Bachelor thesis

The Brunn–Minkowski and log-Brunn–Minkowski inequalities

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Abstract

The Brunn–Minkowski inequality is known as one of the simplest geometric inequalities. But despite its simplicity, its intrinsic motivation is given by several problems that not only arise in geometry. One of these problems is the classical Minkowski problem, which asks for existence and uniqueness conditions of convex bodies with given boundary.

While this problem has been solved in the 19th century by Minkowski himself, an analogue problem arose, the so-called logarithmic Minkowski problem. As an analogue of the classical problem, it asks for existence and uniqueness of convex bodies with given cone volumes.

In this thesis we introduce the basics of convex geometry to establish the Brunn–Minkowski inequality and sketch the solution of the uniqueness problem for the classical Minkowski problem. For that, we will present three well-known proofs of the Brunn–Minkowski inequality. Furthermore, we discuss the advances in the solution of the logarithmic Minkowski problem and present an approach that works for the planar case. This includes recent results about the logarithmic Brunn–Minkowski inequality, a logarithmic analogue of the corresponding classical inequality.

Zusammenfassung

Die Brunn–Minkowski-Ungleichung ist bekannt als eine der einfachsten geometrischen Ungleichungen. Aber abgesehen von ihrer Einfachkeit, wird sie durch verschiedene, teilweise nicht-geometrische, Probleme motiviert. Eines dieser Probleme ist das klassische Minkowski-Problem, welches nach Bedingungen für die Existenz und Eindeutigkeit von konvexen Körpern mit gegebenem Rand fragt.

Während dieses Problem bereits durch Minkowski im 19. Jahrhundert gelöst wurde, ist das sogenannte logarithmische Minkowski-Problem bislang ungelöst. Als analoges Problem zum Minkowski-Problem fragt dieses nach Existenz und Eindeutigkeit von konvexen Körpern mit gegebenem Kegelvolumen.

In dieser Arbeit führen wir die Grundlagen der konvexen Geometrie ein, um zunächst die Brunn–Minkowski-Ungleichung zu zeigen, und skizzieren eine Lösung des Eindeutigkeitsproblems des klassischen Minkowski-Problems. Dazu geben wir drei bekannte Beweise der Brunn–Minkowski-Ungleichung. Weiter diskutieren wir die Fortschritte in der Lösung des logarithmischen Minkowski-Problems und zeigen einen Ansatz, der für planare Körper funktioniert. Dies beinhaltet aktuelle Resultate über die logarithmische Brunn–Minkowski-Ungleichung, ein logarithmisches Analogon der klassischen Ungleichung.

1 Preface

The field of convex geometry offers numerous valuable results. One of the most important and defining results is the Brunn–Minkowski inequality, that is given by

$$\mathrm{V}(K)^{1/n} + \mathrm{V}(L)^{1/n} \leq \mathrm{V}(K+L)^{1/n}$$

for convex sets K, L in n-dimensional Euclidean space. Here, V denotes the elementary volume of geometric bodies, and the sum of two convex sets is built point-wise, i. e. $x + y \in K + L$ for all $x \in K$, $y \in L$.

This inequality has numerous applications and motivates its own theory, the Brunn–Minkowski theory. For example, the isoperimetric problem, which asks for conditions under which a body with given volume has the maximum perimeter, can be solved by applying the Brunn–Minkowski inequality to certain constructions.

Another problem, which is more natural, is the Minkowski problem: It asks for conditions under which a certain convex body with given surface exists and is unique. While the existence follows with elementary methods in geometry, the uniqueness question needs the Brunn–Minkowski inequality, or equivalently Minkowski's first inequality.

Other applications of the Brunn–Minkowski inequality are in numerical mathematics and analysis, as the sets that appear in those fields are often convex, and thus subject to the theory of convex bodies. Especially the latter field, and the field of functional analysis, the theory of analytical properties of linear maps between vector spaces (so-called operators), developed the theory further for infinite-dimensional vector spaces.

While the classical Brunn–Minkowski inequality gives a good lower bound for the volume of the sums of convex bodies, one is interested in a similar theory for other sums. In contemporary convex geometry, one of those theory is the L_p-Brunn–Minkowski theory, which was motivated by the work of Firey in [Fir62], and developed under the name Brunn–Minkowski–Firey theory by Lutwak in [Lut93; Lut96].

This theory searches for similar inequalities for the p-sum, a sum of convex bodies that is motivated by the representation of them through support functions. The L_p-Brunn–Minkowski inequality, the L_p-counterpart of the Brunn–Minkowski inequality, was established by Firey in [Fir62] for the case $p > 1$. Further, the case $p = 1$ is just the classical Brunn–Minkowski inequality. Hence, the only remaining case is $p < 1$, where especially the case $p \to 0$, the so-called log-Brunn–Minkowski inequality, is of special interest, since it solves a determination problem of convex bodies.

In this thesis, we will start with an introduction to convex geometry. We will define convexity, constructions of convex sets and introduce usual characterisations of convex sets. Further, we will discuss the measurability question that arises in combination with geometry and especially convex sets.

In the next step, we will state and prove the Brunn–Minkowski inequality. We will discuss three proof variants of the Brunn–Minkowski inequality, which use methods from various areas of mathematics, namely analysis, geometry and measure theory. This gives us a basis to mention applications of this inequality, in our case the Minkowski problem and Minkowski's first inequality.

As an generalisation of the classical Brunn–Minkowski theory, we will discuss the most basic ideas of L_p-Brunn–Minkowski theory. Those ideas leads us to the logarithmic Brunn–Minkowski inequality, which we introduce in Chapter 5. After proving the trivial case for $\mathbb{R}$ and discussing some technical results, we will state the established cases for $n = 2, 3$ and discuss the application of the log-Brunn–Minkowski inequality, namely the log-Minkowski problem as an analogue of the classical Minkowski problem.

In this thesis, all vector spaces are real. Furthermore, convex sets are denoted by symbols K, L, etc. Functions with the real numbers as domain are denoted by the usual characters f, g, h. Further, we will denote the $(n-1)$-dimensional unit sphere, i. e. the boundary of an n-dimensional unit ball, by S^{n-1}. That balls are denoted by B, whenever the dimension of them will be clear from context. For a complete overview of notation, see the section "Notation" on page 55.

2 An introduction to convex geometry

Convex geometry and its results are tightly coupled with results from measure theory, analysis, linear algebra and topology. In this chapter we will introduce the foundations of convex geometry, which are necessary to state and prove the Brunn–Minkowski theorem and its generalisations.

In the first section of this chapter we will discuss the most basic notion in convex geometry, the notion of convexity as a property of sets. Further, we will explore some special cases of convex sets, in our case polytopes and convex bodies. This part of the theory depends on the notion of convex (and concave, resp.) functions, and we will discuss the properties that hold for that class of functions.

In Section 2.2, we will introduce a more systematic way to understand the space of convex bodies: We endow this set with a metric, the so-called Hausdorff metric. This metric behaves good with regard to several functionals that appear in the characterisation of convex bodies, especially with the support functions. Further, it even motivates the characterisation of convex bodies with supporting hyperplanes and the resulting support functions.

The results of that section are an useful approximation theorem for convex bodies and the characterisation of convex bodies by hyperplanes. Especially, we will give a proof that this characterisation is correct, and discuss a generalisation from functional analysis and describe the alternative method of functional analysis, which even holds for infinite-dimensional spaces.

2.1 Basic definitions and properties

The most basic concept (and definition) in convex geometry is that of the convex set. Convexity appears in many settings in mathematics. For example, it is a important concept in analysis, where many spaces with desirable properties are convex. Further, they appear in applied settings, for example in numerical mathematics, where the convexity of certain sets is crucial.

The following definition of convexity is one of the standard definitions, and can be found in several textbooks about the topic (e. g. [Sch14]). It states that a set A is convex, if any straight line between two points, say $x, y \in A$, is contained in A.

Definition 2.1 (Convexity)**.** A set $A \subseteq \mathbb{R}^n$ in n-dimensional Euclidean space is said to be *convex*, if, for $x, y \in A$, $0 \leq \lambda \leq 1$, $\lambda \cdot x + (1-\lambda) \cdot y \in A$, i. e. the point $\lambda \cdot x + (1-\lambda) \cdot y$ is an element of the set A.

More generally, a subset A of an $\mathbb{R}$-vector space $(V, +, \cdot)$ is called *convex*, if $\lambda \cdot x + (1-\lambda) \cdot y \in A$ for all $x, y \in A$, $0 \leq \lambda \leq 1$.

Before we give examples of convex sets, we are discussing the usual used vector space, in our case the Euclidean space. We identify the n-dimensional Euclidean space with the Hilbert space $\mathbb{R}^n$ with usual addition, scalar multiplication and inner product

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i \text{ for all } v, w \in \mathbb{R}^n.$$

Angles are defined in the usual way as follows: Let $v, w \in \mathbb{R}^n$ be two vectors. Then, the angle θ between x and y is determined by the relation

$$\cos(\theta) = \frac{\langle v, w\rangle}{|v| \cdot |w|},$$

where $|v|$ is a shorthand for $|v| = (\langle v, v\rangle)^{1/2}$.

The distance of two points $x, y \in \mathbb{R}^n$ is given by the induced metric, that is

$$d(x, y) = |x - y| = (\langle x - y, x - y\rangle)^{1/2}.$$

For the Euclidean space, this reduces to

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} \text{ for points } x, y \in \mathbb{R}^n.$$

Now, the first examples for convex sets are the trivial choices that arise from algebraic deliberations.

Example 2.2. The trivial convex sets on a vector space $(V, +, \cdot)$ are the whole set and the zero subspace, i. e. the vector space $\{0\} \subset V$.

That the whole space V is a convex set follows directly from the fact, that the product and the sum are maps $\cdot : K \times V \to V$ and $+ : V \times V \to V$: For $0 \leq \lambda \leq 1$ and $x, y \in V$ we have $\lambda x \in V$ and $(1 - \lambda)y \in V$, and furthermore

$$\lambda x + (1 - \lambda)y \in V.$$

Further, $\{0\}$ is a subspace of V, hence a vector space. Therefore, the same argument applies.

With this result in mind, it follows directly that also the translates of a convex sets are convex: Let $K \subseteq V$ be a convex subset, $v \in V$. Then, the *translate* $v + K := \{v + x \mid x \in K\}$ is a convex subset too, since

$$(1 - \lambda)(v + x) + \lambda(v + y) = v + (1 - \lambda)x + \lambda y \text{ for all } x, y \in K \text{ and } \lambda \in [0, 1].$$

In fact, a much stronger statement holds. Let $K, L \subseteq V$ be two convex subsets. Further, let $K + L := \{x + y \mid x \in K, y \in L\}$, the *Minkowski sum* of A and B. For $\lambda \in [0, 1]$, $x_1 + y_1, x_2 + y_2 \in K + L$ with associated $x_1, x_2 \in K$, $y_1, y_2 \in L$, we have

$$(1 - \lambda)(x_1 + y_1) + \lambda(x_2 + y_2) = (1 - \lambda)x_1 + \lambda x_2 + (1 - \lambda)y_1 + \lambda y_2 \in K + L.$$

This holds, since the summands $(1-\lambda)x_1 + \lambda x_2$ and $(1-\lambda)y_2 + \lambda y_2$ are elements of K and L, respectively, by convexity of K and L. Thus, the Minkowski sum of two convex sets is a convex set, too.

By convexity of a single point subspace $\{x\} \subseteq V$ for $x \in V$, the former statement that translates of convex sets are convex sets follows directly by this result.

Furthermore, a *dilate*, i. e. $\lambda \cdot K := \lambda K := \{\lambda \cdot x \mid x \in K\}$ for $\lambda \in \mathbb{R} \setminus \{0\}$, is also a convex body by similar reasoning. Sometimes, we will extend this definition to arbitrary subsets of $\mathbb{R}^n$ in an analogous way.

Example 2.3. One is interested in the least convex set that contains certain points $x_1, \dots, x_k \in V$.

The convex combinations are given by the following formula: Let $\lambda_1, \dots, \lambda_k \in [0,1]$ such that $\sum_{i=1}^k \lambda_i = 1$. Then the corresponding *convex combination* of $x_1, \dots, x_k$ is the point

$$\sum_{i=1}^k \lambda_i x_i = \lambda_1 x_1 + \cdots + \lambda_k x_k.$$

We denote the set of all such convex combinations by $\operatorname{conv}(x_1, \dots, x_k)$ and call it the *convex hull* of $x_1, \dots, x_k$. It is evident that $\operatorname{conv}(x_1, \dots, x_k)$ is convex.

This gives us an infinite amount of examples for convex sets. For example, in $\mathbb{R}^n$ with standard basis $e_1, \dots, e_n$, the convex hull $\operatorname{conv}(0, e_1, \dots, e_n)$ is called n-dimensional standard simplex. More generally, a convex hull $\operatorname{conv}(a_1, \dots, a_{n+1})$ of points $a_1, \dots, a_{n+1} \in \mathbb{R}^n$ is called a *simplex* if $a_1, \dots, a_{n+1}$ are affine-independent, i. e. they do not lie in a single proper subspace of $\mathbb{R}^n$.

This motivates a general operator, namely the convex hull operator. Let $A \subseteq \mathbb{R}^n$. Further, let

$$\mathcal{X} := \{K \subseteq \mathbb{R}^n \mid K \text{ convex}, A \subseteq K\}$$

the set of convex sets that contain A. Then the intersection

$$\operatorname{conv}(A) := \bigcap_{K \in \mathcal{X}} X$$

is a convex set: Let $x, y \in \operatorname{conv}(A)$, $\lambda \in [0,1]$. Then $x, y \in K$ for all $K \in \mathcal{X}$, hence

$$(1-\lambda)x + \lambda y \in K \text{ for all } K \in \mathcal{X}$$

by convexity. Therefore, the set $\operatorname{conv}(A)$ is convex.

This defines the *convex hull operator* $\operatorname{conv} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$, which assigns to each subset of $\mathbb{R}^n$ its convex hull. We call $\operatorname{conv}(A)$ the convex hull of $A \subseteq \mathbb{R}^n$.

It is simple to show that these two definitions for finite A coincide, and we are referring to [Sch14] for a proof of this statement.

A subset $K \subseteq \mathbb{R}^n$ is bounded precisely when there exists a $\varepsilon > 0$ such that $K \subseteq B(0, \varepsilon)$, i. e. the set is contained in a norm ball of radius ε and centre 0. It is clear that the convex hull of a bounded set is bounded, again.

Theorem 2.4 (Heine–Borel, cf. [Mun00, Theorem 27.3]). *Let $K \subseteq \mathbb{R}^n$. Then, K is compact if and only if K is closed and bounded.*

The Heine–Borel theorem is a characterisation of compact sets in the Euclidean space, and therefore of topological nature. Its proof gives deep insights into the topology of metrical spaces, but would distract our path through the basics of convex geometry. Therefore, we are referring to [Mun00, Theorem 27.3] for a proof of that theorem.

We are mostly interested in compact convex sets, since they have finite volume by the Heine–Borel theorem.

Definition 2.5 (Convex body). A convex subset $K \subseteq \mathbb{R}^n$ is called *convex body*, if K is compact and has non-empty interior, i. e. there exist $x \in K$, $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq K$.

The set of all n-dimensional convex bodies is denoted by $\mathcal{K}^n$.

The case that $0 \in K$ is in most statements a valid restriction that occurs with loss of generality, since most statements are invariant of translation. Therefore, we will denote the set of n-dimensional convex bodies that contain the origin 0 in their interior by $\mathcal{K}_0^n$. That is, $K \in \mathcal{K}_0^n$ if K is a convex body and $0 \in \int K$.

Definition 2.6. A convex body $K \in \mathcal{K}^n$ is called *origin-symmetric* if $K = -K$.

Definition 2.7 (Polytope). Let $v_1, \dots, v_k \in \mathbb{R}^n$. A (convex) *polytope* P with vertices $v_1, \dots, v_k$ is the convex hull of these vertices $v_1, \dots, v_k$.

From the previous definition of a convex polytope P it is evident, that P is bounded. Thereto, consider the maximum $m := \max\{|v_1|, \dots, |v_k|\}$ of vertices $v_1, \dots, v_k$. Then, P is contained in the unit ball with radius m.

Furthermore, a convex polytope P has non-empty interior, if the vertices do not lie in a proper subspace of $\mathbb{R}^n$.

A usual decomposition of P is into its faces of dimensions $0, 1, \dots, n$. This decomposition is not necessary for us, but we restrict to the facets of P. A *facet* is an $(n-1)$-dimensional face of P, i. e. the intersection of the boundary of P and an hyperplane such that this intersection is locally homeomorphic to $\mathbb{R}^{n-1}$.

We denote the set of the outer unit normals of these facets by $F(P)$, and the facet with outer unit normal $u \in S^{n-1}$ by $F(P, u)$. With this notation, $F(P, u)$ is an $(n-1)$-dimensional polytope.

Definition 2.8 (Homothetic bodies). Two convex bodies $K, L \in \mathcal{K}^n$ are said to be *homothetic*, if there are a translation vector $v \in \mathbb{R}^n$ and a dilation factor $\alpha > 0$ such that $K = v + \alpha L$.

