GRÜNBAUM'S INEQUALITY

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Abstract. Grünbaum's inequality states that the centroid divides any convex body into not too small portions. We are stating and proving Grünbaum's inequality by spherical symmetrization and subsequent conification and show that the inequality is sharp for a carefully chosen hyperplane and convex body.

A classical problem in convex geometry is as follows: Given a convex body K in n-dimensional Euclidean space and a point x that models the notion of the center of mass (that point is usually called the *centroid* of K, cf. Definition 2), is there a lower bound for

$$\frac{\operatorname{vol}(K \cap H^-)}{\operatorname{vol}(K)},$$

where H is some hyperplane (i.e. an $(n-1)$-dimensional affine subspace) through the centroid and H^- is the negative half-space of H?

While a simple answer is possible for some classes of convex bodies, e.g. for a symmetric body that ratio is given by $1/2$, a general answer is desirable. Grünbaum's inequality (Theorem 14), first published in [Grü60], states that such lower bound is given by

$$\left(\frac{n}{n+1}\right)^n,$$

an expression that solely depends on the dimension $n \in \mathbb{N}$.

1. Preliminaries

Our situation is as follows: We construct the n-dimensional Euclidean space as the set $\mathbb{R}^n$. To measure angles and distances, we endow that space with the Euclidean inner product given by

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \sum_{i=1}^n x_i y_i. \end{aligned}$$

We denote the standard basis vectors of $\mathbb{R}^n$ by $e_1, \dots, e_n$. Another notion of great importance is the property that a set is convex. We say that a subset $K \subset \mathbb{R}^n$ is convex if any straight line in K is a subset of K. More formally, we would say that K is *convex* if

$$(1-\lambda)x + \lambda y \in K$$

for all points $x, y \in k$ and dilation factors $\lambda \in [0, 1]$.

As we restricted ourselves to the Euclidean norm

$$|\cdot|\colon \mathbb{R}^n \longrightarrow \mathbb{R},$$
$$x \longmapsto \Big(\sum_{i=1}^{n} x_n^2\Big)^{1/2},$$

by choice of the Euclidean inner product, we will denote the unit ball of the Euclidean norm by B_n, i.e.

$$B_n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}.$$

We denote the boundary of B_n by S^{n-1}, i.e.

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

Clearly $S^{n-1} \subset B_n$.

Furthermore we denote *hyperplanes* by

$$H(u,a) = \{x \in \mathbb{R}^n \mid \langle u, x\rangle = a\}$$

for $u \in S^{n-1}$, $a \in \mathbb{R}$. The associated negative and positive half-spaces are denoted by $H^-(u,a)$ and $H^+(u,a)$, respectively.

Definition 1. Let $n \in \mathbb{N}$. A subset $K \subset \mathbb{R}^n$ is called a *convex body* if it is convex and compact. The set of n-dimensional convex bodies is denoted by $\mathcal{K}^n$.

To measure k-dimensional subspaces of the n-dimensional Euclidean space $\mathbb{R}^n$, we use the scaled k-dimensional Hausdorff measure, which we denote by vol_k. Here "scaled" means that we scale the Hausdorff measure such that

$$\mathrm{vol}_k(B_k \times \{0\}^{n-k}) = \lambda^k(B_k)$$

holds, where λ^k denotes the k-dimensional Lebesgue measure on $\mathbb{R}^k$. We also call the function

$$\mathrm{vol}_k\colon \mathcal{K}^n \longrightarrow [0,\infty]$$

the k-dimensional *volume functional.* If $k = n$ then we also write vol instead of vol_n, i.e.

$$\mathrm{vol}(K) = \mathrm{vol}_n(K)$$

for a measurable subset $K \subseteq \mathbb{R}^n$.

We denote the volume of the n-dimensional unit ball B_n by

$$\omega_n := \mathrm{vol}_n(B_n).$$

For a deeper introduction to measure theory we refer to [Hal74].

Definition 2 (Centroid)**.** Let $K \in \mathcal{K}^n$. The *centroid* of K is given by

$$c(K) := \int_K x \; dx.$$

Lemma 3. *Let $K, L \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. Then the centroid satisfies the following properties:*

(i) For disjoint convex bodies K, L, it holds

$$\mathrm{vol}(K \cup L) \cdot c(K \cup L) = \mathrm{vol}(K) \cdot c(K) + \mathrm{vol}(L) \cdot c(L).$$

(ii) $c(K) + c = c(K + x)$.

