alpha(g^{-1}),$$

and because of Lemma 6.7 every representative of $\alpha(h_\kappa^{-1})$ intersects $\mathrm{C}_b^{\mathrm{acc}}$, where $b \coloneqq \mathrm{k}_{\mathrm{acc},\kappa}(\nu(a, g^{-1}))$. This implies

$$\big(\mathrm{f}_+(h_\kappa^{-1}), \mathrm{f}_-(h_\kappa^{-1})\big) \in I_{b,\mathrm{st}} \times J_{b,\mathrm{st}} \subseteq I_b^\circ \times J_b .$$

Thus, $\mathrm{g}_\kappa(a, g^{-1})^{-1} \boldsymbol{.} \alpha(g^{-1})$ intersects $h_\kappa^{-1} \boldsymbol{.} \mathrm{C}_b^{\mathrm{acc}}$ as well, implying $h_\kappa \in \mathrm{Per}_{b,m}$ for some $m \in \mathbb{N}$. Since $g \in \mathrm{Per}_n$ and $h_\kappa \in [g]$, part (III) of Property 3 now yields $m = n$. This yields (6.22) and thereby finishes the proof of Property 4.

6.6 Property 5

Instead of directly establishing the existence of a family $\{\mathcal{E}_a\}_{a \in \widehat{A}}$ of open disks in $\mathbb{C}$ fulfilling the demands of Property 5, we will first provide a family of intervals in $\mathbb{R}$ which satisfies a set of properties corresponding to those requested. Working on the real axis results in a less involved discussion. A first part is provided by Lemma 6.14 below. We then expand the real intervals to complex disks with centers in the real line ("complex disk hull"). Since all considered actions are by Möbius transformations, all inclusion properties are inherited by the complex disks.

We start with a few preparations. Taking advantage of the admissibility of the reduced set of branches $\mathcal{C}$, we may suppose that for each $a \in \widehat{A}$ the interval $\widehat{I}_a$ is a bounded subset of $\mathbb{R}$. For that we possibly need to conjugate the group Γ by some element in $\mathrm{PSL}_2(\mathbb{R})$, which, however, does not affect the validity of any results. Alternatively, we may interpret this step as using a non-standard chart for the relevant part of $\widehat{\mathbb{R}}$. See Remark 4.2(i). For each $a \in \widehat{A}$ let $x_a, y_a \in \mathbb{R}$, $x_a < y_a$, denote the boundary points of $\widehat{I}_a$, thus

$$\widehat{I}_a = [x_a, y_a] . \tag{6.23}$$

Lemma 6.14. *There exists a family* $\{(\varepsilon_a^x, \varepsilon_a^y)\}_{a \in \widehat{A}}$ *in* $\mathbb{R}^2$ *such that for each* $a \in \widehat{A}$,

(i) $\varepsilon_a^x, \varepsilon_a^y > 0$,

and with

$$E_a \coloneqq \big(x_a - \varepsilon_a^x, y_a + \varepsilon_a^y\big)$$

the following statements hold true:

(ii) *For each* $b \in \widehat{A}$ *and each* $g \in C_{a,b}$ *we have*

$$g.\infty \notin \overline{E_a}\,.$$

(iii) *For each* $b \in \widehat{A}$ *and each* $g \in C_{a,b}$ *such that* $g^{-1}.\widehat{I}_a$ *is contained in the interior of* $\widehat{I}_b$ *we have*

$$g^{-1}.\overline{E}_a \subseteq E_b\,.$$

(iv) *For all* $b \in \widehat{A}$ *and all* $p \in P_{a,b}$ *such that* $g_p^{-1}.\widehat{I}_a$ *is contained in the interior of* $\widehat{I}_b$ *we find a compact interval* $K_{a,b,p}$ *of* $\mathbb{R}$ *such that*

$$p^{-n}g_p^{-1}.\overline{E_a} \subseteq K_{a,b,p} \subseteq E_b$$

for all $n \in \mathbb{N}$.

(v) *For all* $b \in \widehat{A}$ *and all* $p \in P_{a,b}$*, the fixed point of* p *is not contained in* $g_p^{-1}.\overline{E_a}$.

Moreover, for any $a \in \widehat{A}$*, there exist thresholds* $\eta_a^x > 0$ *and* $\eta_a^y > 0$ *such that any family* $\{(\varepsilon_a^x, \varepsilon_a^y)\}_{a\in\widehat{A}}$ *that satisfies* $\varepsilon_a^x \in (0, \eta_a^x)$ *and* $\varepsilon_a^y \in (0, \eta_a^y)$ *for all* $a \in \widehat{A}$*, also satisfies* (i)–(v).

Proof. In what follows we will show the existence of the thresholds η_a^x and η_a^y for all $a \in \widehat{A}$ by showing that they only need to obey a finite number of positive upper bounds. These bounds depend on $\widehat{A}$ and a finite number of elements in Γ, but they do not have any interdependencies among each other, i. e., the values for the thresholds are independent of each other.

We start by considering $a, b \in \widehat{A}$ and $g \in C_{a,b}$. Since $g^{-1} \in \mathcal{G}_{\mathrm{acc}}(b, a)$ we have

$$g^{-1}.I_{a,\mathrm{st}} \subseteq I_{b,\mathrm{st}}$$

by (B7$_{\mathrm{acc}}$i). Thus, $g^{-1}.\widehat{I}_a \subseteq \widehat{I}_b$ by continuity of g. By hypothesis, $\infty \notin \widehat{I}_b$ and hence

$$g.\infty \notin \widehat{I}_a\,.$$

Since $\widehat{I}_a$ is compact (see (6.3) and (6.23)), we can find an open neighborhood $\mathcal{U}$ (a "thickening") of $\widehat{I}_a$ in $\widehat{\mathbb{R}}$ that does not contain $g.\infty$. The open neighborhood $\mathcal{U}$ can be chosen uniformly for all $b \in \widehat{A}$ and all $g \in C_{a,b}$, as $\widehat{A}$ and $C_{a,b}$ are finite sets. The first condition on the thresholds η_a^x and η_a^y is that $(x_a - \eta_a^x, y_a + \eta_a^y)$ is contained in $\mathcal{U}$. This implies a positive upper bound for each of η_a^x and η_a^y, which can obviously be optimized (of which we will not take care here).

We now suppose in addition that $g^{-1}.\widehat{I}_a \subseteq \widehat{I}_b^\circ$. Again using that $\widehat{I}_a$, and hence $g^{-1}.\widehat{I}_a$, is compact, we find an open neighborhood $\widetilde{\mathcal{U}}$ of $\widehat{I}_a$ in $\widehat{\mathbb{R}}$ such that

$$g^{-1}.\widetilde{\mathcal{U}} \subseteq \widehat{I}_b^\circ\,.$$

Again taking advantage of the finiteness of $\widehat{A}$ and $C_{a,b}$ for each $b \in \widehat{A}$, we can choose $\widetilde{\mathcal{U}}$ uniformly for all $b \in \widehat{A}$ and all $g \in C_{a,b}$. Our second condition on the thresholds η_a^x and η_a^y is that $(x_a - \eta_a^x, y_a + \eta_a^y)$ is contained in $\widetilde{\mathcal{U}}$. This requirement can obviously be satisfied.

We now consider $a, b = (k, V) \in \widehat{A}$ and $p \in P_{a,b}$. The fixed point of p is

$$\pi_{\mathrm{Z}}(b)_{\pi_A(b)} = V_k\,,$$

which is the unique point contained in

$$\bigcap_{n \in \mathbb{N}_0} p^{-n}.\widehat{I}_b\,.$$

From (5.38), (B7$_{\mathrm{acc}}$i), and (5.37) it follows that

$$g_p^{-1}.I_{a,\mathrm{st}} \subseteq I_{b,\mathrm{st}} \setminus p^{-1}.I_{b,\mathrm{st}}$$

and hence

$$g_p^{-1}.\widehat{I}_a \subseteq \widehat{I}_b \setminus p^{-2}.\widehat{I}_b\,.$$

Therefore, V_k is not contained in $g_p^{-1}.\widehat{I}_a$. Our third condition on the thresholds η_a^x and η_a^y is that

$$g_p^{-1}.\big(x_a - \eta_a^x, y_a + \eta_a^y\big)$$

does not contain V_k for all $b \in \widehat{A}$ and all $p \in P_{a,b}$. Analogously to above, using the compactness of $\widehat{I}_a$ and the finiteness of $\widehat{A}$ and $P_{a,b}$, we deduce that such choices of $\eta_a^x, \eta_a^y > 0$ are possible.

We now suppose in addition that $g_p^{-1}.\widehat{I}_a \subseteq \widehat{I}_b^\circ$. By the compactness of $\widehat{I}_a$ we find an open neighborhood $\mathcal{V}$ of $\widehat{I}_a$ such that

$$g_p^{-1}.\mathcal{V} \subseteq \widehat{I}_b^\circ\,.$$

From (5.37) we obtain that for each subset M of $\widehat{I}_b^\circ$ we have

$$p^{-n}.M \subseteq \widehat{I}_b^\circ$$

for all $n \in \mathbb{N}_0$, and hence in particular

$$p^{-n}g_p^{-1}.\mathcal{V} \subseteq \widehat{I}_b^\circ\,.$$

Thus, for the requested compact set $K_{a,b,p}$ we may pick $\widehat{I}_b$. As before, using the finiteness of $\widehat{A}$ and $P_{a,b}$, we can choose the open neighborhood $\mathcal{V}$ uniformly for all $b \in \widehat{A}$ and all $p \in P_{a,b}$. The fourth, and final condition on the thresholds η_a^x and η_a^y is

$$\big(x_a - \eta_a^x, y_a + \eta_a^y\big) \subseteq \mathcal{V}\,,$$

which can obviously be satisfied. We immediately check that the upper bounds for any of the thresholds η_a^x, η_a^y for any $a \in \widehat{A}$ are independent among each other. This completes the proof. $\square$

Let $\{(\varepsilon_a^x, \varepsilon_a^y)\}_{a\in\widehat{A}}$ be a family satisfying all properties stated in Lemma 6.14 and set, as in Lemma 6.14,

$$E_a := \left(x_a - \varepsilon_a^x, y_a + \varepsilon_a^y\right)$$

for all $a \in \widehat{A}$. In what follows we will show that, by possibly shrinking ε_a^x and ε_a^y for some $a \in \widehat{A}$ and allowing interdependencies among the elements of the family $\{(\varepsilon_a^x, \varepsilon_a^y)\}_{a\in\widehat{A}}$, we can guarantee that Lemma 6.14(iii) is also valid in the case that $g^{-1}.\widehat{I}_a \not\subseteq \widehat{I}_b^\circ$ and that Lemma 6.14(iv) is also valid if $g_p^{-1}.\widehat{I}_a \not\subseteq \widehat{I}_b^\circ$. By taking advantage of the discussion in the proof of Lemma 6.14, we see that both cases can be subsumed to the situation that there exist $a, b \in \widehat{A}$ and $g \in \mathcal{G}_{acc}(b,a)^{-1}$ such that

$$g^{-1}.\widehat{I}_a \subseteq \widehat{I}_b$$

and the two sets $g^{-1}.\widehat{I}_a$ and $\widehat{I}_b$ have a common boundary point. It suffices to show that we can fix $\varepsilon_a^x, \varepsilon_a^y, \varepsilon_b^x, \varepsilon_b^y > 0$ such that

$$g^{-1}.\left(x_a - \varepsilon_a^x, y_a + \varepsilon_a^y\right) \subseteq \left(x_b - \varepsilon_b^x, y_b + \varepsilon_b^y\right).$$

This is clearly possible for any "local" consideration, i. e., for fixed $a, b \in \widehat{A}$ and $g \in \mathcal{G}_{acc}(b,a)^{-1}$. We need to show that global choices are possible.