Definition 2.9 (Convex and concave functions). Let V be a $\mathbb{R}$-vector space, $X \subseteq V$ convex and let $f : X \to \mathbb{R}$ be a function. Then, the function f is said to be *convex*, if it satisfies

$$f(\lambda + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$$

for all $\lambda \in [0, 1]$, $a, b \in X$.

Furthermore, a function f is called *concave*, if $-f$ is convex.

If the above inequality is proper, f is called *strictly convex* and *strictly concave*, respectively.

Example 2.10. The identity function and constant function on any subset $A \subseteq \mathbb{R}^n$ are convex.

More generally, any vector space homomorphism $f : V \to \mathbb{R}$ is convex and concave. Especially, a vector space homomorphism is bijective if, and only if, it is strictly convex and strictly concave.

Lemma 2.11 (Jensen's inequality). *If $f :$ is a convex function, then*

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \leq \sum_{i=1}^{k} \lambda_i f(x_i)$$

for $x_1, \dots, x_k \in \mathbb{R}^k$ and $\lambda_1, \dots, \lambda_n \in [0,1]$ that satisfy $\sum_{i=1}^{k} \lambda_i = 1$.

Proof. The inequality holds for $k = 1$ and $k = 2$ trivially by the hypothesis that f is a convex function.

Therefore, let $k \geq 2$ such that the inequality holds. Then, let $x_1, \dots, x_{k+1} \in \mathbb{R}^k, \lambda_1, \dots, \lambda_{k+1}$ with $\sum_{i=1}^{k+1} \lambda_i = 1$. We may assume without loss of generality, that one of the $\lambda_1, \dots, \lambda_{k+1}$ is not 0, since otherwise

$$\sum_{i=1}^{k+1} \lambda_i = 0 \neq 1.$$

Since this is contrary to the hypothesis, we may assume that $\lambda_i > 0$ for some $i \in \{1, \dots, n\}$. With an appropriate renaming of $x_1, \dots, x_{k+1}$, we may assume that $\lambda_1 > 0$.

Then,

$$\begin{aligned} f\left(\sum_{i=1}^{k} \lambda_i x_i\right) &= f\left(\lambda_1 x_1 + (1-\lambda_1) \sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i\right) \\ &\leq \lambda_1 f(x_1) + (1-\lambda_1) f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i\right) \end{aligned}$$

by the convexity of f. Further,

$$\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} = \frac{\sum_{i=2}^{k+1}}{1-\lambda_1} = 1,$$

since $1 - \lambda_1 = \sum_{i=2}^{k+1} \lambda_i$.

Therefore, we can apply the induction hypothesis and we have

$$\begin{aligned} \lambda_1 f(x_1) + (1-\lambda_1) f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i\right) &\leq \lambda_1 f(x_1) + (1-\lambda_1) \sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} f(x_i) \\ &= \sum_{i=1}^{k+1} \lambda_i f(x_i). \end{aligned}$$

□

The same argument gives rise to a similar statement for convex sets. Let $K \subseteq \mathbb{R}^n$ be convex. Further, let $x_1, \dots, x_k \in K$, $\lambda_1, \dots, \lambda_k \in [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$. Then,

$$\sum_{i=1}^k \lambda_i x_i = \lambda_1 x_1 + (1-\lambda_1) \sum_{i=2}^k \frac{\lambda_i}{1-\lambda_1} x_i,$$

and by the same argument as in the preceding proof,

$$\sum_{i=2}^k \frac{\lambda_i}{1-\lambda_1} x_i \in K$$

by induction hypothesis. This clearly implies, by convexity of K,

$$\sum_{i=1}^k \lambda_i x_i \in K.$$

Now, suppose that K is a set that satisfies $\sum_{i=1}^{k_0} \lambda_i x_i \in K$ for $x_i \in K$, $\lambda_i \in [0,1]$ with a constant $k_0 \geq 2$. With this property, clearly a convexity property holds for all $2 \leq k \leq k_0$, since we can set $\lambda_k = \dots = \lambda_{k_0} = 0$. This motivates these two theorems, where the latter is a result for convex functions that is obtained in a similar manner.

Theorem 2.12. *A set $K \subseteq V$ is convex, if and only if*

$$\sum_{i=1}^k \lambda_i x_i \in K \textit{ for } x_i \in K, \lambda_i \in [0,1], i \in \{1, \dots, k\}$$

for a constant $k \in \mathbb{N}$, $k \geq 2$.

Theorem 2.13. *A function $f : U \to \mathbb{R}$ is convex, if and only if*

$$f\left(\sum_{i=1}^k \lambda_i x_i\right) \leq \sum_{i=1}^k \lambda_i f(x_i).$$

2.2 Support functions

Whenever we mention topological terms, these will be in line with the definitions of [Mun00]. The following definition is taken from [Sch14, p. 60] and is one variant of several equivalent definitions.

We start with the following definition, which endows the set of convex bodies with a topology.

Definition 2.14 (Hausdorff distance)**.** The *Hausdorff distance* $\delta : \mathcal{K}^n \times \mathcal{K}^n \to [0, \infty)$ is given by

$$\delta(K, L) = \max\left\{\sup_{x\in K} \inf_{y \in L} |x-y|, \sup_{x\in L} \inf_{y\in K} |x-y|\right\},$$

where $K, L \in \mathcal{K}^n$ are n-dimensional convex bodies.

Since all the sets are compact, we may replace the suprema by maxima and the infima by minima. Hence, we have

$$\delta(K,L) = \max\left\{\max_{x\in K}\min_{y\in L}\ |x-y|, \max_{x\in L}\min_{y\in K}\ |x-y|\right\}.$$

This distance function is a metric on the set of convex bodies.

Lemma 2.15. *The set of compact subsets $\mathcal{K}^n$ together with the Hausdorff distance $\delta : \mathcal{K}^n \times \mathcal{K}^n \to [0,\infty)$ is a metric space.*

Proof. Let $K, L, M \in \mathcal{K}^n$. That the Hausdorff distance is symmetric, is evident from its definition:

$$\delta(K,L) = \max\left\{\sup_{x\in K}\inf_{y\in L} |x-y|, \sup_{x\in L}\inf_{y\in K}\ |x-y|\right\} = \delta(L,K).$$

Further, δ is positive-definite; when $\delta(K,L) = 0$, we also have

$$\inf_{y\in L} |x-y| \leq 0 \text{ for all } x \in K,$$
$$\inf_{y\in K} |x-y| \leq 0 \text{ for all } y \in K.$$

Therefore, we can find a $y \in L$ for a $x \in K$ such that $|x-y| = 0$, i. e. $x = y$, or in other words $y \in K$. By symmetry, we also find a $y \in K$ for which $y \in L$ holds.

To prove the triangle inequality, we suppose that

$$\delta(K,M) = \max_{x\in K}\min_{z\in M} ||x-z||.$$

Since K and M are compact, we can choose $x \in K$, $z \in M$ such that $\delta(K,M) = ||x-z||$.

Now, we have

$$\delta(K,M) = ||x-z|| \leq ||x-y|| + ||y-z||$$

for all $y \in L$ by the triangle inequality for the norm. Hence,

$$\delta(K,M) \leq \max_{x\in K}\min_{z\in M} ||x-y|| + \max_{x\in K}\min_{z\in M} ||y-z||$$

for all $y \in L$.

Since the minimised terms do not depend on z and x, respectively, we obtain

$$\delta(K,M) \leq \max_{x\in K} ||x-y|| + \min_{z\in M} ||y-z||.$$

Therefore, we obtain the triangle inequality

$$\delta(K,M) \leq \max_{x\in K}\min_{y\in L} ||x-y|| + \max_{y\in L}\min_{z\in M} ||y-z|| \leq \delta(K,L) + \delta(L,M).$$

□

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of convex bodies. Then, this sequence is said to be convergent, if $K_n \to K$ for $n \to \infty$ with respect to the Hausdorff distance.

Lemma 2.16. *The metric space $(\mathcal{K}^n, \delta)$ is complete, i. e. every Cauchy sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{K}^n$ has a limit $K \in \mathcal{K}^n$ such that $K_n \to K$ for $n \to \infty$ w. r. t. the Hausdorff distance δ.*

Proof. See [Sch14, Theorem 1.8.3] and [Sch14, Theorem 1.8.6] in combination with standard topological results. □

Definition 2.17 (Support function)**.** For a convex body $K \in \mathcal{K}^n$, its *support function* $h_K : S^{n-1} \to \mathbb{R}$ is given by

$$h_K : u \longmapsto \max\{\langle u, x\rangle \mid x \in K\}.$$

The idea of the support function is as follows: Choose some boundary point $x \in \partial K$ of a convex body K. Then, there is a hyperplane H that intersects at x such that its associated negative half-space H_- contains K, since otherwise, the point x is cannot be a boundary point of K by convexity of K.

Such hyperplane is formally given by $(n-1)$ vectors that span the hyperplane and an affine vector v. In finite real vector spaces, we can replace this identification by an outer unit normal vector $u \in S^{n-1}$ of the hyperplane an distance $h \in \mathbb{R}$ in direction of u. This distance is given by $\max_{x\in K}\langle u, x\rangle$, which motivates the definition of the support function.

Remark. We can extend the support function $h_K : S^{n-1} \to \mathbb{R}$ of a convex set $K \subseteq \mathbb{R}^n$ uniquely to a function $\tilde{h}_K : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{h}_K(v) = \|v\| h_K(\frac{1}{\|v\|}v).$$

Furthermore, we have

$$\tilde{h}_K(v) = \|v\| \max\{\langle \frac{1}{\|v\|}v, x\rangle \mid x \in K\} = \max\{\langle v, x\rangle \mid x \in K\}.$$

Hence, we could extend the definition of the support function to all vectors in $\mathbb{R}^n$ and obtain the same function.

Thus, the support function is uniquely determined by its values on the unit sphere S^{n-1}. Since this argument does not depend on the choice of the norm, we could choose any norm for the definition of the support function.

In this sense, the geometry, which we develop, does not directly depend up to this point not on the choice of norm.

Example 2.18. Let $Q = [-1, 1]^n$ be the n-dimensional unit hypercube. Further, let $u = (u_1, \dots, u_n) \in S^{n-1}$. Then, the vector $(\mathrm{sgn}(u_1), \dots, \mathrm{sgn}(u_n))$ is an element of Q, and further an element of the boundary of Q, i. e. $(\mathrm{sgn}(u_1), \dots, \mathrm{sgn}(u_n))\partial Q$.

Thus, the support function $h_Q : S^{n-1} \to \mathbb{R}$ of Q is given by

$$h_Q(u) = \sum_{i=1}^{n} |u_i|,$$

the 1-norm function restricted to S^{n-1}.

Example 2.19. Let B be the n-dimensional unit disc. Then, the support function $h_B : S^{n-1} \to \mathbb{R}$ is given by $h_B(u) = 1$ for $u \in S^{n-1}$.

Generally, for $\lambda > 0$, the n-dimensional ball λB with radius λ has the support function $h_{\lambda B}(u) = \lambda$.

Lemma 2.20 (Monotonicity of support functions)**.** *Let $K, L \in \mathcal{K}^n$ be two convex bodies that satisfy $K \subseteq L$, i. e. K is a subbody of L. Then for their support functions $h_K, h_L : S^{n-1} \to \mathbb{R}^+$ it holds the inequality*

$$h_K(u) \leq h_L(u) \text{ for all } u \in S^{n-1}.$$

Furthermore, if $h_K(u) \leq h_L(u)$ for all $u \in S^{n-1}$, then $K \subseteq L$.

Proof. Let $K, L \in \mathcal{K}_0^n$ and $u \in S^{n-1}$. Consider the sets

$$A := \{\langle u, x\rangle \mid x \in K\},$$
$$B := \{\langle u, y\rangle \mid y \in L\}.$$

Then $A \subseteq B$, since $K \subseteq L$. Therefore, we have

$$h_K(u) = \max A \leq \max B = h_L(u).$$

□

A surprising but useful characterisation of the Hausdorff distance is given by the support function.

Lemma 2.21. *Let $K, L \in \mathcal{K}^n$ convex bodies with support functions $h_K, h_L : S^{n-1} \to \mathbb{R}$. Then, their Hausdorff distance is given by*

$$\delta(K, L) = ||h_K - h_L||_\infty,$$

where $||\cdot||_\infty$ denotes the supremum norm on $C(S^{n-1})$, the set of continuous functions $S^{n-1} \to \mathbb{R}$.

Proof. [Sch14, Lemma 1.8.14]. □

Theorem 2.22. *Let $K \in \mathcal{K}^n$. Then, there is a sequence of bounded, convex polytopes $(P_n)_{n\in\mathbb{N}}$ that converges to K.*

Proof. [Sch14, Theorem 1.8.16]. □

The support function clearly contains all information that are necessary to reconstruct the convex body. Since it is a map $h_K : S^{n-1} \to \mathbb{R}$, we can construct all hyperplanes that support K by choosing

$$H_u = \{x \in \mathbb{r}^n \mid \langle x, u\rangle = h(u)\}$$

in direction $u \in S^{n-1}$.

Furthermore, $K \in H_u^-$, where H_u^- is the negative half-space of H_u, for any $u \in f^{n-1}$. This gives rise to the definition of the Aleksandrov body.

Definition 2.23 (Aleksandrov body)**.** Let $h : S^{n-1} \to \mathbb{R}^+$ be a positive function defined on the $n-1$-dimensional unit sphere. Then, the set defined by

$$Q_h := \{x \in \mathbb{R}^n \mid \langle x, u\rangle \leq h(u) \text{ for all } u \in S^{n-1}\}$$

is called the *Aleksandrov body* or *Wulff shape* of h.

Remark. Let $K \in \mathcal{K}^n$ be a convex body with support function $h_K : S^{n-1} \to \mathbb{R}$. Now, consider the Aleksandrov body of h_K, which we denote by Q_K.

Our assertion is that $Q_K = K$. For this, let $x \in K$. Then, $\langle x, u\rangle \leq h_K(u)$ for all $u \in S^{n-1}$ by definition of the support function. Thus, $x \in Q_K$.

For the reverse direction, suppose that $x \in Q_K$, i. e. we have $\langle x, u\rangle \leq h_K(u)$ for all $u \in S^{n-1}$. To obtain a contradiction, suppose that $x \notin K$. But then $\langle x, u\rangle > h_K(u)$ for some $u \in S^{n-1}$, since otherwise x lies in the interior of K. Hence, $x \in K$.

Thus, the Aleksandrov body of the support function of K is the body K. In this sense, the Aleskandrov body construction is an inverse operation to the support function.

Another proof can be obtained by considering the Hausdorff distance $\delta(K, Q_K)$.

One might expect that the reverse holds, i. e., for a function $h : S^{n-1} \to \mathbb{R}$, the Aleskandrov body Q_h has support function h. While this follows indeed for functions $h : S^{n-1} \to \mathbb{R}$ that are support functions of convex bodies, this does not hold in general, as the following example will demonstrate.

Example 2.24. Consider the function

$$\begin{aligned} h : S^1 &\longrightarrow \mathbb{R}, \\ (u_1, u_2) &\longmapsto |u_1| + |u_2|. \end{aligned}$$

Then, the Aleskandrov body Q_h is the unit square $Q := [-1, 1]^2$, since $h_Q(u) = h(u)$ for all $u \in S^1$.

Now, we modify this function at a few points, resulting in a new function $\tilde{h} : S^1 \to \mathbb{R}$. Let $C = 2^{-1/2}\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$. Then, let that function be given by $\tilde{h}(u) = h(u)$ for all $u \in S^1 \setminus C$ and by $\tilde{h}(u) = \sqrt{2}$ for $u \in C$.

Then, the Aleksandrov body of the function $\tilde{h}$ is the unit ball with respect to the 1-norm on $\mathbb{R}^2$, which is depicted in Figure 2.1. This holds, since for all $S^1 \setminus C$, the function $\tilde{h}$ describes the given square Q, but its values on C cuts the resulting body by appropriate hyperplanes.

Thus, the connection between the support function of an Aleksandrov body and its generating function is non-trivial.

Definition 2.25 (Gauss map)**.** Let $K \in \mathcal{K}^n$ be a convex body.

A boundary point $x \in \partial K$ is called *regular* if the supporting hyperplane at x is unique, i. e. the boundary ∂K has an unique outer unit normal vector at x. The set of regular points is denoted by $\partial' K$.

The map $\nu_K : \partial' K \to S^{n-1}$ that assigns the outer unit normal $\nu_K(x)$ of the supporting hyperplane at $x \in \partial K'$ to a regular boundary point $x \in \partial' K$ is called the *Gauss map* of K.

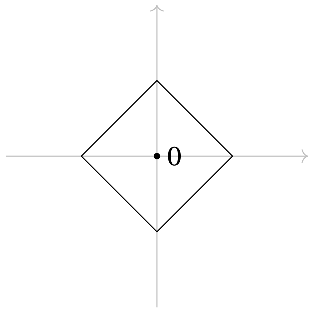

Figure 2.1: Unit ball with respect to the 1-norm on $\mathbb{R}^2$.