Proof. The first property follows from

$$\begin{aligned}\mathrm{vol}(K \cup L) \cdot c(K \cup L) &= \int_{K\cup L} x\ dx \\ &= \int_K x\ dx + \int_L x\ dx \\ &= \mathrm{vol}(K) \cdot c(K) + \mathrm{vol}(L) \cdot c(L).\end{aligned}$$

Furthermore

$$\begin{aligned}c(K) + x &= \frac{1}{\mathrm{vol}(K)} \int_K y\ dy + x \\ &= \frac{1}{\mathrm{vol}(K)} \Big(\int_K y\ dy + x\,\mathrm{vol}(K) \Big) \\ &= \frac{1}{\mathrm{vol}(K)} \int_K y + x\ dy \\ &= \frac{1}{\mathrm{vol}(K)} \int_{K+x} y\ dy = c(K + x)\end{aligned}$$

by the translation-invariance of the volume functional vol. □

Remark 4. Let $K \in \mathcal{K}^n$ be a symmetric convex body, i.e. $K = -K$. Then

$$\int_K x\ dx = c(K) = c(-K) = \int_K -x\ dx,$$

which implies

$$2 \int_K x\ dx = c(K) - c(-K) = 0.$$

Therefore the centroid of K must be located at the origin, i.e.

$$c(K) = \int_K x\ dx = 0.$$

1.1. **The Brunn–Minkowski inequality.** The Brunn–Minkowski inequality is a deep result from convex geometry on the volumes of convex combinations

$$(1 - \lambda)K + \lambda L = \{(1 - \lambda) \cdot x + \lambda \cdot y \mid x \in K, y \in L\}$$

for convex bodies $K, L \in \mathcal{K}^n$. This construction is also called Minkowski sum in the literature.

Theorem 5 (Brunn–Minkowski inequality, classical version)**.** *Let $K, L \in \mathcal{K}^n$ be any convex bodies and $\lambda \in (0, 1)$. Then*

$$\mathrm{vol}^{1/n}((1 - \lambda)K + \lambda L) \geq (1 - \lambda) \mathrm{vol}^{1/n}(K) + \lambda \mathrm{vol}^{1/n}(L).$$

For proofs of the Brunn–Minkowski inequality, we refer to the literature, e.g. [Sch14, Section 7.1].

Before stating and proving an equivalent version of the Brunn–Minkowski inequality, we need the notion of convex and concave functions on $\mathbb{R}^n$.

Definition 6. Let $A \subseteq \mathbb{R}^n$ be a convex set. A function

$$f \colon A \longrightarrow \mathbb{R}$$

is called *convex* if

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

for all $x, y \in A$, $\lambda \in [0,1]$.

Further f is called *concave* if $(-f)\colon A \to \mathbb{R}$, $x \mapsto -f(x)$, is convex.

Theorem 7 (Brunn–Minkowski inequality, functional version)**.** *Let $K, L \in \mathcal{K}^n$ be convex bodies. Then the functional*

$$\begin{aligned}(0,1) &\longrightarrow \mathbb{R},\\ \lambda &\longmapsto \mathrm{vol}^{1/n}((1-\lambda)K + \lambda L),\end{aligned}$$

is concave.

Proof. Let

$$\begin{aligned}f\colon (0,1) &\longrightarrow \mathbb{R},\\ \lambda &\longmapsto \mathrm{vol}^{1/n}((1-\lambda)K + \lambda L).\end{aligned}$$

Further let $\alpha, \beta, \lambda \in (0,1)$. Then

$$\begin{aligned}&(1-(1-\lambda)\alpha - \lambda\beta)K + ((1-\lambda)\alpha - \lambda\beta)L\\ &= (1-\lambda)(1-\alpha)K + (1-\lambda)\alpha L + \lambda(1-\beta)K + \lambda\beta L\\ &= (1-\lambda((1-\alpha)K + \alpha L) + \lambda((1-\beta)K + \beta L).\end{aligned}$$

Thereby

$$\begin{aligned}&f((1-\lambda)\alpha + \lambda\beta)\\ =\ &\mathrm{vol}^{1/n}((1-\lambda)(1-\alpha)K + (1-\lambda)\alpha L + \lambda(1-\beta)K + \lambda\beta L)\\ \geq\ &(1-\lambda)\,\mathrm{vol}^{1/n}((1-\alpha)K + \alpha L) + \lambda\,\mathrm{vol}^{1/n}((1-\beta)K + \beta L)\\ =\ &(1-\lambda)f(\alpha) + \lambda f(\beta).\end{aligned}$$

□

2. Grünbaum's inequality

Grünbaum's inequality is one of the central results of Grünbaum's paper [Grü60]. It states that any cut through the centroid of a convex body K divides that convex body in two halves K^+, K^- such that

$$\mathrm{vol}(K^+) \leq \left(\frac{n}{n+1}\right)^n \mathrm{vol}(K)$$

and

$$\mathrm{vol}(K^-) \leq \left(\frac{n}{n+1}\right)^n \mathrm{vol}(K)$$

hold.

Towards a proof of Grünbaum's inequality that we will state in Theorem 14, we need some results about the centroid of a special class of convex bodies, namely that of the cones.