To that end we note that any such pair of sets $g^{-1}.\widehat{I}_a$ and $\widehat{I}_b$ has a single common boundary point, not two common boundary points, by ($B6_{acc}$). For the boundary point of $\widehat{I}_a$, say z_a, for which $g^{-1}.z_a$ is contained in the interior of $\widehat{I}_b$, we may and shall suppose that the threshold η_a^z is chosen sufficiently small such that $g^{-1}.(z_a \pm \eta_a^z)$ is also contained in $\widehat{I}_b$. (We may restrict here to either $+$ or $-$ as needed. However, we may also require this property for both signs.) We now consider the joint boundary point of $\widehat{I}_b$ and $g^{-1}.\widehat{I}_a$. Without loss of generality, we may suppose that it is y_b. Since the action of Γ preserves orientation, the corresponding boundary point of $\widehat{I}_a$ is then y_a, hence

$$g^{-1}.y_a = y_b\,.$$

Then we need to pick $\varepsilon_a^y > 0$ such that

$$g^{-1}.\left(y_a + \varepsilon_a^y\right) < y_b + \varepsilon_b^y\,.$$

Thus, the threshold for admissible choices for ε_a^y depends on the value of ε_b^y. It might happen that there is $c \in \widehat{A}$ and $h \in \mathcal{G}_{acc}(a,c)^{-1}$ such that

$$h^{-1}.\widehat{I}_c \subseteq \widehat{I}_a \qquad \text{and} \qquad h^{-1}.y_c = y_a\,.$$

In other words, the dependency situation reappears for $\widehat{I}_a$ and we observe a tower of dependency situations: As the threshold for ε_c^y depends on the choice of ε_a^y, and the threshold for ε_a^y depends on the choice of ε_b^y, the threshold for ε_c^y ultimately depends on the choice of ε_b^y. As long as a, b and c are three distinct elements of $\widehat{A}$, we can easily garantee admissible choices for ε_b^y, ε_a^y and ε_c^y by picking them in this very order.

We may organize all occuring dependencies (i. e., considering all elements of $\widehat{A}$ and all situations of coinciding boundary points simultaneously) as a directed graph with

$$\left\{\varepsilon_a^x, \varepsilon_a^y \;\middle|\; a \in \widehat{A}\right\}$$

as set of vertices. As soon as this dependency graph has loops, unsolvable situations may occur. We will now show that the graph is loop-free.

To that end we assume, in order to seek a contradiction, that there exists a finite sequence

$$a_1\,,\ \ldots\,,\ a_{n+1} \in \widehat{A}$$

with $a_{n+1} = a_1$, and

$$g_j \in \mathcal{G}_{\mathrm{acc}}(a_j, a_{j+1}) \qquad \text{for } j \in \{1, \ldots, n\}$$

such that

$$y_{a_j} = g_j \boldsymbol{.} y_{a_{j+1}} \qquad \text{for } j \in \{1, \ldots, n\}\,.$$

Then

$$y_{a_1} = g_1 \cdots g_n \boldsymbol{.} y_{a_{n+1}} = q \boldsymbol{.} y_{a_1}$$

with

$$q \coloneqq g_1 \cdots g_n\,.$$

As the boundary points of $\widehat{I}_{a_1}$ are not hyperbolic fixed points (see Lemma 5.34(iv)), the element q is parabolic and y_{a_1} a parabolic fixed point. In turn, $P_{a_2,a_1} \neq \varnothing$, say $P_{a_2,a_1} = \{p\}$, and y_{a_1} is the fixed point of p and $g_p^{-1} = g_1$. However, $g_p^{-1} \boldsymbol{.} \widehat{I}_{a_2} = g_1 \boldsymbol{.} \widehat{I}_{a_2}$ contains y_{a_1}, which contradicts Lemma 6.14(v). Thus, the dependency graph has no loops.

Finally, for each $a \in \widehat{A}$, we let $\mathcal{E}_a$ be the complex Euclidean disk spanned by the real interval E_a, i. e., $\mathcal{E}_a$ is the unique Euclidean disk in $\mathbb{C}$ with center in $\mathbb{R}$ such that $\mathbb{R} \cap \mathcal{E}_a = E_a$. The family $\{\mathcal{E}_a\}_{a\in\widehat{A}}$ fulfills all requested properties. This finishes the proof that $\mathcal{S}$ fulfills Property 5 and thereby the proof of Theorem 6.1.

Chapter 7

Sets of Branches for Non-Compact Hyperbolic Orbisurfaces

Let Γ be a geometrically finite Fuchsian group that contains hyperbolic elements and is such that its orbit space $\mathbb{X} = \Gamma\backslash\mathbb{H}$ has hyperbolic ends (cusps or funnels). Then $\mathbb{X}$ is a non-compact hyperbolic orbisurface as defined in Section 1.6.

In Chapter 4 we introduced and studied the notion of a set of branches $\mathcal{C}$ for the geodesic flow $\widehat{\Phi}$ on $\mathbb{X}$. In Chapter 5 we reduced $\mathcal{C}$ through various steps in order to extract a structure tuple, which, in Chapter 6, was seen to fulfill all demands of a strict transfer operator approach as defined in Section 3.1. Hence, due to these efforts and the combination of Theorem 6.1 with Theorem 3.1, we are now in the position to conclude that every Fuchsian group admitting the construction of a set of branches also admits the representation of the (meromorphic continuation of the) Selberg zeta function in terms of Fredholm determinants of a family of transfer operators, which arises from $\mathcal{C}$ (see Sections 3.3 and 6.1). This means the single remaining puzzle piece for the proof of Theorem A consists of showing that each of the Fuchsian groups we consider does in fact admit the construction of a set of branches. This is the objective of this final chapter.

To that end we split the realm of non-cocompact orbisurfaces in those with and without cusps. In the former case we will see (Section 7.1) that the groundwork has already been laid by Pohl in [54], as we will show that cross sections emerging from a cusp expansion algorithm (see Chapter 2) can be seen as emerging from a set of branches. More precisely, Theorem 7.1 below shows that $\mathcal{C}_{\mathrm{P}}$ from (2.8) *is* a set of branches. This comes by no surprise, since Pohl's algorithm has been the starting point of our studies. The notion of a set of branches has been introduced in order to identify the key aspects of her approach and subsequently generalize her results to a broader class of Fuchsian groups as well as a wider variety of suitable cross sections. Chapter 2 contains all the necessary background information on the cusp expansion algorithm.

In the case that the orbisurface $\mathbb{X}$ does not have cusps (and thus, by assumption, has at least one funnel), we construct an auxiliary orbisurface $\mathbb{X}_{\mathcal{W}}$ that does so and to which again the cusp expansion algorithm can be applied (Section 7.2). These constructions are completely geometric and the required background information has been provided in Chapter 1. Thus, by virtue of Theorem 7.1, we obtain a set of branches $\mathcal{C}_{\mathcal{W}}$ for the geodesic flow on $\mathbb{X}_{\mathcal{W}}$. We then proceed to show, in Theorem 7.16, how $\mathcal{C}_{\mathcal{W}}$ induces a set of branches for the geodesic flow on the original orbisurface $\mathbb{X}$, which ultimately finishes the proof of Theorem A. As is integral to this thesis, all arguments are completely constructive. This means that for every admissible orbisurface with fundamental group given by means of a full set of generators, a set of branches can be distilled from the discussion in this chapter.

7.1 Orbisurfaces with Cusps

Let $\mathbb{X}$ be a geometrically finite developable hyperbolic orbisurface with cusps. Then $\mathbb{X} = \Gamma\diagdown\mathbb{H}$, where Γ is a geometrically finite Fuchsian group containing parabolic elements (see Corollary 1.33). Without loss of generality we assume that $\pi(\infty)$ is a cusp of $\mathbb{X}$, where π denotes the canonical quotient map from (1.26). We further assume that $\mathbb{X}$ bears periodic geodesics, which is equivalent to Γ containing hyperbolic elements (see Proposition 1.13). We are indifferent to whether or not $\mathbb{X}$ has conical singularities, or equivalently, whether or not Γ contains elliptic elements. We denote by $\widehat{\Phi}$ the geodesic flow on $\mathbb{X}$ (see (1.33)).

Recall from (1.28) that Γ_∞ denotes the stabilizer subgroup of ∞ in Γ and recall the sets of isometric spheres $\mathrm{ISO}(\Gamma)$, the common exterior $\mathcal{K}$ of $\mathrm{ISO}(\Gamma)$, and the subset of relevant isometric spheres $\mathrm{REL}(\Gamma)$ from the Sections 1.9 and 1.10, as well as the relevant part β_{I} of a relevant isometric sphere I from (1.68). As has been discussed in Section 2.1, we are required to impose that for every $\mathrm{I} \in \mathrm{REL}(\Gamma)$ its summit $s(\mathrm{I})$ is contained in β_{I} but no endpoint of it (see (A)). We emphasize again that we do not make any additional use of that restriction here (see also Remark 2.1). The group Γ now fulfills all requirements of the cusp expansion algorithm outlined in Section 2.1, application of which provides a finite family

$$\mathcal{C}_{\mathrm{P}} = \{\mathrm{C}_{\mathrm{P},1}, \ldots, \mathrm{C}_{\mathrm{P},N}\} \tag{7.1}$$

of subsets of $\mathrm{S}\mathbb{H}$ such that

$$\widehat{\mathrm{C}}_{\mathrm{P}} = \pi\big(\bigcup \mathcal{C}_{\mathrm{P}} \cap \mathrm{S}\mathbb{H}_{\mathrm{st}}\big)$$

is a cross section for $\widehat{\Phi}$ with respect to any measure μ on $\mathscr{G}(\mathbb{X})$ fulfilling

$$\mu(\{\widehat{\gamma}_\nu \mid \nu \in \mathrm{S}\mathbb{H} \setminus \mathrm{S}\mathbb{H}_{\mathrm{st}}\}) = 0\,,$$

where $\mathrm{S}\mathbb{H}$ and $\mathrm{S}\mathbb{H}_{\mathrm{st}}$ are as in (1.18) and (2.12), respectively.

We retain all notation from Chapter 2.

Theorem 7.1. *The set $\mathcal{C}_{\mathrm{P}}$ from (7.1) is a set of branches for $\widehat{\Phi}$.*

Proof. The finiteness of $\mathcal{C}_{\mathrm{P}}$ follows immediately from its construction in Section 2.1. More precisely, the set BM from (2.7) is finite by virtue of (2.5), the map

$$\begin{cases} \mathrm{BM} & \longrightarrow & \mathcal{C}(\mathrm{BM}) \\ b & \longmapsto & \mathrm{C}_{\mathrm{P}}(b) \end{cases}$$

is a bijection, and $\mathcal{C}_{\mathrm{P}} \subseteq \mathcal{C}(\mathrm{BM})$.

Recall the set $E(\mathbb{X})$ from (1.41) and its density in $\Lambda(\Gamma) \times \Lambda(\Gamma)$ from Proposition 1.15. Let $j \in A$. The sets $I_{\mathrm{P},j}$ and $J_{\mathrm{P},j}$ are both open, wherefore we find

$$(x, y) \in E(\mathbb{X}) \cap \left(I_{\mathrm{P},j} \times J_{\mathrm{P},j}\right).$$

By Lemma 2.12 there exists $\nu \in \mathrm{C}_{\mathrm{P},j}$ such that

$$\left(\gamma_\nu(+\infty), \gamma_\nu(-\infty)\right) = (x, y)\,,$$

which yields (B1).