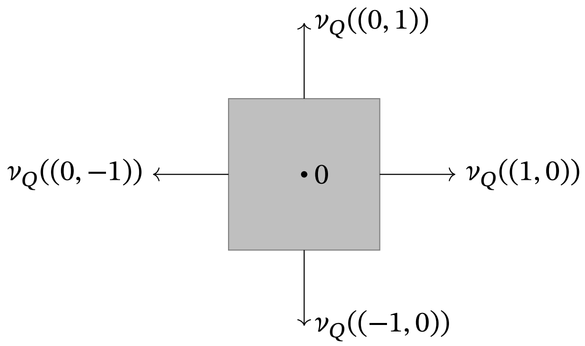

Figure 2.2: Outer unit normals of the unit square in $\mathbb{R}^2$

Example 2.26. Let $Q = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$ be a square in Euclidean plane, which is depicted in Figure 2.2. Its boundary is given by

$$\partial Q = (\{-1,1\} \times [-1,1]) \cup ([-1,1] \times \{1,-1\}).$$

Now, the corners of Q are $(1,1), (-1,1), (-1,-1), (1,-1)$ and these elements do not have an unique outer unit normal. It is easy to see that the remaining boundary points are regular. Thus, the set of regular boundary points of Q is given by

$$\partial' Q = (\{-1,1\}) \times (-1,1) \cup ((-1,1) \times \{1,-1\}).$$

Thus, the Gauss map of Q is given by

$$\begin{aligned} \nu_Q : \partial' Q &\longrightarrow S^1, \\ x &\longmapsto \begin{cases} (1,0) & \text{for } x \in \{1\} \times (-1,1), \\ (0,1) & \text{for } x \in (-1,1) \times \{1\}, \\ (-1,0) & \text{for } x \in \{-1\} \times (-1,1), \\ (0,-1) & \text{for } x \in (-1,1) \times \{-1\}. \end{cases} \end{aligned}$$

For example, $\nu_Q(1, 1/2) = (1,0)$.

2.3 Measurability of convex sets

Before talking about the volume of convex sets, we have to talk about their measurability. While one might expect that convex sets behave well with respect to the usual measures, i. e. Borel measures on $\mathbb{R}^n$, pathological examples in measure theory, e. g. the Vitali construction or even the Cantor set, give rise to a proof that convex sets are, in fact, measurable.

For this procedure, we will not rely on the usual Lebesgue measure, but replace it by the so-called Hausdorff measure. The Hausdorff measure usually appears in fractal geometry as an ingredient for the fractal dimension of sets, but in our treatment, this will be of no interest.

We choose the Hausdorff measure, since it allows us, in contrast to the usual construction of the Lebesgue measure, to get a meaningful measure on subsets that have a lower dimension than their surrounding space.

For example, a well-known theorem (cf. [Els18]) states that, for an injective curve $\gamma : [0,1] \to \mathbb{R}^n$, its length is the 1-dimensional Hausdorff measure of its image $\gamma([0,1])$. A similar result holds for the boundary or surface of convex bodies.

Since proving deep theorems about measure theory is not our aim, we define the outer Hausdorff measure in an effective way. For that, we denote the *diameter* of a subset $A \subseteq \mathbb{R}^n$ by $\operatorname{diam}(A)$, and define it as follows:

$$\operatorname{diam}(A) := \sup\{d(x,y) \mid x,y \in A\} \quad \text{for all } A \subseteq \mathbb{R}^n.$$

Definition 2.27 (Hausdorff measure)**.** Let (M, d) be a metric space, $p \in [0, \infty)$. The p-dimensional *outer Hausdorff measure* is given by

$$\mathcal{H}^p(A) = \lim_{\delta \to 0+} \mathcal{H}^p_\delta(A),$$

where

$$\mathcal{H}^p_\delta = \inf\Big\{ \sum_{i=1}^{\infty} (\operatorname{diam}(A_i))^p \mid (A_i)_{i\in\mathbb{N}} \subseteq B(M), A \subseteq \bigcup_{i=1}^{\infty} A_i, \operatorname{diam}(A_i) < \delta \Big\}.$$

That this is a metric outer measure follows from general theory and is elaborated in [Els18]. To obtain a Borel measure from this definition, the usual path is to choose the Carathéodory construction. We shall sketch this construction here, but for technical details, we refer to [Els18] again.

Definition 2.28 (Measurable set)**.** Let μ be an outer measure on $\mathbb{R}^n$. A set $A \subseteq \mathbb{R}^n$ is called *measurable* if $\mu(B) = \mu(A \cap B) + \mu(A^C \cap B)$ for all $B \subseteq \mathbb{R}^n$.

This definition of a measurable set is due to Carathéodory and the essential component of the Carathéodory construction.

One can show that the set of all measurable subsets is a σ-algebra, and that an outer measure μ on $\mathbb{R}^n$ restricted to the subset of all measurable sets is a measure, cf. [Els18, Satz II.4.4].

By applying the Carathéodory construction to the outer Hausdorff measure, we obtain the *Hausdorff measure*. By general theory it follows that this measure is a Borel measure, i. e. it is defined on the Borel σ-algebra on $\mathbb{R}$. We denote the n-dimensional Hausdorff measure in this section by $\mathcal{H}^n$.

Theorem 2.29. *The n-dimensional Hausdorff measure satisfies the following properties:*

(i) $\mathcal{H}^n$ *is translation-invariant, i. e. for a measurable set* $A \subseteq \mathbb{R}^n$ *and* $x \in \mathbb{R}^n$*, it holds* $\mathcal{H}^n(A + x) = \mathcal{H}^n(A)$*.*

(ii) $\mathcal{H}^n$ *is positive homogeneous, i. e. for a measurable set* $A \subseteq \mathbb{R}^n$ *and a positive constant* $\alpha > 0$*, it holds* $\mathcal{H}^n(\alpha A) = \alpha^n \mathcal{H}^n(A)$*.*

(iii) $\mathcal{H}^n$ *is monotonic, i. e. for measurable sets* $A, B \subseteq \mathbb{R}^n$*,* $A \subseteq B$*, it holds* $\mathcal{H}^n(A) \leq \mathcal{H}^n(B)$*.*

Proof. These properties follow directly from the respective properties of the Hausdorff outer measure, cf. [Els18, p. 78]. □

Theorem 2.30 (Equivalence of Hausdorff and Lebesgue measure)**.** *Let* $A \subseteq \mathbb{R}^n$ *be a Borel set. Then, the n-dimensional Hausdorff measure and n-dimensional Lebesgue measure are equivalent, such that*

$$\lambda^n(A) = \alpha_n \mathcal{H}^n(A),$$

holds, where $\alpha_n > 0$ *depends solely on the dimension n.*

Proof. See [Els18, Satz III.2.9]. □

The actual constant α_n is of no further interest for us, since most of the properties, which we prove, do not depend on the actual measure of some body. Thus, the Hausdorff measure and the Lebesgue measure coincide on Borel sets.

Theorem 2.31. *Let K be a convex set. Then, the boundary ∂K is a $\mathcal{H}^n$-null set.*

Proof. See [Els18, Satz II.7.7] and [Lan86]. □

Since we can partition a convex set $K \subseteq \mathbb{R}^n$ into its interior $\operatorname{int} K$ and its boundary $K \cap \partial K$, the measurability of K is equivalent with the measurability of $K \cap \partial K$. Now, $K \cap \partial K \subseteq \partial K$ is a subset of a null set. Thus, $K \cap \partial K$ is measurable w.r.t. $\mathcal{H}^n$. Hence, a convex set is measurable.

Remark. Let $K \subseteq \mathbb{R}^n$ be a convex set. Now, consider the boundary of K that belongs to K, which is given by the set $L' = \partial K \cap K$.

Then, $L' \subseteq \partial K$, and further, L' is a null set, since ∂K is a null set by the preceding theorem.

Since K is convex, we can partition K as follows: $K = \int(K) \cup L'$. Now, K is the union of two $\mathcal{H}^n$-measurable sets. Hence, a convex set is $\mathcal{H}^n$-measurable.

Definition 2.32 (Volume functional)**.** Let $n \in \mathbb{N}$, $n \geq 1$. Further, let $k \in \mathbb{N}$, $1 \leq k \leq n$ and let $B \subseteq \mathcal{P}(\mathbb{R}^n))$ the set of $\mathcal{H}^k$-measurable sets in $\mathbb{R}^n$.

The k-dimensional volume functional $\mathrm{V}_k : B \to [0, \infty]$ is given by

$$\mathrm{V}_k(A) = \alpha_n \mathcal{H}^n(A) \quad \text{for all } A \in B.$$

We denote the n-dimensional volume functional on $\mathbb{R}^n$ by $\mathrm{V} := \mathrm{V}_n$.

Remark. Let $P \in \mathcal{K}^n$ be a polytope with unit normals $u_1, \ldots, u_k$. Then, the volume of P is given by the formula

$$\mathrm{V}(P) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{V}_{n-1}(F(P, u_i)) h_P(u_i),$$

where V_{n-1} denotes the $(n-1)$-dimensional Hausdorff measure.

This can be seen as follows: For a polytope P, we can always assume that $0 \in P$, i. e. we can translate P such that 0 lies in the polytope. Since the Hausdorff measure is translation-invariant, the volume of P will not change.

Then, the polytope P is the union of k cones of their facets, i. e. the cones

$$P_i := \operatorname{conv}(\{0\}, F(P, u_i)) \quad \text{for } i = 1, \ldots, k.$$

By general properties of the support function $h_P : S^{n-1} \to \mathbb{R}$, the cone P_i has height $h_P(u_i)$, and therefore $\mathrm{V}(P_i) = 1/n\, \mathrm{V}_{n-1}(F(P, u_i)) h_P(u_i)$. In this formula, $F(P, u_i)$ denotes the facet with outer unit normal u_i of P, which is the base of the cone P_i. The volume formula for cones follows from Cavalieri's principle.

Since the boundaries of the cones are null sets, their finite union is also a null set, yielding the general formula for convex polytopes.

When one has a convex body, there are several measures that one can construct. One of such measures is the surface area measure of a convex body.

Definition 2.33 (Surface area measure of convex bodies)**.** Let $K \in \mathcal{K}_0^n$ be a convex body and let $\nu_K : \partial K \to S^{n-1}$ be the associated generalised Gauss map. Then, the *surface-area measure* of K is defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$$

for each Borel set $\omega \subseteq S^{n-1}$.

The surface area measure S_K measures the area of the surface with given outer unit normal vectors. Let $\omega \subseteq \mathbb{S}^{n-1}$ be a Borel set of outer unit normal vectors. Then $S_K(\omega)$ measures the set of all points $x \in \partial K$ that have an outer unit normal vector in ω.

Example 2.34. Let B be an two-dimensional unit ball, i. e. the set $B[0,1]$ with respect to the Euclidean norm on $\mathbb{R}^2$. Then, the surface area measure of B is given by $S_K(\omega) = \mathcal{H}^1(\nu_K^{-1}(\omega))$ for all Borel sets $\omega \subseteq S^1$.

For $x \in \partial B = S^1$, we have $\nu_K(x) = x$, i. e. the outer unit normal at x is given by $x \in S^1$. Thus, the Gauss map $\nu_K : \partial B \to S^1$ is the identity map.

Hence,

$$S_K(\omega) = \mathcal{H}^1(\omega) \quad \text{for all Borel sets } \omega \subseteq S^1.$$

By the same argument, this holds for higher dimensions too.

One problem that is associated with this measure is the so-called Minkowski problem, which we shall treat in Section 3.3.

Another measure is given by the cone-volume measure.

Definition 2.35 (Cone-volume measure, cf. [Bör+12])**.** The *cone-volume measure* of a convex body $K \in \mathcal{K}_0^n$ is defined by

$$V_K(\omega) = \frac{1}{n} \int_{\nu_K^{-1}(\omega)} \langle x, \nu_K(x) \rangle \, d\mathcal{H}^{n-1}(x)$$

for a Borel set $\omega \in \mathcal{B}(S^{n-1})$.

By this definition, the cone-volume measure V_K measures the (not necessarily convex) cone whose base is the set $\nu_K^{-1}(\omega)$ to the origin of the Euclidean space, thus measuring the amount of volume of K with respect to a given set of outer unit normal vectors.

Example 2.36. Let $P \in \mathcal{K}^n$ be a convex polytope with unit normals $u_1, \dots, u_k$. Then, for a unit normal u_i, $i \in \{1, \dots, k\}$, we have $V_P(\{u_i\}) = \frac{1}{n} \mathrm{V}_{n-1}(u_i) h_P(u_i)$.

Thus, the cone-volume measure V_P is given by

$$V_P = h_P(u_1)\delta_1 + \cdots + h_P(u_k)\delta_i,$$

where δ_i is the discrete measure concentrated on $\{u_i\}$, for $i = 1, \dots, k$.

With the polygon P given in Figure 2.3, it follows that the measure V_P is concentrated on $\{\vartheta_1, \dots, \vartheta_5\}$.

With the preceding example, the equality $V_P = \mathrm{V}(P)$ directly follows for polytopes, and by approximation for all convex bodies, i. e. $V_K = \mathrm{V}(K)$ for all $K \in \mathcal{K}_0^n$.

Thus, one denotes by $\bar{V}_K$ the normalised cone-volume measure, which is given by $\bar{V}_K(\omega) = \mathrm{V}(K)^{-1} V_K(\omega)$ for all Borel sets $\omega \subseteq S^{n-1}$. Hence, $\bar{V}_K$ is a probability measure.

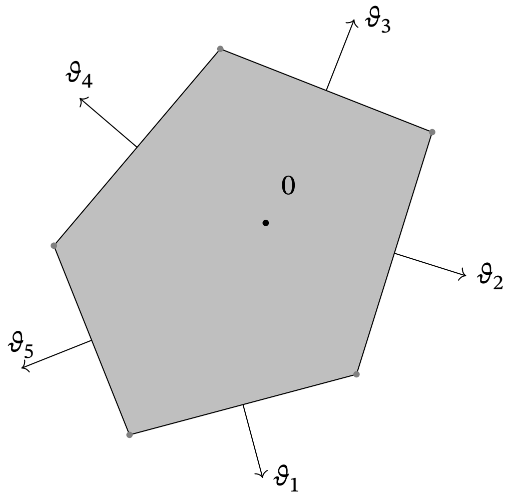

Figure 2.3: A polygon in 2-dimensional Euclidean space and its outer unit normals.

3 The Brunn–Minkowski theory

The Brunn–Minkowski theory is motivated by the work of Brunn and Minkowski on the Brunn–Minkowski inequality in the 19th century, and is a large branch of convex geometry. One of its most famous results, the Brunn–Minkowski inequality for convex bodies, gives a grasp what the theory is about. The most basic objects in the theory are convex bodies, whose general theory we developed in the previous chapter.

The volume functional $\mathrm{V} : \mathcal{K}^n \to [0, \infty)$ that assigns a positive measure to each convex body is of interest: It is used to characterise several subtypes of convex bodies and the connections between them. For example, the result of the Brunn–Minkowski theorem, which we shall prove in Theorem 3.3, states that two convex bodies K and L are homothetic if and only if equality in the Brunn–Minkowski inequality holds.

Nowadays the Brunn–Minkowski theory is a large area of geometry, which offers a massive amount of results about convex bodies. Therefore, we cannot provide a complete survey through the topic, but we refer to [Sch14] as a standard work on Brunn–Minkowski theory. Instead of this, we focus on the development of the Brunn–Minkowski inequality and the necessary theory to state and answer the Minkowski problem, which is one of the most important motivations of the Brunn–Minkowski inequality.

3.1 Brunn–Minkowski inequality

The Brunn–Minkowski inequality is the central inequality of the Brunn–Minkowski theory. It states that, for convex bodies $K, L \in \mathcal{K}^n$, the volumes of K, L and their Minkowski sum $K + L$ satisfy the condition

$$\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda) \cdot \mathrm{V}(K)^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n} \text{ for all } \lambda \in [0, 1].$$

In the literature, there are many different variants of the Brunn–Minkowski inequality, which are mostly equivalent. In this chapter, we will develop both the classical and general Brunn–Minkowski inequality, where the former is the variant that was established by Brunn in the 19th century.

Apart from that, the so-called general Brunn–Minkowski inequality is an extension of the classical inequality to all convex or even measurable set. This inequality is of our interest, since it allows us to prove the classical inequality by analytical and measure theoretic arguments, an approach that is popular among the proofs

The proof techniques used to prove these inequalities are of special interest, since they show a special connection of analysis, convex geometry and measure theory. The obtained result of the general inequality will hold for all $\mathcal{H}^n$-measurable sets, i. e. sets that are measurable with respect to the n-dimensional Hausdorff measure. Thus, the general variant will include the classical Brunn–Minkowski inequality as a special case, whereas convex bodies are measurable by Lemma 2.31.

The following statement of the general Brunn–Minkowski inequality is that from [Gar02, Theorem 4.1], which is a popular survey article on the Brunn–Minkowski inequality and applications.

Theorem 3.1 (General Brunn–Minkowski inequality). *Let K, L be two measurable sets, and $\lambda \in [0,1]$ such that $(1-\lambda)K + \lambda L$ is measurable w. r. t. the n-dimensional Hausdorff measure. Then, the Brunn–Minkowski inequality*

$$\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda) \cdot \mathrm{V}(K)^{1/n} + \lambda\, \mathrm{V}(L)^{1/n}$$

holds.

This form of the inequality, which we also gave in the introduction to the section, is also called the standard form. Another form of this inequality is the so-called multiplicative form, where the dimension-dependent exponents disappear and the sum on the right side transforms to a product of powers.