2.1. The case for cones.

For our purposes, the following definition of a cone is sufficient: A *cone* $C \subset \mathbb{R}^n$ with basis $C_b \in \mathcal{K}^n$ that lies completely in a hyperplane and vertex $v \in \mathbb{R}^n$ is the convex hull of C_b and v, i.e.

$$C := \{(1-\lambda)x + \lambda v \mid x \in C_b, \lambda \in (0,1)\}.$$

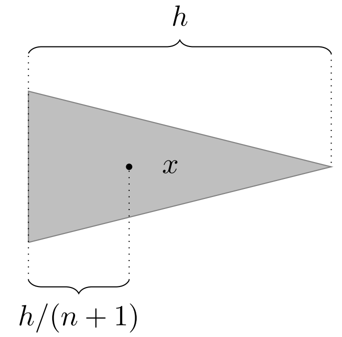

FIGURE 1. A 2-dimensional cone with height h and marked centroid x at height $h/(n+1)$.

Lemma 8. *Let $C \subseteq \mathbb{R}^n$ be the cone with basis $\{0\} \times B_{n-1}$ and vertex $(h, 0, \dots, 0) \in \mathbb{R}^n$. Its volume is given by*

$$\operatorname{vol}(C) = \frac{1}{n} h \operatorname{vol}_{n-1}(B_{n-1}).$$

Proof. The assertion directly follows from Fubini's theorem and integral substitution:

$$\begin{aligned}
\operatorname{vol}(C) = \int_C \, dx &= \int_{x_1=0}^{h} \int_{C_{x_1}} \, d(x_2, \dots, x_n) \, dx_1 \\
&= \int_{x_1=0}^{h} \operatorname{vol}_{n-1}(C_{x_1}) \, dx_1 \\
&= \int_{x_1=0}^{h} \operatorname{vol}_{n-1}\left(\frac{h - x_1}{h} C_0\right) dx_1 \\
&= \frac{\operatorname{vol}_{n-1}(B_{n-1})}{h^{n-1}} \int_{x_1=0}^{h} (h - x_1)^{n-1} \, dx_1 \\
&= \frac{\operatorname{vol}_{n-1}(B_{n-1})}{h^{n-1}} \int_{x_1=0}^{h} x_1^{n-1} \, dx_1 \\
&= \frac{\operatorname{vol}_{n-1}(B_{n-1})}{h^{n-1}} \frac{h^n}{n} = \frac{1}{n} \operatorname{vol}_{n-1}(B_{n-1}) h.
\end{aligned}$$

□

Another application of Fubini's theorem (in the form of Cavalieri's principle) yields a result that only involves the volume of the basis and the height of the cone.

Corollary 9. *Let $C \subseteq \mathbb{R}^n$ be a cone with basis C_0 and height $h > 0$. The volume of C is given by*

$$\operatorname{vol}(C) = \frac{1}{n} h \operatorname{vol}_{n-1}(C_0).$$

Theorem 10 (Centroid of the rotational-symmetric cone)**.** *Let $C \subseteq \mathbb{R}^n$ be the cone with basis $\{0\} \times B_{n-1}$ and vertex $(h, 0, \dots, 0) \in \mathbb{R}^n$. Then the* *centroid of C is given by*

$$c(C) = (c_1(C), \dots, c_n(C)) = \Big(\frac{h}{n+1}, 0, \dots, 0\Big) \in \mathbb{R}^n$$

Proof. By Theorem X the volume of C is given by

$$\operatorname{vol}(C) = \frac{1}{n} h \operatorname{vol}_{n-1}(C_0),$$

where $C_t = \{(x_1, \dots, x_n) \in C \mid x_1 = t\}$ for $0 \le t \le h$. Further the $(n-1)$-dimensional volume of C_t is given by

$$\operatorname{vol}_{n-1}(C_t) = \operatorname{vol}_{n-1}\Big(\frac{h-t}{h} C_0\Big),$$

since C is the convex hull of its basis and vertex.

By substitution and Fubini's theorem, we see that

$$\begin{aligned} c_1(C) \operatorname{vol}_n(C) = \int_C x_1 \; d(x_1, \dots, x_n) &= \int_{x_1=0}^{h} \int_{C_{x_1}} x_1 \; d(x_2, \dots, x_n) \; dx_1 \\ &= \int_{x_1=0}^{h} x_1 \operatorname{vol}_{n-1}(C_{x_1}) \; dx_1 \\ &= \frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}} \int_{x_1=0}^{h} x_1 (h - x_1)^{n-1} \; dx_1 \\ &= \frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}} \int_{x_1=0}^{h} (x_1 - h) x_1^{n-1} \; dx_1 \\ &= \frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}} \Big(\frac{h^{n+1}}{n} - \frac{h^{n+1}}{n+1}\Big) \\ &= \operatorname{vol}_{n-1}(C_0) h^2 \frac{1}{n(n+1)}. \end{aligned}$$

Thus the x_1-coordinate of the centroid is given by

$$c_1(C) = \frac{\operatorname{vol}_{n-1}(C_0) h^2 n}{\operatorname{vol}_{n-1}(C_0) h n (n+1)} = \frac{h}{n+1},$$

where we used the volume formula for cones.