By construction, we have $\mathrm{bp}(\mathrm{C}_{\mathrm{P},j}) = (x_j, \infty)_{\mathbb{H}}$ with $x_j \in \mathbb{Q}$, for all $j \in A$ (see (2.6) and (2.13), or Corollary 2.6). Since ∞ represents a cusp of $\mathbb{X}$ and is therefore an element of $\widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$, property (B2) follows from Lemma 2.5.

Property (B3) is clear by definition: For every $b = (B, \beta) \in \mathrm{BM}$ the set $\mathrm{C}_{\mathrm{P}}(b)$ is the subset of unit tangent vectors based on the vertical side β of $B \in \mathbb{B}$ and pointing into B°. By Lemma 2.2, the set B° is completely contained in one of the two half-spaces H_1, H_2 relative to β, say $B^\circ \subseteq H_1$. Hence, every vector of $\mathrm{C}_{\mathrm{P}}(b)$ points into H_1.

Let again $j \in A$. Then, because of $\{x_j, \infty\} \subseteq \widehat{\mathbb{R}} \setminus \widehat{\mathbb{R}}_{\mathrm{st}}$ and Lemma 2.13, there exists a pair $(k, g) \in A \times \Gamma$ such that

$$\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \mathbb{R} \setminus \{x_j\} = I_{\mathrm{P},j} \cup J_{\mathrm{P},j} = I_{\mathrm{P},j} \cup g.I_{\mathrm{P},k}\,.$$

This yields (B4).

Property (B5) follows immediately from Lemma 2.12 and property (B6) is a consequence of

$$\Gamma.\mathcal{C}_{\mathrm{P}} \supseteq \mathcal{C}(\mathrm{BM})$$

and the minimality of $\mathcal{C}_{\mathrm{P}}$.

Finally, in order to verify (B7), let $j \in A$ and let $\nu \in \mathrm{C}_{\mathrm{P},j,\mathrm{st}}$. Because of Lemma 2.14, the number

$$t_{\mathrm{P}}^{+}(\nu) = \min\left\{t > 0 \;\middle|\; \gamma_\nu'(t) \in \Gamma.\mathrm{C}_{\mathrm{P}}\right\}$$

is well-defined and $\gamma'_\nu(t^+_{\mathrm{P}}(\nu)) \in \mathrm{C}_{\mathrm{P},\mathrm{st}}$. Because of (B6), there exist a unique index $k^+_{\mathrm{P}}(\nu) \in A$ and a unique transformation $g^+_{\mathrm{P}}(\nu) \in \Gamma$ such that

$$\gamma'_\nu(t^+_{\mathrm{P}}(\nu)) \in g^+_{\mathrm{P}}(\nu)\boldsymbol{.}\, \mathrm{C}_{\mathrm{P},k^+_{\mathrm{P}}(\nu)}\,.$$

By construction we have

$$g^+_{\mathrm{P}}(\nu)\boldsymbol{.}\mathrm{H}^{\mathrm{P}}_+(k^+_{\mathrm{P}}(\nu)) \subseteq \mathrm{H}^{\mathrm{P}}_+(j)\,,$$

hence,

$$g^+_{\mathrm{P}}(\nu)\boldsymbol{.}I_{\mathrm{P},k^+_{\mathrm{P}}(\nu)} \subseteq I_{\mathrm{P},j}\,. \tag{7.2}$$

For $k \in A$ we set

$$\mathrm{C}_{\mathrm{P},j}\,|_k \coloneqq \left\{\nu \in \mathrm{C}_{\mathrm{P},j} \;\middle|\; k^+_{\mathrm{P}}(\nu) = k\right\}\,.$$

Then

$$\mathrm{C}_{\mathrm{P},j,\mathrm{st}} = \bigcup_{k\in A} \left(\mathrm{C}_{\mathrm{P},j}\,|_k \cap \mathrm{S}\mathbb{H}_{\mathrm{st}}\right)\,, \tag{7.3}$$

where the union is clearly disjoint. We further define

$$\mathcal{G}(j,k) \coloneqq \bigcup_{\nu\in\mathrm{C}_{\mathrm{P},j}|_k} \{g^+_{\mathrm{P}}(\nu)\}\,.$$

Then the sets $\mathrm{C}_{\mathrm{P},j}\,|_k$ decompose further as

$$\mathrm{C}_{\mathrm{P},j}\,|_k = \bigcup_{g\in\mathcal{G}(j,k)} \left\{\nu \in \mathrm{C}_{\mathrm{P},j}\,|_k \;\middle|\; g^+_{\mathrm{P}}(\nu) = g\right\}\,. \tag{7.4}$$

Again, the union is disjoint. Because of (7.2) we further have

$$J_{\mathrm{P},j,\mathrm{st}} \subseteq g^+_{\mathrm{P}}(\nu)\boldsymbol{.}J_{\mathrm{P},k^+_{\mathrm{P}}(\nu),\mathrm{st}}\,.$$

By combining this with (B5) and (7.2) we obtain

$$\left\{\gamma_\nu(+\infty) \;\middle|\; \nu \in \mathrm{C}_{\mathrm{P},j}\,|_k\,,\; g^+_{\mathrm{P}}(\nu) = g\right\}_{\mathrm{st}} = \left\{\gamma_{g\boldsymbol{.}\eta}(+\infty) \mid \eta \in \mathrm{C}_{\mathrm{P},k}\right\}_{\mathrm{st}}\,.$$

Combination of this with (7.3), (7.4), and Lemma 2.12 in turn yields

$$
\begin{aligned}
I_{\mathrm{P},j,\mathrm{st}} &= \{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_{\mathrm{P},j}\}_{\mathrm{st}} = \bigcup_{k\in A} \{\gamma_\nu(+\infty) \mid \nu \in \mathrm{C}_{\mathrm{P},j}\,|_k\}_{\mathrm{st}} \\
&= \bigcup_{k\in A} \bigcup_{g\in\mathcal{G}(j,k)} \left\{\gamma_\nu(+\infty) \,\middle|\, \nu \in \mathrm{C}_{\mathrm{P},j}\,|_k\,,\ g_{\mathrm{P}}^+(\nu) = g\right\}_{\mathrm{st}} \\
&= \bigcup_{k\in A} \bigcup_{g\in\mathcal{G}(j,k)} g\boldsymbol{.}\{\gamma_\eta(+\infty) \mid \eta \in \mathrm{C}_{\mathrm{P},k}\}_{\mathrm{st}} \\
&= \bigcup_{k\in A} \bigcup_{g\in\mathcal{G}(j,k)} g\boldsymbol{.}I_{\mathrm{P},k,\mathrm{st}}\,,
\end{aligned}
$$

and the union is disjoint since those in (7.3) and (7.4) are. Hence, we obtain the second relation in (B7a). Combining it with (7.2) also yields the first. The definitions of the indices and transformations involved immediately imply (B7b). And for (B7c) we argue analogously by using $t_{\mathrm{P}}^-(\nu)$ from Lemma 2.14 instead of $t_{\mathrm{P}}^+(\nu)$. This completes the proof. □

Since a given set of branches can always be transformed into a simultaneously admissible, finitely ramified, and weakly non-collapsing one (see Proposition 4.35), we do not have to assure those properties here. However, finiteness of ramification is automatically fulfilled for all sets of branches emerging from a cusp expansion procedure.

Proposition 7.2. *The set of branches $\mathcal{C}_{\mathrm{P}}$ is finitely ramified.*

Proof. Let $j \in A$ and let $b_j = (B_j, \beta_j) \in \mathrm{BM}$ be such that $\mathrm{C}_{\mathrm{P},j} = \mathrm{C}_{\mathrm{P}}(b_j)$. Hence, $\beta_j \in S_{\mathbb{B}}^{\mathrm{V}}$ is a side of $B_j \in \mathbb{B}$ and

$$
\mathrm{C}_{\mathrm{P},j} = \left\{\nu \in \mathrm{S}\mathbb{H} \,\middle|\, \mathrm{bp}(\nu) \in \beta_j \text{ and } \nu \text{ points into } B_j^\circ\right\}\,.
$$

Let $\nu \in \mathrm{C}_{\mathrm{P},j,\mathrm{st}}$. By Lemma 2.2 the set B_j is a hyperbolic polygon with finitely many sides. The next intersection time $t_{\mathrm{P}}^+(\nu)$ exists by Lemma 2.14 and because of Lemma 2.7 and Lemma 2.4 we have

$$
\mathrm{bp}(\gamma_\nu'(t_{\mathrm{P}}^+(\nu))) \in \partial B_j\,.
$$

By Lemma 2.13, (B6), and again Lemma 2.7, for every side β of B_j we find a unique pair $(k, g) \in A \times \Gamma$ such that $g\boldsymbol{.}\overline{\mathrm{bp}(\mathrm{C}_{\mathrm{P},k})} = \beta$ and the vectors of $g\boldsymbol{.}\mathrm{C}_{\mathrm{P},k}$ do not point into B_j°. Since this exhausts all possibilities for the location of $\gamma_\nu'(t_{\mathrm{P}}^+(\nu))$, we have

$$
\#\left\{\left(k_{\mathrm{P}}^+(\nu), g_{\mathrm{P}}^+(\nu)\right) \,\middle|\, \nu \in \mathrm{C}_{\mathrm{P},j}\right\} = \#\{\text{sides of } B_j\} - 1 < +\infty\,,
$$

with $k_{\mathrm{P}}^+(\nu)$ and $g_{\mathrm{P}}^+(\nu)$ as in the proof of Theorem 7.1. This yields the claim. □

7.2 Orbisurfaces without Cusps

We retain the notion from Section 7.1, but, for the moment, abandon the orbisurfaces with cusps to study the situation when only funnels are present. Hence, we assume that Γ is a geometrically finite Fuchsian group that contains hyperbolic but no parabolic elements and for which the associated orbisurface $\mathbb{X}$ is not compact. By conjugation in $\mathrm{PSL}_2(\mathbb{R})$ we can always achieve that

($\star$) the ordinary set $\Omega(\Gamma) = \partial_g\mathbb{H} \setminus \Lambda(\Gamma)$ contains a neighborhood of ∞.

Therefore, we may assume that this is the case. Then the stabilizer subgroup Γ_∞ is trivial and $\mathrm{ISO}(\Gamma) \neq \varnothing$. By Proposition 1.42 the common exterior

$$\mathcal{K} = \bigcap_{\mathrm{I}\in\mathrm{ISO}(\Gamma)} \operatorname{ext}\mathrm{I} = \bigcap_{g\in\Gamma} \operatorname{ext}\mathrm{I}(\Gamma)$$

is a geometrically finite exact convex fundamental polygon for Γ.

Our strategy is as follows: We construct a new Fuchsian group $\Gamma_{\mathcal{W}}$ from Γ via a cut-off procedure on the fundamental domain $\mathcal{K}$. The group $\Gamma_{\mathcal{W}}$ then has a cusp represented by ∞ and is seen to fulfill all requirements of the cusp expansion algorithm. Hence, by virtue of Theorem 7.1, we obtain a set of branches $\mathcal{C}_{\mathcal{W}}$ for the geodesic flow on $\mathbb{X}_{\mathcal{W}} = \Gamma_{\mathcal{W}}\backslash\mathbb{H}$. We then return to Γ and see that $\mathcal{C}_{\mathcal{W}}$ induces a set of branches on the orbit space $\mathbb{X}$ of Γ as well.