Theorem 3.2 (General Brunn–Minkowski inequality, multiplicative form). *Let $K, L \subseteq \mathbb{R}^n$ be two measurable sets, $\lambda \in [0,1]$ such that $(1-\lambda)K + \lambda L$ is measurable w. r. t. the n-dimensional Lebesgue measure. Then*

$$\mathrm{V}((1-\lambda)K + \lambda L) \geq \mathrm{V}(K)^{1-\lambda}\, \mathrm{V}(L)^{\lambda}$$

The classical form of the Brunn–Minkowski inequality is given as follows:

Theorem 3.3 (Classical Brunn–Minkowski inequality). *Let $K, L \subseteq \mathbb{R}^n$ be two convex bodies and $\lambda \in [0,1]$. Then*

$$\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda) \cdot (\mathrm{V}(K))^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n}.$$

Before applying the general Brunn–Minkowski inequality to the proof of the classic variant, we have to discuss the measurability of the Minkowski sum $K + L$ for convex bodies $K, L \in \mathcal{K}^n$. It turns out, that with the restriction to convex bodies the result is almost trivial. The following lemma and proof use solely topological results.

Lemma 3.4. *Let $K, L \in \mathcal{K}^n$. Then the Minkowski sum $K + L$ is measurable.*

Proof. Consider the map

$$\begin{aligned} f\colon\ K \times L &\longrightarrow K + L, \\ (x, y) &\longmapsto x + y. \end{aligned}$$

This map is continuous, since $\mathbb{R}^n$ is a topological vector space. Since K and L are compact, their Cartesian product $K \times L$ is compact. Then the image $f(K \times L)$ must be compact, cf. [Mun00]. The map f is onto by definition, hence $f(K \times L) = K + L$ is compact, a Borel-set, and therefore measurable. □

Although the same result holds also for Borel sets $K, L \subseteq \mathbb{R}^n$, the Minkowski sum of two measurable sets $K, L \subseteq \mathbb{R}^n$ is not measurable again.

Example 3.5. Let $A \subseteq [0,1]$ be a non-measurable subset. For example, we can obtain such set by the well-known Vitali construction, cf. [Els18].

now, consider the sets

$$K := A \times \{0\}, L := \{0\} \times [0,1],$$

which are subsets of $[0,1] \times [0,1]$. These sets are measurable, since K and L are null sets with respect to the usual measure on $[0,1]^2$.

Our aim is to show that their Minkowski sum $S := K + L$ is not measurable. Indeed, this sum is given by $S = A \times \{0\} + \{0\} \times [0,1] = A \times [0,1]$. Now, suppose that S is measurable. Then, by Fubini's theorem, we would have

$$\mathrm{V}(S) = \int_0^1 \int_0^1 1_S(x,y)\, dx\, dy = \int_0^1 1_[0,1](y) \int_0^1 1_K(x)\, dx\, dy.$$

But since the inner integral is not defined by assumption, the set S cannot be measurable. Thus, there exist sums of measurable spaces that are not measurable.

Despite the measurability of sums does not hold in general, it holds for all Borel sets $K, L \subseteq \mathbb{R}^n$. This follows from the decomposition

$$K + L = \bigcup_{x \in K} x + L.$$

With the general Brunn–Minkowski inequality it is almost trivial to prove its classical version. Certainly, the equality cases need another treatment, which does not directly follow from the general variant. Accordingly, we defer the proof of the classical Brunn–Minkowski inequality including its equality cases to the end of this section.

Now, we will present two proofs of the general Brunn–Minkowski inequality. The first proof will use an analytic inequality that is called the Prékopa–Leindler inequality. In the second proof, we choose a measure-theoretic approach.

In the following procedure, the term "measurable" in conjunction with a subset $A \subseteq \mathbb{R}^n$ means the measurable with respect to the n-dimensional Hausdorff measure. In other words, A is called measurable, if A is $\mathcal{H}^n$-measurable. The same holds for the term "integrable".

Theorem 3.6 (Prékopa–Leindler inequality, cf. [Gar02, Theorem 7.1])**.** *Let $f, g, h \colon \mathbb{R}^n \to \mathbb{R}$ be integrable maps that satisfy*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(x)^{\lambda}$$

for all $\lambda \in [0,1]$, $x, y \in \mathbb{R}^n$.

Then

$$\int_{\mathbb{R}^n} h(x)\, dx \geq \left(\int_{\mathbb{R}^n} f(x)\, dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\, dx \right)^{\lambda}.$$

In the proof of the Prékopa–Leindler inequality, we will use the inequality of arithmetic and geometric means. This inequality is a standard result from analysis, and is obtained by induction or as an application of the concavity of the logarithm function $\log\colon (0,\infty) \to \mathbb{R}$.

Lemma 3.7 (Inequality of the arithmetic and geometric means). *For non-negative real numbers $x_1, \dots, x_n \in \mathbb{R}^n$ it holds*

$$\frac{\sum_{i=1}^n x_i}{n} \geq \left(\prod_{i=1}^n x_i\right)^{1/n}$$

where equality holds precisely when $x_1 = \cdots = x_n$.

The following proof of the Prékopa–Leindler inequality (Theorem 3.6) is due to [Sch14, p. 374f], and is by induction.

Proof of Theorem 3.6. The proof of the Prékopa–Leindler inequality is by induction over n, although the base case $n = 1$ is not even trivial.

Let $n = 1$ and $F := \int f(x)\,dx$, $G := \int g(x)\,dx$. Further, we suppose that $F \geq 0$, $G \geq 0$. Then, define the functions $u, v\colon (0,1) \to \mathbb{R}$ by the fixed-point problem

$$\frac{1}{F}\int_{-\infty}^{u(t)} f(x)\,dx = \frac{1}{G}\int_{-\infty}^{v(t)} g(x)\,dx = t.$$

Since f, g are integrable and positive and therefore $\int f < \infty$, $\int g < \infty$, these functions exist.

Further, the derivatives of u, v are determined by

$$\frac{f(u(t))u'(t)}{F} = \frac{g(v(t))v'(t)}{G} = 1 \text{ for all } t \in (0,1)$$

by the fundamental theorem of calculus and differentiation rules. Now, we consider the linear combination

$$\begin{aligned} w\colon (0,1) &\longrightarrow \mathbb{R}, \\ t &\longmapsto (1-\lambda)u(t) + \lambda v(t). \end{aligned}$$

This function is differentiable, and by the arithmetic-geometric inequality (Lemma 3.7), we have

$$\begin{aligned} w'(t) &= (1-\lambda)u'(t) + \lambda v'(t) \\ &\geq (u'(t))^{1-\lambda}(v'(t))^{\lambda} \\ &= \left(\frac{F}{f(u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda}. \end{aligned}$$

By plugging the preceding formulas together, we obtain the estimate

$$\begin{aligned}\int_{\mathbb{R}} h(x)\,dx &\ge \int_0^1 h(w(t))w'(t)\,dt \\ &\ge \int_0^1 f(u(t))^{1-\lambda} g(v(t))^{\lambda} \left(\frac{F}{f(u(t))}\right)^{1-\lambda} \left(\frac{G}{g(v(t))}\right)^{\lambda} \\ &= F^{1-\lambda} G^{\lambda}.\end{aligned}$$

When we have non-positive functions f, g, we consider the functions $f^+, g^+, f^-, g^- : \mathbb{R} \to \mathbb{R}$ that are positive and negative, respectively, such that $f = f^+ + f^-$ and $g = g^+ + g^-$. Since the Prékopa–Leindler inequality holds for f^+, g^+, f^-, g^-, it also holds for the linear combinations. Thus, the Prékopa–Leindler inequality holds for the one-dimensional case

Now, let $n \ge 1$ such that the Prékopa–Leindler inequality holds for appropriate functions $\mathbb{R}^n \to \mathbb{R}$. Let $f, g, h : \mathbb{R}^{n+1} \to \mathbb{R}$ be functions that satisfy the precondition, i. e.

$$h((1-\lambda)x + \lambda y) \ge f(x)^{1-\lambda} g(y)^{\lambda} \quad \text{for all } x, y \in \mathbb{R}^{n+1}, \lambda \in [0,1].$$

Further, let $x = (x_1, \dots, x_n, x_{n+1}), y = (y_1, \dots, y_n, y_{n+1}) \in \mathbb{R}^{n+1}$. Then, we have

$$h((1-\lambda)x + \lambda y) \ge f(x)^{1-\lambda} g(y)^{1-\lambda},$$

and with the notation $z := (1-\lambda)x_{n+1} + \lambda y_{n+1}$, we have

$$h((1-\lambda)\cdot(x_1, \dots, x_n) + \lambda \cdot (y_1, \dots, y_n), z) \ge f(x)^{1-\lambda} g(y)^{1-\lambda}.$$

But this is the precondition for the n-dimensional case, thus we obtain

$$\int_{\mathbb{R}^n} h(\hat{x}, z)\,d\hat{x} \ge \left(\int_{\mathbb{R}^n} f(\hat{x}, x_{n+1})\,d\hat{x}\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(\hat{x}, x_{n+1})\,d\hat{x}\right)^{\lambda}.$$

Now, consider the functions

$$\begin{aligned} F &: \mathbb{R} \longrightarrow \mathbb{R}, & G &: \mathbb{R} \longrightarrow \mathbb{R}, & H &: \mathbb{R} \longrightarrow \mathbb{R}, \\ y &\longmapsto \int_{\mathbb{R}^n} f(\hat{x}, y)\,d\hat{x}, & y &\longmapsto \int_{\mathbb{R}^n} g(\hat{x}, y)\,d\hat{x}, & y &\longmapsto \int_{\mathbb{R}^n} h(\hat{x}, y)\,d\hat{x}. \end{aligned}$$

With this naming, the preceding condition says that we have $H((1-\lambda)a + \lambda b) \ge F(a)^{1-\lambda} G(b)^{1-\lambda}$ for all $a, b \in \mathbb{R}, \lambda > 0$.

This is the precondition of the one-dimensional Prékopa–Leindler inequality. Therefore, we obtain by application of the one-dimensional, established case the inequality

$$\int_{\mathbb{R}} H(x)\,dx \ge \left(\int_{\mathbb{R}} F(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} G(x)\,dx\right)^{\lambda}.$$

By Fubini's theorem, the left side is the integral $\int h(x)\,dx)$ and on the right sides, there are the integrals $\int f(x)\,dx$ and $\int g(x)\,dx$. Hence, the Prékopa–Leindler inequality holds in $(n+1)$-dimensional space. □

We continue with the proof of the general Brunn–Minkowski inequality (Theorem 3.1) by using the Prékopa–Leindler inequality (Theorem 3.6).

First proof of the general Brunn–Minkowski inequality (Theorem 3.1). Let $M := (1-\lambda)K + \lambda L$. Further, let $1_A : \mathbb{R}^n \to \{0,1\}$ denote the characteristic function of the set $A \subseteq \mathbb{R}^n$.

Now, consider the characteristic functions $1_K, 1_L, 1_M : \mathbb{R}^n$ of K, L and M, respectively. Then the precondition of the Prékopa–Leindler inequality (Theorem 3.6) is

$$1_M((1-\lambda)x + \lambda y) \geq 1_K(x)^{1-\lambda} 1_L(y)^{\lambda}$$

for all $x, y \in \mathbb{R}^n$. Suppose that $1_K(x)^{1-\lambda} 1_L(y)^{\lambda} = 1$ holds. Then, $x \in K$ and $y \in K$, and therefore also $(1-\lambda)x + \lambda y \in M$.

Then

$$\begin{aligned} \mathrm{V}((1-\lambda)K + \lambda L) &= \int_{\mathbb{R}^n} 1_M \, dx \\ &\geq \left(\int_{\mathbb{R}^n} 1_K \, dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} 1_L \, dx \right)^{\lambda} \\ &= (\mathrm{V}(K))^{1-\lambda} (\mathrm{V}(L))^{\lambda} \end{aligned}$$

by the Prékopa–Leindler inequality. That this is the Brunn–Minkowski inequality follows from Theorem 3.11. □

The second proof approach of the general Brunn–Minkowski inequality is by approximation. This method first arose in [HO56], where the arguments that have been used were mostly elementary. We follow mostly the proof that is given in [Gar02], with amendments from [BZ88].

Suppose that $A \subseteq \mathbb{R}^n$ is a measurable set. Then, by the inner regularity of the Lebesgue measure, and thus of the volume functional V_n, we can approximate A by countably compact sets $(A_k)_{k \in \mathbb{N}}$, $A_k \subseteq A$ for $k \in \mathbb{N}$. Thus, we can assume that A is compact.

Definition 3.8 (Cuboids and elementary sets)**.** A *cuboid* Cin $\mathbb{R}^n$ is the product of n compact intervals in $\mathbb{R}$, i. e. there are $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ that satisfy $a_i < b_i$ for $i = 1, \dots, n$ such that

$$C = \prod_{i=1}^{n} [a_i, b_i].$$

A set $A \subseteq \mathbb{R}^n$ is called elementary, if it is the union of finitely many cuboids.

Furthermore, we can approximate such compact set A by cuboids, as the following lemma states.

Lemma 3.9. *Let $K \subseteq \mathbb{R}^n$ be a compact set. Then, there exists a sequence of elementary sets $(K_i)_{i \in \mathbb{N}}$ such that*

$$\mathrm{V}(K_i) \to \mathrm{V}(K) \textit{ for } i \to \infty.$$

Proof. Let $\varepsilon > 0$. Then, consider the set $K + B_\infty[0, \varepsilon] \supset K$, which is compact since the sum of two compact sets is compact.

Further, $K + B_\infty[0, \varepsilon] \to K$ for $\varepsilon \to 0$. Especially, this convergence is monotonically when $\varepsilon \to 0$ monotonically. Thus,

$$\lambda^n(K + B_\infty[0, \varepsilon]) \to \lambda^n(K)$$

monotonically.

To construct the approximation of K, consider the following open cover of K:

$$\mathcal{U}_\varepsilon = \{B_\infty(x, \varepsilon) \mid x \in K\}.$$

By compactness, there exists a finite sub cover $\tilde{\mathcal{U}}_\varepsilon \subseteq \mathcal{U}_\varepsilon$. Further, we have $\mathcal{U}_\varepsilon \subseteq K + B_\infty[0, \varepsilon]$ by construction.

Now, the set $K_\varepsilon := \overline{\bigcup_{U \in \tilde{\mathcal{U}}_\varepsilon} U}$, which is the closure of the union of all open cuboids in $\tilde{\mathcal{U}}_\varepsilon$, consists of finitely many hypercubes, and there exists a partition into cuboids. With the chain of inclusions

$$K + B[0, \varepsilon] \subseteq K_\varepsilon \subseteq \tilde{\mathcal{U}}_\varepsilon,$$

the convergence

$$\lambda^n(K_\varepsilon) \longrightarrow \lambda^n(K) \text{ for } \varepsilon \to 0$$

follows by monotonicity of the volume functional. □

This approximation lemma is the crucial ingredient of the following proof. We show that the Brunn–Minkowski inequality holds for sets that are unions of finitely many cuboids. Since we can approximate any compact set by such elementary sets, the Brunn–Minkowski inequality extends to all compact sets.

Further, with the inner regularity of the Hausdorff measure in mind, we can approximate a measurable subset $A \subseteq \mathbb{R}^n$ by compact subsets. This implies especially that the Brunn–Minkowski inequality holds for all measurable sets when it holds for compact subsets.

This decomposition is clearly not unique, e. g. consider any n-dimensional cuboid A_1. Furthermore, the cuboids do not need to be almost disjoint, i. e. they only intersect on their boundaries. That this restriction is not necessary and we can assume without loss of generality that there exists such almost disjoint decomposition, follows from the fact, that the intersection of two cuboids is either empty, a point, or a cuboid. Therefore, let denote $n(A)$ the minimum number of almost disjoint cuboids necessary to construct the elementary set $A \subseteq \mathbb{R}^n$.

Lemma 3.10. *Let K be an n-dimensional elementary set with $n(K) > 1$. Then, there exists a decomposition of K into $n(K)$ cuboids and a hyperplane H parallel to the coordinate axes such that*

$$n(K \cap H_+) < n(K) \quad \text{and} \quad n(K \cap H_-) < n(K).$$

Proof. Let K be an elementary set as above and set $m := n(K)$. Further, let $K_1, \dots, K_m \subseteq \mathbb{R}^n$ be almost disjoint compact cuboids that are a decomposition of K, i. e. $K = \bigcup_{i=1}^m K_i$.

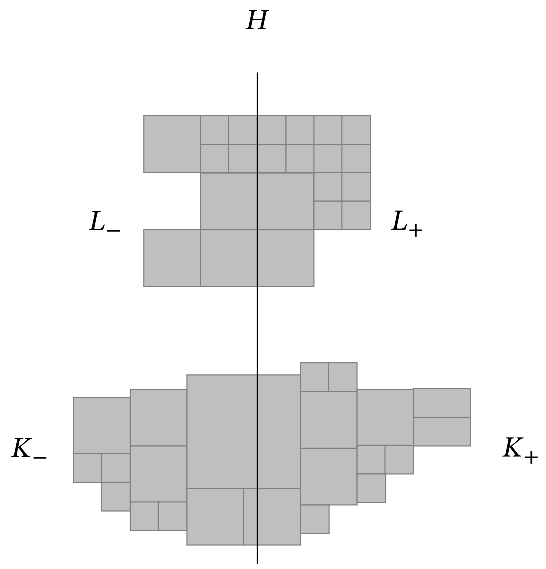

Figure 3.1: Partition of sets K, L that are unions of cuboids by a hyperplane H.