For the other coordinates $2 \le i \le n$ of the centroid, we have

$$\int_C x_i \; d(x_1, \dots, x_n) = 0,$$

since C is symmetric w.r.t. the hyperplanes $H(e_i, 0)$. This implies the representation of the centroid of C as stated above. □

Theorem 11. *Let $C \subseteq \mathbb{R}^n$ be the cone with basis $\{0\} \times B_{n-1}$ and vertex $(h, 0, \dots, 0) \in \mathbb{R}^n$. Further let*

$$C^- := C \cap H^-(e_1, c_1(C))$$

be the portion of C that lies in the negative half-space.

Then the ratio of the volumes of C^- and C is given by

$$\frac{\operatorname{vol}(C^-)}{\operatorname{vol}(C)} = \Big(\frac{n}{n+1}\Big)^n.$$

Proof. Let $a := c_1(C) = \frac{h}{n+1}$. Then we see that the assertion holds by a simple calculation:

$$\begin{aligned}\operatorname{vol}(C^-) &= \int_{x_1=0}^{a} \int_{C_{x_1}} d(x_2,\dots,x_n)\, dx_1 \\ &= \frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}} \int_{x_1=0}^{a} (h-x_1)^{n-1}\, dx_1 \\ &= \frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}} \int_{x_1=h-a}^{h} x_1^{n-1}\, dx_1 \\ &= \frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}\cdot n}(h^n-(h-a)^n) \\ &= \frac{1}{n}\frac{\operatorname{vol}_{n-1}(C_0)}{h^{n-1}}(h^n-(h-\frac{h}{n+1})^n) \\ &= \frac{1}{n} h \operatorname{vol}_{n-1}(C_0)\Big(\frac{n}{n+1}\Big)^n.\end{aligned}$$

With the volume formula for cones (cf. Corollary 9)

$$\operatorname{vol}(C) = \frac{1}{n} h \operatorname{vol}_{n-1}(C_0),$$

our calculation yields

$$\operatorname{vol}(C^-) = \operatorname{vol}(C)\cdot\Big(\frac{n}{n+1}\Big)^n.$$

□

2.2. **The general case.** Before we state Grünbaum's inequality and prove it, we construct the spherical symmetrization of a convex body $K \in \mathcal{K}^n$ along the x_1-axis. This has the following reason: The first step in Grünbaum's inequality is to replace a convex body K by its spherical symmetrization $\hat{K}$ that has the same volume ratios as K, since spherical symmetrization will not alter the centroid if it lies on the x_1-axis.

The idea of spherical symmetrization is as follows: Given a convex body $K \in \mathcal{K}^n$, we replace each section along the x_1-axis, i.e.

$$K_t = K \cap H(e_1, t) = \{x \in K \mid x_1 = t\}$$

for $t \in \mathbb{R}$, by an $(n-1)$-dimensional disc $\hat{K}_t$ scaled such that

$$\operatorname{vol}_{n-1}(\hat{K}_t) = \operatorname{vol}_{n-1}(K_t).$$

An example of a spherical symmetrization of a convex body is depicted in Figure 2.

Theorem 12 (Spherical symmetrization)**.** *Let $K \in \mathcal{K}^n$ and let denote*

$$K_t := K \cap H(e_1, t) = K \cap \{x \in \mathbb{R}^n \mid x_1 = t\}$$

the slice of K along the x_1-axis at $x_1 = t$, for $t \in \mathbb{R}$. Further define $L = \min_{x\in K} x_1$ and $H = \max_{x\in K} x_1$.

Then

$$\hat{K} := \bigcup_{L\le t\le H} \{t\} \times (\omega_{n-1}^{1/(n-1)} \cdot \operatorname{vol}_{n-1}^{1/(n-1)}(K_t)^{1/(n-1)} \cdot B^{n-1})$$

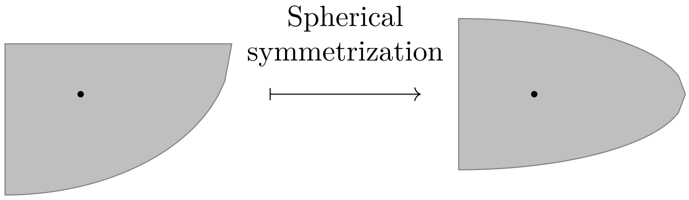

FIGURE 2. The process of spherical symmetrization of a convex body K with marked centroids. Note that K and its spherical symmetrization $\hat{K}$ have the same volume.

is a convex body and has the same volume as K, i.e.

$$\operatorname{vol}(\hat{K}) = \operatorname{vol}(K).$$

The set $\hat{K}$ is called the spherical symmetrization *(also called* Schwarz symmetrization *in the literature, cf. [BF34, p. 73]).*