Because of ($\star$) there exist $a, b \in \mathbb{R}$, $a < b$, such that

$$\bigcup \mathrm{ISO}(\Gamma) \subseteq \mathrm{Re}|_{\mathbb{H}}^{-1}\left([a,b]\right), \tag{7.5}$$

and we may assume that a and b are chosen optimal for that purpose, i. e., such that for every choice of $\varepsilon_1, \varepsilon_2 \geq 0$, not both equal to 0, the pair $(a+\varepsilon_1, b-\varepsilon_2)$ does not fulfill (7.5). Then there exist unique spheres $\mathrm{I}_1, \mathrm{I}_2 \in \mathrm{REL}(\Gamma)$ such that

$$a \in \partial_g \mathrm{I}_1 \qquad \text{and} \qquad b \in \partial_g \mathrm{I}_2\ .$$

This implies

$$(b,a)_c \subseteq \Omega(\Gamma)\,. \tag{7.6}$$

Denote by Γ_{REL} the set of generators of isometric spheres as in Section 1.10. The following result is immediate from Lemma 1.20(i) and $\Gamma_\infty = \{\mathrm{id}\}$.

Lemma 7.3. *The maps*

$$\left\{\begin{array}{ccc} \Gamma\setminus\{\mathrm{id}\} & \longrightarrow & \mathrm{ISO}(\Gamma) \\ g & \longmapsto & \mathrm{I}(g) \end{array}\right. \qquad \textit{and} \qquad \left\{\begin{array}{ccc} \Gamma_{\mathrm{REL}} & \longrightarrow & \mathrm{REL}(\Gamma) \\ g & \longmapsto & \mathrm{I}(g) \end{array}\right.$$

are both bijections.

Corollary 7.4. *There exist unique transformations* $g_1, g_2 \in \Gamma_{\mathrm{REL}}$ *such that*

$$a \in \underline{g}\mathrm{I}(g_1) \qquad \text{and} \qquad b \in \underline{g}\mathrm{I}(g_2)\,.$$

From Lemma 7.3 we further obtain that Γ_{REL} is the unique side-pairing for $\mathcal{K}$ in Γ (see Definition 1.30).

Recall the sets of (finite and infinite) vertices $V_{\mathcal{F}}$ and $V_{\mathcal{F}}^{g}$ of a geometrically finite polygon, the cycle transformations c_v, and the vertex cycles $C(v)$ for $v \in V_{\mathcal{K}}$ from Section 1.10, as well as the angle sum $\theta(C(v))$ from (1.63). Since $\mathcal{K}$ is a geometrically finite fundamental domain for Γ, we obtain from Lemma 1.34 that, for every $v \in V_{\mathcal{K}}$,

$$\frac{2\pi}{\theta(C(v))} \in \mathbb{N}\,. \tag{7.7}$$

Furthermore, $V_{\mathcal{K}}^{g} = \varnothing$ (see Remark 1.35).

7.2.1 An Auxiliary Group

We now define the group $\Gamma_{\mathcal{W}}$ described above. To that end we fix choices of

$$a' \in (-\infty, a)_{\mathbb{R}} \qquad \text{and} \qquad b' \in (b, +\infty)_{\mathbb{R}}$$

and set

$$\lambda := b' - a'\,. \tag{7.8}$$

The domain

$$\mathcal{W} := \mathcal{K} \cap \mathrm{Re}|_{\mathbb{H}}^{-1}\big((a', b')\big) \subseteq \mathbb{H} \tag{7.9}$$

will play the role of a fundamental domain for $\Gamma_{\mathcal{W}}$. Indeed, it is immediately clear that $\mathcal{W}$ is again a geometrically finite convex polygon. For $S_{\mathcal{M}}$ denoting the set of sides of $\mathcal{M} \in \{\mathcal{K}, \mathcal{W}\}$ we find

$$S_{\mathcal{W}} = S_{\mathcal{K}} \cup \big\{(a', \infty)_{\mathbb{H}}, (b', \infty)_{\mathbb{H}}\big\}\,.$$

Thus, a side-pairing for $\mathcal{W}$ is given by

$$G_{\mathcal{W}} := \Gamma_{\mathrm{REL}} \cup \{\mathrm{t}_{\lambda}^{\pm 1}\}\,,$$

with t_{λ} as in (1.7). We further infer

$$V_{\mathcal{W}} = V_{\mathcal{K}} \qquad \text{and} \qquad V_{\mathcal{W}}^{g} = \{\infty\}\,.$$

Lemma 7.5. *The subgroup* $\Gamma_{\mathcal{W}} := \langle G_{\mathcal{W}} \rangle$ *of* $\mathrm{PSL}_2(\mathbb{R})$ *is a geometrically finite Fuchsian group whose orbit space* $\mathbb{X}_{\mathcal{W}} := \Gamma_{\mathcal{W}} \backslash \mathbb{H}$ *bears a single cusp and* $\mathcal{W}$ *is a convex fundamental polygon for* $\Gamma_{\mathcal{W}}$.

Proof. We want to apply Poincaré's theorem (Proposition 1.36). We have already seen that $\mathcal{W}$ is a convex polygon with a side-pairing. Thus, it remains to check that $G_{\mathcal{W}}$ fulfills the conditions (I) and (II) of Proposition 1.36. Condition (I) immediately follows from $V_{\mathcal{W}} = V_{\mathcal{K}}$ and (7.7). The sides of $\mathcal{W}$ adjacent to ∞ are $(a', \infty)_{\mathbb{H}}$ and (b', ∞), which are paired by $\mathrm{t}_\lambda^{\pm 1}$. Thus, $c_\infty \in \{\mathrm{t}_\lambda^{\pm 1}\}$, depending on choice of sign. Either way, c_∞ is parabolic, which, because of $V_{\mathcal{W}}^{g} = \{\infty\}$, implies (II). Hence, $\Gamma_{\mathcal{W}}$ is a Fuchsian group.

Since $\#S_{\mathcal{W}} = \#S_{\mathcal{K}} + 2 < +\infty$, the polygon $\mathcal{W}$ is geometrically finite. As we have already established, it is further convex and exact. Thus, the group $\Gamma_{\mathcal{W}}$ is geometrically finite. And since ∞ is the sole infinite vertex of $\mathcal{W}$, Proposition 1.43(ii) implies that $\mathbb{X}_{\mathcal{W}}$ has exactly one cusp. □

Denote by $\widehat{\mathbb{R}}_{\mathrm{st},\mathcal{W}}$ the set $\widehat{\mathbb{R}}_{\mathrm{st}}$ with respect to $\Gamma_{\mathcal{W}}$, that is

$$\widehat{\mathbb{R}}_{\mathrm{st},\mathcal{W}} \coloneqq \Lambda(\Gamma_{\mathcal{W}}) \setminus \Gamma_{\mathcal{W}}.\infty\,.$$

Definition 7.5 effectively defines $\Gamma_{\mathcal{W}}$ as the group that emerges from Γ via the addition of t_λ to the set of generators. Or in other words, Γ is a non-trivial subgroup of $\Gamma_{\mathcal{W}}$. From this the following result is immediate.

Corollary 7.6. *We have $\Lambda(\Gamma) \subseteq \Lambda(\Gamma_{\mathcal{W}})$, $\widehat{\mathbb{R}}_{\mathrm{st}} \subseteq \widehat{\mathbb{R}}_{\mathrm{st},\mathcal{W}}$, and $E(\mathbb{X}) \subseteq E(\mathbb{X}_{\mathcal{W}})$.*

In order to apply the cusp expansion algorithm to $\Gamma_{\mathcal{W}}$ we require the common exterior with respect to $\Gamma_{\mathcal{W}}$, i. e., the set

$$\mathcal{K}_{\mathcal{W}} \coloneqq \mathcal{K}_{\Gamma_{\mathcal{W}}} = \bigcap_{\mathrm{I} \in \mathrm{ISO}(\Gamma_{\mathcal{W}})} \operatorname{ext} \mathrm{I} = \bigcap_{g \in \Gamma_{\mathcal{W}} \setminus \Gamma_{\mathcal{W},\infty}} \operatorname{ext} \mathrm{I}(g)\,, \tag{7.10}$$

where $\Gamma_{\mathcal{W},\infty}$ denotes the stabilizer of ∞ in $\Gamma_{\mathcal{W}}$. If $\mathcal{F}_{\mathcal{W}}$ is a Ford fundamental domain for $\Gamma_{\mathcal{W}}$, then we can re-obtain $\mathcal{K}_{\mathcal{W}}$ by means of the $\Gamma_{\mathcal{W},\infty}$-invariance of $\mathcal{K}_{\mathcal{W}}$ (see (1.74)). We show that $\mathcal{W}$ is a Ford fundamental domain for $\Gamma_{\mathcal{W}}$, starting with the verification that λ from (7.8) is the cusp width of the one cusp of $\mathbb{X}_{\mathcal{W}}$.

Lemma 7.7. *The stabilizer $\Gamma_{\mathcal{W},\infty}$ of ∞ in $\Gamma_{\mathcal{W}}$ is generated by t_λ.*

Proof. By construction, $\mathrm{t}_\lambda \in \Gamma_{\mathcal{W}}$, t_λ is parabolic and fixes ∞, and thus every non-identity transformation in $\Gamma_{\mathcal{W}}$ fixing ∞ must be parabolic by virtue of Lemma 1.8. In particular, every non-identity element in $\Gamma_{\mathcal{W},\infty}$ has the same fixed point set, which, by Lemma 1.7, implies that $\Gamma_{\mathcal{W},\infty}$ is cyclic and thus generated by some element t_κ with $|\kappa| \leq \lambda$. We suppose for contradiction that $|\kappa| < \lambda$ and we may assume $\kappa > 0$ without loss of generality. Because of (7.5) (or by Proposition 1.24) the set $\{r(\mathrm{I}) \mid \mathrm{I} \in \mathrm{ISO}(\Gamma)\}$ is bounded from above. Since the non-vertical sides of $\mathcal{W}$ coincide with those of $\mathcal{K}$, there exists $M > 0$ such that

$$\mathcal{W}_M \coloneqq \left\{ z \in \mathrm{Re}|_{\mathbb{H}}^{-1}\big((a', b')\big) \;\middle|\; \mathrm{Im}\, z \geq M \right\} \subseteq \mathcal{W}\,.$$

Let $\eta := (\lambda - \kappa)/2$ and $y > M$. Then $\eta < \lambda/2$ and thus, $z^* := a' + \eta + \mathrm{i}y \in \mathcal{W}_M$. But also

$$\mathrm{t}_\kappa.z^* = a' + \eta + \kappa + \mathrm{i}y = a' + \lambda - \eta + \mathrm{i}y = b' - \eta + \mathrm{i}y \in \mathcal{W}_M\,,$$

which contradicts $\mathcal{W}$ being a fundamental domain for $\Gamma_\mathcal{W}$. Thus, $|\kappa| = \lambda$, which yields the assertion. □

Because of Lemma 7.7, the strip

$$\mathcal{F}_{\mathcal{W},\infty} := \mathrm{Re}|_{\mathbb{H}}^{-1}\big((a', b')\big)$$

is a fundamental domain for $\Gamma_{\mathcal{W},\infty}$ in $\mathbb{H}$. As before, we denote by $\mathrm{REL}(\Gamma_\mathcal{W})$ the set of relevant isometric spheres of $\Gamma_\mathcal{W}$. In addition, we denote by $\mathrm{REL}_\mathcal{W}$ the subset of isometric spheres of $\Gamma_\mathcal{W}$ that contribute non-trivially to the boundary of $\mathcal{W}$.

Lemma 7.8. $\mathrm{REL}(\Gamma) = \mathrm{REL}_\mathcal{W}$.