If K is not connected, it is trivial to choose the hyperplane H. Therefore, we suppose that K is connected. Then, there are two cuboids K_i, K_j, $1 \leq i, j \leq m$, $i \neq j$, such that $K_i \cap K_j$ lies in the interior of K and has non-zero $(n-1)$-dimensional volume.

Since K_i, K_j are compact cuboids, $K_i \cap K_j$ lies in a hyperplane parallel to some coordinate axis. Thus, choose this $(n-1)$-dimensional hyperplane H with $K_i \cap K_j \subseteq H$.

Now, let C be any n-dimensional compact cuboid. The section of H through C decomposes C into two cuboids $C_+ = H_+ \cap C$ and $C_- = H_- \cap C$, where H_+, H_- denote the closed half spaces associated with H.

Therefore, the hyperplane H satisfies the asserted properties. □

Second proof of Theorem 3.1. Let K, L be cuboids with side lengths $k_1, \dots, k_n > 0$, $l_1, \dots, l_n > 0$. Then their volumes are given by

$$\mathrm{V}(K) = \prod_{i=1}^{n} k_i \quad \text{and} \quad \mathrm{V}(L) = \prod_{i=1}^{n} l_i.$$

Further, their Minkowski sum $K + L$ is also a cuboid with side lengths $k_1 + l_1, \dots, k_n + l_n > 0$. Thus, it has the volume

$$\mathrm{V}(K+L) = \prod_{i=1}^{n} k_i + l_i.$$

Now, let $m \in \mathbb{N}$ such that an Brunn–Minkowski inequality holds for any two sets K, L that are unions of at most m closed cuboids $K_1, \dots, K_m \subseteq K$, $L_1, \dots, L_m \subseteq L$. Then, extend these sets by two cuboids K_{m+1}, L_{m+1}, i. e.

$$K = \bigcup_{i=1}^{m+1} K_i, \quad L = \bigcup_{i=1}^{m+1} L_i.$$

We shall denote the sets that are unions of these $(m+1)$ cuboids hereafter by K and L.

Let H be a hyperplane such that $K_+ := K \cap H_+, K_- := K \cap H_-$ contain at most m cuboids. This hyperplane H exists by Lemma 3.10.

Furthermore, with the translation invariance of the volume functional, we can translate L such that $L_+ := L \cap H_+, L_- := L \cap H_-$ contain at most m cuboids.

Without loss of generality, we can further assume that L is translated such such that

$$\frac{\mathrm{V}(K_+)}{\mathrm{V}(K)} = \frac{\mathrm{V}(L_+)}{\mathrm{V}(L)} \quad \text{and} \quad \frac{\mathrm{V}(K_-)}{\mathrm{V}(K)} = \frac{\mathrm{V}(L_-)}{\mathrm{V}(L)}$$

hold.

To clarify the idea of this process of partition, it is depicted in Figure 3.1.

Now, $K_+ + L_+ \subseteq H_+$ and $K_- + L_- \subseteq H_-$, and further $(K_+ + L_+) \cup (K_- + L_-) \subseteq K + L$. Thus, we have

$$\begin{aligned} \mathrm{V}(K+L) &\geq \mathrm{V}(K_+ + L_+) + \mathrm{V}(K_- + L_-) \\ &\geq (\mathrm{V}(K_+)^{1/n} + \mathrm{V}(L_+)^{1/n})^n + (\mathrm{V}(K_-)^{1/n} + \mathrm{V}(L_-)^{1/n})^n, \end{aligned}$$

where we applied the induction hypothesis in the second step.

Now, let $K, L \subseteq \mathbb{R}^n$ be compact sets that we approximate by elementary sets $(K_i)_{i\in\mathbb{N}}$, $(L_i)_{i\in\mathbb{N}}$. Then,

$$\mathrm{V}(K_i)^{1/n} \to \mathrm{V}(K)^{1/n} \quad \text{and} \quad \mathrm{V}(L_i)^{1/n} \to \mathrm{V}(L)^{1/n}$$

as $i \to \infty$. On the other side,

$$\mathrm{V}(K_i + L_i)^{1/n} \to \mathrm{V}(K+L)^{1/n} \quad \text{for } i \to \infty,$$

since $K_i + L_i$ is an approximation of $K + L$. Hence,

$$\mathrm{V}(K+L)^{1/n} \geq \mathrm{V}(K)^{1/n} + \mathrm{V}(L)^{1/n}.$$

Thus, the inequality also holds for compact sets. For measurable sets K and L, we have

$$\begin{aligned} \mathrm{V}(K) &= \sup\{\mathrm{V}(F) \mid F \subseteq K \text{ compact }\}, \\ \mathrm{V}(L) &= \sup\{\mathrm{V}(F) \mid F \subseteq K \text{ compact }\}, \end{aligned}$$

by the inner regularity of the volume functional. This directly yields the inequality for all measurable sets.

To obtain the standard form, we replace K and L by the dilates $(1-\lambda)K$ and λL, for $\lambda > 0$, and use the positive homogeneity of the volume functional. □

In the last two proofs of the Brunn–Minkowski inequality, we have encountered several variants of the same inequality. That these variants are all equivalent, i. e. if one of them holds for all measurable subsets $K, L \subseteq \mathbb{R}$ and $\lambda > 0$ if applicable, is trivial for most implications.

The following lemma establishes theses equivalences. Further, it finishes the proof of the Brunn–Minkowski inequality using the Prékopa–Leindler inequality by proving the last ingredient that we have deferred.

Lemma 3.11. *The following variants of the general Brunn–Minkowski inequality are equivalent:*

(i) $\mathrm{V}((1-\lambda)K+\lambda L)^{1/n} \geq (1-\lambda)\cdot \mathrm{V}(K)^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n}$ *for all measurable* $K, L \subseteq \mathbb{R}^n$ *and* $\lambda \in [0,1]$,

(ii) $\mathrm{V}((1-\lambda)K+\lambda L) \geq \mathrm{V}(K)^{1-\lambda}\cdot \mathrm{V}(L)^{\lambda}$ *for all measurable* $K, L \subseteq \mathbb{R}^n$ *and* $\lambda \in [0,1]$.

(iii) $\mathrm{V}(K) = \mathrm{V}(L) = 1 \implies \mathrm{V}(K+L)^{1/n} \geq 1$ *for all measurable* $K, L \subseteq \mathbb{R}^n$,

(iv) $\mathrm{V}(K+L)^{1/n} \geq \mathrm{V}(K)^{1/n} + (\mathrm{V}(L)^{1/n}$ *for all measurable* $K, L \subseteq \mathbb{R}^n$,

Proof. (i) $\implies$ (ii): This follows from the arithmetic-geometric mean inequality (Lemma 3.7).

(ii) $\implies$ (iii): Let $K, L \subseteq \mathbb{R}^n$ measurable such that $\mathrm{V}(K) = \mathrm{V}(L) = 1$. Further, choose $\lambda = \frac{1}{2}$. Then,

$$\mathrm{V}\Big(\frac{1}{2}K + \frac{1}{2}L\Big) = \frac{1}{2^n}\,\mathrm{V}(K+L) \geq 1,$$

and consequently

$$\mathrm{V}(K+L)^{1/n} \geq \frac{1}{2} > 1.$$

(iii) $\implies$ (iv): Let

$$\lambda' := \frac{\mathrm{V}(K)^{1/n}}{\mathrm{V}(K)^{1/n} + \mathrm{V}(L)^{1/n}}$$

and $K' = \mathrm{V}(K)^{-1/n}K$, $L' = \mathrm{V}(L)^{-1/n}L$. Now, $\mathrm{V}(K') = 1$, $\mathrm{V}(L') = 1$ by the positive homogeneity of the volume functional.

Thus,

$$\begin{aligned} 1 \leq \mathrm{V}((1-\lambda')K' + \lambda' L') &= \mathrm{V}\Big(\frac{\mathrm{V}(L)^{1/n}}{\mathrm{V}(K)^{1/n}+\mathrm{V}(L)^{1/n}}\,\mathrm{V}(K)^{-1/n}K + \frac{\mathrm{V}(K)^{1/n}}{\mathrm{V}(K)^{1/n}+\mathrm{V}(L)^{1/n}}\,\mathrm{V}(L)^{-1/n}L\Big) \\ &= \mathrm{V}\Big(\frac{1}{\mathrm{V}(K)^{1/n}+\mathrm{V}(L)^{1/n}}(K+L)\Big) \\ &= \frac{\mathrm{V}(K+L)}{(\mathrm{V}(L)^{1/n}+\mathrm{V}(K)^{1/n})^n}, \end{aligned}$$

which implies (iv) when replacing K by $(1-\lambda)K$ and L by λL.

(iv) $\implies$ (i): This follows from the positive homogeneity of the volume functional. □

We are at a point where we have collected enough evidence to conclude that the general Brunn–Minkowski inequality for measurable sets holds. Its classical variant is a direct implication, since convex bodies are measurable.

Proof of the classical Brunn–Minkowski inequality (Theorem 3.3). Let $K, L \in \mathcal{K}^n$ be two convex bodies and $\lambda \in [0,1]$. Then, the sets K and L are measurable and further, the set $(1-\lambda)K + \lambda L$ is measurable.

Thus, we can apply the general Brunn–Minkowski inequality (Theorem 3.1), and obtain the inequality

$$\mathrm{V}((1-\lambda)K+\lambda L)^{1/n} \geq (1-\lambda)\cdot \mathrm{V}(K)^{1/n} + \lambda\cdot \mathrm{V}(L)^{1/n}.$$

□

Incidentally, we obtain the Brunn–Minkowski inequality for all convex subsets, since these are measurable by Lemma 2.31.

A remarkable and important result is, that the Brunn–Minkowski inequality is sharp, i. e. the equality cases of the inequality are well-known. These equality cases are important for applications of the inequality, where they give rise to equality cases of other inequalities, e. g. Minkowski's first inequality (Theorem 3.18), which we shall discuss in the next section.

One may try to obtain these equality cases by discovering the equality cases of the Prékopa–Leindler inequality. While the equality cases for this inequality are known for the one-dimensional case, they are largely unknown for the general case, cf. [].

Another way to obtain these equality cases is to suppose that we are in the setting of the last proof of the Brunn–Minkowski inequality, i. e. the proof by approximation through cuboids. Then, we note that equality holds precisely when there holds equality in all given intermediate inequalities.

Although this only yields the equality cases for finite unions of cuboids, it is possible to extend this result to all compact sets, giving a much stronger formulation of equality cases than that that we shall prove. The process to obtain such stronger equality cases is given in [BZ88].

We will chose another direction to obtain the equality cases, and use a classical proof method to obtain these.

Theorem 3.12. *Let $K, L \in \mathcal{K}^n$ be two convex bodies. Then*

$$\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} = (1-\lambda)\cdot \mathrm{V}(K)^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n}$$

for some $\lambda \in (0,1)$ if and only if K and L are homothetic.

Lemma 3.13. *Let $K, L \in \mathcal{K}^n$ homothetic convex bodies, i. e. there is a translation vector $v \in \mathbb{R}^n$ and a dilation factor $\alpha > 0$ such that $K = v + \alpha L$. Then,*

$$\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} = (1-\lambda)\,\mathrm{V}(K)^{1/n} + \lambda\,\mathrm{V}(L)^{1/n}$$

for all $\lambda \in [0,1]$.

Proof. This follows directly from the homtheticity of K and L:

$$\begin{aligned}
\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} &= \mathrm{V}((1-\lambda)(v + \alpha L) + \lambda L)^{1/n} \\
&= \mathrm{V}((1-\lambda)v + (1-\lambda)\alpha L + \lambda L)^{1/n} \\
&= \mathrm{V}(((1-\lambda)\alpha + \lambda)L)^{1/n} \\
&= (1-\lambda)\alpha\,\mathrm{V}(L)^{1/n} + \lambda\,\mathrm{V}(L)^{1/n} \\
&= (1-\lambda)\,\mathrm{V}(K)^{1/n} + \lambda\,\mathrm{V}(L)^{1/n}
\end{aligned}$$

by the translation invariance and the homogenity of the volume functional. □

The last proof that we shall discuss is the geometric proof that is due to Kneser and Süss, which has been reproduced in [Sch14] and [BF34]. We mostly follow the reasoning from [Sch14].

In contrast to the preceding proofs, i. e. using Prékopa–Leindler and approximation by cuboids, this proof directly establishes equality cases for the Brunn–Minkowski inequality.

Proof of Theorem 3.3. Let $n = 1$. Then K and L are closed intervals with non-empty interior and therefore homothetic. Thus, equality and especially the inequality hold.

Now, suppose that the Brunn–Minkowski inequality holds for some $n \in \mathbb{N}$ and let $K, L \in \mathcal{K}^{n+1}$ be $(n+1)$-dimensional convex body. We assume without loss of generality that $\mathrm{V}(K) = \mathrm{V}(L) = 1$, i. e. K, L have the same volume despite of their shape.

Choose an unit vector $u \in S^n \subseteq \mathbb{R}^{n+1}$; this choice may be arbitrary. Then, we define the functions

$$v_M\colon \mathbb{R} \to [0,\infty], \quad t \mapsto \mathrm{V}_{n-1}(M \cap H_{u,t}), \qquad w_M\colon \mathbb{R} \to [0,\infty], \quad t \mapsto \mathrm{V}_n(M \cap H^-_{u,t}),$$

for a convex body $M \in \mathcal{K}^{n+1}$. We write $H(t) := H_{u,t}$ for the hyperplane with outer unit normal u and affinity t, and $H^-(t) := H^-_{u,t}$ for its negative half-space.

While the function v_M measures the n-dimensional volume of the section of M with the affine hyperplane $H(t)$, the function w_M measures the $(n+1)$-dimensional volume of the intersection $M \cap H(t)^-$. Therefore, there are $s_M, t_M \in \mathbb{R}$ such that

$$\mathrm{V}(M) = \int_{s_M}^{t_M} dt = \mathrm{V}(M).$$

Hence, the function w_M is differentiable with derivative v_M, i. e.

$$\frac{d}{dt} w_M(t) = v_M(t).$$

Furthermore, the function v_M is continuous by the continuity of the volume functional. Thus, we have

$$w_M(t) = \int_{s_M}^{t_M} v_M(t)\, dt$$

and its derivative is strictly positive for $s_M < t < t_M$. Thus, the function w_M is injective and the derivative of its inverse function $z_M := w_i^{-1}$ is given by

$$z'_M(t) = (w_M^{-1})'(t) = (v_M(w_M^{-1}(t))) \qquad \text{for all } s_M < t < t_M.$$

For brevity, we set $K_\lambda := (1-\lambda)K + \lambda L$ for $\lambda \in [0,1]$, $z_\lambda(t) := z_{K_\lambda}(t)$, $s_\lambda := s_{K_\lambda}$ and $t_\lambda := t_{K_\lambda}$. Further, we write $K(t) := K \cap H(z_K(t))$ and $L(t) := L \cap H(z_L(t))$.

Then, we have the relation

$$(1-\lambda)K(t) + \lambda L(t) \subseteq K_\lambda \cap H(z_\lambda(t)),$$

and therefore,

$$\begin{aligned}
\mathrm{V}(K_\lambda) &= \int_{s_\lambda}^{t_\lambda} \mathrm{V}_n(K_\lambda \cap H(t))\,dt \\
&= \int_0^1 \mathrm{V}_n(K_\lambda \cap H(z_\lambda(t)))z'_\lambda(t)\,dt \\
&\geq \int_0^1 \mathrm{V}_n((1-\lambda)K(t) + \lambda L(t))\Big(\frac{1-\lambda}{v_K(z_K(t))} + \frac{\lambda}{v_L(z_L(t))}\Big)\,dt \\
&\geq \int_0^1 \big((1-\lambda)\,\mathrm{V}(K(t))^n + \lambda\,\mathrm{V}(L(t))^n\big)^{1/n}\left(\frac{1-\lambda}{v_K(z_K(t))} + \frac{\lambda}{v_L(z_L(t))}\right)dt,
\end{aligned}$$

where we used the induction hypothesis in the last step. With the inequality

$$\big((1-\lambda)\,\mathrm{V}(K(t))^n + \lambda\,\mathrm{V}(L(t))^n\big)\left(\frac{1-\lambda}{v_K(z_K(t))} + \frac{\lambda}{v_L(z_K(t))}\right) \geq 1,$$

which follows from the concavity and monotonicity of the logarithm, the assertion $\mathrm{V}_{n+1}(K_\lambda) \geq 1$ follows.

For the equality conditions, suppose that $\mathrm{V}(K_\lambda) = 1$ holds for some $\lambda \in (0,1)$. Then, we have $v_K(z_K(t)) = v_L(z_L(t))$ for all $t \in 0,1]$. Without loss of generality, we assume that K and L have their centre at the origin. Then

$$0\int_K \langle x,u\rangle dx = \int_{s_K}^{t_K} \mathrm{V}(K \cap H(t))t\,dt = \int_0^1 z_K(t)\,dt = \int_0^1 z_L(t)\,dt,$$

since $t \mapsto z_K(t) - z_L(t)$ is constant for $t \in [0,1]$. Therefore $z_K = z_L$, and since $u \in S^{n-1}$ was arbitrary, the bodies must be homothetic. □

A convenient interpretation of the Brunn–Minkowski inequality is given by the concavity of certain functions.