Proof. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \hat{K} \subset \mathbb{R}^n$ and $\lambda \in [0,1]$. By definition of $\hat{K}$, it hold $x_1, y_1 \in [L, H]$ and

$$|(x_2, \ldots, x_n)| \leq \operatorname{vol}_{n-1}^{1/(n-1)}(K_{x_1}).$$

We want to show that

$$(1-\lambda)x + \lambda y \in \hat{K},$$

or equivalently

$$(1-\lambda)x_1 + \lambda y_1 \in [L, H]$$

and

$$|(1-\lambda)(x_2, \ldots, x_n) + \lambda(y_2, \ldots, y_n)| \leq \operatorname{vol}_{n-1}^{1/(n-1)}(K_{(1-\lambda)x_1+\lambda y_1}).$$

Define the function

$$\begin{aligned} \rho\colon [L, H] &\longrightarrow \mathbb{R}, \\ t &\longmapsto \operatorname{vol}_{n-1}^{1/(n-1)}(\hat{K}_t). \end{aligned}$$

Now we have

$$\begin{aligned} |(1-\lambda)(x_2, \ldots, x_n) + \lambda(y_2, \ldots, y_n)| &\leq (1-\lambda)|(x_2, \ldots, x_n)| + \lambda|(y_2, \ldots, y_n)| \\ &\leq (1-\lambda)\rho(x_1) + \lambda\rho(y_1), \end{aligned}$$

since $(x_2, \ldots, x_n) \in \operatorname{vol}_{n-1}^{1/(n-1)}(K_{x_1}) \cdot B_{n-1}$ and $(y_2, \ldots, y_n) \in \operatorname{vol}_{n-1}^{1/(n-1)}(K_{y_1}) \cdot B_{n-1}$ by definition of the spherical symmetrization $\hat{K}$.

Moreover, $\rho\colon [L, H] \to \mathbb{R}$ is a concave function by the functional version of Brunn–Minkowski inequality (Theorem 7). Thus

$$\begin{aligned} |(1-\lambda)(x_2, \ldots, x_n) + \lambda(y_2, \ldots, y_n)| &\leq (1-\lambda)\rho(x_1) + \lambda\rho(y_1) \\ &= \rho((1-\lambda)x_1 + \lambda y_1), \end{aligned}$$

which is, together with the trivial fact that

$$(1-\lambda)x_1 + \lambda y_1 \in [L, H],$$

equivalent to $(1-\lambda)x + \lambda y \in \hat{K}$. Hence $\hat{K}$ is convex.

The volume identity follows from a simple application of Fubini's theorem:

$$\begin{aligned}
\operatorname{vol}(\hat{K}) &= \int_{\hat{K}} 1 \; dx \\
&= \int_{x_1=L}^{H} \operatorname{vol}_{n-1}(\hat{K}_{x_1}) \; dx_1 \\
&= \int_{x_1=L}^{H} \operatorname{vol}_{n-1}(\omega_{n-1}^{1/(n-1)} \operatorname{vol}_{n-1}^{1/(n-1)}(K_{x_1}) B^{n-1}) \; dx_1 \\
&= \int_{x_1=L}^{H} \operatorname{vol}_{n-1}(K_{x_1}) \; dx_1 = \operatorname{vol}(K).
\end{aligned}$$

□

Remark 13. Let $K \in \mathcal{K}^n$ be a convex body such that $c(K) = 0$. Consider its spherical symmetrization

$$\hat{K} = \bigcup_{L \le t \le H} \{t\} \times \operatorname{vol}_{n-1}^{1/(n-1)}(K_t) \cdot B_{n-1}.$$

Now

$$\begin{aligned}
\operatorname{vol}(\hat{K}) \cdot c(\hat{K}) &= \int_{\hat{K}} x \; dx \\
&= \int_{x_1=L}^{H} \int_{\hat{K}_{x_1}} x \; d(x_2, \ldots, x_n) \; dx_1
\end{aligned}$$

by Fubini's theorem. Thus

$$\begin{aligned}
\operatorname{vol}(\hat{K}) \cdot c_1(\hat{K}) &= \int_{x_1=L}^{H} \int_{\hat{K}_{x_1}} x_1 \; d(x_2, \ldots, x_n) \; dx_1 \\
&= \int_{x_1=L}^{H} x_1 \operatorname{vol}_{n-1}(K_{x_1}) = \operatorname{vol}(K) \cdot c_1(K),
\end{aligned}$$

since $\operatorname{vol}_{n-1}(K_{x_1}) = \operatorname{vol}_{n-1}(\hat{K}_{x_1})$ for $L \le x_1 \le H$ by construction of $\hat{K}$.

Furthermore $\hat{K}$ is symmetric w.r.t. the hyperplanes $H(e_2, 0), \ldots, H(e_n, 0)$, since $\{t\} \times B_{n-1}$ is invariant under unitary transformations with eigenvector e_1 for all $t \in [L, H]$. Therefore

$$\operatorname{vol}(\hat{K}) \cdot c_i(\hat{K}) = 0 = \operatorname{vol}(K) \cdot c_i(K)$$

for $i \in \{2, \ldots, n\}$.