Proof. By construction we have $\Gamma_{\mathrm{REL}} \subseteq \Gamma_\mathcal{W}$. Since the non-vertical sides of $\mathcal{W}$ coincide with those of $\mathcal{K}$, it follows that

$$\mathrm{REL}(\Gamma) \subseteq \mathrm{REL}_\mathcal{W}\,.$$

For $\mathrm{I} \in \mathrm{REL}_\mathcal{W}$ the geodesic segment $\mathrm{I} \cap \partial\mathcal{K}$ contains more than one point, implying

$$\mathrm{REL}(\Gamma) \supseteq \mathrm{REL}_\mathcal{W}\,.$$ □

Proposition 7.9. *The fundamental domain $\mathcal{W}$ for $\Gamma_\mathcal{W}$ is of the Ford type.*

Proof. Because of Lemma 7.7 it remains to show that

$$\mathcal{W} = \mathcal{F}_{\mathcal{W},\infty} \cap \bigcap_{g \in \Gamma_\mathcal{W} \setminus \Gamma_{\mathcal{W},\infty}} \operatorname{ext} \mathrm{I}(g) = \mathcal{F}_{\mathcal{W},\infty} \cap \bigcap_{\mathrm{I} \in \mathrm{REL}(\Gamma_\mathcal{W})} \operatorname{ext}\, \mathrm{I}\,.$$

From Lemma 7.8 we obtain

$$\begin{aligned}\mathcal{W} = \mathrm{Re}|_{\mathbb{H}}^{-1}\big(a', b'\big) \cap \mathcal{K} &= \mathcal{F}_{\mathcal{W},\infty} \cap \bigcap_{\mathrm{I} \in \mathrm{REL}(\Gamma)} \operatorname{ext}\, \mathrm{I} \\ &= \mathcal{F}_{\mathcal{W},\infty} \cap \bigcap_{\mathrm{I} \in \mathrm{REL}_\mathcal{W}} \operatorname{ext}\, \mathrm{I} = \mathcal{F}_{\mathcal{W},\infty} \cap \bigcap_{\mathrm{I} \in \mathrm{REL}(\Gamma_\mathcal{W})} \operatorname{ext}\, \mathrm{I}\,,\end{aligned}$$

and the claim follows. □

Corollary 7.10. *If Γ fulfills condition (A), then so does $\Gamma_\mathcal{W}$.*

Assume that Γ fulfills condition (A). Then, by Theorem 7.1, Section 2.1 yields a set of branches for the geodesic flow on $\mathbb{X}_{\mathcal{W}}$, which here we denote by

$$\mathcal{C}_{\mathcal{W}} = \{\mathrm{C}_{\mathcal{W},1}, \ldots, \mathrm{C}_{\mathcal{W},N}\}\,,$$

and by Lemma 2.17 we may assume that

$$\mathrm{bp}(\mathrm{C}_{\mathcal{W},j}) \subseteq \mathrm{Re}|_{\mathbb{H}}^{-1}\big([a', b']\big)\,, \tag{7.11}$$

for every $j \in A := \{1, \ldots, N\}$. We further denote by $\mathcal{G}_{\mathcal{W}}(j,k)$ the transition set for $j,k \in A$ given by (B7) as well as by $I_{\mathcal{W},j}$ and $J_{\mathcal{W},j}$ the intervals associated to $\mathrm{C}_{\mathcal{W},j}$ by (B3). Because of (2.13), for every $j \in A$ the set $\mathrm{Re}(\mathrm{bp}(\mathrm{C}_{\mathcal{W},j}))$ is a singleton in $\mathbb{R}$, and we denote, as before, by $x_j \in \mathbb{R}$ the unique point it contains.

7.2.2 A Set of Branches for Γ

We now transfer the set of branches $\mathcal{C}_{\mathcal{W}}$ back to the orbit space $\mathbb{X}$ of the initial group Γ whose hyperbolic ends are all funnels. We emphasize again that the set of branches $\mathcal{C}_{\mathcal{W}}$ emerged by a cusp expansion procedure for $\Gamma_{\mathcal{W}}$ and therefore bears additional structure beyond that provided in Definition 4.1, and that we do exploit this additional structure. Hence, we do not claim that an arbitrary set of branches for $\Gamma_{\mathcal{W}}$ induces a set of branches for Γ, neither in the manner presented in this section, nor anyhow.

Not all of the branches $\mathrm{C}_{\mathcal{W},j}$ "survive" the transfer to Γ. We clarify what we mean by that: Since Γ contains no parabolic elements and the ordinary set is assumed to contain a neighborhood of ∞, we have

$$\mathbb{R}_{\mathrm{st}} = \widehat{\mathbb{R}}_{\mathrm{st}} = \Lambda(\Gamma)\,.$$

For $j \in A$ define

$$\mathrm{C}_{\mathcal{W},j,\mathrm{st}} := \mathrm{C}_{\mathcal{W},j} \cap \mathrm{S}\mathbb{H}_{\mathrm{st}}\,,$$

with $\mathrm{S}\mathbb{H}_{\mathrm{st}}$ as in (2.12). Then $\mathrm{C}_{\mathcal{W},j,\mathrm{st}} = \varnothing$ whenever

$$I_{\mathcal{W},j} \cap \mathbb{R}_{\mathrm{st}} = \varnothing \qquad \text{or} \qquad J_{\mathcal{W},j} \cap \mathbb{R}_{\mathrm{st}} = \varnothing\,.$$

This is the case, for instance, if $x_j \in \{a', b'\}$. Since the cusp expansion algorithm for $\Gamma_{\mathcal{W}}$ does indeed establish branches with that property—note that

$$\{a', b'\} \subseteq \widetilde{W}_{\mathcal{K}_{\mathcal{W}}}\,,$$

with $\widetilde{W}_{\mathcal{K}}$ as in (2.1)—and those branches are not intersected by periodic geodesics of Γ, it is necessary to exclude those from the set of branches in order to fulfill the demands of (B1). Therefore, we define

$$A' := \{j \in A \mid \mathrm{C}_{\mathcal{W},j,\mathrm{st}} \neq \varnothing\} \ .$$

Proposition 7.11. *We have $A' \neq \varnothing$.*

The proof of Proposition 7.11 makes use of Proposition 1.26. There it was assumed that Γ is either non-elementary or a hyperbolic cylinder. By the discussion in Section 1.8, the groups which are excluded by that and contain hyperbolic elements are exactly the groups conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to

$$\left\langle \mathrm{s}_{\frac{\pi}{2}}, \mathrm{h}_\ell \,\middle|\, \mathrm{s}_{\frac{\pi}{2}}^2 = \mathrm{id} \right\rangle \ ,$$

for any $\ell > 1$, where h_* and s_* are as in (1.6) and (1.8), respectively. We therefore have to treat these groups separately, which is done in the following example. This example further serves to illustrate the strategy of the ensuing proof of Proposition 7.11: Utilizing Proposition 1.26 and the density of the set $E(\mathbb{X})$ from (1.41) in $\Lambda(\Gamma) \times \Lambda(\Gamma)$, we find interrelated hyperbolic fixed points underneath the outermost isometric spheres (more precisely, in the intervals $\mathcal{W}(g_{1/2})$ from (1.55), for $g_{1/2}$ the unique transformations from Corollary 7.4). We then identify a branch copy separating the hyperbolic fixed points, which is then seen to be intersected by the associated hyperbolic axis.

Example 7.12. We consider the conjugation of the aforementioned group by the transformation $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$, which leads to the generators

$$h_\ell := \frac{1}{2\sqrt{\ell}} \begin{bmatrix} \ell+1 & \ell-1 \\ \ell-1 & \ell+1 \end{bmatrix} \qquad \text{and} \qquad s := \mathrm{s}_{\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ,$$

for $\ell > 1$. Then

$$h_\ell.1 = 1\,, \quad h_\ell.(-1) = -1 \quad \text{and} \quad h_\ell.\mathrm{i} = 1 - \frac{2}{\ell^2+1} + \frac{2\mathrm{i}\ell}{\ell^2+1} ,$$

which, because of $\ell > 1$, identifies 1 as the attractor of h_ℓ. A fundamental domain is indicated in Figure 23.

We proceed as described above in order to find a set of branches. Choose, for instance,

$$a' := \frac{\ell+1+3\sqrt{\ell}}{1-\ell} \qquad \text{and} \qquad b' := \frac{\ell+1+3\sqrt{\ell}}{\ell-1} \,.$$

A set of branches as constructed by the cusp expansion algorithm is indicated in Figure 24. From Figure 24 it already becomes apparent that the axis of h_ℓ inter-

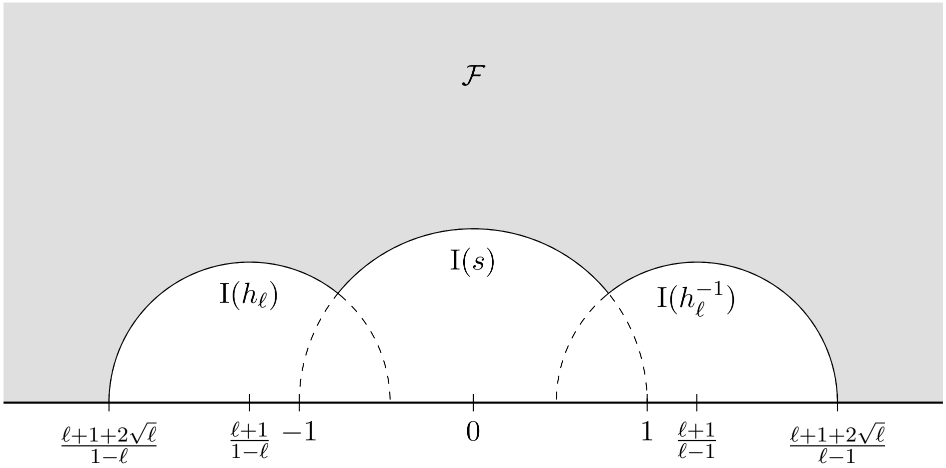


Figure 23: A Ford fundamental domain $\mathcal{F}$ for $\langle h_\ell, s\rangle$. Since h_ℓ fixes 1 and -1, we have $\alpha(h_\ell)(\mathbb{R}) = \mathrm{I}(s)$. Thus, the angle that $\mathcal{F}$ subtends at the intersection points of the isometric spheres is $\pi/2$ each (see Lemma 1.21(iv)), which implies that $\mathcal{F}$ fulfills all requirements of Proposition 1.36.

sects $\mathrm{C}_{\mathcal{W},4}$. Indeed,

$$\mathrm{C}_{\mathcal{W},4} = \{\nu \in \mathrm{S}\mathbb{H} \mid \mathrm{bp}(\nu) \in (0,\infty)_{\mathbb{H}}\,,\ \gamma_\nu(+\infty) \in (0,+\infty)\}$$

and thus,

$$-1 \in J_{\mathcal{W},4,\mathrm{st}} \qquad \text{and} \qquad 1 \in I_{\mathcal{W},4,\mathrm{st}}\,.$$

Hence, by (B5) there exists $\nu \in \mathrm{C}_{\mathcal{W},4}$ such that

$$[\gamma_\nu] = \alpha(h_\ell)\,.$$

It follows that $4 \in A'$. In fact, in this example we find $A' = \{4\}$.

We further require the following observation.