Remark. Consider the function

$$\begin{aligned}
g\colon\ [0,1] &\longrightarrow \mathbb{R}, \\
\lambda &\longmapsto \mathrm{V}((1-\lambda)K + \lambda L)^{1/n}.
\end{aligned}$$

For $\alpha, \beta, \lambda \in [0,1]$, we have

$$\begin{aligned}
g((1-\lambda)\alpha + \lambda\beta) &= \mathrm{V}((1-(1-\lambda)\alpha - \lambda\beta)K + (1-\lambda)\alpha L + \lambda\beta L)^{1/n} \\
&= \mathrm{V}((1-\lambda)(1-\alpha)K + (1-\lambda)\alpha L + \lambda(1-\beta)K + \lambda\beta L)^{1/n} \\
&\leq (1-\lambda)\,\mathrm{V}((1-\alpha)K + \alpha L)^{1/n} + \lambda\,\mathrm{V}((1-\beta)K + \beta L)^{1/n} \\
&= (1-\lambda)g(\alpha) + \lambda g(\beta).
\end{aligned}$$

Hence, the function g is concave.

Furthermore, this concavity is equivalent to the Brunn–Minkowski inequality and therefore a crucial property of that inequality.

In a similar fashion, the function

$$\begin{aligned} h\colon [0,1] &\longrightarrow \mathbb{R}, \\ \lambda &\longmapsto \mathrm{V}((1-\lambda)K + \lambda L)^{1/n} - (1-\lambda)\,\mathrm{V}(K)^{1/n} - \lambda\,\mathrm{V}(L)^{1/n}, \end{aligned}$$

is concave, too. Further, this function h is positive by the Brunn–Minkowski inequality.

3.2 Mixed volumes and quermassintegrals

Mixed volumes and the related quermassintegrals are an intrinsic extension of classical volumes. In this section, we introduce mixed volumes for polytopes and, more generally, convex bodies. Furthermore, we develop the necessary tools to answer the Minkowski problem in Section 3.3.

Before discussing the geometric notion of certain mixed volumes, we will establish a result that is sometimes called *Minkowski's theorem*. It characterises the volume of a Minkowski combination $\lambda_1 K_1 + \cdots + \lambda_s K_s$ as a polynomial in s variables.

We shall use the proof idea from [Ale05, pp. 36] to define the mixed volume on polytopes, and we use an approximation result to extend this mixed volume to all convex bodies.

Two polytopes $P_1, P_2 \subseteq \mathbb{R}^n$ are called *strongly isomorphic*, if the faces in direction $u \in S^{n-1}$ of P_1 and P_2 have the same dimension, i. e. $\dim(F(P_1,u)) = \dim(F(P_2,u))$ for all $u \in S^{n-1}$, where $F(P_i,u)$ denotes the (unique) face of P_i in direction u. Clearly, this defines an equivalence relation on polytopes.

The following approximation result is necessary to extend the mixed volume, which we will define on polytopes, to all convex bodies. It is given in [Sch14, Theorem 2.4.15] and is an extension of the approximation result of Theorem 2.22.

Lemma 3.14. *Let $K_1, \dots, K_s \in \mathcal{K}^n$ and $\varepsilon > 0$ Then, there are strongly isomorphic polytopes $P_1, \dots, P_s$ such that $\delta(K_i, P_i) \le \varepsilon$ for $i = 1, \dots, s$.*

Proof. See [Sch14, Theorem 2.4.15]. □

Theorem 3.15 (Minkowski, cf. [BZ88, p. 136])**.** *Let $K_1, \dots, K_s \in \mathcal{K}^n$ be convex bodies. Then, the volume of the sum body $\lambda_1 K_1 + \cdots + \lambda_s K_s$, where $\lambda_1, \dots, \lambda_s \in [0,\infty)$, is a polynomial in s variables of degree n with respect to the variables $\lambda_1, \dots, \lambda_s$.*

That is, the volume of the body $\sum_{i=1}^{s} \lambda_i K_i$ is determined by

$$\mathrm{V}\left(\sum_{i=1}^{s} \lambda_i K_i\right) = \sum_{m \in [s]^n} \mathrm{V}(K_{m_1}, \dots, K_{m_n}) \lambda_{m_1} \cdots \lambda_{m_n},$$

where $\mathrm{V}(K_{m_1}, \dots, K_{m_n})$ are the coefficients of the polynomial.

Proof. We proof the statement for polytopes and extend it to convex bodies by the preceding approximation lemma. For the approximation, let $(P_n^{(1)}, \dots, P_n^{(s)})_{n\in\mathbb{N}}$ be a sequence of s-tuples of strongly isomorphic polytopes that approximate $(K_1, \dots, K_s)$ in each component. This sequence exists as per Lemma 3.14.

Then, consider the sequence $(\mathrm{V}(\lambda_1 P_n^{(1)} + \cdots + \lambda_s P_n^{(s)}))_{n\in\mathbb{N}}$ of polynomials. Since these all have degree n, and the volume functional is continuous, these must converge point-wise to an polynomial of degree n.

The remaining proof is by induction. The base case $n = 1$ follows from the result, that all bounded polytopes in $\mathbb{R}$ are intervals. Therefore, all polytopes are homothetic, and from the Brunn–Minkowski theorem (Theorem 3.3), we obtain

$$\mathrm{V}\left(\sum_{i=1}^{s} \lambda_i P_i\right) = \sum_{i=1}^{s} \lambda_i \,\mathrm{V}(P_i)$$

for polytopes $P_1, \dots, P_s \subseteq \mathbb{R}$.

This is clearly a polynomial of degree 1 in the variables $\lambda_1, \dots, \lambda_s$.

For the induction step, we suppose that the result holds for polytopes of dimension $(n-1)$. Let $P_1, \dots, P_s \subseteq \mathbb{R}^n$ be polytopes of dimension n. Further, let $h_1, \dots, h_s : S^{n-1} \to \mathbb{R}$ denote the support functions of $P_1, \dots, P_s$.

Then, the volume of $P := \lambda_1 P_1 + \cdots + \lambda_s P_s$ is by

$$\mathrm{V}\left(\sum_{i=1}^{s} \lambda_s P_s\right) = \frac{1}{n} \sum_{u\in F(P)} \left(\sum_{i=1}^{s} \lambda_s h_s(u)\right) \mathrm{V}_{n-1}(F(P, u)),$$

where $F(P)$ denotes the set of unit normals of the facets of P.

With the assumption that the facets of P are linear combinations of facets in $P_1, \dots, P_s$, a result that we shall not prove here, it is evident by induction hypothesis, that their volume is a polynomial of degree $(n-1)$. Hence, the volume of P is a polynomial of degree n. □

The coefficients $\mathrm{V}(K_1, \dots, K_n)$ in the preceding theorem are called *mixed volumes*, and only depend on the choice of convex bodies $K_1, \dots, K_n \in \mathcal{K}^n$. This follows directly from the result that the volume $\mathrm{V}(\sum_{i=1}^{N} \lambda_i K_i)$ is a polynomial for all $K_1, \dots, K_N \in \mathcal{K}^n$. We write

$$V(K[n-k], L[k]) := V(\underbrace{K, \dots, K}_{(n-k)\text{ times}}, \underbrace{L, \dots, L}_{k\text{ times}})$$

to denote that we take the mixed volume of $(n-k)$-times K and k-times L.

With this result, it is evident that the function

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{R}, \\ \lambda &\longmapsto \mathrm{V}((1-\lambda)K + \lambda L), \end{aligned}$$

is differentiable and therefore continuous. This is an extension of the trivial differentiability of $\lambda \mapsto \mathrm{V}(\lambda K)$ for a convex body K, which is essentially the positive homogenity of the volume functional.

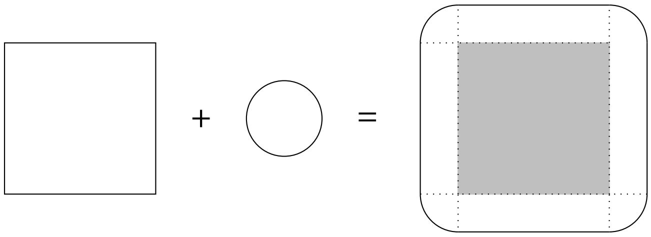

Figure 3.2: Minkowski sum of a cuboid and a disc in $\mathbb{R}^2$

Example 3.16. Let B denote the 2-dimensional unit disc and let Q denote the unit square in the plane that has side lengths $q_1, q_2 = 2$.

Then, the convex body $Q + \varepsilon B$ consists of the original body Q, four rectangles with side lengths $(2, \varepsilon)$ and the body εB that is split and positioned to round the corners of Q. The situation is illustrated in figure 3.2.

Thus, the volume is given by

$$\mathrm{V}(Q + \varepsilon B) = \mathrm{V}(Q) + 8\varepsilon + \mathrm{V}(\varepsilon B).$$

From this result, we can deduce that the mixed volumes of Q and B are given by

$$\begin{aligned} \mathrm{V}(Q, Q) &= \mathrm{V}(Q), \\ \mathrm{V}(Q, B) &= 4, \\ \mathrm{V}(B, Q) &= 4 \text{ and} \\ \mathrm{V}(B, B) &= \mathrm{V}(B). \end{aligned}$$

The mixed volume on polytopes, and therefore the mixed volume on convex bodies, satisfies several properties that are crucial for its applications. All of them are relatively straightforward to verify by using the formula from Theorem 3.15.

Theorem 3.17. *Let $K_1, \dots, K_n \in \mathcal{K}^n$. Then, the mixed volume functional* $\mathrm{V} : (\mathcal{K}^n)^n \to \mathbb{R}$ *fulfils the following properties:*

(i) $\mathrm{V}(K, \dots, K) = \mathrm{V}(K)$.

(ii) The mixed volume is order-invariant, i. e. $\mathrm{V}(K_{\sigma(1)}, \dots, K_{\sigma(n)})$ *for all permutations* $\sigma \in S_n$.

(iii) $\mathrm{V}(\alpha K_1, K_2, \dots K_n) = \alpha \mathrm{V}(K_1, \dots, K_n)$ *for* $\alpha > 0$*, i. e. it is positive homogeneous.*

(iv) $\mathrm{V}(K_1 + L_1, K_2, \dots, K_n) = \mathrm{V}(K_1, K_2, \dots, K_n) + \mathrm{V}(L_1, K_2, \dots, K_n)$ *for* $L_1 \in \mathcal{K}^n$*, i. e. it is linear with respect to the Minkowski addition.*

Now, consider the volume $\mathrm{V}((1 - \lambda)K + \lambda L)$ of two convex bodies $K, L \in \mathcal{K}^n$. With Theorem 3.15, the map $(s, t) \mapsto \mathrm{V}(sK + tL)$ is a polynomial of degree n and is determined by all mixed volumes of K, L.

This leads to the idea to replace the volume on the left side of the Brunn–Minkowski inequality (Theorem 3.3) by this polynomial expression, and motivates the following theorem.

Theorem 3.18 (Minkowski's first inequality). *Let $K, L \in \mathcal{K}^n$. Then*

$$\mathrm{V}(K[n-1], L) \geq \mathrm{V}(K)^{(n-1)/n}\,\mathrm{V}(L)^{1/n},$$

where equality holds precisely when K and L are homothetic.

Proof. Let $\lambda \in [0,1]$ and consider the function

$$\begin{aligned} f\colon\ \mathbb{R} &\longrightarrow [0,\infty], \\ t &\longmapsto (\mathrm{V}((1-t)K + tL))^{1/n} - (1-t)\,\mathrm{V}(K)^{1/n} - t\,\mathrm{V}(L)^{1/n}. \end{aligned}$$

By the Brunn–Minkowski inequality, this function is concave and positive. Further, we have

$$\mathrm{V}((1-t)K + tL) = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k\, \mathrm{V}(K[n-k], L[k]).$$

Thus, the function f is differentiable everywhere, and $f'(0) \geq 0$. If K, L are homothetic, then f is constant and $f = 0$.

By applying differentiation rules, we obtain

$$f'(0) = (\mathrm{V}(K))^{(n-1)/n}\big(\,\mathrm{V}(K, L, \dots, L) - (\mathrm{V}(K))^{(n-1)/n}(\mathrm{V}(L))^{1/n}\big).$$

Because $f'(0)$ is positive, the right multiplicand must be positive. Thus,

$$(\mathrm{V}(K[n-1], L))^n \geq \mathrm{V}(K)^{n-1}\,\mathrm{V}(L).$$

The equality cases are harder to establish. Therefore, we refer to [Sch14, Theorem 7.6.19] for a full proof of these equality conditions. □

Remark. Minkowski's first inequality and the Brunn–Minkowski inequality are equivalent. This was conjectured by Minkowski and proved in Bol [Bol43].

Now, let $K, L \in \mathcal{K}^n$. Then, the following formula follows from the general mixed-volume equality:

$$\mathrm{V}(\alpha K + \beta L) = \sum_{i=0}^{n} \binom{n}{k} \alpha^{n-k}\beta^k\, \mathrm{V}(K[n-k], L[k]).$$

This purely algebraic computation motivates the so-called Steiner formula, which in turn defines the quermassintegrals.

Theorem 3.19 (Steiner formula). *Let $K \in \mathcal{K}^n$ be a convex body. Then, the* Steiner formula *holds, which is given by*

$$\mathrm{V}(K + \varepsilon B) = \sum_{k=0}^{n} \binom{n}{k} \varepsilon^k W_k(K),$$

where $\varepsilon \in \mathbb{R}$ and B is the n-dimensional unit ball.

Definition 3.20 (Quermassintegral). For a convex body $K \in \mathcal{K}^n$, the factors $W_0(K) \dots, W_n(K)$ in the Steiner formula are called the *quermassintegrals* of K.

Remark. Minkowski's definition for the surface area of a convex body is as follows: Let $K \in \mathcal{K}^n$ be a convex body. Then, the surface area is given by

$$S(K) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{V}(K + \varepsilon B) - \mathrm{V}(K)}{\varepsilon},$$

where B denotes the n-dimensional unit ball.

With the Steiner formula, we have

$$\begin{aligned} S(K) &= \lim_{\varepsilon \to 0^+} \frac{\sum_{k=0}^{n} \binom{n}{k} \varepsilon^k W_i(K) - \mathrm{V}(K)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \sum_{k=1}^{n} \binom{n}{k} \varepsilon^{k-1} W_i(K) = nW_1(K), \end{aligned}$$

since $W_n(K) = \mathrm{V}(K)$.

Thus, the Minkowski surface area of a convex body K is the determined by the first quermassintegral $W_1(K)$, or the mixed volume $\mathrm{V}(K, B, \dots B)$.

That the Minkowski surface area equals the surface area measure follows by approximation through polytopes. For elementary bodies, i. e. circles and rectangles, this result is simple to verify.

Example 3.21. Let Q be the 2-dimensional unit square with side lengths 2, as in Example 3.16. Then, by the previous remark, we have $S_Q = 2W_1(Q) = 8$.

This coincides with our intuitive perception that the circumference of Q equals to 8.

Example 3.22. Let B be the 2-dimensional unit ball, i. e. the set

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}.$$

Now, $W_0(B) = W_1(B) = W_2(B) = \mathrm{V}(B)$, as $V(B, \dots, B) = \mathrm{V}(B)$. Thus, the surface area of B is given by $S_B = 2\,\mathrm{V}(B) = 2\pi$.

If we lift the restriction that B is 2-dimensional, this further generalises to $S_B = n\,\mathrm{V}(B)$.

3.3 Minkowski problem

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The *classical Minkowski problem* is one of the most important applications of the Brunn–Minkowski theorem and the Minkowski theorem and asks for uniqueness and existence of a convex body with given surface area measure.

For that, let μ be a Borel measure on S^{n-1}. Then, the existence question of the Minkowski problem asks under which conditions there exists a convex body with surface area measure μ. Furthermore, the uniqueness question asks for the conditions under which the surface area measure of two convex bodies is the same.

Those questions were posed and solved by Minkowski and we mostly follow the exposition in [Sch14, Chapter 8].

Both questions also arise in the following discrete settings: For the existence question, let $u_1, \dots, u_k \in S^{n-1}$ be unit vectors and $h_1, \dots, h_k \in [0, \infty)$. Then, the existence questions asks for an n-dimensional polytope $P \in \mathcal{K}^n$ with k facets and support numbers $h_P(u_1), \dots, h_P(u_k)$ in directions $u_1, \dots, u_k$. The uniqueness questions asks for the conditions under which two polytopes have same support numbers and facet unit normals.

Especially the first question, i. e. the existence question, is of fundamental importance, as it motivates the extension to general surface area measures and convex bodies by approximation. But this approximation process is quite complex and requires crucial effort.

Theorem 3.23 (Minkowki's existence theorem, discrete setting). *Let $u_1, \dots, u_k \in S^{n-1}$ be unit normal vectors that span the $\mathbb{R}^n$. Further, let $\alpha_1, \dots, \alpha_k \in \mathbb{R}^+$ such that*

$$\sum_{i=0}^{k} \alpha_i u_i = 0.$$

Then there exists a polytope $P \in \mathcal{K}^n$ with outer unit normals $u_1, \dots, u_k$ and support function $h_P(u_i) = \alpha_i$ for $i = 1, \dots, k$.