Theorem 14 (Grünbaum's inequality, [Grü60, Theorem 2])**.** *Let $K \subseteq \mathbb{R}^n$ be a convex body and let denote*

$$x = \int_K y \; dy$$

the centroid of K. For any hyperplane H through x, the portion of K contained in the negative half-space H^-, namely $K \cap H^-$, satisfies the inequality

$$\frac{\operatorname{vol}(K \cap H^-)}{\operatorname{vol}(K)} \ge \left(\frac{n}{n+1}\right)^n.$$

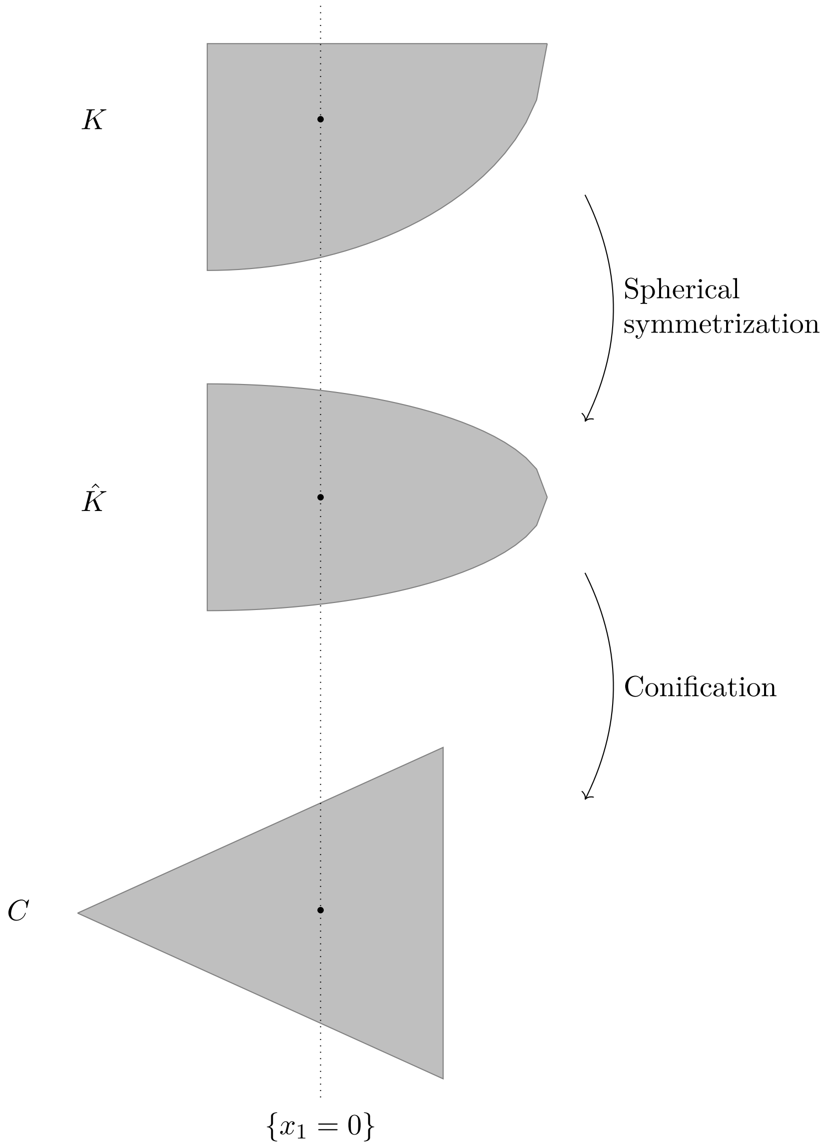

FIGURE 3. A sketch of the process of substitutions for a 2-dimensional convex body K in the proof of Theorem 14. The first step is a spherical symmetrization (cf. Theorem 12) that replaces K by $\hat{K}$ that shares its volume ratios and centroid with K. In the second step, called *conification*, we replace $\hat{K}$ by a cone with its basis in the positive half-space and vertex in the negative half-space. We note that this conification shifts the centroid, previously located at the origin, towards the positive half-space. This fact, together with prior knowledge about centroids of cones, yields the inequality to be proven.