Lemma 7.13. *Let $j \in A$ and $g \in \Gamma_{\mathcal{W}}$ be such that $g.\mathrm{bp}(\mathrm{C}_{\mathcal{W},j})$ is vertical and contained in $\mathrm{Re}|_{\mathbb{H}}^{-1}((a',b'))$. Then $g \in \Gamma$.*

Proof. Assume $g \neq \mathrm{id}$, for otherwise there is nothing to show. Since $\mathrm{bp}(\mathrm{C}_{\mathcal{W},j})$ and $g.\mathrm{bp}(\mathrm{C}_{\mathcal{W},j})$ are both vertical, for $y \in \mathrm{Re}(g.\mathrm{bp}(\mathrm{C}_{\mathcal{W},j}))$ either

$$\begin{aligned} &y = g.\infty \qquad \text{and} \qquad \infty = g.x_k\,, \qquad \text{or}\\ &y = g.x_k \qquad \text{and} \qquad g \in \Gamma_{\mathcal{W},\infty}\,. \end{aligned}$$

Because of (7.11) and $g \neq \mathrm{id}$ the latter case implies that $\{y, x_k\} = \{a', b'\}$, which contradicts the choice of j and g. Hence, the former must hold, which implies in

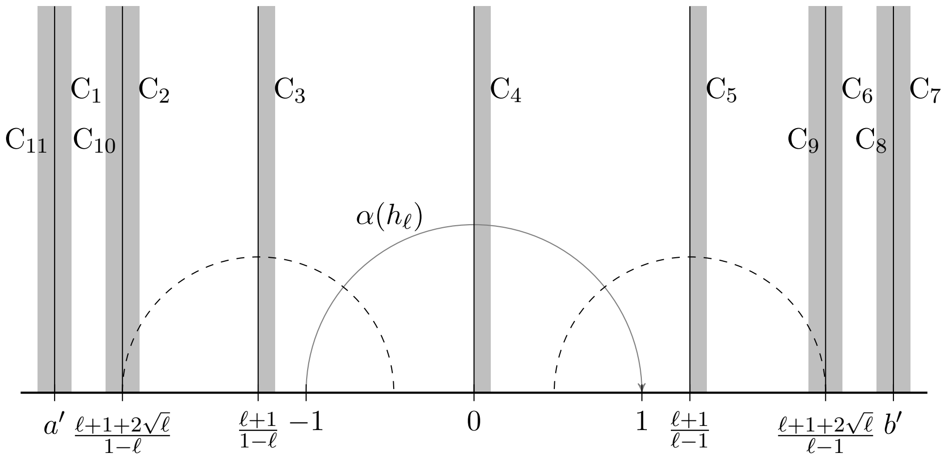


Figure 24: A set of branches obtained by application of the cusp expansion algorithm. As before, the gray stripes indicate that the respective set $\mathrm{C}_j = \mathrm{C}_{\mathcal{W},j}$ consists of unit tangent vectors based on the adjacent vertical geodesic and pointing into the indicated half-space. The subscript "$\mathcal{W}$" is omitted in favor of readability.

particular that y equals the center of $\mathrm{I}(g^{-1})$ and x_k equals the center of $\mathrm{I}(g)$ (see Lemma 1.19(i)). Because of that, Lemma 2.15 implies

$$\{\mathrm{I}(g), \mathrm{I}(g^{-1})\} \subseteq \mathrm{REL}(\Gamma_{\mathcal{W}})\,.$$

Combining this with (7.11) and Lemma 7.8 yields

$$\{\mathrm{I}(g), \mathrm{I}(g^{-1})\} \subseteq \mathrm{REL}_{\mathcal{W}} = \mathrm{REL}(\Gamma)\,.$$

This together with Lemma 7.3 yields a unique $h \in \Gamma_{\mathrm{REL}}$ such that $\mathrm{I}(h) = \mathrm{I}(g)$. By Lemma 1.20(i) this implies $g = \mathrm{t}_\lambda^n h$ with some $n \in \mathbb{Z}$ and λ as in (7.8). Now Lemma 1.20(ii) yields

$$\mathrm{I}(h^{-1}) = \mathrm{I}(g^{-1}\mathrm{t}_\lambda^n) = \mathrm{t}_\lambda^{-n}.\mathrm{I}(g^{-1})\,.$$

Because of Proposition 1.41 we also have $\mathrm{I}(h^{-1}) \in \mathrm{REL}_{\mathcal{W}}$, hence in particular $\mathrm{I}(h^{-1}) \in \mathrm{Re}|_{\mathbb{H}}^{-1}([a', b'])$. This leaves $n = 0$ as the only possibility, implying $g \in \Gamma$. $\square$

Recall the set $\widetilde{W}_{\mathcal{K}}$ for $\mathcal{K}$ the common exterior from (2.1). We write $\widetilde{W}_{\mathcal{K}_{\mathcal{W}}}$ for this set with respect to the common exterior $\mathcal{K}_{\mathcal{W}}$ of $\Gamma_{\mathcal{W}}$ (see (7.10)).

Proof of Proposition 7.11. Because of Example 7.12 it suffices to consider groups Γ non-conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to $\langle \mathrm{s}_{\frac{\pi}{2}}, \mathrm{h}_\ell \rangle$, $\ell > 0$.

Since Γ is geometrically finite, the set $\mathrm{REL}(\Gamma)$ is finite. By the choice of the points $a, b \in \mathbb{R}$ there exist $\mathrm{I}_1, \mathrm{I}_2 \in \mathrm{REL}(\Gamma)$ such that

$$a \in \not{g}\, \mathrm{I}_1 \qquad \text{and} \qquad b \in \not{g}\, \mathrm{I}_2 \,. \tag{7.12}$$

Since Γ is assumed to contain hyperbolic elements, $\mathrm{REL}(\Gamma)$ is not a singleton by virtue of Proposition 1.41 and hyperbolic elements being of infinite order (see the discussion before Lemma 1.9), and thus, $\mathrm{I}_1 \neq \mathrm{I}_2$. Because of Corollary 7.4 there exist uniquely determined $g_1, g_2 \in \Gamma$, $g_1 \neq g_2$, such that $\mathrm{I}_\iota = \mathrm{I}(g_\iota)$ for $\iota \in \{1, 2\}$. Denote by

$$\beta_\iota \coloneqq \beta_{\mathrm{I}_\iota} = \partial\mathcal{W} \cap \mathrm{I}_\iota$$

the relevant part of I_ι. If $\beta_1 \cap \beta_2 \neq \varnothing$, then $\mathrm{I}(g_1) \cap \mathrm{I}(g_2) \neq \varnothing$ and (7.12) implies

$$\mathrm{REL}(\Gamma) = \{\mathrm{I}_1, \mathrm{I}_2\} \,.$$

The combination of Lemma 1.21(i) and Proposition 1.41 implies that $g_1 = g_2^{-1}$ and both are elliptic. Hence, Γ is cyclic, generated by an elliptic transformation, and thus void of hyperbolic elements. Since this contradicts the assumption, we conclude

$$\beta_1 \cap \beta_2 = \varnothing \,. \tag{7.13}$$

From here on we distinguish the cases $g_1 = g_2^{-1}$ and $g_1 \neq g_2^{-1}$, starting with the latter. Because of (7.12) the geodesic segments $\beta_1, \beta_2 \subseteq \partial\mathcal{W}$ have at least one endpoint in $\partial_g \mathbb{H}$. Since every element of Γ fixes $\partial_g \mathbb{H}$, the geodesic segments $g_1.\beta_1$ and $g_2.\beta_2$ have one endpoint in $\partial_g \mathbb{H}$ as well. Furthermore, because of Proposition 1.41,

$$g_1.\beta_1 \,,\ g_2.\beta_2 \subseteq \partial\mathcal{W} \,.$$

This implies $\not{g}(g_1.\beta_1) \subseteq (a, b)$, and thus $g_1.(b, a)_c \subseteq (a, b)$. Hence, there exists an interval $I \subseteq (a, b)$, say $I = (g_1.a, x)$ with $x > g_1.a$, such that

$$\mathrm{Re}|_{\mathbb{H}}^{-1}(I) \subseteq \mathcal{W} \quad \text{and} \quad \mathrm{Re}|_{\mathbb{H}}^{-1}(I) \cap \mathrm{Re}|_{\mathbb{H}}^{-1}(\mathscr{W}(g_\iota)) = \varnothing \,, \tag{7.14}$$

for $\iota \in \{1, 2\}$ and $\mathscr{W}(g_\iota)$ as in (1.55). Hence, $g_1.a \in \widetilde{W}_{\mathcal{K}_\mathcal{W}}$ and $\mathrm{Re}|_{\mathbb{H}}^{-1}(I)$ is contained in a cell B (see Lemma 2.2). This implies

$$\big(B, \mathrm{Re}|_{\mathbb{H}}^{-1}(g_1.a)\big) \in \mathrm{BM} \,,$$

with BM as in (2.7), and thus there exits a tuple $(k, h) \in A \times \Gamma_\mathcal{W}$ such that

$$h.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k}) \subseteq \mathrm{Re}|_{\mathbb{H}}^{-1}(g_1.a) \qquad \text{and} \qquad I \subseteq h.I_k \,.$$

Lemma 7.13 yields $h \in \Gamma$ and from (7.14) and the choice of g_1, g_2 we obtain

$$\mathscr{W}(g_1) \subseteq h.J_k \qquad \text{and} \qquad \mathscr{W}(g_2) \subseteq h.I_k \,. \tag{7.15}$$

Because of Proposition 1.26 we have $\Lambda(\Gamma) \cap \mathscr{W}(g_\iota) \neq \varnothing$, for $\iota \in \{1,2\}$. Since the sets $\mathscr{W}(g_\iota)$ are open, we find $x_1, x_2 \in \Lambda(\Gamma)$ and $\varepsilon > 0$ such that

$$(x_\iota - \varepsilon, x_\iota + \varepsilon) \subseteq \mathscr{W}(g_\iota)\,,$$

for $\iota \in \{1,2\}$. From Proposition 1.15 we obtain $\gamma \in \mathscr{G}_{\mathrm{Per},\Gamma}(\mathbb{H})$ such that

$$\big(\gamma(+\infty), \gamma(-\infty)\big) \in (x_2 - \varepsilon, x_2 + \varepsilon) \times (x_1 - \varepsilon, x_1 + \varepsilon)\,. \tag{7.16}$$

The combination of (7.15) and (7.16) with $\gamma(+\infty), \gamma(-\infty) \in \Lambda(\Gamma)$ implies

$$\big(h^{-1}.\gamma(+\infty), h^{-1}.\gamma(-\infty)\big) \in I_{k,\mathrm{st}} \times J_{k,\mathrm{st}}\,.$$

Because of Corollary 7.6 this remains valid in the context of $\Gamma_{\mathcal{W}}$, and therefore Lemma 2.12 yields $k \in A'$.

Now assume that $g_1 = g_2^{-1}$. If Γ is cyclic, then $\mathrm{REL}(\Gamma) = \{\mathrm{I}_1, \mathrm{I}_2\}$. By Lemma 1.21(i) we again find an interval $I \subseteq (a,b)$ fulfilling (7.14), and from there on may argue as before.