Theorem 3.24 (Minkowski's existence theorem). *Let μ be a Borel measure on S^{n-1} with*

$$\int_{S^{n-1}} u \, d\mu(u) = 0$$

and $\mu(\sigma) < \mu(S^{n-1})$ for each closed hemisphere $\sigma \subseteq S^{n-1}$.

Then, there is a convex body $K \in \mathcal{K}^n$ with surface area measure μ, i. e. $S_K = \mu$.

Since these two results are not further related to the Brunn–Minkowski inequality, we refer for the proofs of them briefly to [Sch14].

The interesting problem is clearly the uniqueness question: An approximation through polytopes does not solve the general problem. The usual proof uses the Minkowski inequality (Theorem 3.18), where the equality conditions are of special importance.

Theorem 3.25 (Minkowski's uniqueness theorem). *Let μ be a Borel measure on S^{n-1}. Suppose that two convex bodies $K, L \in \mathcal{K}^n$ have surface area measure μ, i. e. $S_K = S_L = \mu$. Then, K, L are translates.*

Proof. Suppose that $S_K(\omega) = S_L(\omega)$ for all Borel sets $\omega \subseteq S^{n-1}$. Then

$$\mathrm{V}(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) \, dS_K(u) = \frac{1}{n} \int_{S^{n-1}} h_K(u) \, dS_L(u).$$

Therefore,

$$\mathrm{V}(K)^n = \mathrm{V}(K, L, \dots, L)^n \geq \mathrm{V}(K)\,\mathrm{V}(L)^{n-1}$$

by the Minkowski inequality (Theorem 3.18). From this inequality, we deduce $\mathrm{V}(K) \geq \mathrm{V}(L)$.

Analogously, we obtain $\mathrm{V}(L) \geq \mathrm{V}(K)$, and therefore we have the equality

$$\mathrm{V}(K, L, \dots, L)^n = \mathrm{V}(K)\,\mathrm{V}(L)^{n-1}.$$

With the equality conditions of the Minkowski inequality, the bodies must be homothetic. Since $\mathrm{V}(K) = \mathrm{V}(L)$, K and L must be translates. □

4 L_p-Brunn–Minkowski theory

The L_p-Brunn–Minkowski theory is a branch of convex geometry, that was motivated by the classical Brunn–Minkowski theory, which we introduced in the previous chapters. It is a generalisation of the classical theory, where we replace the usual Minkowski sum $K + L$ by a p-weighted variant $K +_p L$, for $p \geq 0$. This operation is called Firey addition and is supplemented by a Firey multiplication, resulting in an analogue of the classical Minkowski combination.

From this chapter on, p will always be a non-negative real number, i. e. $p \geq 0$.

The notion of p-sums, i. e. Firey sums of convex bodies with parameter p, was introduced by Firey in [Fir62]. For this reason, they are also called Firey sums, a term coined by Erwin Lutwak in [Lut93].

Definition 4.1 (Firey combinations)**.** Let $K, L \in \mathcal{K}_0^n$, in other words two convex bodies that contain the origin in their interior. Further, let $p > 0$. Then, the *p-sum* of K and L, which is denoted by $K +_p L$, is the Aleksandrov body of the function

$$\begin{aligned} h\colon S^{n-1} &\longrightarrow \mathbb{R}, \\ u &\longmapsto \left(h_K^p(u) + h_L^p(u)\right)^{1/p}, \end{aligned}$$

where $h_K, h_L\colon S^{n-1} \to \mathbb{R}$ are the support functions of K and L, respectively.

Further, the *p-scalar multiplication* of $K \in \mathcal{K}_0^n$, $s \geq 0$, is given by $s \cdot K = s^{1/p}K$, where the right side is the traditional Minkowski scalar multiplication of s and K.

This definition directly motivates the construction of Firey combinations $s \cdot K +_p t \cdot L$ for $K, L \in \mathcal{K}_0^n$, $s, t \geq 0$. By definition of the Aleskandrov body, this combination is given by

$$s \cdot K +_p t \cdot L = \{x \in \mathbb{R}^n \mid \langle x, u\rangle \leq \left(sh_K(u)^p + th_L(u)^p\right)^{1/p}\}.$$

The aim of the L_p-Brunn–Minkowski theory is to investigate these combinations and find an inequality for these combinations that is similar to the classical Brunn–Minkowski inequality.

That the L_p-Brunn–Minkowski theory is an generalisation of the classical theory follows from the evident result, that the Firey combination with parameter $p = 1$ is just the Minkowski combination. That is, it holds

$$\mathrm{V}((1-\lambda) \cdot K +_1 \lambda \cdot L),$$

and the Brunn–Minkowski inequality follows for this case, as we have proven in Theorem 3.3. We shall call the generalisation of the Brunn–Minkowski inequality to general Firey combinations

With the following monotonicity result, which is similar to the well-known monotonicity property of L_p-spaces, the L_p-Brunn–Minkowski inequality follows for the case that $p \geq 1$. The following result is a specialisation of a result obtained by Firey in [Fir62], where the author proved that the Firey combinations $(1-\lambda)\cdot K +_\lambda \cdot L$ depend continuously on $K, L \in \mathcal{K}_0^n$, $p > 0$ and $\lambda \geq 0$.

Lemma 4.2 (Monotonicity of Firey combinations). *Suppose that $K, L \in \mathcal{K}_0^n$ and $\lambda \in [0,1]$. For $1 \leq p < q < \infty$, it holds*

$$(1-\lambda)\cdot K +_p \lambda \cdot L \subseteq (1-\lambda)\cdot K +_q \lambda \cdot L,$$

i. e. the Firey sum depends monotonic on the chosen parameter.

Proof. Suppose that $h_K, h_L \colon S^{n-1} \to \mathbb{R}$ are the support functions of K and L, respectively. Then, consider the functions

$$h_p \colon S^{n-1} \longrightarrow \mathbb{R}, \quad u \longmapsto \left((1-\lambda)h_K(u)^p + \lambda h_L(u)^p\right)^{1/p}, \qquad h_q \colon S^{n-1} \longrightarrow \mathbb{R}, \quad u \longmapsto \left((1-\lambda)h_K(u)^q + \lambda h_L(u)^q\right)^{1/q}.$$

By definition the body $(1-\lambda)\cdot K +_p \lambda \cdot L$ is the Aleksandrov body of h_p, and $(1-\lambda)\cdot K +_q \lambda \cdot L$ is the Aleskandrov body of h_q. Further, these bodies have support functions h_p and h_q almost everywhere w.r.t. the $(n-1)$-dimensional Hausdorff measure.

Our aim is to show that $h_p(u) \leq h_q(u)$ holds for all $u \in S^{n-1}$. Then, by Lemma 2.20, we have $Q_{h_p} \subseteq Q_{h_q}$ and therefore

$$(1-\lambda)\cdot K +_p \lambda \cdot L \subseteq (1-\lambda)\cdot K +_q \lambda \cdot L.$$

This is equivalent to the result that the function

$$f_u \colon [1,\infty) \longrightarrow \mathbb{R}^n, \quad p \longmapsto ((1-\lambda)h_K(u)^p + \lambda h_L(u)^p)^{1/p},$$

is monotonic, for each $u \in S^{n-1}$. Since K, L contain the origin in their interior, the support functions h_K, h_L are strictly positive. Thus,

$$f_u(p) = ((1-\lambda)h_K(u)^p + \lambda h_L(u)^p)^{1/p} \leq ((1-\lambda)h_K(u)^q + \lambda h_L(u)^q)^{1/q}.$$

□

Now, let $K, L \in \mathcal{K}_0^n$, $\lambda > 0$. Then, the Brunn–Minkowski inequality (Theorem 3.3) states that

$$\mathrm{V}((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)\cdot \mathrm{V}(K)^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n}.$$

Since $(1-\lambda)K + \lambda L = (1-\lambda)\cdot K +_1 \lambda \cdot L$, where the scalar multiplication is with respect to the parameter 1, we have

$$\mathrm{V}((1-\lambda)\cdot K +_1 \lambda \cdot L)^{1/n} \geq (1-\lambda)\,\mathrm{V}(K)^{1/n} + \lambda\,\mathrm{V}(L)^{1/n}.$$

Now, let $p \geq 1$. Then, by monotonicity of the volume functional and by the preceding monotonicity result, we have

$$\mathrm{V}((1-\lambda)\cdot K +_p \lambda \cdot L)^{1/n} \geq (1-\lambda)\cdot \mathrm{V}(K)^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n},$$

where the scalar multiplication is with respect to the chosen parameter $p \geq 1$.

This proves the following theorem, a generalisation of the Brunn–Minkowski inequality to Firey combinations of bodies.

Theorem 4.3 (L_p-Brunn–Minkowski inequality, $p \geq 1$)**.** *Suppose $K, L \in \mathcal{K}_0^n$ and $p \geq 1 < \infty$. Then,*

$$\mathrm{V}((1-\lambda)\cdot K +_p \lambda \cdot L)^{1/n} \geq (1-\lambda)\cdot \mathrm{V}(K)^{1/n} + \lambda \cdot \mathrm{V}(L)^{1/n}.$$

This result is a special case of a more general result, which states a Brunn–Minkowski type inequality on p-quermassintegrals. This approach is taken in [Lut93].

In general, one is interested for a similar inequality of Brunn–Minkowski type for the general case $0 \leq p$. This turns out to be hard to prove, since this would on the one side imply a stronger Brunn–Minkowski inequality than the classical version. Furthermore, the support function of $(1-\lambda)\cdot K +_p \lambda \cdot L$ is, in general, not given by $((1-\lambda)h_K^p + \lambda h_L^p)^{1/p}$, when considering factors $0 < p < 1$.

One step in this direction has been taken in [Bör⁺12], where the authors have proven such stronger inequality for the planar case.

We shall not track this path further down, but we are interested in the limit case as $p \to 0$. This leads to the so-called logarithmic Firey combinations, which have some interesting applications that are similar to the classical Minkowski problem.

Definition 4.4 (Logarithmic Firey combination)**.** Let $K, L \in \mathcal{K}_0^n$, $\lambda \in [0,1]$. Then the *logarithmic Firey combination* $(1-\lambda)\cdot K +_0 \lambda \cdot L$ is the Aleksandrov body of the function

$$\begin{aligned} h\colon S^{n-1} &\longrightarrow \mathbb{R}, \\ u &\longmapsto h_K^{1-\lambda} h_L^{\lambda}, \end{aligned}$$

where $h_K, h_L\colon S^{n-1} \to \mathbb{R}$ are the support functions of K and L, respectively.

5 The log-Brunn–Minkowski inequality

The log-Brunn–Minkowski inequality is the logarithmic analogue of the L_p-Brunn–Minkowski inequality. It arises naturally in the investigation of the logarithmic Minkowski problem and has been introduced in Böröczky et al. [Bör⁺12].

Our aim is to show that the log-Brunn–Minkowski inequality is equivalent to the log-Minkowski inequality and to present the proof idea of the log-Brunn–Minkowski inequality for the 2-dimensional case, a result that was established in [Bör⁺12].

Furthermore, we will discuss the log-Minkowski problem, which asks for the existence and uniqueness of a convex body with given cone-volume measure, a problem that is treated in [Bör⁺12; Bör⁺13].

The log-Brunn–Minkowski inequality arises when one replaces the Firey combination in the usual L_p-Brunn–Minkowski inequality by a logarithmic Firey combination.

In classical theory, there is a strong relationship between the Brunn–Minkowski-type inequality and a Minkowski-type inequality, cf. Theorem 3.18. This phenomenon also appears in a similar way for the logarithmic Brunn–Minkowski inequality. While it is hard to establish the inequality, the equivalence with an inequality of Minkowski-type is known.

Lemma 5.1 (Böröczky et al. [Bör⁺12, Lemma 3.2])**.** *The log-Brunn–Minkowski inequality, which is given by*

$$\mathrm{V}((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq \mathrm{V}(K)^{1-\lambda}\,\mathrm{V}(L)^{\lambda},$$

holds for all origin symmetric convex bodies $K, L \in \mathcal{K}_0^n$, $\lambda > 0$, if and only if the log-Minkowski inequality, which is given by

$$\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\bar{V}_K(u) \geq \frac{1}{n}\log\left(\frac{\mathrm{V}(L)}{\mathrm{V}(K)}\right),$$

holds for all $K, L \in \mathcal{K}_0^n$, $\lambda > 0$.

Proof. Let $K, L \in \mathcal{K}_0^n$ be origin-symmetric convex bodies and suppose that the logarithmic Minkowski inequality holds. Further, let

$$Q_\lambda = (1-\lambda)\cdot K +_0 \lambda \cdot L,$$

the Aleksandrov body of $u \mapsto h_K^{1-\lambda}(u)\cdot h_L^{\lambda}(u)$, for $\lambda \in [0,1]$, and denote by $h_\lambda : S^{n-1} \to \mathbb{R}$ the support function of Q_λ.

Then,

$$\begin{aligned}
&\frac{1}{n\,\mathrm{V}(Q_\lambda)}\int_{S^{n-1}} h_\lambda \log\left(\frac{h_K^{1-\lambda}(u)h_L^{\lambda}(u)}{h_\lambda(u)}\right) dS_{Q_\lambda}(u)\\
&= \frac{1}{n\,\mathrm{V}(Q_\lambda)}\left((1-\lambda)\int_{S^{n-1}} \log\Big(\frac{h_K(u)}{h_\lambda(u)}\Big)h_\lambda(u)\, dS_{Q_\lambda}(u) + \lambda\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_\lambda(u)}\right)h_\lambda(u)\, dS_{Q_\lambda}(u)\right)\\
&= (1-\lambda)\int_{S^{n-1}} \log\left(\frac{h_K(u)}{h_\lambda(u)}\right) d\bar{V}_{Q_\lambda}(u) + \lambda\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_\lambda(u)}\right) d\bar{V}_{Q_\lambda}(u),
\end{aligned}$$

where $dV_K = (1/n)h_K\, dS_K$ has been used in the second equality, cf. [Bör+12]. Now, it follows

$$\begin{aligned}
&(1-\lambda)\int_{S^{n-1}} \log\left(\frac{h_K(u)}{h_\lambda(u)}\right) d\bar{V}_{Q_\lambda}(u) + \lambda\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_\lambda(u)}\right) d\bar{V}_{Q_\lambda}(u)\\
&\geq (1-\lambda)\frac{1}{n}\log\left(\frac{\mathrm{V}(K)}{\mathrm{V}(Q_\lambda)}\right) + \lambda\frac{1}{n}\log\left(\frac{\mathrm{V}(L)}{\mathrm{V}(Q_\lambda)}\right)\\
&= \frac{1}{n}\big(\log(\mathrm{V}(K)^{1-\lambda}\,\mathrm{V}(L)^{\lambda}) - \log(\mathrm{V}(Q_\lambda))\big)
\end{aligned}$$

from the logarithmic Minkowski inequality.

Since $h_\lambda(u) = h_K^{1-\lambda}(u)h_L^{\lambda}(u)$ for almost all $u \in S^{n-1}$ with respect to the surface-area measure S_{Q_λ} of Q_λ, we have

$$h_\lambda(u)\log\left(\frac{h_K^{1-\lambda}(u)h_L^{\lambda}(u)}{h_\lambda(u)}\right) = 0$$

for almost all $u \in S^{n-1}$. Therefore,

$$\frac{1}{n\,\mathrm{V}(Q_\lambda)}\int_{S^{n-1}} h_\lambda(u)\log\left(\frac{h_K^{1-\lambda}(u)h_L^{\lambda}(u)}{h_\lambda(u)}\right) dS_{Q_\lambda}(u) = 0,$$

and the logarithmic Brunn–Minkowski inequality follows.

For the reverse direction, suppose that the log-Brunn–Minkowski inequality holds for K and L. Then, consider the function

$$\begin{aligned}
f\colon\ [0,1] &\longrightarrow (0,\infty),\\
\lambda &\longmapsto \log(\mathrm{V}(Q_\lambda)).
\end{aligned}$$

Our aim is to show that f is concave.

For given $\sigma, \tau \in [0,1]$, let

$$K_\sigma = (1-\sigma)\cdot K +_0 \sigma\cdot L, \quad K_\tau = (1-\tau)\cdot K +_0 \tau\cdot L.$$

Since K_σ is the Aleksandrov body of $u \mapsto h_K^{1-\sigma}(u)h_L^{\sigma}(u)$ for $u \in S^{n-1}$, we have $h_{K_\sigma}(u) \leq h_K^{1-\sigma}(u)h_L^{\sigma}(u)$ for all $u \in S^{n-1}$.