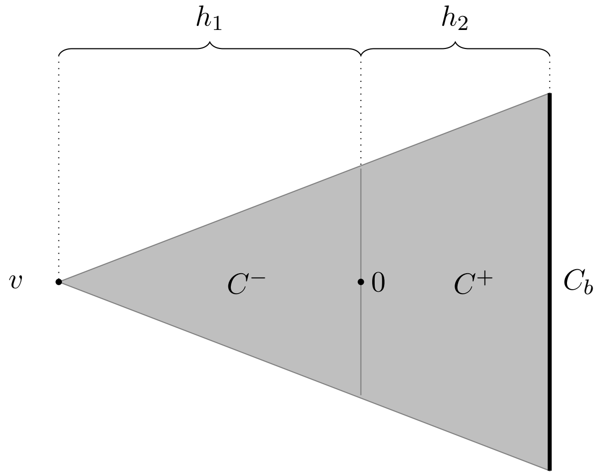

FIGURE 4. Desired situation for our cone with vertex $v \in H^-$ and basis $C_b \in H^+$ in the proof of Theorem 14.

Proof. Without loss of generality, we suppose that

$$H = H(e_1, 0) = \{y \in \mathbb{R}^n \mid y_1 = 0\}$$

and $c(K) = x = 0$. Further, for an arbitrary convex body $K \in \mathcal{K}^n$, we write

$$K^- = \{x \in K \mid x_1 \leq 0\}, K^+ = \{x \in K \mid x_1 \geq 0\}.$$

We consider the spherical symmetrization $\hat{K}$ of K, cf. Theorem 12. It holds

$$\frac{\mathrm{vol}(\hat{K} \cap H^-)}{\mathrm{vol}(\hat{K})} = \frac{\mathrm{vol}(K \cap H^-)}{\mathrm{vol}(K)},$$

therefore it is sufficient to show the result for spherical symmetrizations.

In the following step we construct a cone $C \subset \mathbb{R}^n$ with the same volume ratios as $\hat{K}$ relative to the hyperplane H. Our constructed cone C shall have vertex $(-h_1, 0, \ldots, 0) \in C^-$ and a basis $C_b := C_{h_2} \subset C^+$. Further it shall satisfy $\mathrm{vol}_{n-1}(C_0) = \mathrm{vol}_{n-1}(\hat{K}_0)$. This involves solving the following system of equations, where $h_1, h_2 > 0$ are unknown:

$$\frac{1}{n} h_1 \, \mathrm{vol}_{n-1}(C_0) = \mathrm{vol}(\hat{K}^-),$$
$$\frac{1}{n}(h_1 + h_2) \, \mathrm{vol}_{n-1}(C_b) = \mathrm{vol}(\hat{K}).$$

That desired situation is depicted in Figure 4. As

$$\mathrm{vol}_{n-1}(C_0) = \mathrm{vol}_{n-1}\left(\frac{h_1}{h_1 + h_2} C_b\right),$$

the second equation is equivalent to

$$\frac{1}{n}(h_1 + h_2)\left(\frac{h_1 + h_2}{h_1}\right)^{n-1} \mathrm{vol}_{n-1}(C_0) = \mathrm{vol}(K).$$

Therefore, a solution for that problem is given by

$$h_1 = \frac{n}{\mathrm{vol}_{n-1}(K_0)} \mathrm{vol}(K^-)$$
$$h_2 = \frac{n}{\mathrm{vol}_{n-1}(K_0)} \big((\mathrm{vol}(K)\, \mathrm{vol}^{n-1}(K^-))^{1/n} - \mathrm{vol}(K^-) \big)$$

Our aim is to show that the centroid of C is pushed towards the negative half-space H^- by conification. Then we have

$$c_1(C) = h_2 - \frac{h_1 + h_2}{n+1} \leq 0,$$

which is equivalent to the desired inequality.

For that step, we first note that, for

$$L = |\min_{x \in \hat{K}} x_1| > 0, \quad H = \max_{x \in \hat{K}} x_1 > 0,$$

we have $h_1 > L$ and $h_2 < H$. Now we decompose $\hat{K}^+$ and $\hat{K}^-$ by C and $\mathbb{R}^n \setminus C$: Let

$$\hat{K}_1^+ = \hat{K}^+ \cap C, \qquad \hat{K}_2^+ = \hat{K}^+ \cap (\mathbb{R}^n \setminus C),$$
$$\hat{K}_1^- = \hat{K}^- \cap C, \qquad \hat{K}_2^- = \hat{K}^- \cap (\mathbb{R}^n \setminus C).$$

Now the centroids of $\hat{K}^+$ and $\hat{K}^-$ are given by

$$\mathrm{vol}(\hat{K}^+) \cdot c_1(\hat{K}^+) = \mathrm{vol}(\hat{K}_1^+) \cdot c_1(\hat{K}_1^+) + \mathrm{vol}(\hat{K}_2^+) \cdot c_1(\hat{K}_2^+) \text{ and}$$
$$\mathrm{vol}(\hat{K}^-) \cdot c_1(\hat{K}^-) = \mathrm{vol}(\hat{K}_1^-) \cdot c_1(\hat{K}_1^-) - \mathrm{vol}(\hat{K}_2^-) \cdot c_1(\hat{K}_2^-)$$

by Lemma 3. Since conification retains the masses of $\hat{K}_1^+$ and $\hat{K}_1^-$ but shifts the masses of $\hat{K}_2^+$ and $\hat{K}_2^+$ towards $-\infty$ along the x_1-axis, the masses of C^- and C^+ are shifted towards $-\infty$ along the x_1-axis. As the centroid of $\hat{K}$ was at the origin $0 \in \mathbb{R}^n$, the centroid of C can only move to the negative half-space $H^- = \{x_1 \leq 0\}$.