Now assume that Γ is non-cyclic. Let $\xi_1, \xi_2 \in \overline{\mathbb{H}}^g$ be such that

$$\overline{\beta_1}^g = [a, \xi_1]_{\mathbb{H}} \qquad \text{and} \qquad \overline{\beta_2}^g = [\xi_2, b]_{\mathbb{H}}\,.$$

Because of Proposition 1.41 we have $g_1.\beta_1 = \beta_2$. Since $\beta_1, \beta_2 \subseteq \partial_g \mathcal{W}$, the combination of that with Proposition 7.9 and Lemma 1.44 implies

$$\operatorname{Im} \xi_1 = \operatorname{Im} \xi_2\,. \tag{7.17}$$

Since Γ is non-cyclic, the boundary of $\mathcal{W}$ consists of further segments besides the segments β_1, β_2 (for otherwise $\Gamma = \langle g_1, g_2 \rangle = \langle g_1 \rangle$ by Proposition 1.36). Because of Lemma 1.45, at least one of these further segments contains the summit of its associated isometric sphere. More precisely, there exists $\mathrm{I}_3 \in \mathrm{REL}(\Gamma) \setminus \{\mathrm{I}_1, \mathrm{I}_2\}$ such that $s(\mathrm{I}_3) \in \beta_{\mathrm{I}_3}$ and

$$c(\mathrm{I}_1) = \operatorname{Re} s(\mathrm{I}_1) < \operatorname{Re} s(\mathrm{I}_3) = c(\mathrm{I}_3) < \operatorname{Re} s(\mathrm{I}_2) = c(\mathrm{I}_2)\,. \tag{7.18}$$

By Lemma 2.13 and Lemma 2.16, there exists a pair $(k,h) \in A \times \Gamma_{\mathcal{W}}$ such that

$$h.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k}) = \big(c(\mathrm{I}_3), \infty\big)_{\mathbb{H}} \qquad \text{and} \qquad \big(c(\mathrm{I}_2), c(\mathrm{I}_1)\big) \in h.I_k \times h.J_k\,.$$

Again, Lemma 7.13 yields $h \in \Gamma$. By following the structure of the argument above and taking the Γ-invariance of $\Lambda(\Gamma)$ into account, we see that it suffices to show that $c(\mathrm{I}_3)$ separates at least some points in $\mathscr{W}(g_1) \cap \Lambda(\Gamma)$ from at least some points in $\mathscr{W}(g_2) \cap \Lambda(\Gamma)$, or in other words,

$$\mathscr{W}(g_1) \cap \Lambda(\Gamma) \cap h.J_k \neq \varnothing \qquad \text{and} \qquad \mathscr{W}(g_2) \cap \Lambda(\Gamma) \cap h.I_k \neq \varnothing\,. \tag{7.19}$$

In order to see this we distinguish several cases, starting with the assumption that g_1 (and thus also g_2) is hyperbolic. By Lemma 1.21(i) we have $\mathrm{I}_1 \cap \mathrm{I}_2 = \varnothing$, and therefore, $\mathscr{W}(g_1) \cap \mathscr{W}(g_2) = \varnothing$. If $c(\mathrm{I}_3) \notin \mathscr{W}(g_1) \cup \mathscr{W}(g_2)$, then (7.19) follows immediately from (7.18). Thus, assume that this is not the case. Without loss of generality we may assume that $c(\mathrm{I}_3) \in \mathscr{W}(g_1)$. Then $\mathscr{W}(g_2) \subseteq h.I_k$ by construction, and hence Proposition 1.26 implies that $\Lambda(\Gamma) \nsubseteq h.J_k$. Suppose for contradiction that $\Lambda(\Gamma) \subseteq h.I_k$. Denote by g_3 the generator of I_3, which is unique by Lemma 7.3. By construction we have $g_3 \notin \{g_1, g_2\}$. The transformation g_3 cannot be an involution, for then $g_3 h.I_k = h.J_k$, and since Γ-action preserves $\Lambda(\Gamma)$, we would obtain a contradiction to the assumption. Therefore, $c(g_3) \neq c(g_3^{-1})$, and we show that

$$c(g_3^{-1}) \in \big(c(g_1), c(g_2) - r\big)\,, \tag{7.20}$$

with $r := r(g_1) = r(g_2)$. To that end we first show that

$$\{c(g_3), c(g_3^{-1})\} \subseteq \big(c(g_1), c(g_2)\big)\,. \tag{7.21}$$

Let $x \in \{c(g_3), c(g_3^{-1})\}$. Since $r' := r(g_3) = r(g_3^{-1})$, we then have

$$x + \mathrm{i}r' \in \{s(g_3), s(g_3^{-1})\}\,.$$

From (1.53), $s(\mathrm{I}_3) \in \beta_{\mathrm{I}_3}$, and Proposition 1.41 we obtain $\{s(g_3), s(g_3^{-1})\} \subseteq \partial\mathcal{W}$. In particular, neither summit is contained in $\operatorname{int} \mathrm{I}_1 \cup \operatorname{int} \mathrm{I}_2$. Since $\mathrm{I}_3 \notin \{\mathrm{I}_1, \mathrm{I}_2\}$, it follows from Lemma 7.3 that neither summit is contained in $\mathrm{I}_1 \cup \mathrm{I}_2$ either. But then, for $x \leq c(g_1)$ we find

$$x - r' < c(g_1) - r = a\,,$$

while for $x \geq c(g_2)$ we find

$$x + r' > c(g_2) + r = b\,.$$

Thus, either case entails a contradiction to (7.12). This yields (7.21). By the assumption $c(g_3) \in \mathscr{W}(g_1)$, the geodesic arc $\operatorname{bp}(\mathrm{C}_{\mathcal{W},k}) = (c(g_3), \infty)_{\mathbb{H}}$ intersects I_1 in exactly one point in $\mathbb{H}$, say ξ_3. Therefore, the geodesic arc $g_1.\operatorname{bp}(\mathrm{C}_{\mathcal{W},k})$ intersects $\mathrm{I}_2 = g_1.\mathrm{I}_1$ exactly in $g_1.\xi_3$. Since $\operatorname{Re}\xi_3 \in (c(g_1), c(g_1) + r)$ by (7.21), (1.53), and $g_1.(c(g_1) + r) = c(g_2) - r$, we find $\operatorname{Re}(g_1.\xi_3) \in (c(g_2) - r, c(g_2))$. By combining this with $g_1.\infty = c(g_1^{-1}) = c(g_2)$, we conclude that $g_1.\operatorname{bp}(\mathrm{C}_{\mathcal{W},k})$ is non-vertical and

$$\big(c(g_2) - r, c(g_2)\big) \subsetneq \operatorname{Re}\big(g_1.\operatorname{bp}(\mathrm{C}_{\mathcal{W},k})\big)\,.$$

Thus, if $c(g_3^{-1}) \in (c(g_2) - r, c(g_2))$, then, because of

$$g_3.\operatorname{bp}(\mathrm{C}_{\mathcal{W},k}) = \big(g_3.c(\mathrm{I}_3), \infty\big)_{\mathbb{H}} = \big(\infty, c(g_3^{-1})\big)_{\mathbb{H}}\,, \tag{7.22}$$

the geodesic arcs $g_3.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k})$ and $g_1.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k})$ intersect each other without coinciding. Since $\Gamma \subseteq \Gamma_{\mathcal{W}}$ and $\mathcal{C}_{\mathcal{W}}$ is a set of branches for the geodesic flow on $\mathbb{X}_{\mathcal{W}}$, this yields a contradiction by violation of (B6). Because of (7.21), this yields (7.20). Now, by combination of (7.20) with

$$g_3h.I_k = g_3.\big(c(g_3), +\infty\big) = \big(-\infty, c(g_3^{-1})\big)\,,$$

the assumptions $\mathscr{W}(g_2) \subseteq h.I_k$, $c(g_3) \in \mathscr{W}(g_1)$, and $\mathscr{W}(g_1) \cap \mathscr{W}(g_2) = \varnothing$, and the identity (7.22), we infer

$$\mathscr{W}(g_2) \subseteq g_3h.J_k\,.$$

Hence, the same argument which showed that g_3 cannot be an involution again yields a contradiction. Hence, $\Lambda(\Gamma) \subseteq h.I_k$ cannot hold true, which in turn implies (7.19). This yields the assertion in the case $g_1 = g_2^{-1}$ and g_1 being hyperbolic.

Finally, assume that g_1 is elliptic of some order $\sigma = \sigma(g_3)$. Since $\mathrm{I}_1 \neq \mathrm{I}_2$, we have $\sigma \geq 3$. By Lemma 1.22 the angle between I_1 and I_2 at the fixed point $\mathrm{f}(g_1)$ exceeds $2\pi/3$ (measured above the spheres). Since $\mathcal{W}$ is geometrically finite, we may enumerate its sides as $\alpha_1, \ldots, \alpha_m$ from left to right, i. e., such that

$$\beta_1 = \alpha_1\,, \quad \alpha_m = \beta_2\,, \quad \text{and} \quad \alpha_i \cap \alpha_{i+1} \neq \varnothing\,,$$

for $i = 1, \ldots, m-1$. Analogously, we may enumerate the elements of $V_{\mathcal{W}}$, the finite vertices of $\mathcal{W}$, by $v_1, \ldots, v_{m-1}$ such that $\{v_i\} = \alpha_i \cap \alpha_{i+1}$, for all i. Finally, denote the angle that $\mathcal{W}$ subtends at v_i by θ_i. Since we have $\mathrm{I}_1 \cap \mathrm{I}_2 \neq \varnothing$ and

$$\mathrm{Re}(\alpha_i) \subseteq \mathscr{W}(g_1) \cup \mathscr{W}(g_2)$$

for every $i \in \{2, \ldots, m-1\}$, we conclude that $\mathrm{I} \cap \mathrm{I}_\iota \neq \varnothing$ for some $\iota \in \{1, 2\}$, for every $\mathrm{I} \in \mathrm{ISO}(\Gamma)$ for which $\beta_{\mathrm{I}} = \alpha_i$ for some i. Since $v_i \in \overline{\mathrm{ext}\,\mathrm{I}_1 \cap \mathrm{ext}\,\mathrm{I}_2}$ for every i, this implies

$$\frac{2\pi}{3} < \theta_i < \pi \tag{7.23}$$

for all i. Consider the vertex cycle $C(v_1) = \{v_{i_1}, \ldots, v_{i_\ell}\}$ with $v_{i_1} = v_1$. Then

$$g_1.v_1 = v_{m-1} \in C(v_1)$$

and hence $\ell > 1$. Because of Lemma 1.34 there exists $\omega \in \mathbb{N}$ such that

$$\frac{2\pi}{\omega} = \theta(C(v_1)) = \sum_{\kappa=1}^{\ell} \theta_{i_\kappa} \overset{(7.23)}{>} \frac{2\ell\pi}{3}\,,$$

which implies $\ell\omega < 3$. Since $\ell, \omega \in \mathbb{N}$ and $\ell > 1$, this leaves

$$(\ell, \omega) = (2, 1)$$

as the only possible configuration. But this implies

$$C(v_1) = \{v_1, v_{m-1}\} \qquad \text{and} \qquad \theta_1 + \theta_{m-1} = 2\pi\,,$$

which means that at least one of the two angles equals or exceeds π, in violation of the second relation in (7.23). Hence, this final case is contradictory and the proof is finished. $\square$

In the proof of Proposition 7.11, for any given constellation, we identified a hyperbolic transformation $g \in \Gamma$ with fixed points $\mathrm{f}_+(g)$ and $\mathrm{f}_-(g)$ sufficiently far apart such that there exists $k \in A$ and $h \in \Gamma_{\mathcal{W}}$ for which $h.\,\mathrm{C}_{\mathcal{W},k}$ is intersected by $\alpha(g)$. This then yielded $k \in A'$, and because (k, h) could be chosen such that $h.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k})$ is vertical and, necessarily,

$$\mathrm{Re}\big(h.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k})\big) \in (a', b')\,,$$

Lemma 7.13 yields $h \in \Gamma$. The same argumentation also applies for g^{-1}, with the roles of g_1 and g_2 in the proof of Proposition 7.11 interchanged. Hence, we obtain a second branch copy $h'.\,\mathrm{C}_{\mathcal{W},k'}$, $(k', h') \in A' \times \Gamma$, pointing in the opposite direction of $h.\,\mathrm{C}_{\mathcal{W},k}$, i. e.,

$$h.I_{\mathcal{W},k} = (h.x_k, +\infty) \qquad \text{and} \qquad h'.I_{\mathcal{W},k'} = (-\infty, h'.x_{k'})\,.$$

Therefore, the union $h.I_{\mathcal{W},k} \cup h'.I_{\mathcal{W},k'}$ covers $\mathbb{R}$ except, perhaps, for a bounded interval. Since

$$\big(\mathrm{f}_+(g), \mathrm{f}_-(g)\big) \in h'.J_{\mathcal{W},k'} \times h'.I_{\mathcal{W},k'}\,,$$

iterated application of g contracts $h'.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k'})$ towards $\mathrm{f}_+(g)$. In other words, there exists $n \in \mathbb{N}$ such that

$$g^n h'.\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k'}) \subseteq h.\mathrm{H}_+(k)\,.$$

This means

$$h.J_{\mathcal{W},k} \subseteq g^n h'.I_{\mathcal{W},k'} \qquad \text{and} \qquad g^n h'.J_{\mathcal{W},k'} \subseteq h.I_{\mathcal{W},k}\,,$$

which in turn yields the following result.