Let $\alpha := (1-\lambda)\sigma + \lambda\tau$ for $\lambda \in [0,1]$. Then,

$$\begin{aligned} h_{K_\sigma}^{1-\lambda} h_{K_\tau}^{\lambda} &\leq (h_K^{1-\sigma} h_L^{\sigma})^{1-\lambda} (h_K^{1-\tau} h_L^{\tau})^{\lambda} \\ &= h_K^{(1-\sigma)\cdot(1-\lambda)} h_L^{\sigma\cdot(1-\lambda)} h_K^{\lambda\cdot(1-\tau)} h_L^{\lambda\cdot\tau} \\ &= h_K^{1-\alpha} h_L^{\alpha}, \end{aligned}$$

and therefore

$$(1-\lambda)\cdot K_\sigma +_0 \lambda \cdot K_\tau \subseteq (1-\alpha)\cdot K +_0 \alpha \cdot L.$$

Thus,

$$\begin{aligned} f((1-\lambda)\sigma + \lambda\tau) &= \log\big(\mathrm{V}(1-\alpha)\cdot K +_0 \alpha L\big) \\ &\geq \log\big(\mathrm{V}((1-\lambda)\cdot K_\sigma +_0 \lambda\cdot K_\tau)\big) \\ &\geq (1-\lambda)\log(\mathrm{V}(K_\sigma)) + \lambda\log(\mathrm{V}(K_t au)) \\ &= (1-\lambda) f(\sigma) + \lambda f(\tau) \end{aligned}$$

by the log-Brunn–Minkowski inequality. Hence, $f\colon [0,1] \to (0,\infty)$ is concave.

By the technical result [Bör+12, Lemma 2.1], it holds

$$\frac{d}{d\lambda} Q_\lambda\Big|_{\lambda=0} = \int_{S^{n-1}} h_K(u) \log\left(\frac{h_L(u)}{h_K(u)}\right) dS_K(u).$$

Since f is concave, we further have

$$\mathrm{V}(Q_0)^{-1}\,\frac{d}{d\lambda} Q_\lambda\Big|_{\lambda=0} \geq \mathrm{V}(Q_1) - \mathrm{V}(Q_0) = \log\left(\frac{\mathrm{V}(L)}{\mathrm{V}(K)}\right).$$

Since $d\bar{V}_K = \mathrm{V}(K)^{-1}(1/n) h_K\, dS_K$ holds for all origin-symmetric convex bodies $K \in \mathcal{K}_0^n$, we have

$$n\,\mathrm{V}(K) \int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) \frac{1}{n\,\mathrm{V}(K)} h_K(u)\, dS_K(u) = n\,\mathrm{V}(K) \int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\bar{V}_K(u).$$

This implies the logarithmic Minkowski inequality:

$$\int_{S^{n-1}} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\bar{V}_K(u) = \frac{1}{n\,\mathrm{V}(K)}\,\frac{d}{d\lambda} Q_\lambda\Big|_{\lambda=0} \geq \frac{1}{n}\log\left(\frac{\mathrm{V}(L)}{\mathrm{V}(K)}\right).$$

□

In this sense, the log-Brunn–Minkowski and log-Minkowski inequalities are equivalent. A similar result holds with similar arguments for the L_p-Brunn–Minkowski and an L_p-Minkowski inequality, cf. [Bör+12, Lemma 3.1].

5.1 Established cases

The logarithmic Brunn–Minkowski and its equality cases are still unknown for the general case. The trivial case in $\mathbb{R}^1$ is almost trivial.

Theorem 5.2 (log-Brunn–Minkowski for $n = 1$)**.** *Let $K, L \in \mathcal{K}_0^1$ be compact origin-symmetric intervals. Then,*

$$\mathrm{V}((1-\lambda)\cdot K +_0 \lambda \cdot L) = \mathrm{V}(K)^{1-\lambda}\,\mathrm{V}(L)^{\lambda} \quad \textit{for all } \lambda \in [0,1].$$

Proof. Since K, L are compact origin-symmetric intervals, there are $a, b \in (0, \infty)$ such that $K = [-a, a], L = [-b, b]$.

The support functions of K and L are given by

$$h_K\colon \{-1,+1\} \longrightarrow \mathbb{R}, \quad s \longmapsto a, \qquad h_L\colon \{-1,+1\} \longrightarrow \mathbb{R}, \quad s \longmapsto b,$$

i. e. they are constant maps. Thus, $(1-\lambda)\cdot K +_0 \lambda \cdot L$ is the Aleksandrov body of

$$h\colon \{-1,+1\} \longrightarrow \mathbb{R}, \quad s \longmapsto h_K(s)^{1-\lambda} h_L(s)^{\lambda} = a^{1-\lambda} b^{\lambda}.$$

Therefore, with $c := a^{1-\lambda}b^{\lambda}$, we have

$$(1-\lambda)\cdot K +_0 \lambda \cdot L = [-c, c],$$

and we have $\mathrm{V}((1-\lambda)\cdot K +_0 \lambda \cdot L) = 2a^{1-\lambda}b^{\lambda}$.

On the other side, we have $\mathrm{V}(K)^{1-\lambda} = (2a)^{1-\lambda}$ and $\mathrm{V}(L)^{\lambda} = (2b)^{\lambda}$, thus the equality holds. ☐

For this case, the questions regarding the equality cases vanish, since all convex bodies on $\mathbb{R}$ are homothetic, and thus have a particular relation under volumes.

Furthermore, with Lemma 5.1, the log-Minkowski inequality for $\mathbb{R}$ is established, i. e. it holds

$$\int_{S^0} \log\left(\frac{h_K(u)}{h_L(u)}\right) d\bar{V}_K(u) \geq \log\left(\frac{\mathrm{V}(K)}{\mathrm{V}(L)}\right) = \log(a) - \log(b)$$

for all compact origin-symmetric intervals $K = [-a, a], L = [-b, b] \in \mathcal{K}_0^1$.

On the left side, we have

$$\int_{S^0} \log\left(\frac{h_K(u)}{h_L(u)}\right) d\bar{V}_K(u) = (\bar{V}_K(-1) + \bar{V}_K(1)) \cdot \log\left(\frac{h_K(-1)}{h_L(-1)}\right) = \log(a) - \log(b)$$

by the origin-symmetry of K and L. Hence, equality also holds for the log-Minkowski inequality in $\mathbb{R}$.

In Böröczky et al. [Bör+12], the authors established the planar case of the logarithmic Brunn–Minkowski inequality.

Theorem 5.3 (log-Minkowski inequality in the plane). *Let $K, L \in \mathcal{K}_0^2$ be two planar origin-symmetric convex bodies in the Euclidean space, and $\lambda > 0$. Then*

$$\int_{S^1} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\bar{V}_K(u) \geq \frac{1}{2} \log\left(\frac{\mathrm{V}(L)}{\mathrm{V}(K)}\right).$$

Equality in the inequality holds if and only if K and L are dilates or parallelograms with parallel sides.

Theorem 5.4 (log-Brunn–Minkowski inequality in the plane). *Let $K, L \in \mathcal{K}_0^2$ be two planar origin-symmetric convex bodies in the Euclidean space $\mathbb{R}^n$, and $\lambda > 0$. Then,*

$$\mathrm{V}((1-\lambda) \cdot K +_o \lambda L) \geq \mathrm{V}(K)^{1-\lambda}\, \mathrm{V}(L)^{\lambda}.$$

Proof. This follows directly from the equivalence in Lemma 5.1. □

Another proof of the planar case of the log-Brunn–Minkowski inequality is given in Ma [Ma15], where the authors use an integral-geometric approach.

It is conjectured that a general version of the log-Brunn–Minkowski inequality holds, cf. [Bör$^+$13; YZ18; Ma15]. By Lemma 5.1, this conjecture is connected with a general version of the log-Minkowski inequality. If one of these conjectures is established, the other follows.

Despite the inequality, the equality cases are necessary for important applications of the logarithmic Brunn–Minkowski inequality, as we see in Section 5.2.

In Yang and Zhang [YZ18], the authors established a 3-dimensional version of the log-Brunn–Minkowski inequality.

Theorem 5.5 (log-Brunn–Minkowski inequality for $n = 3$, Yang and Zhang [YZ18]). *Let $K, L \in \mathcal{K}_0^3$ be origin-symmetric convex bodies in $\mathbb{R}^3$, and $\lambda > 0$. Then,*

$$\mathrm{V}((1-\lambda) \cdot K +_0 \lambda \cdot L) \geq \mathrm{V}(K)^{1-\lambda)\, \mathrm{V}(L)^{\lambda}}$$

with equality precisely when K and L are dilates, i. e. there is a number $\alpha > 0$ such that $K = \alpha L$ holds.

By Lemma 5.1, this clearly implies the logarithmic Minkowski inequality for $\mathbb{R}^3$:

Theorem 5.6 (log-Minkowski inequality for $n = 3$). *Let $K, L \in \mathcal{K}_0^3$ be origin-symmetric convex bodies in $\mathbb{R}^3$, $\lambda > 0$. Then,*

$$\int_{S^2} \log\left(\frac{h_L(u)}{h_K(u)}\right) d\bar{V}_K(u) \geq \frac{1}{3} \log\left(\frac{\mathrm{V}(L)}{\mathrm{V}(K)}\right).$$

5.2 Application to the logarithmic Minkowski problem

Remember the classical Minkowski problem: Let μ be a Borel measure on the unit sphere S^{n-1}. Under which conditions does a convex body with surface area measure μ exists and under which conditions this convex body is unique with that property?

While both the uniqueness and existence questions are established, as we have seen in Theorem 3.24 and Theorem 3.25, this result gives rise to the question what happens when we ask for a convex body with given cone-volume measure instead of given surface area measure. This problem is called *logarithmic Minkowski problem*, or short *log-Minkowski problem*, and is of special interest, since its aim is to characterise convex bodies by an associated measure.

We mostly follow [Bör+12, Section 5] and [Bör+13], where the theory of the logarithmic Minkowski problem is established as an application of the log-Brunn–Minkowski inequality.

The logarithmic Minkowski problem is divided into two subproblems: First, the existence question asks for the conditions under which a convex body $K \in \mathcal{K}^n$ with given cone-volume measure μ exists. Secondly, the *uniqueness question* asks for the conditions under which there is exactly one convex body $K \in \mathcal{K}^n$ with cone-volume measure μ.

Since progress is slow, only the existence question of the so-called even logarithmic Minkowski problem has been solved in Böröczky et al. [Bör+13]. For that, a Borel measure μ on S^{n-1} is called *even* if $\mu(\omega) = \mu(-\omega)$ for a Borel set $\omega \subseteq S^{n-1}$. It is clear that a body $K \in \mathcal{K}_0^n$ whose cone-volume measure is even, must be symmetric. Thus, the even log-Minkowski problem asks for the existence and uniqueness of symmetric convex bodies with given cone-volume measure.

This existence question is linked to the subspace concentration condition. A Borel measure μ as above is said to satisfy the *subspace concentration condition* if, for every proper and non-trivial subspace $U \subseteq \mathbb{R}^n$,

$$\mu(U \cap S^{n-1}) = \frac{1}{n}\mu(S^{n-1})\dim(U),$$

and there exists a complementary subspace $U' \subseteq \mathbb{R}^n$ to some subspace $U \subseteq \mathbb{R}^n$ with

$$\mu(U' \cap S^{n-1}) = \frac{1}{n}\mu(S^{n-1})\dim(U').$$

Here, a subspace $U \subseteq \mathbb{R}^n$ is proper and non-trivial if $0 < \dim(U) < n$. Further, a subspace $U' \subseteq \mathbb{R}^n$ is said to be *complementary* to $U \subseteq \mathbb{R}^n$ if $U + U' = \mathbb{R}^n$.

The subspace concentration condition is a necessary and sufficient condition for a measure μ to be the cone-volume measure of o-symmetric body $K \in \mathcal{K}_0^n$. This result was established in Böröczky et al. [Bör+13] and answers the existence question of the log-Minkowski problem for o-symmetric convex bodies.

Theorem 5.7 (Böröczky et al. [Bör+13, Theorem 1.1])**.** *Let μ be a even Borel measure on S^{n-1}. Then, there exists a symmetric convex body $K \in \mathcal{K}_0^n$ with cone-volume measure μ if and only if μ satisfies the subspace concentration condition.*

Regarding the uniqueness question of the even log-Minkowski problem, there is substantial progress in its solution for the planar case. The solution, in the general case, depends on the logarithmic Minkowski inequality and its equality cases. While the inequality satisfies some independence of dimensions, the equality cases of the cases $n = 2$ and $n = 3$ differ in subtle ways.

Furthermore, the equality cases do not come with the usual equivalence of log-Minkowski and log-Brunn–Minkowski (Lemma 5.1). For these reasons, the equality cases are a harder problem than the inequality alone.

The following technical lemma, whose inequality has been proved in [Gag93] and whose equality cases were established in [Bör+12], is necessary to establish the equality cases of the log-Minkowski inequality for the planar case.

Lemma 5.8 (Böröczky et al. [Bör+12, Lemma 5.1]). *Let $K, L \in \mathcal{K}_0^2$ be origin-symmetric convex bodies. Then*

$$\int_{S^1} \frac{h_K(u)^2}{h_L(u)} \, dS_K(u) \leq \frac{\mathrm{V}(K)}{\mathrm{V}(L)} \int_{S^1} h_L(u) \, dS_K(u),$$

where $h_K, h_L : S^1 \to \mathbb{R}$ are the support functions of K and L, respectively.

Furthermore, equality holds precisely when K and L are dilates or parallelograms with parallel sides.

Proof. See [Bör+12, Lemma 5.1] for a proof of the inequality and the equality cases. □

Theorem 5.9 (Böröczky et al. [Bör+12, Theorem 5.2]). *Let $K, L \in \mathcal{K}_0^2$ be origin-symmetric convex bodies with $V_K = V_L$, i. e. their cone-volume measures coincide. Then either $K = L$ or K and L are parallelograms with parallel sides.*

Proof. Assume that $K \neq L$. Since the volumes of K and L coincide by the relation

$$\mathrm{V}(K) = V_K(S^1) = V_L(S^1) = \mathrm{V}(L),$$

they cannot be dilates by the homogenity of the volume functional. □

Corollary 5.10. *Let $K, L \in \mathcal{K}_0^2$ be origin-symmetric convex bodies. Then their cone-volume measures coincide if and only if $K = L$ or K and L are parallelograms with parallel sides.*

6 Conclusions and outlook

The Brunn–Minkowski inequality is a well-studied geometric inequality. Most of the proofs make use of more general cases of the same inequality, and therefore give insights in measure theory and analysis. Yet, there are important questions linked with the Brunn–Minkowski inequality.

One of them are inequality conditions for the Prékopa–Leindler inequality. These are still unknown for the general case, i. e. for integrable functions $f : \mathbb{R}^n \to [-\infty, \infty]$, and seem to be quite complicated even for the one-dimensional case.

Furthermore, while the equality conditions of the Brunn–Minkowski inequality are known, their proofs are mostly technical and use non-trivial properties of the volume functional. An elementary and accessible method for these proofs is desirable, and is given in parts in Klain [Kla11]. Although this question is not of great importance, more elementary proofs of the equality conditions are to be expected.

Other questions arise when one replaces the usual volume functional, i. e. the Lebesgue or Hausdorff measure, by another measure. For example, one asks for the conditions under which the Brunn–Minkowski inequality holds for other measures, a problem that seems similar to the Minkowski problem. Further, one could ask for the equality conditions and their dependence on the chosen measure, under the assumption that the conditions for inequality are known. This branch is, in combination with the non-classical theory, very active and regularly yields new results.

While these problems are open in the classical Brunn–Minkowski theory, there are dozens of problems that are open in the L_p-Brunn–Minkowski theory, a theory that we investigated in Chapter 4.

Under these problems, the most prominent is the L_p-Minkowski problem, where we discussed the cases $p = 0, 1$. Considering the importance of the posed problems, there is continuous progress in this branch of convex and differential geometry. While the discrete version of the logarithmic Minkowski problem was solved in Böröczky, Hegedus, and Zhu [BHZ16], the general continuous version of the problem is still a conjecture for arbitrary dimensions.

With regards to this problem, the log-Brunn–Minkowski and L_p-Brunn–Minkowski inequalities, where we investigated the former one in Chapter 5, are of fundamental importance, as the proof of the logarithmic Minkowski problem for $n = 2$ shows.

Thus, although the Brunn–Minkowski theory is well-known in its traditional exposition, there are still open questions, which are linked with important applications in geometry and analysis.

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Notation

$\langle x, y\rangle$	Scalar product of x, y
$\overline{K}$	Closure of K
∂K	Boundary of K
$\operatorname{int} K$	Interior of K
$[n]$	Set of integers $0, \dots, n$
B	Unit ball
$B[x, r]$	Closed ball with centre x and radius r
$B(x, r)$	Open ball with centre x and radius r
$F(P)$	Outer unit normals of the facets of a polytope P
$F(P, u)$	Facet of P with outer unit normal u
$\mathcal{H}^n$	n-dimensional Hausdorff measure
h_K	Support function of $K \in \mathcal{K}^n$
$\mathcal{K}^n$	Set of n-dimensional convex bodies
$\mathcal{K}_0^n$	Set of n-dimensional convex bodies that contain the origin in their interior
ν_K	Gauss map of K
$+_p$	Firey combination for $p > 0$
$+_0$	logarithmic Firey combination
S^{n-1}	$(n-1)$-dimensional unit sphere in $\mathbb{R}^n$
$S(K)$	Surface area of K
S_K	Surface area measure of K
$\mathrm{V}(K)$	Volume of the measurable set K
$\mathrm{V}_k(K)$	k-dimensional volume of the measurable set K
$V(K_1, \dots, K_n)$	Mixed volume of $K_1, \dots, K_n$
V_K	Cone-volume measure of K
$\bar{V}_K$	Normalised cone-volume measure of K
$W_k(K)$	k-th quermassintegral of K