Thus

$$c_1(C) = h_2 - \frac{h_1 + h_2}{n+1} \leq 0.$$

By substituting h_1 and h_2 by their respective solutions, we see that this inequality is equivalent to

$$\frac{n}{n+1} \mathrm{vol}^{1/n}(K)\, \mathrm{vol}^{(n-1)/n}(K^-) \leq \mathrm{vol}(K^-),$$

which in turn is equivalent to

$$\left(\frac{n}{n+1}\right)^n \leq \frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}.$$

□

Remark 15. By symmetry in the proof of Theorem 14, we get that

$$\left(\frac{n}{n+1}\right)^n \leq \frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}.$$

With $\mathrm{vol}(K) = \mathrm{vol}(K^+) + \mathrm{vol}(K^-)$ we obtain upper bounds for the volume ratios:

$$1 - \left(\frac{n}{n+1}\right)^n \geq \frac{\mathrm{vol}(K^\pm)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n.$$

A direct consequence of our preliminary considerations is that cones satisfy equality in Grünbaum's inequality:

Corollary 16. *The bound in Grünbaum's inequality (Theorem 14) is sharp, i.e. there is a convex body $K \subseteq \mathbb{R}^n$ with centroid x satisfying*

$$\frac{\operatorname{vol}(K \cap H^-)}{\operatorname{vol}(K)} = \left(\frac{n}{n+1}\right)^n$$

for a hyperplane H through x.

Proof. Such a convex body is given by a cone, cf. Theorem 11. □

It is important to note that the result of the previous corollary depends heavily on the choice of the hyperplane H given a convex body K. Let

$$H = \{x \in \mathbb{R}^n \mid x_1 = 0\}$$

and $C \subset \mathbb{R}^n$ a cone with circular basis parallel to H and centroid $c(C) = 0 \in \mathbb{R}^n$. Then by Theorem 11 the hyperplane H is suitable to obtain the worst ratio.

Consider the hyperplane

$$\tilde{H} = \{x \in \mathbb{R}^n \mid x_2 = 0\}$$

that is orthogonal to H. Now the cone C is axial symmetric w.r.t. $\tilde{H}$ and therefore

$$\frac{\operatorname{vol}(C \cap \tilde{H}^-)}{\operatorname{vol}(C)} = \frac{1}{2}.$$

Thus the hyperplane $\tilde{H}$ is suitable to obtain the best ratio, contrary to the hyperplane H.

Therefore the attained volume ratio depends heavily on the choice of the hyperplane H that is used to divide a convex body $K \subset \mathbb{R}^n$ into two portions.

Furthermore, we are able to state lower and upper bounds in Grünbaum's inequality that are independent of the dimension $n \in \mathbb{N}$ by the definition of Euler's number.

Corollary 17. *Let $K \in \mathcal{K}^n$ be a convex body and H be some hyperplane through the centroid $c(K) \in \mathbb{R}^n$ of K. Then*

$$\frac{2}{3} > \frac{e-1}{e} > \frac{\operatorname{vol}(K \cap H^-)}{\operatorname{vol}(K)} > \frac{1}{e} > \frac{1}{3}.$$

Proof. Consider the sequence defined by

$$a_n := \left(\frac{n+1}{n}\right)^n.$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing and positive, i.e. $a_{n+1} > a_n > 0$ for all $n \in \mathbb{N}$. Thus

$$e > a_n$$

for all $n \in \mathbb{N}$, where $e \in \mathbb{R}$ is Euler's number.

Hence

$$\frac{\operatorname{vol}(K \cap H^-)}{\operatorname{vol}(K)} \geq \left(\frac{n+1}{n}\right)^n = \frac{1}{a_n} > \frac{1}{e} > \frac{1}{3}.$$

The upper bound follows from considering the positive half-space H^+ and noting that

$$\mathrm{vol}(K) = \mathrm{vol}(K \cap H^+) + \mathrm{vol}(K \cap H^-).$$

□

References

[BF34] T. Bonnesen and W. Fenchel. *Theorie der konvexen Körper.* Vol. 3. 1. Springer, Berlin, 1934.

[Grü60] Branko Grünbaum. "Partitions of mass-distributions and of convex bodies by hyperplanes." In: *Pac. J. Math.* 10 (1960), pp. 1257–1261.

[Hal74] Paul R. Halmos. *Measure theory. 2nd printing.* English. Vol. 18. Springer, 1974.

[Sch14] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory.* 2nd expanded edition. Vol. 151. Cambridge: Cambridge University Press, 2014.

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