Corollary 7.14. *There exist (not necessarily distinct) $j, k \in A'$ and $g, h \in \Gamma$ such that*

$$\mathbb{R} = g.I_{\mathcal{W},j} \cup h.I_{\mathcal{W},k}\,.$$

Recall the transition sets $\mathcal{G}_{\mathcal{W}}(.,.)$ associated to $\mathcal{C}_{\mathcal{W}}$ by (B7). The following lemma is key.

Lemma 7.15. *For every choice of $j, k \in A'$ we have $\mathcal{G}_{\mathcal{W}}(j,k) \subseteq \Gamma$.*

Proof. Fix $j \in A'$ and let $k \in A'$ be such that $\mathcal{G}_{\mathcal{W}}(j,k) \neq \varnothing$. Let $g \in \mathcal{G}_{\mathcal{W}}(j,k)$ and consider

$$\beta_{(k,g)} := g\boldsymbol{.}\,\mathrm{bp}(\mathrm{C}_{\mathcal{W},k}) = (g\boldsymbol{.}x_k, g\boldsymbol{.}\infty)_{\mathbb{H}}\,.$$

This is a complete geodesic segment contained in the half-space $\mathrm{H}_+(j)$. A priori, it might be vertical or non-vertical. Since $x_k \in \{a', b'\}$ implies that one of the sets $I_{\mathcal{W},k,\mathrm{st}}, J_{\mathcal{W},k,\mathrm{st}}$ is empty and thus $k \notin A'$ in violation of the assumption, the assertion in the vertical case has already been shown in Lemma 7.13.

Thus, assume that $\beta_{(k,g)}$ is non-vertical. Then there exists a cell $B \in \mathbb{B}$ for $\Gamma_{\mathcal{W}}$ such that $\beta_{(k,g)}$ and $\beta_{(j,\mathrm{id})}$ are sides of B (see also the Lemmas 2.4 and 2.7). Because of Lemma 2.2 and $\beta_{(k,g)}$ being non-vertical, the cell B is a hyperbolic polygon with $\beta_{(j,\mathrm{id})}$ being one of its two vertical sides. Assume first that B is a hyperbolic triangle. Then either

$$x_j = g\boldsymbol{.}\infty\,, \qquad \text{or} \qquad x_j = g\boldsymbol{.}x_k\,.$$

In the former case, application of Lemma 2.15 yields $\mathrm{I}(g^{-1}) \in \mathrm{REL}(\Gamma_{\mathcal{W}})$, and, by taking Proposition 1.41 into account, we can proceed as in the proof of Lemma 7.13 to conclude $g \in \Gamma$. In the latter case, consider the other vertical side of B. By the constructions in Section 2.1 and Lemma 2.13 there exists a pair $(j', h) \in A \times \Gamma_{\mathcal{W}}$ such that this side is given by $\beta_{(j',h)}$ and we have

$$h^{-1}g \in \mathcal{G}_{\mathcal{W}}(j', k)\,.$$

Then either

$$g\boldsymbol{.}\infty = h\boldsymbol{.}x_{j'}\,, \qquad \text{or} \qquad g\boldsymbol{.}\infty = h\boldsymbol{.}\infty\,.$$

Since $\beta_{(j',h)}$ is vertical, the former case implies $h \in \Gamma_{\mathcal{W},\infty}$, which, with the same argument as above, can only hold true if $h = \mathrm{id}$. Hence, $g\boldsymbol{.}\infty = x_{j'}$ and we argue as before with j' in place of j and thereby obtain $g \in \Gamma$. Because of Lemma 1.19(i) and Proposition 1.25, the latter case implies $\mathrm{I}(g) = \mathrm{I}(h)$, which in turn implies $h^{-1}g \in \Gamma_{\mathcal{W},\infty}$ by Lemma 1.20(i). Hence, by the above,

$$\mathcal{G}_{\mathcal{W}}(j', k) \cap \Gamma_{\mathcal{W},\infty} \neq \varnothing\,.$$

But because of (7.11), this can only be the case if $\{x_k, x_{j'}\} = \{a', b'\}$, which, as we have seen before, leads to $\mathrm{C}_{\mathcal{W},k,\mathrm{st}} = \varnothing$, thereby contradicting the choice of k. Hence, this case is impossible.

Now assume that B is not a hyperbolic triangle. Then, by Lemma 2.2, every side of B is of the form $(s^{\ell}\boldsymbol{.}\infty, s^{\ell+1}\boldsymbol{.}\infty)_{\mathbb{H}}$ for some elliptic transformation $s \in \Gamma_{\mathcal{W}}$

and $\ell \in \{0, \dots, \sigma(s) - 1\}$. Hence, in particular,

$$g = s^{\ell'}$$

for one $\ell' \in \{0, \dots, \sigma(s) - 1\}$, and furthermore,

$$\beta_{(j,\mathrm{id})} = (s^{\iota}.\infty, \infty)_{\mathbb{H}}\,,$$

for one $\iota \in \{\pm 1\}$, meaning $s^{-\iota}.x_j = \infty$. As before, the latter implies $s \in \Gamma$, from which we obtain $g \in \Gamma$. This finishes the proof. $\square$

We are now ready to prove our third and final main result, identifying a set of branches for the geodesic flow on $\mathbb{X}$. Evidently, the proof makes use of $\mathcal{C}_{\mathcal{W}}$ being a set of branches for the geodesic flow on $\mathbb{X}_{\mathcal{W}}$. In order to distinguish between the defining properties from Definition 4.1 in the two different contexts, we denote those fulfilled by $\mathcal{C}_{\mathcal{W}}$ with respect to $\Gamma_{\mathcal{W}}$ by (B1$_{\mathcal{W}}$)–(B7$_{\mathcal{W}}$), respectively.

Theorem 7.16. *$\mathcal{C}'_{\mathcal{W}} \coloneqq \{\mathrm{C}_{\mathcal{W},j} \mid j \in A'\}$ is a set of branches for the geodesic flow on $\mathbb{X}$.*

Proof. From Proposition 7.11 and $A' \subseteq A$ we see that $\mathcal{C}'_{\mathcal{W}}$ is a finite and non-empty set. The definition of A' combined with (B1$_{\mathcal{W}}$) further assures validity of (B1). Let $j \in A'$. Since $[\infty]_{\Gamma_{\mathcal{W}}}$ is the only cusp of $\mathbb{X}_{\mathcal{W}}$, Corollary 2.6 implies that the point x_j either equals the center of some relevant isometric sphere, or it is contained in a representative of a funnel of $\mathbb{X}_{\mathcal{W}}$. Since ∞ is contained in a representative of a funnel of $\mathbb{X}$, so is every center of an isometric sphere for Γ, by virtue of Lemma 1.19(i). Therefore, (B2) follows directly from (B2$_{\mathcal{W}}$). Property (B3) is immediate from (B3$_{\mathcal{W}}$) and property (B4) follows from Corollary 7.14 and ∞ being contained in a funnel interval. The properties (B5) and (B6) follow from (B5$_{\mathcal{W}}$) and (B6$_{\mathcal{W}}$), respectively, by taking Corollary 7.6 into account. Finally, (B7) follows from (B7$_{\mathcal{W}}$) and Lemma 7.15. $\square$

Open Questions

In this thesis we have seen that strict transfer operator approaches exist for every non-cocompact geometrically finite Fuchsian group with hyperbolic elements that fulfills Condition (A). Because of Lemma 1.17 and (B2), the approach via sets of branches cannot easily be extended to include cocompact groups. Since (B6) is also unfulfillable for cocompact groups due to the density of hyperbolic fixed points everywhere on the real line, we do not expect that a unified approach for both types of Fuchsian groups is feasible.

A first immediate open question is concerned with Condition (A): It is a technical assumption, which, to date, is required for the construction of branches in the cusp expansion algorithm. But there are no concerns tied to it in terms of geometric or spectral properties of the hyperbolic orbisurface. Hence, we expect this condition to be completely expendable. For that reason the constructions in this thesis did not utilize it beyond the application of the cusp expansion algorithm. This means that, once a modification of this algorithm has been shown to work regardless of it—and does so in a way such that all statements of Chapter 2 remain valid—the assumption of Condition (A) may be removed from all statements of this thesis as well, without the need for further adjustments.

What would require adjustments, namely in the strict transfer operator approach, is a transfer operator construction for infinitely ramified sets of branches. We excluded them from our studies in this thesis, since we wanted to avoid the necessity of modifications to the results of [22], which are central to our approach. In applications one can easily be faced with infinite ramification. For instance, if one considers a sequence of hyperbolic orbisurfaces all admitting the same set of branches, it might happen that, on the "limit surface," that set of branches becomes infinitely ramified. A study of these sets might also prove fruitful, for some of them appear to not require a cuspidal acceleration, being "fast and slow" at the same time in that sense (but not in the sense that slow transfer operators are assumed to be free of infinite sums). This might shed new light on the relation between eigenfunctions of slow and fast transfer operators (see the next paragraph). But one would have to face questions regarding convergence of the operator itself. We do not know whether or not there is any hope that infinitely ramified sets of branches give rise to nuclear transfer operators.

A further question revolves around the eigenspaces of the (slow and fast) transfer operators. In their paper [1] Adam and Pohl showed that, for Hecke triangle groups and finite-dimensional unitary representations χ, the eigenfunctions with eigenvalue 1 of the fast transfer operator $\widetilde{\mathcal{L}}_{s,\chi}$ are isomorphic to the real-analytic eigenfunctions with eigenvalue 1 of the slow transfer operator $\mathcal{L}_{s,\chi}$ that satisfy a certain growth condition, for every s in the right half-plane. We expect a similar relationship between the 1-eigenfunctions of the two families of transfer operators to hold true in the general case.

On a related note, by building on seminal work by Lewis, Bruggeman, Mühlenbruch, and Zagier [36, 12, 37, 14, 13], Möller and Pohl [44] established an (explicit) isomorphism between Maass cusp forms (certain eigenforms of the Laplacian) and highly regular 1-eigenfunctions of the slow transfer operator family for cofinite Hecke triangle groups. Recently, Bruggeman and Pohl [15] developed similar isomorphisms for automorphic forms associated to Hecke surfaces of infinite volume. This raises the question to what level of generality such identifications might be feasible. Together with the relation between the eigenfunctions of the two transfer operator families, the factorization of Selberg zeta functions revealing their sets of zeros to contain the resonances of the respective Laplacian (see Section 1.12), and the results presented in this thesis, one would obtain a bridge from the number theoretical field of automorphic forms to the spectral theory for hyperbolic orbifolds. For Hecke triangle groups such a bridge now exists by virtue of the work of Pohl et. al. One would like to have it in the most general case feasible.

$$\begin{tikzcd}
\boxed{\textbf{Automorphic forms}} \arrow[d, leftrightarrow] & \boxed{\textbf{Resonances of the Laplacian}} \arrow[d, leftrightarrow] \\
\boxed{\begin{array}{c}\text{1-eigenspaces of}\\ \text{slow/fast transfer operators}\end{array}} \arrow[r, leftrightarrow] & \boxed{\begin{array}{c}\text{Zeros of the}\\ \text{Selberg zeta function}\end{array}}
\end{tikzcd}$$

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